TTU-ES-70-5

# DYNAMIC ELASTICITY BY THE THEORY OF CHARACTERISTICS

by

GEORGE R. BUCHANAN

CHI-HUONG PHUNG

and

JU-CHIN HUANG



Prepared For

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

GRANT NUMBER NGR 43-003-007

SEPTEMBER, 1970

TENNESSEE TECHNOLOGICAL UNIVERSITY

DEPARTMENT OF ENGINEERING SCIENCE

COOKEVILLE, TENNESSEE 38501

TTU-ES-70-5

# DYNAMIC ELASTICITY BY THE THEORY OF CHARACTERISTICS

by

George R. Buchanan

Chi-Houng Phung

and

Ju-Chin Huang

Prepared For

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

GRANT NUMBER NGR 43-003-007

September, 1970

TENNESSEE TECHNOLOGICAL UNIVERSITY

DEPARTMENT OF ENGINEERING SCIENCE

COOKEVILLE, TENNESSEE 38501

## CONTENTS


| | Page |
|---|---|
| ABSTRACT | |
| INTRODUCTION . . . . . . . . . . . . . . . . . . . . . . . . . . | 1 |
| THE DYNAMIC ELASTICITY PROBLEM . . . . . . . . . . . . . . . . . . . . | 1 |
| ANALYSIS USING THE THEORY OF CHARACTERISTICS . . . . . . . . . . . . | 3 |
| CONCLUSIONS . . . . . . . . . . . . . . . . . . . . . . . . . . | 9 |
| REFERENCES . . . . . . . . . . . . . . . . . . . . . . . . . . | 10 |


## ILLUSTRATIONS


Figure 1. The Direction Cosines in a Spherical Coordinate System 11

## ABSTRACT

A characteristic analysis is presented for the equations of elasticity in Cartesian coordinates. The characteristic slope equations are derived, and it is verified that two types of waves exist. The compatibility equations are developed in relation to the direction cosines of a spherical coordinate system. A brief discussion of the method of analysis is included.

## INTRODUCTION

The application of the theory of characteristics and subsequent numerical solution of the characteristic equations is increasing in popularity as a technique for solving wave propagation problems. The sophisticated development of modern digital computers is responsible for the increase in research effort to develop and extend the technique to more complicated problems. This report deals with the development of a characteristic analysis for the three dimensional dynamic elasticity problem.

## THE DYNAMIC ELASTICITY PROBLEM

The equations of motion for a linear, elastic, isotropic and homogeneous medium in Cartesian coordinates are,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 V_x}{\partial t^2} \qquad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \frac{\partial^2 V_y}{\partial t^2} \qquad (2)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 V_z}{\partial t^2} \qquad (3)$$

where

$x$, $y$, and $z$ = Cartesian coordinates

$\sigma_x$, $\sigma_y$ and $\sigma_z$ = the normal stresses in the yz, xz, and xy planes, respectively

$\tau_{xy}$, $\tau_{yz}$, and $\tau_{xz}$ = the shear stresses in the xy, yz, and xz planes, respectively

$\rho$ = density

$V_x$, $V_y$, and $V_z$ = velocities in the x-, y-, and z-directions, respectively

t = the time dimension.

The stress-displacement relations can be written as

$$\sigma_x = AB \frac{\partial u_x}{\partial x} + AC \frac{\partial u_y}{\partial y} + AC \frac{\partial u_z}{\partial z} \tag{4}$$

$$\sigma_y = AC \frac{\partial u_x}{\partial x} + AB \frac{\partial u_y}{\partial y} + AC \frac{\partial u_z}{\partial z} \tag{5}$$

$$\sigma_z = AC \frac{\partial u_x}{\partial x} + AC \frac{\partial u_y}{\partial y} + AB \frac{\partial u_z}{\partial z} \tag{6}$$

$$\tau_{xy} = \frac{A}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{7}$$

$$\tau_{xz} = \frac{A}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \tag{8}$$

and

$$\tau_{yz} = \frac{A}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{9}$$

where

$$A = E/(1 + \nu)$$

$$B = (1 - \nu) / (1 - 2\nu) \tag{10}$$

$$C = \nu/(1 - 2\nu)$$

and

$u_x$, $u_y$, and $u_z$ = displacements in the x-, y-, and z-directions respectively.

