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THE TRANSFORMATION OF EXTERNAL HARMONIC SERIES UNDER A TRANSLATION OF ORIGIN

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ABSTRACT

The behavior of inverse-power spherical harmonic expansions is presented when the origin of the coordinate system undergoes a translation. A seeming paradox, involving large translations is discussed.

THE TRANSFORMATION OF EXTERNAL
HARMONIC SERIES UNDER A TRANSLATION
OF ORIGIN*

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1. Introduction

The object of this report is to present an elementary derivation of the behavior of external spherical harmonics of the form

$$\frac{P_n^m(\cos\theta)e^{im\phi}}{r^{n+1}} \tag{1}$$

when the origin of the coordinate system defining the spherical coordinates r,θ,ϕ undergoes a general translation. The corresponding problem for spherical harmonics of positive degree has been solved by Aardoom [1], as has the rotational problem, where the spherical coordinate system undergoes a rotation. The rotational problem had been previously solved by many authors and a most complete presentation is found in Courant and Hilbert [2].

The spherical harmonics of the form (1) are of particular interest in the fields of satellite theory and geodesy which are concerned with the gravitational potential external to the body which generates it. A common potential

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representation in these fields is

$$V(r,\theta,\phi) = \frac{\mu}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} P_n^m(\cos\theta)\{C_{nm}\cos m\phi + S_{nm}\sin m\phi\}, \tag{2}$$

where μ and a are the gravitational constant for the generating body and a characteristic radius for the body, respectively. The quantities r,θ,ϕ are the spherical coordinates of the field point in a reference system. The angle θ is the co-latitude and ϕ is the east longitude. The constants C_{nm} and S_{nm} characterize the gravitational field of the body and its deviation from a point source field.

The result of this report allows the series (2), usually expressed in a coordinate system located at the center of mass of the generating body, to be recast into a form which is valid in a coordinate system whose origin may be more conveniently located. An example of this procedure is given by Lee [6] where the translations described are used to compute the gravitational force between two bodies where neither body can be considered as a point mass.

It is conceivable that the lunar potential can be expressed in a coordinate system located at the center of mass of the earth and thus facilitate, in some cases, the earth satellite problem.

The method to be described is a direct generalization of one due to Hobson [4], who has considered the special case of translations along the z-axis.

2. The Harmonic Representation

It is possible to put the potential expression (2) into a form in which terms such as (1) appear. To do this we introduce complex coefficients

$$\alpha_{nm} = C_{nm} - iS_{nm}, \tag{3}$$

and (2) becomes

$$V = \text{Re}\ \mu \sum_{n=0}^{\infty} \sum_{m=0}^{n} a^n \alpha_{nm} V_{nm}(r,\theta,\phi), \tag{4}$$

where

$$V_{nm}(r,\theta,\phi) = \frac{P_n^m(\cos\ \theta)e^{im\phi}}{r^{n+1}} . \tag{5}$$

This form of the series eliminates separate consideration of the sin $m\phi$ and cos $m\phi$ cases.

A simple examination of the functions V_{nm}, especially when one considers that

$$P_n^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \tag{6}$$

with $P_n(x)$ the Legendre polynomial, shows that any direct approach to the translation problem with a substitution of translated quantities for r, θ, and ϕ leads to an extremely complex expression. It is therefore desirable to find an alternate representation for $V_{nm}(r,\theta,\phi)$ which allows translations to be applied. Such a representation, due to Maxwell, can be found in Hobson [4] or Cunningham [3].

If the notation of Cunningham is modified to agree with (6),

$$V_{nm} = \frac{(-1)^{n-m}}{(n-m)!} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z}\right)^{n-m} \left(\frac{1}{|\underline{r}|}\right), \qquad (7)$$

with the x,y,z derivatives taken in the reference coordinate system, where $\underline{r}$ is the position vector of the field point in this system. It will be shown in the following paragraphs that this representation for the spherical harmonics is useful for the translation problem and leads to tractable expressions.

