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SOLAR ACTIVITY PREDICTION

*by Ralph J. Slutz, Thomas B. Gray, Marie L. West,
Frank G. Stewart, and Margo Leftin*

Prepared by

ENVIRONMENTAL RESEARCH LABORATORIES, NOAA

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16. ABSTRACT Part I of this report is a statistical study of formulas for predicting the sunspot number several years in advance. It was found that the first seven solar cycles in the Zurich listing have different statistics from the following twelve, and they were rejected from the analysis. By using a data lineup with cycle maxima coinciding, and by using multiple and nonlinear predictors, we found a new formula which gives better error estimates than former formulas derived from the work of McNish and Lincoln. In Part II a statistical analysis is conducted to determine which of several mathematical expressions best describes the relationship between 10.7 cm solar flux and Zurich sunspot numbers. Attention is given to the autocorrelation of the observations, and confidence intervals for the derived relationships are presented. The accuracy of predicting a value of 10.7 cm solar flux from a predicted sunspot number is discussed.					
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SOLAR ACTIVITY PREDICTION
FINAL REPORT on NASA order H-54409A

Ralph J. Slutz, Thomas B. Gray, Marie L. West,
Frank G. Stewart and Margo Leftin

PART I: SUNSPOT NUMBER PREDICTION

1. INTRODUCTION

This is the final report on a study of solar activity prediction which was undertaken in response to NASA order H-54409A. A good summary of the background of the study is given in the order:

"Accurate predictions of atmospheric density several years in advance at operating altitudes of orbiting vehicles are essential for optimum orbital planning and spacecraft lifetime studies for several high-priority MSFC programs, including the Apollo Applications Program, the Saturn V Workshop, and the proposed Logistics Vehicle. It would be desirable to predict the atmospheric density directly, but this is not yet possible since density measurements have been obtained for only about the past ten years. Fortunately, variations in density at high altitudes show a high correlation with variations in solar activity; i.e., sunspot number (Wolfe number) or measurements of the solar radio flux at decimetric wavelengths (10.7 cm). Measurements of the 10.7 cm solar radio flux have been made only since 1947; however, the Zurich relative sunspot numbers constitute a highly homogeneous series extending back to 1849, while less reliable data extend the series back to 1749. Since the correlation is very high between sunspot numbers and atmospheric density at satellite altitudes for the period during which the latter have been observed, conclusions reached from the study of sunspot data may be assumed to apply to high altitude atmospheric density as well.

Sunspot prediction methods to date have primarily been based on an analysis of past data as a function of time, the prediction being an extrapolation of the statistical analysis to some future

time. In 1949, Lincoln and McNish (then employees of the Central Radio Propagation Laboratory) suggested that using the mean of all past solar cycles as an approximation for the next cycle could be improved after the beginning of the cycle by adding to the mean a correctional proportional to the departure of the earlier values of the cycle from the mean. This is the basis for the Lockheed modification of the Lincoln-McNish prediction method, which is currently used by MSFC for solar prediction. More accurate forecasts of solar activity are needed than have been available in the past. A modification to present methods or development of new techniques is required to provide more accurate solar activity forecasts. A detailed theoretical investigation of the physical theory of sunspot formation is beyond the scope of this study; however, physical reasons for the existence of sunspots and any constraints associated with their prediction should, whenever possible, be incorporated into the development of a model to forecast solar activity."

This report is made in two parts because the work was carried out by two separate research teams. Part I, authored by Ralph J. Slutz, Thomas B. Gray, and Marie L. West, describes the studies undertaken to improve and extend the statistical prediction of sunspot numbers. Part II, authored by Frank G. Stewart and Margo Leftin, describes the studies undertaken to relate measurements of solar radio flux to sunspot numbers.

The NASA order called for studies in five areas:

1. Investigate, and identify where possible, improved methods of predicting solar activity. This should include consideration of more sophisticated mathematical techniques, such as use of non-linear predictors.

This is reported in Part I, and an improved formula making use of nonlinear predictors is given in Part I, section 6.

2. Investigate the feasibility of developing techniques for preparing direct predictions of solar parameters more directly related to atmospheric density, such as the solar radio flux.

This was examined, but the conclusion reached was that the available data sample was too small to lead to meaningful predictions other than through its correlation with sunspot number. Thus the effort was concentrated on such correlations.

3. Investigate the relationship between sunspot numbers and solar radio flux with a goal of improving the mathematical relationship between these parameters. This should include studying separately the relationship during the decreasing portion of the cycle.

This is reported in Part II.

4. Develop a model (quantitative) to forecast solar activity for at least one solar cycle (about 11 years) in advance and suggest possible areas in which future improvements may be expected. .

This is reported in sections 9 and 11 of Part I.

5. Compare statistically this model with the Lockheed modification of the Lincoln-McNish technique as used by MSFC.

This is reported in section 10 of Part I.

From the time that solar activity was first recognized as having cyclical variations, there have been many studies of these variations and many attempts to predict them. When they were found to have close correlation with ionization density in the earth's upper atmosphere, the studies gained impetus from their practical application to predicting radio communications characteristics. Now the relationship to atmospheric density gives a further practical impetus because of its effect on satellite lifetimes.

There are two particularly complete surveys of these studies, "Solar-Activity Forecasting" by Vitinskii (1962), and "Survey of Solar Cycle Prediction Models" by Scissum (1967). Vitinskii covers in great detail the studies that have been made on qualitative and statistically quantitative relationships among the sunspot numbers themselves. Scissum also includes some of the early work searching for relationships between sunspot numbers and planetary motions.

The work being reported here has emphasized statistical relationships among the sunspot numbers themselves, particularly extending the earlier studies into a search for multiple predictors and nonlinear predictors. A new formula is presented (section 6) which appears to be an improvement over former prediction methods.

Solar activity variations have been found to be particularly difficult to predict. The generally accepted series of observations extends back for over 200 years, although the first 80 years of that time were pieced together from historical records of observations made before the "sunspot number" was defined or the observations standardized in any way. Even with so long a period, when the data are separated into cycles there are only 19 of them, and they vary so much that it almost seems as though each cycle is unique in some way or other. For instance, cycle number 19, the latest one completed, was the highest in history and its extreme height was only very poorly predicted. Now the current cycle, number 20, has been decreasing from its maximum much more slowly than any cycle in the well-observed set. The variability of the cycles is so great, and the number of cycles is so small, that many prediction methods have been proposed and then found to fail shortly after being proposed.

For this reason we have tried to be particularly conservative in the use of statistics in this study. Relationships have been accepted as significant only if they show correlations that fall in the significant range over a considerable period of time. Relationships have been rejected if they pass statistical significance tests for only a short portion of a cycle. Still, only time will tell whether sufficient conservatism has been used to make the relationships that have been accepted continue to be useful in the future.

2. DATA BASE

Sunspot number data cover an unusually long period of time. The Zurich series is continuous from the year 1749 to the present, and much effort has gone into keeping the methods of handling the observations and data as nearly uniform as possible. It is indeed just this feature, the length and uniformity of the series, that makes it particularly useful as a forecasting tool, even if often an indirect one. Other geophysical observations have much shorter histories. For example, direct observations of atmospheric density in the outer atmosphere and of solar radio-frequency noise flux have been made for only something like one solar activity cycle. Even ionospheric electron density observations cover something less than four solar cycles. Thus these direct observations cover so few cycles that it is nigh to impossible to determine from them statistical relationships among successive cycles. Instead, it has been noted that during their short histories they exhibit a high correlation with the sunspot number data. Accordingly, to get some basis for long-term predictions of these shorter series, it is generally assumed that the observed correlations will be equally good over the long term. This permits using the longer series of sunspot number data as a means of forecasting the other much shorter series.

