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TECHNICAL MEMORANDUM

CODED SCANNING OF OPTICAL IMAGES

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ABSTRACT

Scanning optical systems such as T.V. in effect make one measurement of light intensity for each resolution element in the optical scene. When the incident light levels are low or the resolution desired very high, the energy incident on one resolution element is very low and system performance is limited by the noise level of the detector element. For these conditions a system is suggested here which makes the same number of measurements per line but each is across the entire scan line through a coded mask. When decoded the same resolution is achieved with a significantly improved signal-to-noise ratio. The theory, based upon Hadamard transforms, is developed and it is shown that the improvement in signal-to-noise ratio over single resolution element scanning is $N^{\frac{1}{2}}$ where N is the number of resolution elements per line. Difference equations are given for computing high resolution masks with desirable cyclic properties for simplified implementation.

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Introduction

Optical systems which use a device other than a photographic emulsion as an energy detector commonly scan the image as in television. Under conditions of low ambient lighting and/or high desired resolution the energy incident on a single resolution element is very low and the limiting parameter is the noise at the sensor element itself. Infrared systems, in particular are most often so limited. In such a system, each scan line is sampled and encoded, say N times per line, then these N samples can be considered to be N distinct measurements made through an aperture whose width is L/N where L is the line width. For dispersive spectrometers it has been shown that a significant improvement in signal-to-noise ratio can be achieved by making the N measurements through N different coded masks across the entire scan line in the exit pupil of the instrument.[1][2][3] These ideas are applied below in a system concept for a scanning infrared camera. Since the improvement in signal-to-noise ratio is obtained at the expense of computation time in decoding, the maximum advantage is obtained in a system in which a computer is inherently a part. The ten-band multispectral scanner under development for Skylab produces 8-bit digital data from a circular scan which will have to be reproduced by a computer-like device. It would thus appear typical of the type of device to which coded scanning would be applicable.

Construction and Basic Theory

The basic optical structure will be assumed to be a Schmidt-Cassegrain[4] as shown in Fig. 1. A narrow horizontal slot aperture which defines the scan line is placed on the focal plane. Vertical scanning is accomplished by either the motion of the satellite or aircraft or motion of the entire slot as in a focal plane shutter. For simplicity, we assume the former. The coded mask consisting of a pattern of vertical slots cut into a completely reflecting material is placed at an angle of 45° directly behind the horizontal slot. A typical pattern for the mask is shown in Fig. 2b, but the mathematical theory behind the mask will be covered later. Exactly half of the mask is reflecting so that approximately half of the radiation incident on the horizontal slot is transmitted through the mask to a set of collimating optics which condenses all of the transmitted light onto the detecting element. The other half of the radiation incident upon the mask is reflected and focused upon a second detector as shown in Fig. 1. The output of each detector, with suitable buffering goes to both a sum amplifier and a difference amplifier. The sum amplifier is used for AGC and another purpose to be covered later. The difference amplifier output is digitally encoded.

In the usual raster-like scan the horizontal slot would be sampled say N times through a single aperture as shown in Fig. 2b. In other words, N separate measurements would be made, and as N increases, the energy available to the detector decreases with the area of the aperture. Since we are only interested in the comparison of two different methods of scanning across the horizontal slot, the width of the slot will be assumed to be fixed. Thus, in this case the area of the single slot aperture and the energy decreases as 1/N. To obtain the same resolution, using coded masks again, at least N measurements must be made through N different masks. However, the radiant energy incident on a detector in a single measurement through a mask is a weighted average of the total energy across the slot and is independent of N and is approximately $\frac{T}{2}$ for each detector. Hence, a significant improvement in signal-to-noise ratio can be achieved. By utilizing the cyclic properties of the Hadamard matrices below, a single mask can be constructed so that each measurement is made after shifting the mask one indentation from the preceding measurement. The computer decoding involves matrix multiplication, but because the elements of the matrix are either +1 or -1 it can be accomplished by using only addition and subtraction.