3. Description of the Translation

In addition to the spherical harmonic representation, we must specify the translation of the coordinate system. This will be used to transform the scalar $|\underline{r}|$ and the derivatives in (7). The translation is described in Figure 1,

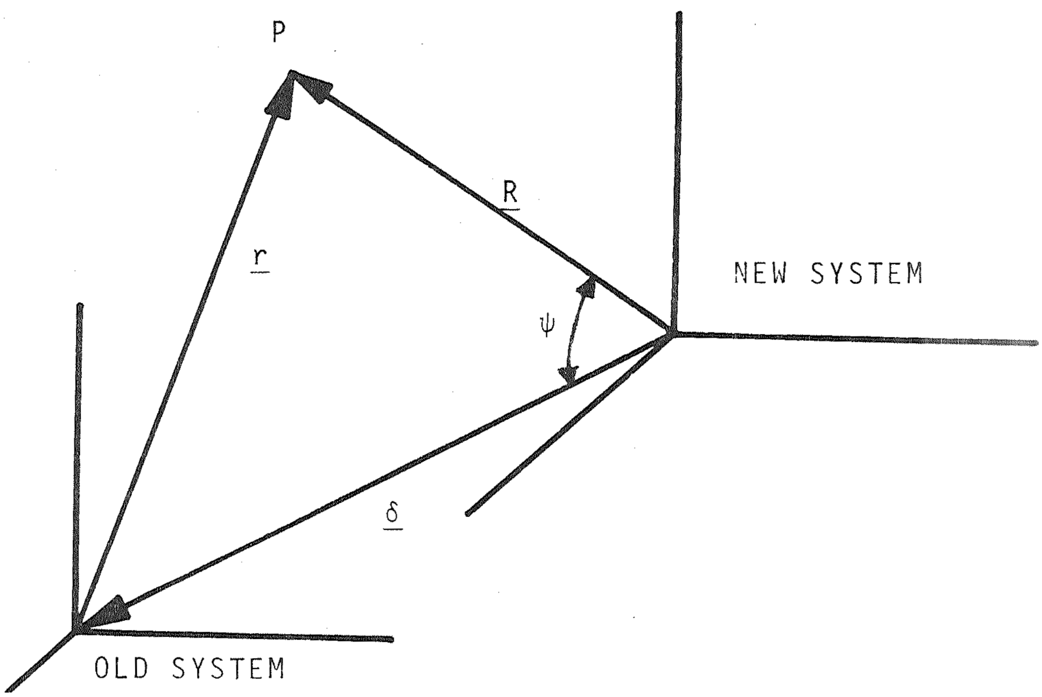

Figure 1. Description of the Translation

The vector $\underline{\delta}$ gives the position of the origin of the old system with respect to the new. A typical field point, P, is described by the vectors $\underline{r}$ and $\underline{R}$ in the old and new systems respectively. In what follows we will associate the spherical coordinates in the new system, R, θ', ϕ' with $\underline{R}$ and δ,θ,ϕ with $\underline{\delta}$ where θ and θ' are colatitudes.

At this point, all the necessary notation has been introduced with the exception of the positive degree spherical harmonic corresponding to (5)

$$H_{k\ell} = r^k P_k^\ell(\cos\ \theta) e^{im\ell}. \tag{8}$$

This will be of use later.

4. The Transformation Process

In order to use only quantities which are defined with respect to the new system, we first examine the quantity

$$\frac{1}{|\underline{r}|} = \frac{1}{|\underline{R}-\underline{\delta}|} \quad .$$

If ψ is the angle between $\underline{R}$ and $\underline{\delta}$, then according to the generating function for Legendre polynomials,

$$\frac{1}{|\underline{R}-\underline{\delta}|} = \begin{cases} \dfrac{1}{R} \displaystyle\sum_{n=0}^{\infty} \left(\frac{\delta}{R}\right)^n P_n(\cos\ \psi) & R > \delta \\[2em] \dfrac{1}{\delta} \displaystyle\sum_{n=0}^{\infty} \left(\frac{R}{\delta}\right)^n P_n(\cos\ \psi) & R < \delta \end{cases}$$

If we use the addition theorem for spherical harmonics, Jackson [5], and some elementary manipulations, then

$$\frac{1}{|\underline{R}-\underline{\delta}|} = \begin{cases} \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{\delta}{R}\right)^{n} \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\theta) P_n^m(\theta') e^{im(\phi-\phi')}, & R>\delta \\ R & \\ \frac{1}{\delta} \sum_{n=0}^{\infty} \left(\frac{R}{\delta}\right)^{n} \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\theta) P_n^m(\theta') e^{im(\phi-\phi')}, & R<\delta \end{cases}$$

where $P_n^m(\theta) = P_n^m(\cos\theta)$, for the sake of brevity. This depends also on the definition of the associated Legendre polynomials for negative order, MacRobert [7],

$$P_n^{-m}(x) = \frac{(n-m)!}{(n+m)!} P_n^m(x) \quad , \quad m > 0 .$$

If the definitions (5) and (8) are used, we find the result.