When Rudolph Wolf of the Zurich observatory introduced his sunspot number formula in 1848, he applied it to records of old observations as far back as 1749. But as one goes further and further back in the series one finds that the observations contain less and less of the detail necessary to make an accurate *ex post facto* determination of the Wolf number. The observations also become increasingly sparse, as can be seen clearly from an examination of Wolf's original data tabulation which is preserved at the Zurich observatory.

McNish and Lincoln (1949) felt that before 1834 the historical uncertainties were great enough to question the uniformity of the series before that time as compared with the series after 1834.

McNish and Lincoln considered the series in two parts: (a) from 1755 to 1834 (cycles 1 through 7 of the Zurich tabulation), and (b) from 1834 to 1944 (cycles 8 through 17 of the Zurich tabulation). They fitted the two parts separately with a Type VI Pearsonian distribution and calculated the chi-square test of the hypothesis that this distribution was valid. They pointed out some uncertainties in determining the number of degrees of freedom that should be used in the calculation, but arrived at the conclusion that the second part of the data is consistent with a Type VI Pearsonian distribution (probability value about 0.65) and that the first part of the data is not consistent with a Type VI distribution (probability value less than 0.001). Essentially the first seven cycles had a much broader distribution about the mean than did the later 10 cycles.

Because of these statistical results, coupled with the historical uncertainties of the early data, McNish and Lincoln decided to exclude the data before 1834 from their prediction analysis, and they carried out the analysis on only the data from 1834 to 1944. However, since 1944 two more full cycles of sunspot data have become available. Both these cycles, numbers 18 and 19 in the Zurich tabulation, had higher maxima than any of the cycles accepted by McNish and Lincoln. The total duration of cycle 18 was as short as any of those in the accepted group of cycles 8 through 17, but for neither 18 nor 19 was the risetime unusually short compared with 8 through 17. Thus, inclusion of the two new cycles as "good" data markedly increased the breadth of the distribution about the mean of the rest of

the "good" part of the series. This has raised the question of whether the McNish-Lincoln rejection would still be justified if these two latest cycles were to be included in the analysis. Indeed, preliminary consideration of this question led the Central Radio Propagation Laboratory to reintroduce in 1968 the earlier cycles into the analyses used for their publication *Ionospheric Predictions*.

The question of whether or not to include the data before 1834 is far from trivial for this analysis. McNish and Lincoln showed that the regression coefficients for forecasting the sunspot number vary throughout the cycle. Thus one cannot lump together all of the data taken during the cycle, but when deriving the regression coefficients one gets essentially only a single observation pair per cycle --- that taken at a time which corresponds for each cycle. Thus if all of the data from 1755 to date were to be used, one would have 19 observation pairs from which to calculate each regression coefficient. On the other hand, rejecting data before 1834 leaves only 12 pairs. In the former case, one might consider that a portion of the data could be set aside as a verification sample on which to test relationships developed from the remaining data. In the latter case it would not seem reasonable to reduce the 12 pairs any further by deleting a verification portion. As a matter of fact, in this study several preliminary analyses were carried out on the full set of data before it became clear that only the reduced set should be used.

A chi-square statistical test was carried out to determine whether the early data should be included in the analysis or not. More precisely, there was tested the null hypothesis that the two distributions from which the early data and the later data are respectively drawn are identical. The data being studied were the "smoothed annual relative sunspot numbers," defined as

$$S_{ij} = \frac{R_{i,j-6} + 2 \sum_{k=j-5}^{j+5} R_{i,k} + R_{i,j+6}}{24},$$

where $R_{i,j}$ is the observed monthly sunspot number derived and published by the Zurich observatory for the j -th month of the i -th cycle. To reduce the effect of the autocorrelation introduced by the smoothing process, data were used centered on months 1, 13, 25, ... 121 of each cycle. Not all of the 19 cycles last as long as 121 months; because of short cycles, month 109 consists of data from only 18 cycles, and month 121 consists of data from only 16 cycles. Thus there are 205 data in the sample covering the 19 cycles.

To put the data on a comparable basis, each month's set was reduced to have a mean of zero and a standard deviation of unity, i.e., the analysis was carried out on

$$V_{ij} = \frac{S_{ij} - \bar{S}_j}{s_j},$$

where

$$\bar{S}_j = \frac{1}{n} \sum_{i=1}^n S_{ij}$$

$$s_j = \sqrt{\frac{\sum_{i=1}^n (S_{ij} - \bar{S}_j)^2}{n-1}}$$

and $n=19$ for all of those months that have full data sets. Thus it was assumed that all months of a cycle have the same shape of distribution, at least approximately so relative to the difference being tested between early and late data.

Boundaries on the V_{ij} were then established which separated the entire set into six cells, each having about the same number of entries, and C_{ik} was determined as the number of observations from the i -th sunspot cycle that fell within the k -th cell of this distribution function. For instance, $C_{5,1}$ is the number of observations from the fifth cycle that fall in the lowest sixth of the overall distribution function.

Then the contingency table of table 1 was prepared in which the six segments of the frequency distribution are compared for each of the sunspot cycle groups being tested and for the overall group of cycles. That is,

$$n_{1,k} = \sum_{i=1}^7 C_{ik} \text{ gives the entries from cycles 1 through 7,}$$

$$n_{2,k} = \sum_{i=8}^{19} C_{ik} \text{ gives the entries from cycles 8 through 19,}$$

$$n_{3,k} = \sum_{i=1}^{19} C_{ik} \text{ gives the entries from all cycles, 1 through 19.}$$

The basic method is described by Cramer (1946).

Table 1. Frequency Distribution of Smoothed Annual Sunspot Numbers

	Cell						Total
	1	2	3	4	5	6	
$n_{1,k}$	24	6	10	6	15	13	(74)
$n_{2,k}$	10	28	24	28	19	32	(131)
$n_{3,k}$	(34)	(34)	(34)	(34)	(34)	(35)	(205)

From the row and column totals the expected (i.e., expected if the null hypothesis is true) number of entries in each box was calculated from

$$E_{ik} = \frac{(\sum_i \eta_{ik})(\sum_k \eta_{ik})}{\sum_i \sum_k \eta_{ik}},$$

and finally chi-square was calculated from

$$\chi^2 = \sum_{i=1}^2 \sum_{k=1}^6 \frac{(\eta_{ik} - E_{ik})^2}{E_{ik}},$$

giving

$$\chi^2 = 29.193.$$

With this we can examine the likelihood that a table such as table 1 would arise from data where the two groupings $\eta_{1,k}$ and $\eta_{2,k}$ come from the same overall population. The customary calculation of the degrees of freedom of table 1 would be $(6 - 1)(2 - 1) = 5$. Entering chi-square tables with this value (Abramowitz and Stegun, 1964), we find that for two distributions drawn from the same overall population a χ^2 of 29 or greater would be expected to occur only 0.002% of the time --- only 1 time out of 50,000. Thus we would reject the hypothesis that the two groups of sunspot numbers belong to the same population.