The pattern on each mask corresponds to the rows of a Hadamard matrix. A N×N Hadamard matrix is a matrix whose elements are either + or -1 with the property:

$$H^T H = HH^T = NI \qquad (1)$$

Furthermore, it is shown in the appendix that if $N=2^n$ then Hadamard matrices, H, can be constructed from a matrix, H^-, whose rows (and columns) are cyclic permutations of one of the rows (columns). (It is this property, of course, which allows simple mask interchange.) The mask is divided into $N=2^n$ equal areas across the length of the horizontal slot and each area is labeled to correspond to a column of the Hadamard matrix. Take any row i of the matrix except the first which consists of all +1's. If the $j^{\underline{th}}$ component of the row vector i is -1, the $j^{\underline{th}}$ area of the mask is made transmitting. If it is +1, it is made reflecting. A typical Hadamard matrix for N=8 is given below as an illustration.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \qquad (2)$$

Note that the first row and column are all 1's. Consider the matrix H^- below formed by deleting the first row and column of H.

$$H^- = \begin{bmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \qquad (3)$$

Each row is a cyclic permutation of the row preceding it. Each row of H^- can be made into a row of H by adding a +1 as the first component. The mask can now be made from H^- by repeating the first row of H^- twice. The first +1 component is added by a fixed single area reflector as shown in Fig. 2c. For each measurement the mask needs to be moved only one slit position from the preceding position because of the cyclic structure of the code. Since the first row of H is all +1's, the encoding for this measurement is taken from the sum amplifier.

Let $I(x)$ be the intensity of illumination of the slot as a function of the distance x measured along it with the slot length normalized to 1. If h_{ij} is the entry in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of H, y_i is the output of the differential amplifier at the $i^{\underline{th}}$ measurement then we have

$$y_1 = \int_0^1 I(x)dx = \int_0^{\frac{1}{N}} I(x)dx + \int_{\frac{1}{N}}^{\frac{2}{N}} I(x)dx + \ldots + \int_{\frac{j-1}{N}}^{\frac{j}{N}} I(x)dx + \ldots + \int_{\frac{N-1}{N}}^{N} I(x)dx$$

$$y_2 = \int_0^{\frac{1}{N}} I(x)h_{21}dx + \int_{\frac{1}{N}}^{\frac{2}{N}} I(x)h_{22}dx + \ldots + \int_{\frac{j-1}{N}}^{\frac{j}{N}} I(x)h_{2j}dx + \ldots + \int_{\frac{N-1}{N}}^{1} I(x)h_{2N}dx$$

$$y_i = \int_0^{\frac{1}{N}} I(x)h_{i1}dx \quad \ldots\ldots \quad \int_{\frac{N-1}{N}}^{1} I(x)h_{iN}dx,$$

$$y_N = \int_0^{\frac{1}{N}} I(x) h_{N1} dx \quad .... \quad \int_{\frac{N-1}{N}}^{1} I(x) h_{NN} dx \qquad (4)$$

Since the h_{ij} are all either + or -1, they can be brought outside the integral. Denote by E_j the integral $\int_{\frac{j-1}{N}}^{j/N} I(x) dx$ which is simply the noise-free response of a detector to the illumination through a mask of a single resolution element.

$$\begin{aligned} y_1 &= \sum_{j=1}^{N} E_j \\ &\vdots \\ y_i &= \sum_{j=1}^{N} h_{ij} E_j \\ &\vdots \\ y_N & \quad \sum_{j=1}^{N} h_{Nj} E_j \end{aligned} \qquad (4a)$$

Hence, to decode the measurements equations (4a) must be solved for the E_j. In matrix notation, where $\hat{y}$ is the column vector of the y_i, $\hat{E}$ a similar vector of the E_j equations (4a) are:

$$\hat{y} = H\hat{E} \qquad (5)$$

thus

$$\hat{E} = \frac{1}{N} H^T \hat{y} \qquad (6)$$

Since H^T consists of only +1's and -1's and N is a power of 2, the matrix multiplication requires only additions and subtractions and a shift of the binary point.