$$\frac{1}{|\underline{R}-\underline{\delta}|} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} \begin{cases} H^*_{nm}(\underline{\delta}) V_{nm}(\underline{R}) & R>\delta \\ H^*_{nm}(\underline{R}) V_{nm}(\underline{\delta}) & \delta > R . \end{cases} \tag{9}$$

The asterisk superscript denotes the complex conjugate.

It is at this point that the convenience of the representation (7) becomes more apparent. With the result given in (9), the term $1/|\underline{r}|$ in (7) has been dealt with. All that remains is the transformation of derivatives. From the definition of the transformation,

$$\underline{r} = \underline{R} - \underline{\delta} \,, \tag{10}$$

and since the coordinate axes remain parallel, it is a simple process to transform derivatives from those with respect to the old coordinates, x,y,z to those with respect to the new quantities, components of $\underline{R}$ or $\underline{\delta}$.

Consider first the case where $R > \delta$. Symbolically, from (10),

$$\frac{\partial}{\partial \underline{r}} = \frac{\partial}{\partial \underline{R}} \,,$$

and if we look at (7) the derivatives of interest are

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^{m} \left(\frac{\partial}{\partial z}\right)^{n-m} = \left(\frac{\partial}{\partial X} + i\frac{\partial}{\partial Y}\right)^{m} \left(\frac{\partial}{\partial Z}\right)^{k-m} \,,$$

where the derivatives with respect to capital letters refer to derivatives with respect to components of $\underline{R}$. We can thus use this result, and the first part of equation (9), to find

$$V_{k\ell}(\underline{r}) = \frac{(-1)^{k-\ell}}{(k-\ell)!} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} H^{*}_{nm}(\underline{\delta}) \left(\frac{\partial}{\partial X} + i\frac{\partial}{\partial Y}\right)^{\ell} \left(\frac{\partial}{\partial Z}\right)^{k-\ell} V_{nm}(\underline{R}) \,.$$

The derivatives acting on $V_{nm}(\underline{R})$ can be simplified with the additional use of (7), and finally

$$V_{k\ell}(\underline{r}) = \frac{1}{(k-\ell)!} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n+k-m-\ell)!}{(n+m)!} H^{*}_{nm}(\underline{\delta}) V_{n+k,m+\ell}(\underline{R}) \,. \tag{11}$$

This result holds for $R > \delta$.

Similarly, if $R < \delta$, we find from (10) that

$$\frac{\partial}{\partial \underline{r}} = - \frac{\partial}{\partial \underline{\delta}} .$$

If an analogous procedure is followed, we find

$$V_{k\ell}(\underline{r}) = \frac{(-1)^k}{(k-\ell)!} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n+k-m-\ell)!}{(n+m)!} H^*_{nm}(\underline{R}) V^{m+\ell}_{n+k}(\underline{\delta}) . \tag{12}$$

This holds if $R < \delta$.

5. Discussion

If the expressions (11) and (12) are examined, then we see that the character of an inverse harmonic series such as (2) may change if the translation distance is large enough. The series may change from one in inverse powers to one in positive powers. However, as was pointed out by Lee (private communication), this is a seeming paradox. If we take a single large ($R < \delta$) translation, then the series changes from one in negative powers to one in positive powers. But if we consider a finite sequence of small translations, each of which keeps $R > \delta$, we may obtain an expansion with the same origin but with an inverse power expansion. The explanation for this can be found in Figure 2. There are in certain circumstances, two series which are meaningful, and a choice must be made between them depending on the circumstances under consideration. In part A of Figure 2 we see the usual situation where a spherical harmonic expansion converges outside the smallest sphere, centered at the center of mass, containing a planet. In part B, we see the case where the origin has been shifted through a distance which is smaller than the radius of the sphere of convergence. In this case there is still only one series,