However, there are definite questions about whether the above calculation of degrees of freedom is appropriate for the sunspot numbers. Even at the spacing chosen for this analysis, the numbers are well known not to be independent. Distinct correlation exists for much wider spacing, as is shown in section 5. At first, one would think that such a correlation would be expected to reduce the number of degrees of freedom in the calculation, and this would make it even less likely that

so large a value of χ^2 would come from a single population. A little more thought, however, reveals that this correlation is almost entirely within the individual groupings being tested, and very small from one group to the other. That is, the entries in $\eta_{1,k}$ have correlation among them, as do the entries in $\eta_{2,k}$, but there is practically no correlation from $\eta_{1,k}$ to $\eta_{2,k}$. This might well be expected to increase the likelihood of finding large values for χ^2 .

Consequently, several statistical experiments were carried out to get a better indication of the meaning of the χ^2 calculated above. First a random selection of the sunspot cycles to be included in each of the two groups was made. That is, rather than putting the first seven cycles in the first group and the last 12 cycles in the second group, seven cycles were drawn at random from the entire set of 19, and these seven were taken as the first group, with the remaining cycles as the second group. From this grouping a value for χ^2 was calculated. This whole process was then repeated 1000 times to give 1000 values for χ^2 . The resulting distribution function of the χ^2 is plotted in figure 2.1, labeled "Random Groupings". It was found that 5% of the observed χ^2 were greater than 29.193, rather than 0.002% as is expected from the standard analysis using 5 degrees of freedom.

A second statistical experiment was made in the same way, but using only the last 12 sunspot cycles (which presumptively are from essentially the same population). These 12 cycles were broken into two groups of six each, with the members of each group selected at random. This was repeated 100 times to get 100 values for χ^2 . The resulting distribution function is also plotted in figure 2.1, labeled "Last 12 Cycles Only". This distribution function is similar to that for the "Random Groupings" of the 19 cycles.

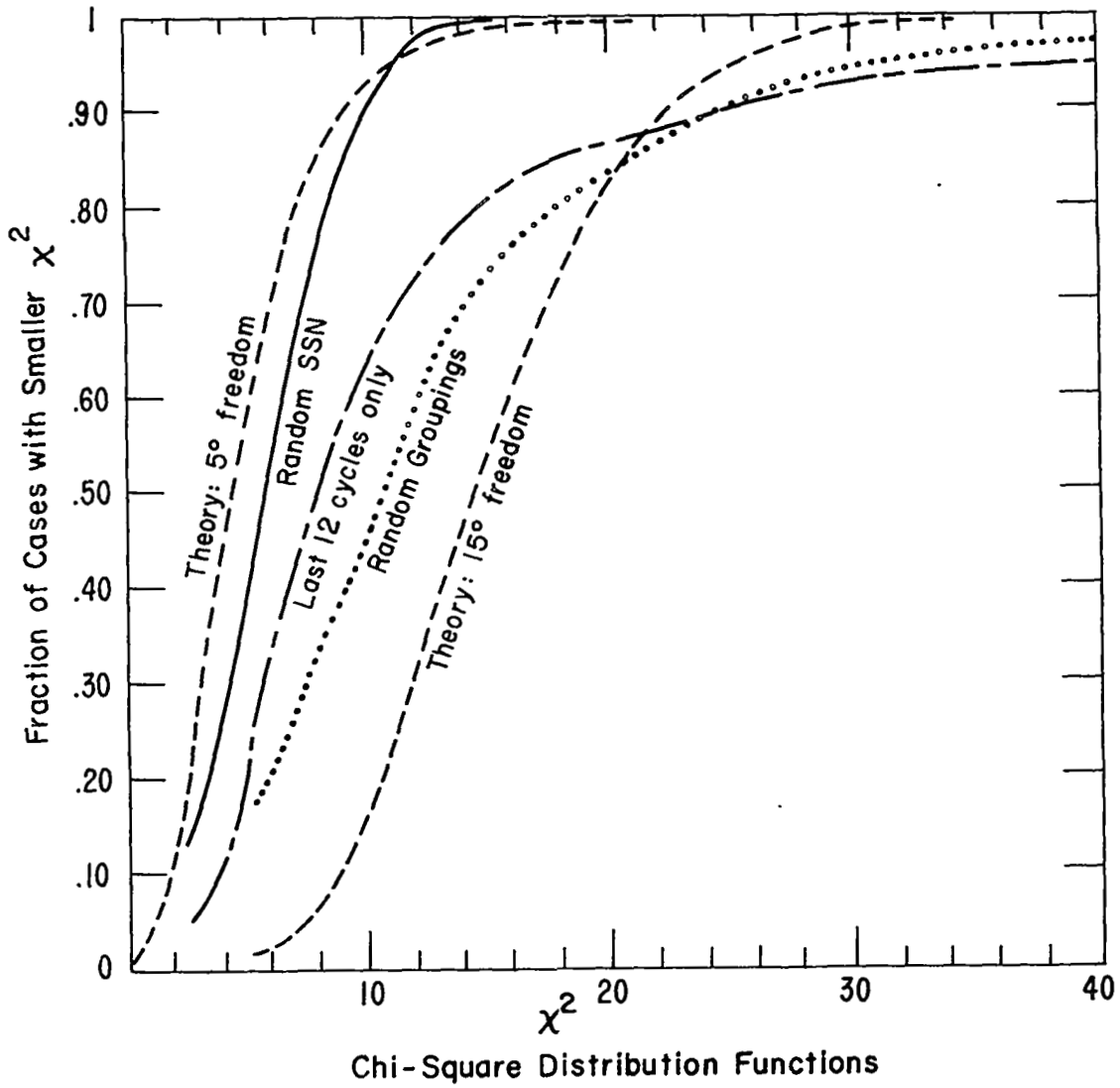


Figure 2.1

A third statistical experiment was to select 19×11 random (uniformly distributed) numbers to simulate the S_{ij} for the 19 cycles of 11 months each. From these random "data" the C_{ik} were determined, and then 100 random groupings of the cycles were taken to get 100 values for χ^2 , as in the second experiment. The resulting distribution function is plotted in figure 2.1, labeled "Random SSN". It can be seen that this compares well with the theoretical curve for 5 degrees of freedom, rather than with the actual sunspot number curve.

A fourth statistical experiment was carried out the same as the third, but without reducing the random "data" to zero mean and unity standard deviation. The resulting distribution function was similar to that from the third experiment.

From these experiments we see that when random numbers were substituted for the sunspot numbers the resulting distribution functions agreed well with the theory of the chi-square test. However, both experiments that used actual sunspot numbers gave distributions with much higher proportions of large values for χ^2 . In fact, the 50% point from the first experiment corresponds roughly with a chi-square distribution with 10 degrees of freedom, and its upper 5% point corresponds roughly to a chi-square distribution with 18 degrees of freedom, rather than 3 to 5 degrees of freedom as was expected from approximate theory. This may well be explained because the random sorting of the 19 observed cycles does not constitute a simulation of the distribution of chi-square for contingency tables, particularly since the sorting by cycles leaves significant correlations within each of the two groups but not from one group to the other. The random sorting of the cycles provides a "randomization test" that is of considerable interest, because we may well conclude that it is a more appropriate distribution with which to compare the chi-square observed for the true chronological order of cycles.

For the practical result, however, we conclude in either case that the first seven cycles should be rejected from the data base. If the calculated chi-square of 29.193 were interpreted on the basis of approximate theory with 3 to 5 degrees of freedom, we would reject them with a confidence of 0.99998. On the other hand, interpreting it on the basis of the random grouping of the sunspot cycles, either all 19 or the last 12, we still would reject them with a confidence of about 0.95. Thus there is no alternative to excluding them.