In the theory of characteristics, it is convenient to treat first-order partial differential equations; therefore, Eqs. 4 through 9 will be differentiated with respect to time to yield

$$\frac{\partial \sigma_x}{\partial t} = AB \frac{\partial V_x}{\partial x} + AC \frac{\partial V_y}{\partial y} + AC \frac{\partial V_z}{\partial z} \tag{11}$$

$$\frac{\partial \sigma_y}{\partial t} = AC \frac{\partial V_x}{\partial x} + AB \frac{\partial V_y}{\partial y} + AC \frac{\partial V_z}{\partial z} \tag{12}$$

$$\frac{\partial \sigma_z}{\partial t} = AC \frac{\partial V_x}{\partial x} + AC \frac{\partial V_y}{\partial y} + AB \frac{\partial V_z}{\partial z} \tag{13}$$

$$\frac{\partial \tau_{xy}}{\partial t} = \frac{A}{2}\frac{\partial V_x}{\partial y} + \frac{A}{2}\frac{\partial V_y}{\partial x} \qquad (14)$$

$$\frac{\partial \tau_{xz}}{\partial t} = \frac{A}{2}\frac{\partial V_x}{\partial z} + \frac{A}{2}\frac{\partial V_z}{\partial x} \qquad (15)$$

and

$$\frac{\partial \tau_{yz}}{\partial t} = \frac{A}{2}\frac{\partial V_y}{\partial z} + \frac{A}{2}\frac{\partial V_z}{\partial y} \qquad (16)$$

Equations 1 through 3 and 11 through 16 make up a set of nine linear first-order partial differential equations which govern the three dimensional dynamic elasticity problem.

## ANALYSIS USING THE THEORY OF CHARACTERISTICS

The analysis used herein is similar to that used by Sauerwein (1) and Madden (2) and will be briefly outlined. The nine governing equations can be represented in a convenient form using the index notation as follows,

$$a_{ijk}\frac{\partial B_j}{\partial x_k} = 0 \qquad (17)$$

where $B_j$ represents the dependent variables; $x_k$ represents the independent variables and the $a_{ijk}$ are constants. The characteristic (slope) equations for Eq. (17) can be developed by changing the independent variables from $x_k$ to some arbitrary coordinate system, say $\eta_1$, $\eta_2$, and $\eta_3$. That is

$$a_{ijk}\frac{\partial B_j}{\partial \eta_m}\frac{\partial \eta_m}{\partial x_k} = 0 \quad , \quad m = 1, 2, \text{and } 3. \qquad (18)$$

Assuming that values of all dependent variables and their derivatives with respect to $\eta_2$ and $\eta_3$ are specified on a surface $\eta_1$ = constant, these transformed partial differential equations would be expected to yield the derivatives with respect to $\eta_1$, if they exist. These derivatives are

$$a_{ijk} \frac{\partial B_j}{\partial \eta_1} \frac{\partial \eta_1}{\partial x_k} = -a_{ijk} \frac{\partial B_j}{\partial \eta_n} \frac{\partial \eta_n}{\partial x_k} \tag{19}$$

where n = 2,3.

To determine the characteristic equations it is desirable to determine the conditions under which the derivatives normal to $\eta_1$ do not exist, that is, the normal derivatives to the surface, $\eta_1$ = a constant, are discontinuous. These discontinuity surfaces which have been mentioned at the beginning of this chapter are also called characteristic surfaces. The requirement for discontinuity in the derivatives with respect to $\eta_1$ is then the vanishing of the determinant of the coefficients of the derivatives with respect to $\eta_1$ in Eqs. (19), or

$$\det \left| a_{ijk} \frac{\partial \eta_1}{\partial x_k} \right| = 0 \tag{20}$$