<u>Signal-to-Noise Ratio</u>

Equation (5) must be modified to account for the effects of noise. The development follows very closely that of Sloan, et al.[1] The noise will be assumed to be primarily due to the detector or sensor element and the first electronic stages. Consider a measurement x_i made through a single slot of width $\frac{1}{N}$. Then

$$x_i = E_i + n_i \tag{7}$$

where n_i is the component of noise contributed by the detector to the $i^{\underline{th}}$ measurement. All measurements are assumed to be made for the same length of time including the measurements through the masks. The mean $\langle n_i \rangle$ averaged over the ensemble of measurements is assumed 0 with variance $\langle n_i^2 \rangle = \sigma^2$. Furthermore, noise in different measurements is uncorrelated, i.e., $\langle n_i, n_j \rangle = 0$ for $i \neq j$. Thus, if the primary noise is behind the aperture (e.g., in the detector) then the noise contribution is independent of the coded mask and each measurement y_i is then,

$$y_i = \sum_{j=1}^{N} h_{ij} E_j + n_j \tag{8}$$

and equations (5) and (6) become:

$$\hat{y} = H\hat{E} + \hat{n} \tag{9}$$

$$\hat{E}^* = \frac{1}{N} H^T \hat{y} \tag{10}$$

where $\hat{E}^*$ is now an estimate of $\hat{E}$. Now

$$\langle \hat{E}^* - \hat{E} \rangle = \langle \frac{1}{N} H^T H\hat{E} + \frac{1}{N} H^T \hat{n} - \hat{E} \rangle$$

$$= \frac{1}{N} H^T \langle \hat{n} \rangle = 0$$

and hence, $\langle\hat{E}^*\rangle = \langle\hat{E}\rangle = \hat{E}$ ($\hat{E}$ is deterministic) and the estimate is unbiased. The variance $\sigma_T^2 = \langle(\hat{E}^*-\hat{E})^T, (\hat{E}^*-\hat{E})\rangle$ is

$$\frac{1}{N^2} \langle\hat{n}^T H, H^T\hat{n}\rangle = \frac{1}{N} \langle\hat{n}^T,\hat{n}\rangle = \frac{N}{N}\sigma^2 = \sigma^2 \tag{11}$$

The signal-to-noise ratio S_T is given by

$$S_T = \frac{\langle\hat{E}\rangle}{\sigma_T} = \frac{\hat{E}}{\sigma} \tag{12}$$

The variance of the measurements through a single slot is computed from (7) and is clearly $\langle\hat{n}^T,\hat{n}\rangle = N\sigma^2$ with signal-to-noise ratio

$$S = \frac{\hat{E}}{N^{\frac{1}{2}}\sigma}$$

The improvement in signal-to-noise ratio is thus

$$\frac{S_T}{S} = N^{\frac{1}{2}}$$

For N=1024, a reasonable number for high resolution scanning, the improvement is a significant gain of 32 times.

Clearly, the improvement is obtained by increasing the average signal level while the noise for different measurements is uncorrelated and adds in an RMS way. This raises the question of dynamic range. In each measurement exactly half of the mask is transmitting, half reflecting. The output from the sum amplifier will be the average across any scan line. Over any given scene (but excluding contrived scenes such as a bar chart) the peak-to-peak variation of the line averages is less than the peak-to-peak variation of the entire scene. Therefore, the dynamic range of the sum amplifier can be used as an AGC to set the operating point of the differential amplifier so that it does not overload. Thus, dynamic range is not likely to be a problem in amplifier design but only in the sensor elements. The input upper level will increase over that of single slot scanning by the number of slots in the horizontal line. The lower bound is of course, zero. In single slot scanning the entire dynamic range will be realized in a single scan as long as there is a very strong source and a very weak source in the same scan line which is not unrealistic. However, this situation can occur in coded scanning only if the distribution of light intensity across the entire slot corresponds exactly to the pattern of one of the rows of H, a most unrealistic and unlikely happenstance. In fact, there is considerable empirical evidence from related work on dispersive spectrometers that the dynamic range is <u>reduced</u> by using coded scanning. However, the dynamic range is a function of the spatial frequencies present in the scene and there is at present insufficient information available for an analytic solution.