As a matter of interest, this method of testing, which differs from that used by McNish and Lincoln, was applied to their data. Using exactly the data included in their report, the value of chi-square was calculated at 20.89, which would reject the first seven cycles with a confidence of only 0.87 if based on the random grouping distribution functions. It was noted, though, that there are several discrepancies between the official Zurich sunspot numbers and those published in the McNish-Lincoln paper. In their table, six of the 12 numbers for the first cycle seem to be displaced 1 year, and four other discrepancies appear at other locations in the table. When these are corrected, the value of chi-square becomes 27.81, which would reject the first seven cycles with a confidence of at least 0.93. Finally, leaving out the 12th year of each cycle because of its frequent overlap with the next cycle, the calculation gives 27.13, which would reject the first seven cycles with a confidence of at least 0.92. Thus all of these conclusions agree with that arrived at by McNish and Lincoln in 1949: the first seven cycles should be rejected. The addition of the two cycles subsequent to 1949 does not change the conclusion. However, our method of testing has the advantage over McNish and Lincoln's that no assumption is made concerning the shape of the hypothetical common distribution.

3. STATISTICAL ANALYSIS

In analyzing the selected data base, we used the standard statistical techniques of multiple linear regression (Crow et al., 1960). Most of the calculations were performed by a master computer program that was developed with enough flexibility to handle all of the different analyses made. It contained controls for conveniently using a single data base and determining the number of predictors to be considered, the lags of each, and for calculating arbitrary functions of any predictor or combination of predictors. These features were used extensively in sections 5 and 6. Further, the program checked whether the data were missing for the predictand or any of the predictors and adjusted its calculations accordingly. This made it possible near the end of a solar cycle to distinguish between the data for cycles that were still decaying and the data for cycles that had passed their minimum and were starting up again. This distinction was carried throughout the analyses reported in sections 4 through 6, but turned out not to provide helpful information. For the prediction past the cycle minimum, in section 9, all data were used, whether or not the cycle minimum had occurred.

In addition to the regression equation fitting the data that were analyzed, the program gave the estimated rms error of the forecast and the multiple correlation coefficient. As discussed in section 4, the former was used as a quality measure of the resulting formula, and competing formulas were compared largely on its basis. The latter was used to test the statistical significance of the resulting formula, as also discussed in section 4. The program also contained controls for separating the data base into developmental and verification samples, and these were used extensively in the analyses reported in section 7. Finally, in accordance with Crow et al. (1960), there was provision for calculating the confidence interval of a specific prediction, as was used in sections 7, 8, and 9.

4. SINGLE PREDICTOR

It is not easy to make a graphical representation of differing forecast formulas which will give a good visual impression of the many different comparisons that need to be made. Not only are there the differing forecast formulas, but also for each one it is necessary to consider the accuracy achieved as a function of the time when the forecast is made and as a function of the time that has elapsed since then. The full representation would be multidimensional, and any reduction to two dimensions will emphasize some aspects at the expense of others. In this study we were particularly concerned with the question of how far into the future a given forecast will be valid. Thus a representation was chosen that emphasizes that question. A forecast formula was assumed to have been used at a given time within the sunspot cycle to predict the sunspot number for as far into the future as possible. Then the estimated error of that prediction was plotted for comparison with that of other forecasts, either from other formulas or from the same formula but prepared at a different time.

Also, this study has been carried out at a time when it is known that the current sunspot cycle is in its declining phase, and the declining phase will almost certainly continue for at least 3 or 4 more years. Thus this phase is of greater practical interest than the rising phase, and for the declining phase it is possible to consider not only formulas that apply at all times of the cycle, but, in addition, formulas that may use specific information about the maximum of the cycle --- its amplitude, its time, the risetime leading up to it, and the like. Accordingly, a representation was also chosen which emphasizes the declining phase of the cycle and which facilitates comparisons between formulas that do and those that do not include specific information about the maximum.

Figure 4.1 shows the results obtained when a single predictor is used consisting of the latest available observation. The formula used is

$$\hat{S}_j = b_0 + b_1 S_k, \quad (F1)$$

where

\hat{S}_j is the forecast for the month j ,

S_k is the smoothed annual SSN observed for month k ,

b_0 and b_1 are constants chosen for each combination of j and k ,

and the constants are to be derived from the data lined up with cycle minima coinciding.

In the figure is plotted the estimated rms error of the forecast, calculated as

$$\text{ESTRMS} = \sqrt{\frac{\text{RSS}}{N_{\text{obs}} - N_{\text{tor}} - 1}}$$

where

RSS = residual sum of squares after fitting,

N_{obs} = the number of observations fitted,

N_{tor} = the number of predictors,

and the -1 is present because there is one more constant to be determined than the number of predictors.

This is plotted vs. the number of months after cycle minimum, and use was made of data from the 12 cycles (numbers 8 through 19), lined up with their minima coinciding as was done by McNish.