The nine governing equations (1) through (3) and (11) through (16), after transformation, may be written in the form of Eq. (20) as follows,

$$\begin{vmatrix}
\eta,_x & 0 & 0 & \eta,_y & \eta,_z & 0 & -\rho\eta,_t & 0 & 0 \\
0 & \eta,_y & 0 & \eta,_x & 0 & \eta,_z & 0 & -\rho\eta,_t & 0 \\
0 & 0 & \eta,_z & 0 & \eta,_x & \eta,_y & 0 & 0 & -\rho\eta,_t \\
\eta,_t & 0 & 0 & 0 & 0 & 0 & -AB\eta,_x & -AC\eta,_y & -AC\eta,_z \\
0 & \eta,_t & 0 & 0 & 0 & 0 & -AC\eta,_x & -AB\eta,_y & -AC\eta,_z \\
0 & 0 & \eta,_t & 0 & 0 & 0 & -AC\eta,_x & -AC\eta,_y & -AB\eta,_z \\
0 & 0 & 0 & \eta,_t & 0 & 0 & -\frac{A}{2}\eta,_y & -\frac{A}{2}\eta,_x & 0 \\
0 & 0 & 0 & 0 & \eta,_t & 0 & -\frac{A}{2}\eta,_z & 0 & -\frac{A}{2}\eta,_x \\
0 & 0 & 0 & 0 & 0 & \eta,_t & 0 & -\frac{A}{2}\eta,_z & -\frac{A}{2}\eta,_y
\end{vmatrix} = 0 \tag{21}$$

where the comma indicates partial differentiation with respect to the variable following the comma. Also, for convenience the subscript 1 has been omitted for the variable $\eta$. Expanding Eq. (21) gives the following

$$\{\rho\eta,_t^2 - \frac{E}{1+\nu}\frac{1-\nu}{1-2\nu}(\eta,_x^2 + \eta,_y^2 + \eta,_z^2)\}\,\{\rho\eta,_t^2 - \frac{E}{2(1+\nu)}(\eta,_x^2 + \eta,_y^2 + \eta,_z^2)\}^2\,\eta,_t^3 = 0 \tag{22}$$

or

$$\rho\eta,_t^2 - \frac{E}{1+\nu}\frac{1-\nu}{1-2\nu}(\eta,_x^2 + \eta,_y^2 + \eta,_z^2) = 0 \tag{23}$$

$$\{\rho\eta,_t^2 - \frac{E}{2(1+\nu)}(\eta,_x^2 + \eta,_y^2 + \eta,_z^2)\}^2 = 0 \tag{24}$$

$$\eta,_t^3 = 0 \tag{25}$$

Equations (23) and (24) show that this three-spatial dimensional problem involves two kinds of waves, namely, a longitudinal and a shear wave. Letting $c_L$ and $c_s$ represent the longitudinal and shear wave velocities, respectively, it follows that

$$c_L^2 = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)} \tag{26}$$

$$c_s^2 = \frac{E}{2\rho(1+\nu)} \tag{27}$$

The extra factor $\eta,_t$ in Eq. (25) indicates the particle path which is a characteristic surface with zero velocity. By expressing Eqs. (23) and (24) as

$$F = \frac{\eta,_t^2}{c^2} - \eta,_x^2 - \eta,_y^2 - \eta,_z^2 \tag{28}$$

where

$$c = c_L \text{ or } c_s$$

and introducing a new parameter $\psi$, the characteristic slope equations become

$$\frac{dx}{d\psi} = \frac{\partial F}{\partial \eta,_x} = -2\eta,_x \tag{29}$$

$$\frac{dy}{d\psi} = \frac{\partial F}{\partial \eta,_y} = -2\eta,_y \tag{30}$$

$$\frac{dz}{d\psi} = \frac{\partial F}{\partial \eta,_z} = -2\eta,_z \tag{31}$$

$$\frac{dt}{d\psi} = \frac{\partial F}{\partial \eta,_t} = \frac{2}{c^2}\eta,_t \tag{32}$$

Eliminating the parameter $\psi$ gives

$$\frac{dx}{dt} = -c^2\,\frac{\eta,_x}{\eta,_t} = \mp c\,\frac{\eta,_x}{\eta,_x^2 + \eta,_y^2 + \eta,_z^2} \tag{33}$$

$$\frac{dy}{dt} = -c^2\,\frac{\eta,_y}{\eta,_t} = \mp c\,\frac{\eta,_y}{\eta,_x^2 + \eta,_y^2 + \eta,_z^2} \tag{34}$$

$$\frac{dz}{dt} = -c^2\,\frac{\eta,_z}{\eta,_t} = \mp c\,\frac{\eta,_z}{\eta,_x^2 + \eta,_y^2 + \eta,_z^2} \tag{35}$$