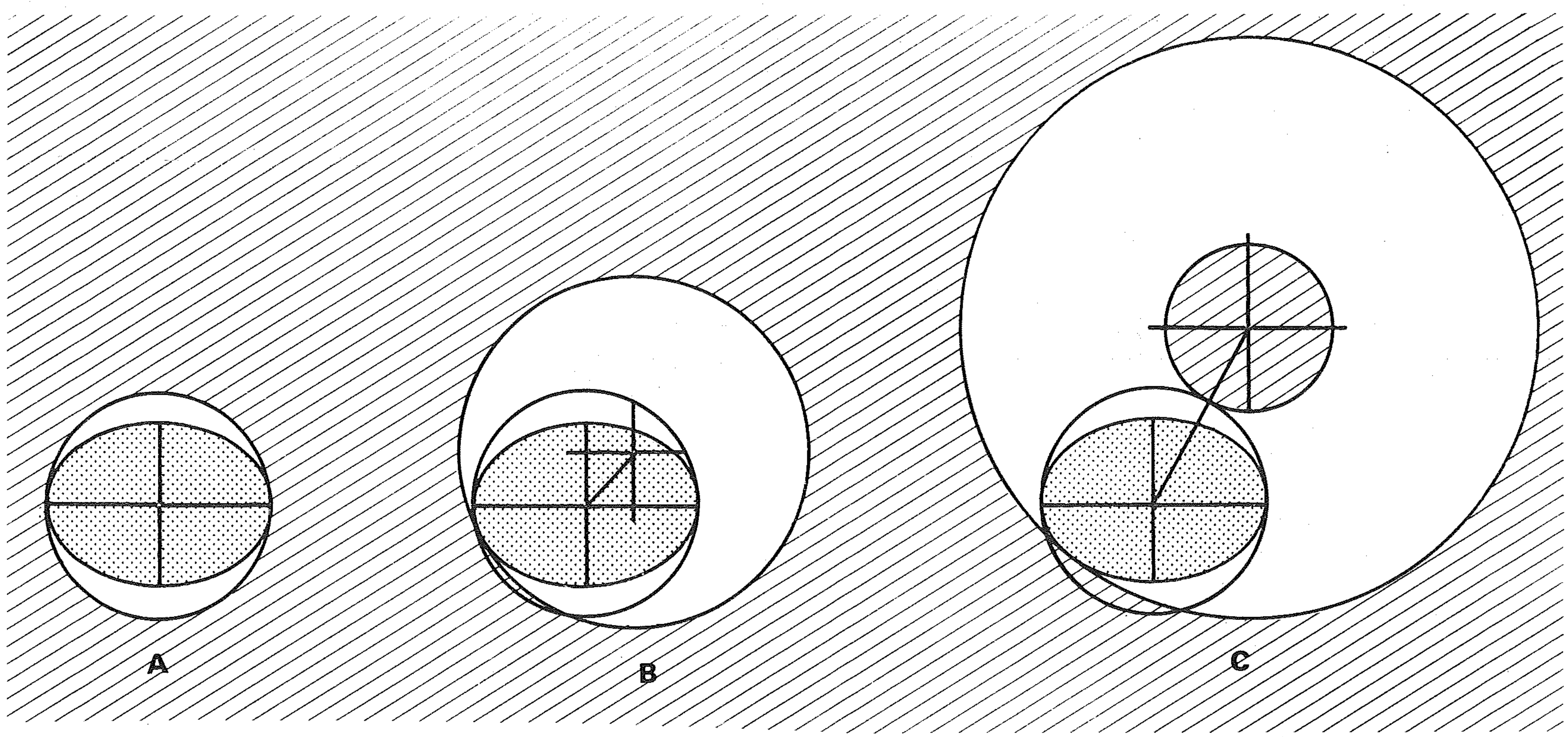

Figure 2. Spheres of Convergence

which is applicable. If, however, the origin is shifted through a distance which is greater than the radius of the original sphere of convergence, there are two applicable series. One series converges outside the smallest sphere centered at the new origin which contains the planet and the other converges inside the sphere about the new origin which is tangent to the original sphere of convergence. Whether this second series converges down to the surface of the planet seems to be an open question at the moment in spite of a discussion due to Moritz [8].

The analytical expressions, (11) and (12), can be substituted into the series (2) and the summations interchanged so that a new spherical harmonic expansion, referred to a new origin, is obtained. This interchange of summations is, however, valid only in those regions where convergence of the original series is guaranteed. In any practical case, the numerical behavior of the summations involving the original coefficients should be investigated to insure that measurement errors are not magnified to the point where the results are insignificant.

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