FORECAST FROM LINEUP OF CYCLE MINIMA
 ONE PREDICTOR = LATEST OBSERVATION

19

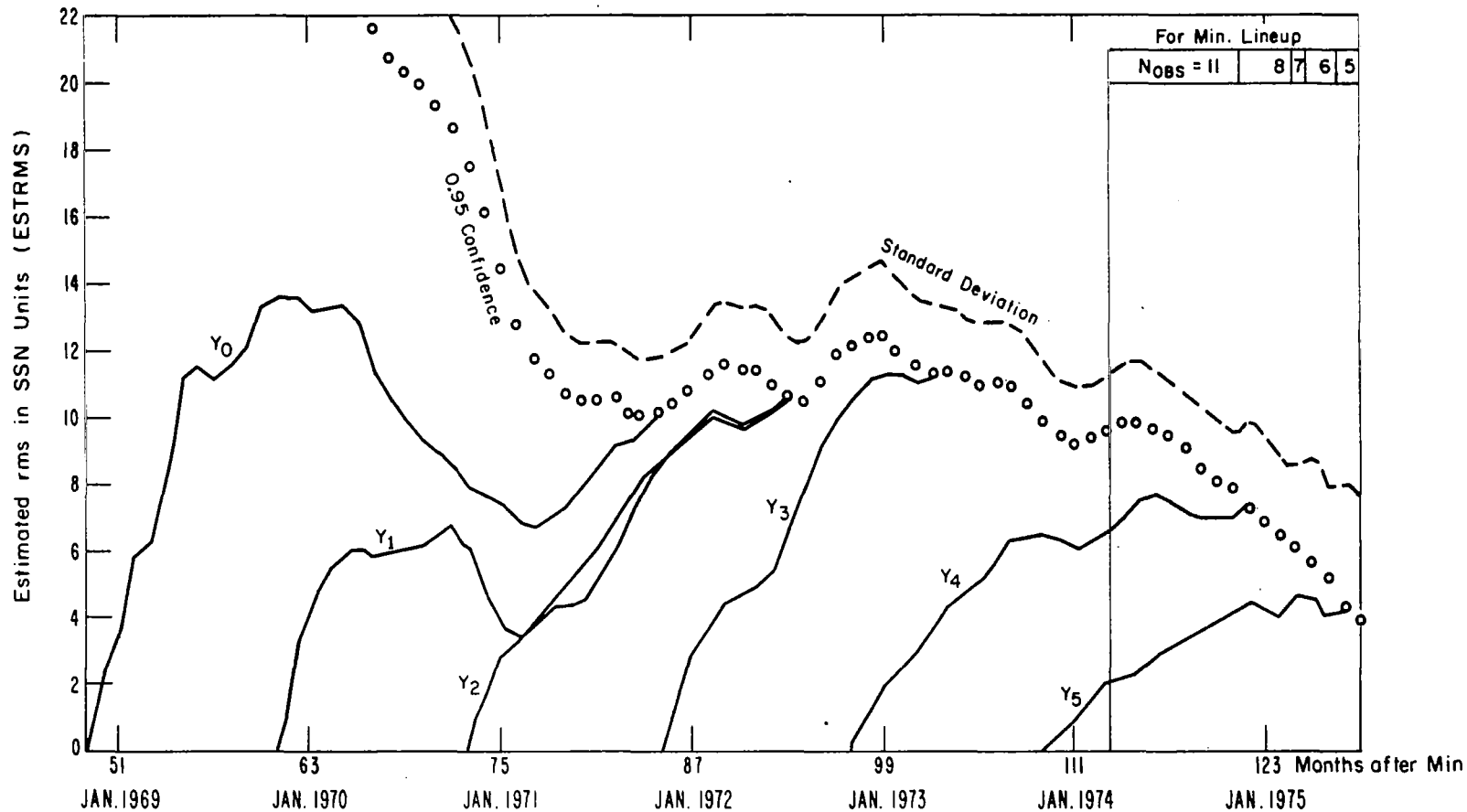


Figure 4.1

The figure shows results for months 49 through 129 after the minimum, and in this form it is applicable to any future cycle. However, to clarify its application to the current cycle, the abscissa has also been labeled with the corresponding dates for this cycle, January 1969 through January 1975. The six curves in the lower part of the figure show the results for forecasts made at 1-year intervals, using as a predictor the smoothed annual SSN for months 49, 61, 73, etc. The curves are labeled Y0, Y1, Y2, etc., for a reason explained later. Remember that the smoothed annual SSN for any particular month is calculated from observations made in the future as well as in the past. Six future monthly observations of the SSN are used in the calculation, so it is logically impossible to know the value of the smoothed annual SSN until at least the end of the sixth month later on. In practice, the time for processing and distributing the data makes this more like seven or eight months. Thus the smoothed annual SSN for month 49 is not known until about month 56 or 57. Thus in application, the forecast cannot be prepared until seven or eight months after the start of each of the curves in the figure.

The validity of a regression fit can be tested by using the standard statistical test for the confidence that the correlation coefficient calculated in the regression is nonzero. For example, when fitting 12 observations with one predictor, the total number of variables in the calculation is 2, and the number of degrees of freedom is $12 - 2 = 10$, since there are two adjustable constants in the calculation. Thus, from Crow et al. (1960), a calculated correlation coefficient greater than $r = 0.576$ will give greater than 0.95 confidence that

the true correlation coefficient is greater than zero; i.e., that the regression is meaningful. Thus in the previous equation for ESTRMS, we can calculate the residual sum of squares, RSS, from

$$RSS = TSS (1 - r^2)$$

where

TSS = the total sum of squares before fitting,
and we have for the standard deviation before fitting, STD,

$$STD = \sqrt{\frac{TSS}{N_{obs} - 1}} .$$

Thus, putting these together gives

$$\frac{ESTRMS}{STD} = \sqrt{1 - r^2} \times \sqrt{\frac{N_{obs} - 1}{N_{obs} - N_{tor} - 1}} .$$

For the above case, r of 0.576 gives

$$\frac{ESTRMS}{STD} = 0.857 .$$

Both r and ESTRMS/STD change when N_{obs} changes.

In figure 4.1 there has been plotted both the standard deviation of the data and the value of ESTRMS calculated from the above formula; the latter is labeled "0.95 confidence". Thus wherever the curves plotted for the various forecasts lie below this latter curve, there is greater than 0.95 confidence that the regression is meaningful. Where the curve plotted for the forecast crosses the "0.95 confidence" line is the limit for meaningfulness with this confidence. In figure 4.1 the forecast curves have been terminated at this point. In actual application of a particular forecast formula, one would probably continue to use the forecast beyond that point, but should remember that there is less confidence that the forecast result is meaningfully different from the mean of the raw data.

While the average duration of a sunspot cycle is about 11 years, there is a great variation from cycle to cycle. In the group from cycle 12 through 19, the longest was cycle 9 with 153 months, and the shortest was cycle 8 with 114 months. (In the earlier cycles which were rejected from this analysis there was an even greater spread, with cycle 4 at 170 months and cycle 3 at 108 months.) To avoid mixing data from the falling part of one cycle with those from the rising part of the following cycle, in the analysis shown in figure 4.1 the data were terminated at the time of minimum at the end of each cycle. Thus all 12 cycles were present for only 113 months after minimum, only 11 cycles had durations up to 121 months after minimum, and so on. This results in a reduction in the number of observations available for the analysis, and is shown in the figure.

Some features of particular interest can be noted in figure 4.1. The forecasts made from the data of month 49 (the time of maximum of the current cycle), remain significant for about 3 years, i.e., the curve of ESTRMS remains below the curve of "0.95 confidence." Forecasts made from data 2 years later (month 73) remain significant for only about a year and a half. But data 2 years later still (month 97), give forecasts retaining their significance for about 2 years. Thus there is a particularly difficult time to forecast, stretching from about month 85 to about month 95 (for the current cycle, from about 3 years after the maximum to about 3.5 years after the maximum).

It may also be noted that for forecasts prepared from data at month 49, the estimated error of the forecast rises very rapidly, reaching in 1 year a value almost as high as for any future time. Thereafter the estimated error falls until at 2 years it is only half what it was at 1 year. In fact, at about 2 years, about month 76 or 77, there appears to be a minimum in the error estimates for those forecasts prepared earlier. The standard deviation of the original data falls rapidly to about this time and then remains nearly constant for 2 more years.

It may also be noted that beyond month 76 the forecasts, made from month 61 are as good as those made from month 73 --- there is no advantage gained by making the forecasts from the later data.

Because the primary practical interest of this analysis deals with the falling part of the present cycle, consideration was given to using some feature of that portion of the cycle which would give additional information about the given cycle being forecast. The smoothed annual value for the sunspot number contains observations made during the 6 months following the month on which the number is centered, so there is good opportunity to observe the falling monthly averages that follow the maximum, and thus within a very few months be assured that the maximum has been passed. Once it is passed, the value of the maximum and the rise time from the preceding minimum become available and give information about the shape of the cycle.

The most obvious way to make use of the information that the maximum has been passed is to measure time within the cycle from the maximum rather than from the minimum. That is, when assembling the data from the preceding cycles for comparison with the present one, to line them up with their maxima coinciding rather than with their minima coinciding. This has historical justification from Waldmeier's studies showing that the shape of the whole cycle is strongly dependent on the value of the maximum, with amplitudes at selected later times dependent largely on that maximum value (Vitinskii, 1962). Also, when the cycles are lined up with their minima coinciding, there is a time from 38 months to 63 months after the start when some cycles are still increasing while others are already decreasing after their maximum. Lined up with maxima coinciding, all of the cycles are generally decreasing thereafter.

This leads us to formula (F2), which is an equation that looks the same as (F1), except that its constants are to be derived from data lined up with the cycle maxima coinciding, rather than with the minima coinciding, as was the case for formula (F1).