Conclusions

It has been shown that coded scanning offers a significant improvement in signal-to-noise ratio at the possible, but unlikely, expense of increased dynamic range. The optical system in front of the focal plane is completely independent of the scanning system and hence, can be optimized for any application. Mechanically, the scanning system is simple rectilinear motion located where its interference with the optical path is minimal except for its designed coding function. The precision of the slots in the coded shutter is to micrometer tolerances not optical. The associated electronics would be unchanged except to take advantage of the significant signal-to-noise ratio improvement. The binary difference equations to calculate the aperture codes for $128 \leq N \leq 1024$ are given in the appendix.

H A Helm

1033-HAH-cds

H. A. Helm

Attachments
  Figures 1 and 2
  Appendix

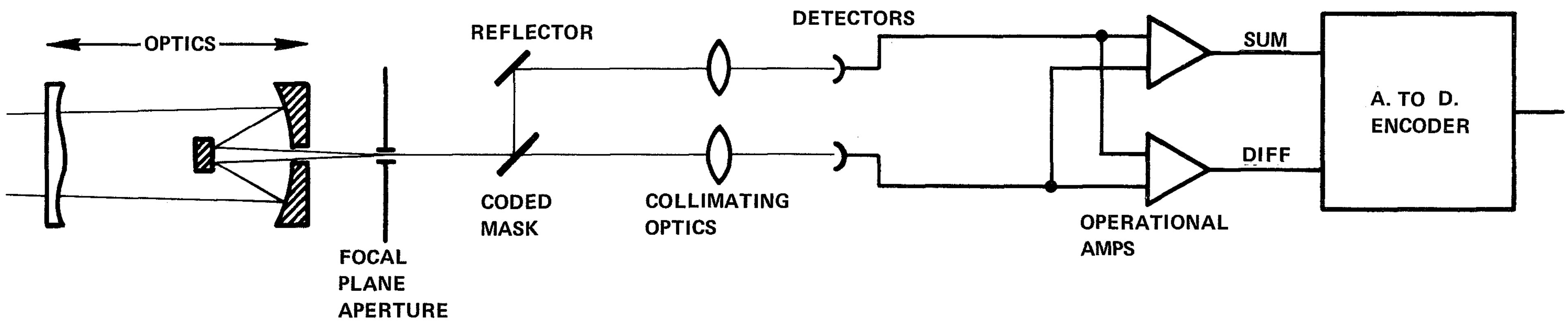
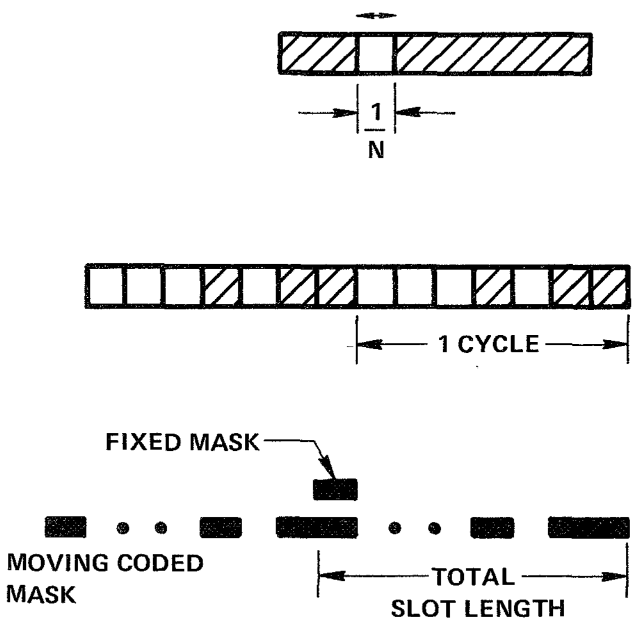