Equations (33), (34), and (35) can be reduced to a simple form by using the direction cosines between the normal to the surface $\eta$ = constant and the x-, y-, and z-axes as

$$\begin{aligned} \text{Cos }(\eta,x) &= \frac{\eta,_x}{\eta,_x^2 + \eta,_y^2 + \eta,_z^2} \\ \text{Cos }(\eta,y) &= \frac{\eta,_y}{\eta,_x^2 + \eta,_y^2 + \eta,_z^2} \\ \text{Cos }(\eta,z) &= \frac{\eta,_z}{\eta,_x^2 + \eta,_y^2 + \eta,_z^2} \end{aligned} \tag{36}$$

Equations (36) can be written in terms of spherical coordinates $\theta$ and $\phi$ according to Figure 1 as

$$\begin{aligned} \text{Cos}\ (\eta,x) &= \text{Sin}\ \theta\ \text{Cos}\ \phi \\ \text{Cos}\ (\eta,y) &= \text{Sin}\ \theta\ \text{Sin}\ \phi \\ \text{Cos}\ (\eta,z) &= \text{Cos}\ \theta \end{aligned} \tag{37}$$

Substituting into Eqs. (33), (34), and (35) yields

$$\begin{aligned} \frac{dx}{dt} &= \pm\ c\ \text{Sin}\ \theta\ \text{Cos}\ \phi \\ \frac{dy}{dt} &= \pm\ c\ \text{Sin}\ \theta\ \text{Sin}\ \phi \\ \frac{dz}{dt} &= \pm\ c\ \text{Cos}\ \theta \end{aligned} \tag{38}$$

In Eqs. (38) only the positive sign need be considered since the negative sign may be obtained by changing the reference for $\theta$ and $\phi$. Therefore, the final form of characteristic slope equations is

$$\begin{aligned} \frac{dx}{dt} &= c\ \text{Sin}\ \theta\ \text{Cos}\ \phi \\ \frac{dy}{dt} &= c\ \text{Sin}\ \theta\ \text{Sin}\ \phi \\ \frac{dz}{dt} &= c\ \text{Cos}\ \theta \end{aligned} \tag{39}$$

In Eqs. (39), a given value for $\theta$ and $\phi$ $(0 \rightarrow 2\pi)$ defines one of the characteristic directions at a point. These characteristics are termed "Bicharacteristics". Considering the entire range of $\theta$ and $\phi$ $(0 \rightarrow 2\pi)$, Eqs. (39) describe a general sphere in space, namely, a "characteristic sphere". The family of bicharacteristics are the generators of the sphere.

The compatibility equation corresponding to the bicharacteristics given by Eqs. (39) is obtained by combining the transformed equations in a manner such that the indeterminable derivatives with respect to $\eta$ do not appear. Multiplying the transformed equations by weighting factors $\lambda_1$, $\lambda_2$, . . ., and $\lambda_9$ respectively and summing will yield such a relation. Relations between the $\lambda$'s are found by equating to zero the coefficients with respect to $\eta$ in the transformed equations. The derivatives can be written in the form:

$$\lambda_i a_{ijk} \frac{\partial \eta}{\partial x_k} = 0 \tag{40}$$

or,

$$\begin{aligned}
\lambda_1 \eta,_x + \lambda_4 \eta,_t &= 0 \\
\lambda_2 \eta,_y + \lambda_5 \eta,_t &= 0 \\
\lambda_3 \eta,_z + \lambda_6 \eta,_t &= 0 \\
\lambda_1 \eta,_y + \lambda_2 \eta,_x + \lambda_7 \eta,_t &= 0 \\
\lambda_1 \eta,_z + \lambda_3 \eta,_x + \lambda_8 \eta,_t &= 0 \\
\lambda_2 \eta,_z + \lambda_3 \eta,_y + \lambda_9 \eta,_t &= 0 \\
- \lambda_1 \rho \eta,_t - \lambda_4 AB\eta,_x - \lambda_5 AC\eta,_x - \lambda_6 AC\eta,_x - \lambda_7 A/2\eta,_y - \lambda_8 A/2\eta,_z &= 0 \\
- \lambda_2 \rho \eta,_t - \lambda_4 AC\eta,_y - \lambda_5 AB\eta,_y - \lambda_6 AC\eta,_y - \lambda_7 A/2\eta,_x - \lambda_9 A/2\eta,_z &= 0 \\
- \lambda_3 \rho \eta,_t - \lambda_4 AC\eta,_z - \lambda_5 AC\eta,_z - \lambda_6 AB\eta,_z - \lambda_8 A/2\eta,_x - \lambda_9 A/2\eta,_y &= 0
\end{aligned} \tag{41}$$