The results from formula (F2) are shown in figure 4.2. This figure has been prepared to facilitate direct comparison with figure 4.1; the abscissas of the two figures are lined up appropriately for the current cycle. Again to facilitate comparisons among the different formulas, the curves are not labeled with the month number k which has an awkward difference of 49 between the minimum lineup formulas and the maximum lineup formulas. Instead, the corresponding curves are labeled Y_0, Y_1, Y_2 , etc., which represent the number of years after the maximum of *the present cycle* to the latest observational data used for the forecasts represented by that curve. As applied to figure 4.2 these designations are straightforward. As applied to figure 4.1, however, it must be remembered that the straightforward meaning of the designations applies only to the present cycle. That is, in the present cycle (No. 20) the maximum occurred at 49 months after the minimum. Thus for minimum lineup data the designation Y_0 really means that the latest data used were those for 49 months after the minimum, and that interpretation needs to be kept in mind for the general application of figure 4.1 to arbitrary future cycles. For such a general application, the designations are indeed awkward to use, but they greatly simplify the comparisons of greatest practical interest for the next several years, those pertaining to the latter portion of the present cycle.

The most obvious comparison of figure 4.2 with figure 4.1 is that the estimated rms error for curve Y_0 increases much less rapidly, but without the dip about 2 years after its start. That is, at 12 months after the maximum, the ESTRMS of figure 4.1 is 13.5 SSN units, while for figure 4.2 it is only 6.1 SSN

FORECAST FROM LINEUP OF CYCLE MAXIMA
 ONE PREDICTOR = LATEST OBSERVATION

25

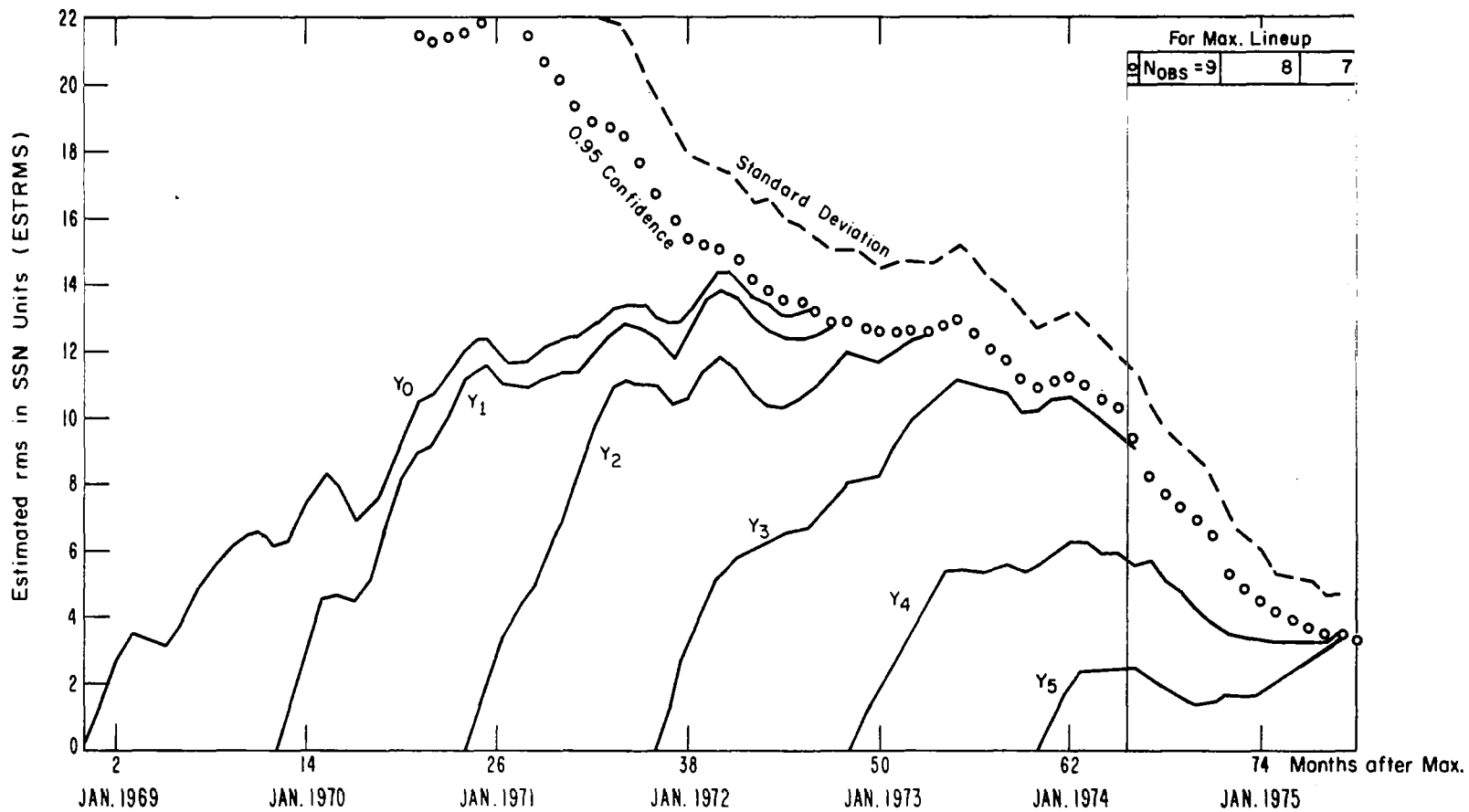


Figure 4.2

units; however, at 28 months after the maximum it is only 6.8 SSN units for figure 4.1 but 11.8 units for figure 4.2. It may also be noted for curve Y0 that the duration from the start of the forecast until it crosses the "0.95 confidence" line is almost a year longer than in figure 4.1, but (unfortunately for the precision of the forecasts) throughout most of the figure the standard deviation of figure 4.2 is higher than that of figure 4.1. Consequently, the Y0 curves are about equally high when that of figure 4.2 crosses the "0.95 confidence" line. In other words, is it any better to have confidence in a forecast, if its resulting estimated error is no better than the mean of the data of figure 4.1?

These comparisons between the corresponding curves of these two figures are so difficult that the pairs of curves have been extracted and are shown in figures 4.3 through 4.8. In figure 4.3 is shown the comparison of the Y0 curves, those made from data at zero time after the maximum. It can be seen that for the first 20 months the maximum lineup curve gives distinctly less error than the minimum lineup curve. Thereafter, however, the situation is reversed, with the minimum curve giving the lesser error for the next 24 months. The limits of 0.95 confidence that these forecasts are significant are shown by flags on the curves at the points where they cross the 0.95 confidence lines, and the longer duration for the maximum curve is also evident.

For forecasts made from data 1 year after maximum, the advantage of the maximum lineup has vanished. Figure 4.4 shows that while both methods start out about the same, by 8 months after the forecasts have been made, the error from the minimum lineup is markedly lower than the other, and remains so until the expiration of the 0.95 confidence regions.