FIGURE 1 - FUNCTIONAL BLOCK DIAGRAM



a) ACROSS SLOT SCANNING WITH APERTURE SIZE EQUAL TO SIZE OF RESOLUTION

b) CYCLIC CODED MASK OF 7 RESOLUTION ELEMENTS

c) ADDITION OF FIXED MASK TO GIVE FULL 8 RESOLUTION ELEMENTS

FIGURE 2 - EXAMPLE OF CODED MASKS

APPENDIX

We shall use the theory of cyclic error correcting codes to construct Hadamard matrices of order 2^k. First, a matrix of the desired dimension with components of 0's and 1's from GF(2) (the Galois field of characteristic 2), is constructed with the property that both the rows and columns form a group of order 2^k. Making the usual correspondences $0 \leftrightarrow 1$, $1 \leftrightarrow -1$ between GF(2) and the reals converts the matrix into a character table and the orthogonality relations for characters gives the desired Hadamard property.

Given any polynomial $\phi(x)$ of degree n, the <u>reciprocal</u> polynomial is defined as $x^n\phi(\frac{1}{x})$. The reciprocal of the product of two polynomials is the product of the reciprocals. It is well known that any polynomial over a finite field is a factor of x^n+1 for some n sufficiently large. Below all polynomials are to be considered over GF(2). Let the irreducible factors of x^n+1 be

$$\phi_1(x)\ \phi_2(x)\ \ldots\ \phi_m(x) = x^n+1 \qquad \text{(A-1)}$$

Since the reciprocal of x^n+1 is x^n+1 if $\phi_j(x)$ is a factor of x^n+1 of degree r then its reciprocal $x^r\phi_j(\frac{1}{x})$ also is a factor of x^n+1. The <u>exponent</u> of a polynomial $\phi(x)$ is the <u>smallest</u> n for which $\phi(x)$ is a factor of x^n+1. Clearly, a polynomial and its reciprocal have the same exponent.

Polynomials over GF(2) are often written as n-tuples or vectors where the components represent the coefficients. Thus

$$(1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0) \leftrightarrow 1x^0 + 1x^1 + 1x^2 + 0x^3 + 1x^4 + 0x^5 + 0x^6 = 1 + x + x^2 + x^4 \qquad \text{(A-2)}$$

Here it is understood that the term of highest degree (x^6) is on the right. It could equally as well be on the left in which case:

$$(1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0) \leftrightarrow x^6 + x^5 + x^4 + x^2 \qquad \text{(A-3)}$$

but (A-3) is the reciprocal of (A-2). Clearly the same result could be obtained by leaving the interpretation the same and writing the vector from right to left as

$$(0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1)$$

As is very well known,[5][6] the elements of a cyclic code form an ideal in the ring of polynomials modulo (x^n+1). Let $n = 2^k-1$. Then the ring is semi-simple and the minimal ideals are generated by primitive orthogonal idempotents[7]. We shall be concerned here only with the minimal ideals all of which are isomorphic to finite fields. That is, to the field of polynomials modulo $\phi_k(x)$ where $\phi_k(x)$ is irreducible and of degree k. There are 2^k-1 non-zero elements in the field and, likewise, in the ideal. Let p(x) be the idempotent generating a minimal ideal where

$$p(x) = a_0x^0 + a_1x + \ldots a_ix^i + \ldots + a_{n-1}x^{n-1} \qquad \text{(A-4)}$$

for $a_i \varepsilon GF(2)$, $n = 2^k-1$. Obviously, the polynomial xp(x) is given $\bmod(x^n+1)$ by

$$xp(x) = a_{n-1}x^0 + a_0x^1 + \ldots a_{i-1}x^i + \ldots a_{n-2}x^{n-1} \qquad \text{(A-5)}$$