Only eight of these homogeneous equations are required to find the following relations

$$\begin{aligned}
\lambda_2 &= \tan \phi \, \lambda_1 \\
\lambda_3 &= P\lambda_1 \\
\lambda_4 &= \pm \frac{\mathrm{Sin}\ \theta\ \mathrm{Cos}\ \phi}{c} \lambda_1 \\
\lambda_5 &= \pm \frac{\mathrm{Sin}\ \theta\ \mathrm{Sin}\ \phi\ \tan \phi}{c} \lambda_1 \\
\lambda_6 &= \pm \frac{P\ \mathrm{Cos}\ \theta}{c} \lambda_1 \\
\lambda_7 &= \pm \frac{\mathrm{Sin}\ \theta}{c} (\mathrm{Sin}\ \phi + \mathrm{Cos}\ \phi \tan \phi)\lambda_1 \\
\lambda_8 &= \pm \frac{\mathrm{Cos}\ \theta + P\ \mathrm{Sin}\ \theta\ \mathrm{Cos}\ \phi}{c} \lambda_1
\end{aligned} \tag{42}$$

and

$$\lambda_9 = \pm \frac{\cos\theta \tan\phi + P \sin\theta \sin\phi}{c} \lambda_1$$

where

$$P = \frac{c^2 - c_L^2 \sin^2\theta - c_s^2 \cos^2\theta}{(c_L^2 - c_s^2) \sin\theta \cos\theta \cos\phi}$$

Thus the compatibility equations can be obtained from the sum of the weighted equations (1) through (3) and (11) through (16) as follows

$$\begin{aligned}
& c(\sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} - \rho V_{x,t}) \\
& + c \tan\phi\, (\tau_{xy,x} + \sigma_{y,y} + \tau_{yz,z} - \rho V_{y,t}) \\
& + cP\, (\tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} - \rho V_{z,t}) \\
& \pm \sin\theta \cos\phi \{\sigma_{x,t} - \rho c_L^2 V_{x,x} - 2\rho\nu(c_L^2 - c_s^2)(V_{y,y} + V_{z,z})\} \\
& \pm \sin\theta \sin\phi \tan\phi \{\sigma_{y,t} - \rho c_L^2 V_{y,y} - 2\rho\nu(c_L^2 - c_s^2)(V_{x,x} + V_{z,z})\} \\
& \pm P \cos\theta \{\sigma_{z,t} - \rho c_L^2 V_{z,z} - 2\rho\nu(c_L^2 - c_s^2)(V_{x,x} + V_{y,y})\} \\
& \pm \sin\theta (\sin\phi + \cos\phi \tan\phi) \{\tau_{xy,t} - \rho c_s^2 (V_{x,y} + V_{y,x})\} \\
& \pm (\cos\theta + P \sin\theta \cos\phi) \{\tau_{xz,t} - \rho c_s^2 (V_{x,z} - V_{z,x})\} \\
& \pm (\cos\theta \tan\phi + P \sin\theta \sin\phi) \{\tau_{yz,t} - \rho c_s^2 (V_{y,z} + V_{z,y})\} = 0
\end{aligned} \tag{43}$$

These are the general compatibility equations for a three-spatial dimensional dynamic elasticity problem. Specific equations can be obtained by choosing values of $\theta$ and $\phi$, and specifying c to be $c_L$ or $c_s$.

## CONCLUSIONS

The characteristic slope equations and corresponding compatibility equations have been developed in Cartesian coordinates for the three dimensional dynamic elasticity problem. It is verified that two types of waves are present, namely, longitudinal and shear waves.