COMPARISON OF MAX. AND MIN. LINEUPS

Y_0

27

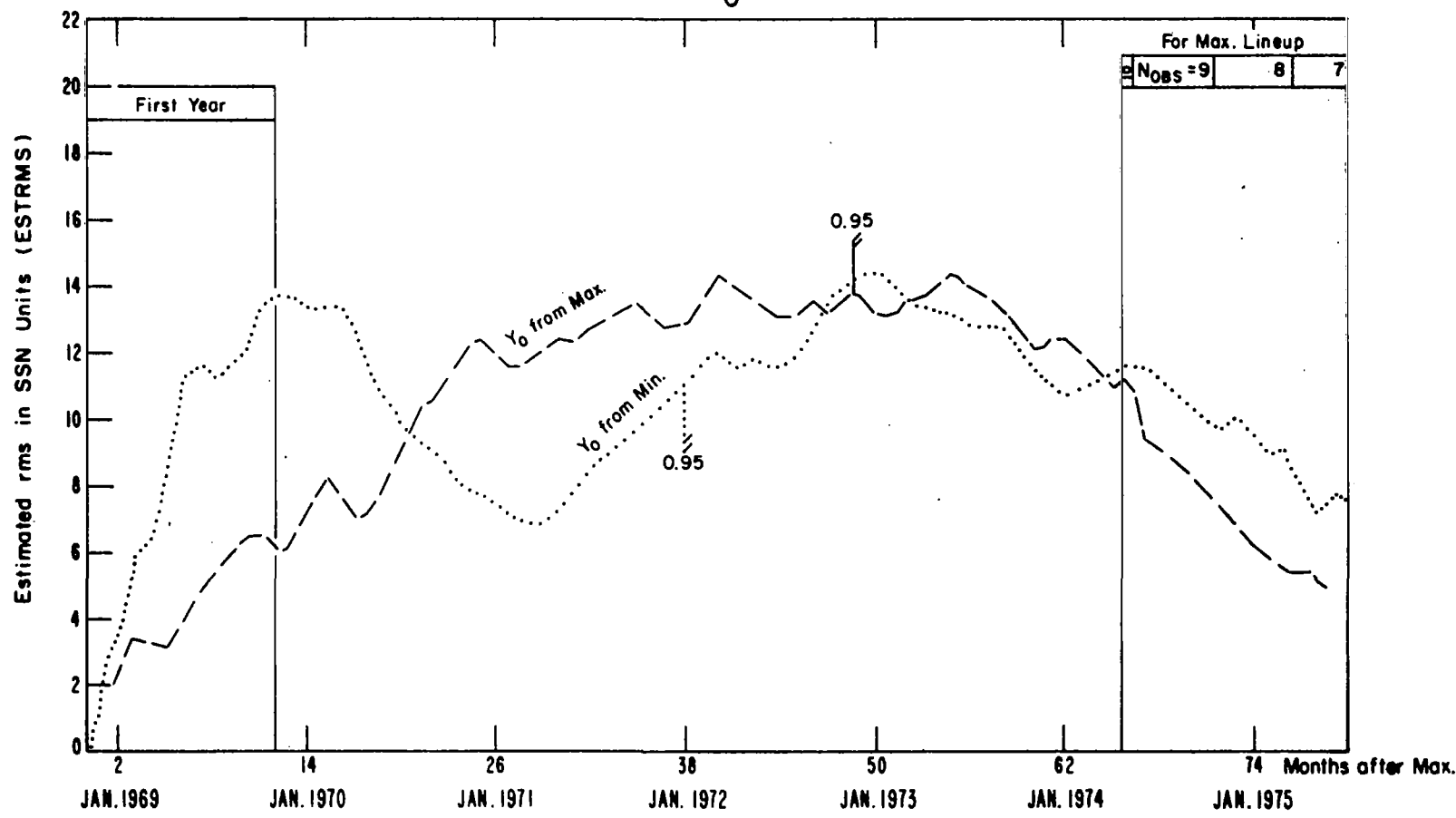


Figure 4.3

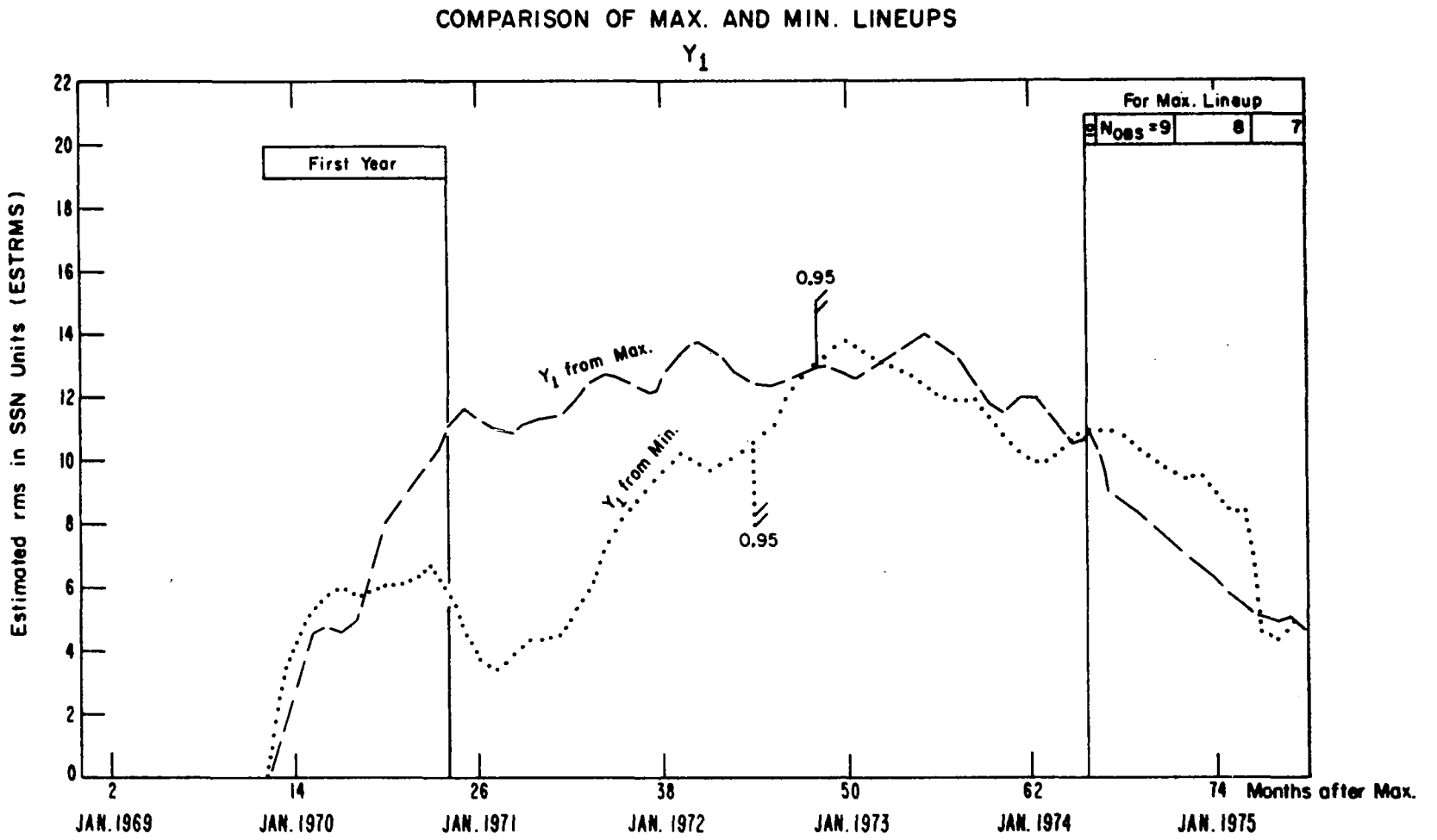


Figure 4.4