a cyclic permutation of (A-4). Writing the 2^k-1 polynomials $x^ip(x)$ as row vectors one obtains the circulent matrix where the a_i are 0 or 1:

$$
\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_i & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_0 & a_1 & a_2 \cdots & a_{n-1} & a_i & a_{n-3} & a_{n-2} \\
a_{n-2} & a_{n-1} & a_0 & \cdots & & & a_{n-4} & a_{n-3} \\
\cdot & & & \ddots & & & \cdot & \cdot \\
\cdot & & & & \ddots & & \cdot & \cdot \\
\cdot & & & & & \ddots & \cdot & \cdot \\
\cdot & & & & & & a_0 & a_1 \\
a_1 & a_2 & & \cdots & & & a_{n-1} & a_0
\end{bmatrix} \qquad (A-6)
$$

If one considers columns as polynomials with high-degree terms on the bottom, then the last column is the reciprocal polynomial of the first row considered as a polynomial with highest degree terms to the right. Also, the other columns are cyclic permutations of it. From the above properties of polynomials and their reciprocals, it can be easily shown that if $p_1(x)$ is the primitive idempotent generating a minimal ideal, then if $p_2(x)$ is its reciprocal, $p_2(x)$ also generates a minimal ideal of the same dimension. Therefore, the columns of (A-6) considered as polynomials are elements of an ideal of dimension 2^k-1 and with the addition of a vector of all 0's form an additive group under addition of order 2^k generated by k independent elements each of order 2. By adding a row and column of all 0's to (A-6) one obtains a modular representation[8] table in which both the rows and columns display the group operations. As an example, consider the case k=3, n=7 and the ring of polynomials modulo (x^7+1). One minimal ideal is generated by $p(x) = 1 + x + x^2 + x^4$ and the elements are:

$$p(x) = 1 + x + x^2 + x^4$$

$$xp(x) = x + x^2 + x^3 + x^5$$

$$x^2p(x) = x^2 + x^3 + x^4 + x^6$$

$$x^3p(x) = 1 + x^3 + x^4 + x^5$$

$$x^4p(x) = x + x^4 + x^5 + x^6$$

$$x^5p(x) = 1 + x^2 + x^5 + x^6$$

$$x^6p(x) = 1 + x + x^3 + x^6$$

Writing these polynomials in matrix form gives

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Here the first three rows or columns may be chosen as the generators of the group. Writing e for the identity and a, b, and c for either the first three rows or first three columns gives the following modular representation table.

TABLE I

e	e	a	b	c	(a+c)	(a+b+c)	(a+b)	(b+c)
e	0	0	0	0	0	0	0	0
a	0	1	1	1	0	1	0	0
b	0	0	1	1	1	0	1	0
c	0	0	0	1	1	1	0	1
(a+b)	0	1	0	0	1	1	1	0
(b+c)	0	0	1	0	0	1	1	1
(a+b+c)	0	1	0	1	0	0	1	1
(a+c)	0	1	1	0	1	0	0	1

Making the substitution $0 \leftrightarrow 1$, $1 \leftrightarrow -1$ converts the above table to the Hadamard matrix (2). We shall now show that the process holds in general. By the above methods, it is clear that a modular representation table of size 2^k can be constructed for arbitrary k and the substitution of the real numbers +1 and -1 for the elements 0 and 1 of GF(2) made. Note that multiplication of +1's and -1's, i.e.,

$$-1 \times -1 = 1 \leftrightarrow 1 + 1 = 0$$

$$-1 \times 1 = -1 \leftrightarrow 1 + 0 = 1$$

Thus the modular representation table of the example becomes the character table

TABLE II

	e	a	b	c	(a+c)	(a+b+c)	(a+b)	(b+c)
e	1	1	1	1	1	1	1	1
a	1	-1	-1	-1	1	-1	1	1
b	1	1	-1	-1	-1	1	-1	1
c	1	1	1	-1	-1	-1	1	-1
(a+b)	1	-1	1	1	-1	-1	-1	1
(b+c)	1	1	-1	1	1	-1	-1	-1
(a+b+c)	1	-1	1	-1	1	1	-1	-1
(a+c)	1	-1	-1	1	-1	1	1	-1