# REFERENCES

1. Sauerwein, Harry. "A General Numerical Method of Characteristics," AIAA Second Aerospace Sciences Meeting, New York, January, 1965, AIAA Paper No. 65-25.

2. Madden, Richard. "Hypervelocity Impact Analysis by the Method of Characteristics," NASA TR-298, January, 1969.

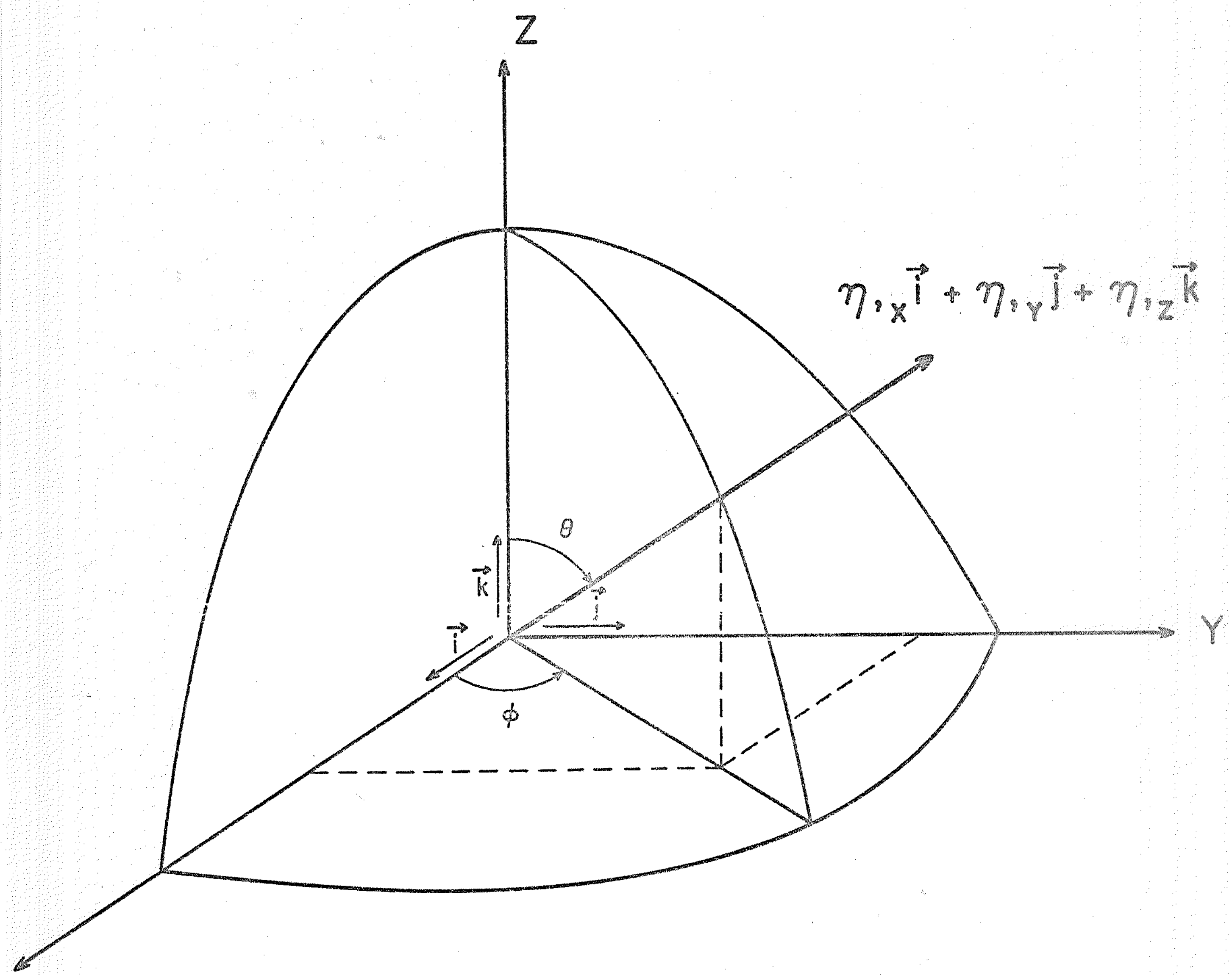


Figure 1. The Direction Cosines in a Spherical Coordinate System