Vector addition of two rows (or columns) modulo 2 in Table I becomes component-wise multiplication of the same rows (columns) in Table II and holds for any size of such tables. Each row (column) of a character table can be made into a diagonal matrix. Thus for the column labeled a in Table II:

$$a: \begin{matrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{matrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = [A]$$

Similarly for c. It is obvious from the above remarks that matrix multiplication is isomorphic to group multiplication, i.e.,

$$a \leftrightarrow [a] \qquad c \leftrightarrow [C]$$

$$(a+c) \leftrightarrow [A][C]$$

and the matrices are a group representation of the additive group of the ideal. Furthermore, the irreducible representations are obviously of dimension 1 and there are precisely 2^k of them. The characters of a group representation are the traces of the matrices in that representation. The characters of the irreducible representations are given by the one dimensional +1's and -1's of the above matrices. Hence, a <u>modular representation table</u> can be made directly into a <u>character table</u> by the same substitution that took Table I into Table II.

In the general theory of group representations, there are several theorems establishing orthogonality relations between the characters of the irreducible representations of the group. For the above additive group C_k, these relations become (using Slepian's notation)

$$\sum_{a\varepsilon C_k} \chi^{\alpha}(A)\ \chi^{\beta}(A) = 2^k \delta_{\alpha\beta} \qquad \text{(A-8)}$$

$$\sum_{\alpha\varepsilon C_k} \chi^{\alpha}(A)\ \chi^{\alpha}(B) = 2^k \delta_{AB} \qquad \text{(A-9)}$$

where $\chi^{\alpha}(A)$ is the entry in the character table in row A and column α and $\delta_{\alpha\beta}$ is the Kronecker delta. Thus, (A-8) says that any two columns of the character table multiplied together component-wise and summed will be zero unless a column is multiplied and summed with itself. Equation (A-9) expresses the same relation for rows. Consider the character table as a matrix [K] and take its transpose $[K]^T$. Then (A-8) and (A-9) express the ordinary row by column matrix multiplication of [K] and $[K]^T$ and thus imply that [K] has the Hadamard property

$$[K]^T[K] = 2^k[I]$$

the desired result.

Finally, it is desirable to have a simpler method of generating the minimal ideals. As noted before, the minimal ideals are isomorphic to the finite fields of the polynomial ring modulo an irreducible polynomial. To generate a sequence of 0's and 1's which interpreted as a polynomial will be a cyclic permutation of the primitive idempotent select from a suitable table an irreducible polynomial of degree k whose exponent is 2^k-1. It has been shown[7] that these polynomials can be considered as delay operators and used to construct binary difference equations or "shift registers." Let y(n) be a sequence of 0's and 1's and $xy(n) = y(n-1)$. Thus, $p(x)y(n) = 0$ is a homogeneous binary difference equation. As an example, take the polynomial $x^3 + x^2 + 1$ of exponent 7. Then,

$$(x^3 + x^2 + 1)\ y(n) = 0$$

and

$$y(n-3) + y(n-2) + y(n) = 0$$

or $y(n) = y(n-2) + y(n-3)$. The first k=3 places of the sequence are arbitrary so pick {1, 0, 0,...}. Using the difference equation the other terms are computed as

{1, 0, 0, 1, 0, 1, 1, 1, 0, 0,...}

the sequence repeating after the first 7 places. Written as a column vector it can be seen that the sequence corresponds to column a of Table I. Appropriate difference equations for various N are given below.

N	Difference Equation
127	$y(n) = y(n-1) + y(n-7)$
255	$y(n) = y(n-1) + y(n-3) + y(n-4) + y(n-8)$
511	$y(n) = y(n-4) + y(n-9)$
1027	$y(n) = y(n-3) + y(n-10)$
2047	$y(n) = y(n-2) + y(n-11)$

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