COMPARISON OF MAX. AND MIN. LINEUPS

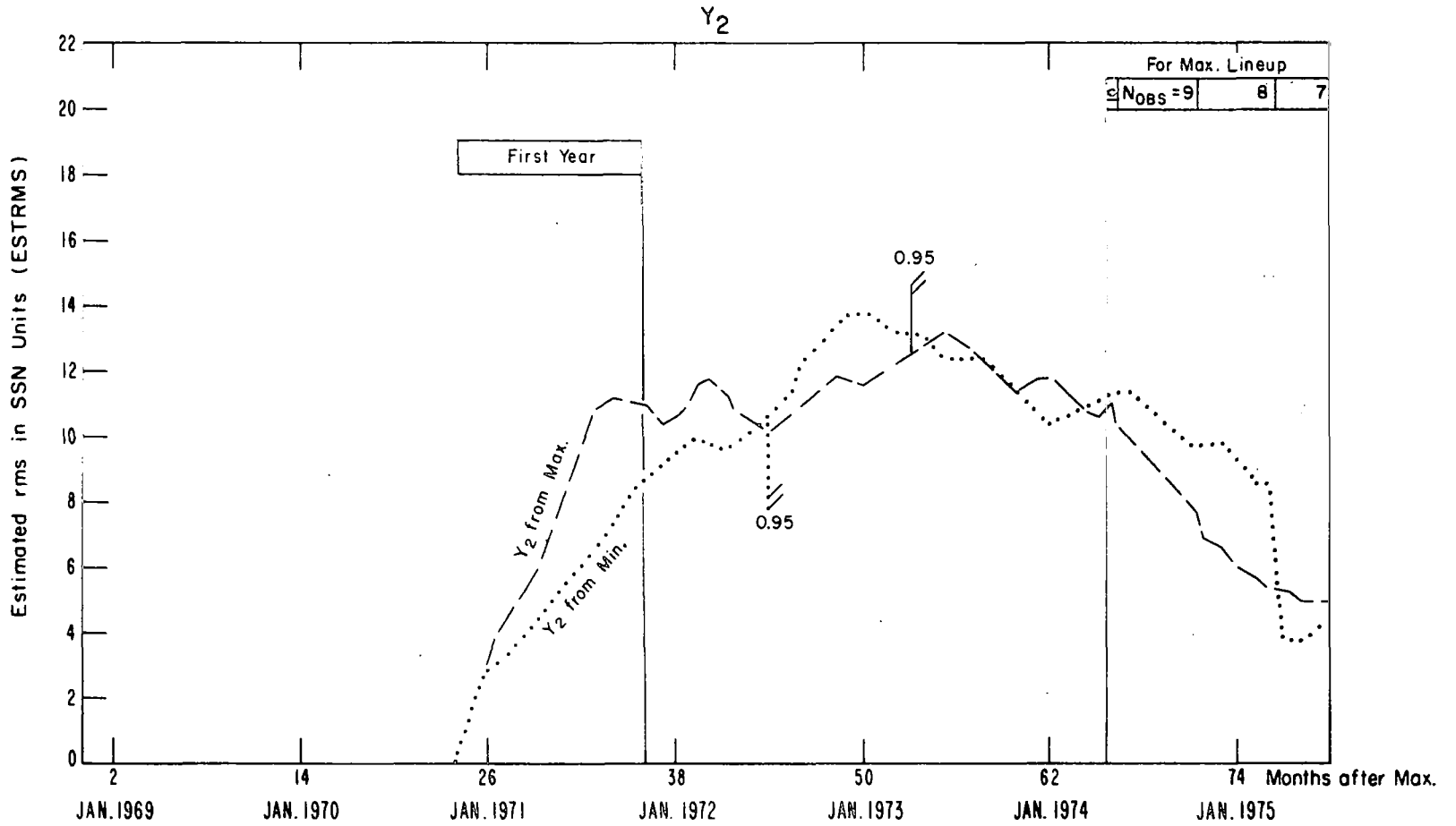


Figure 4.5

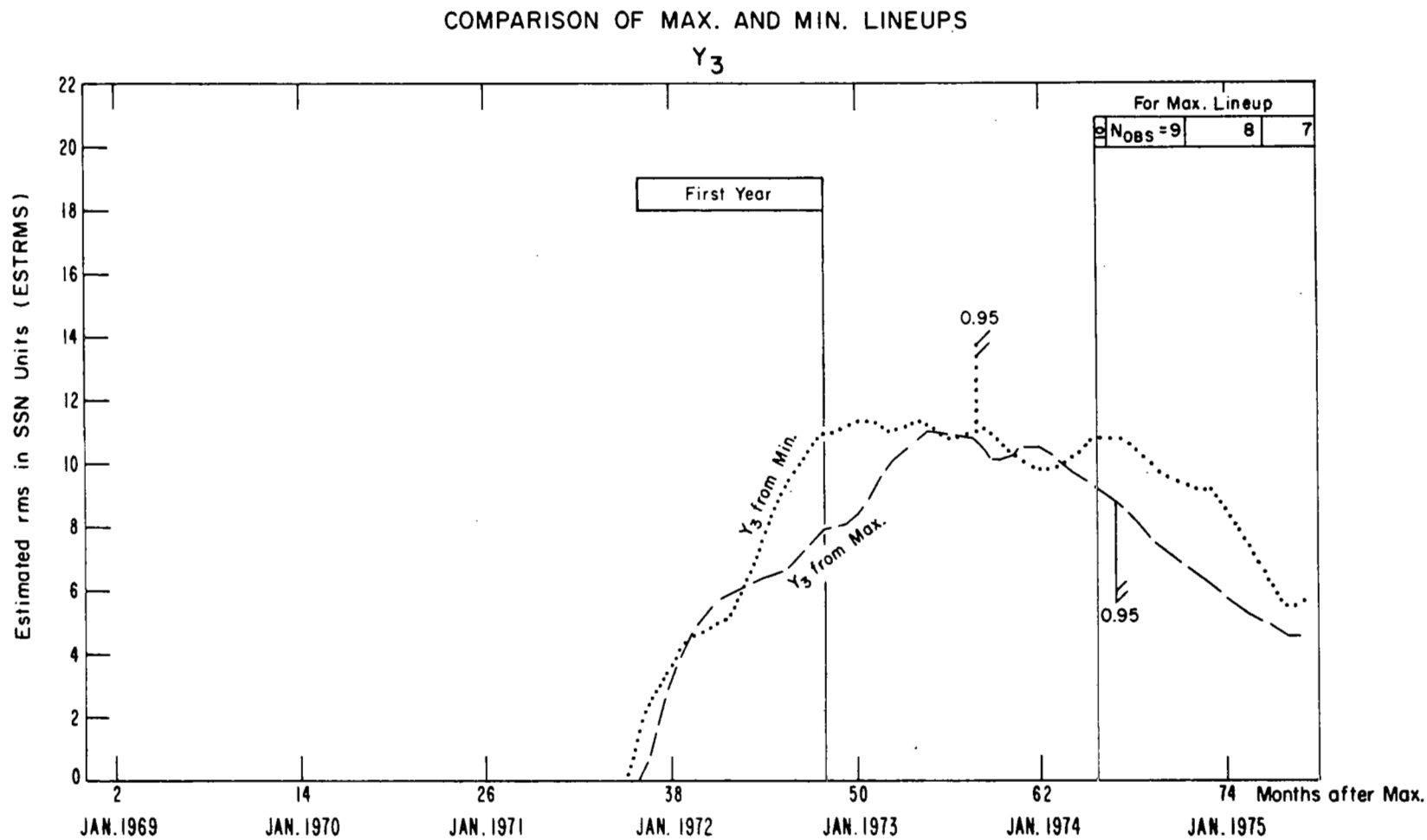


Figure 4.6

COMPARISON OF MAX. AND MIN. LINEUPS

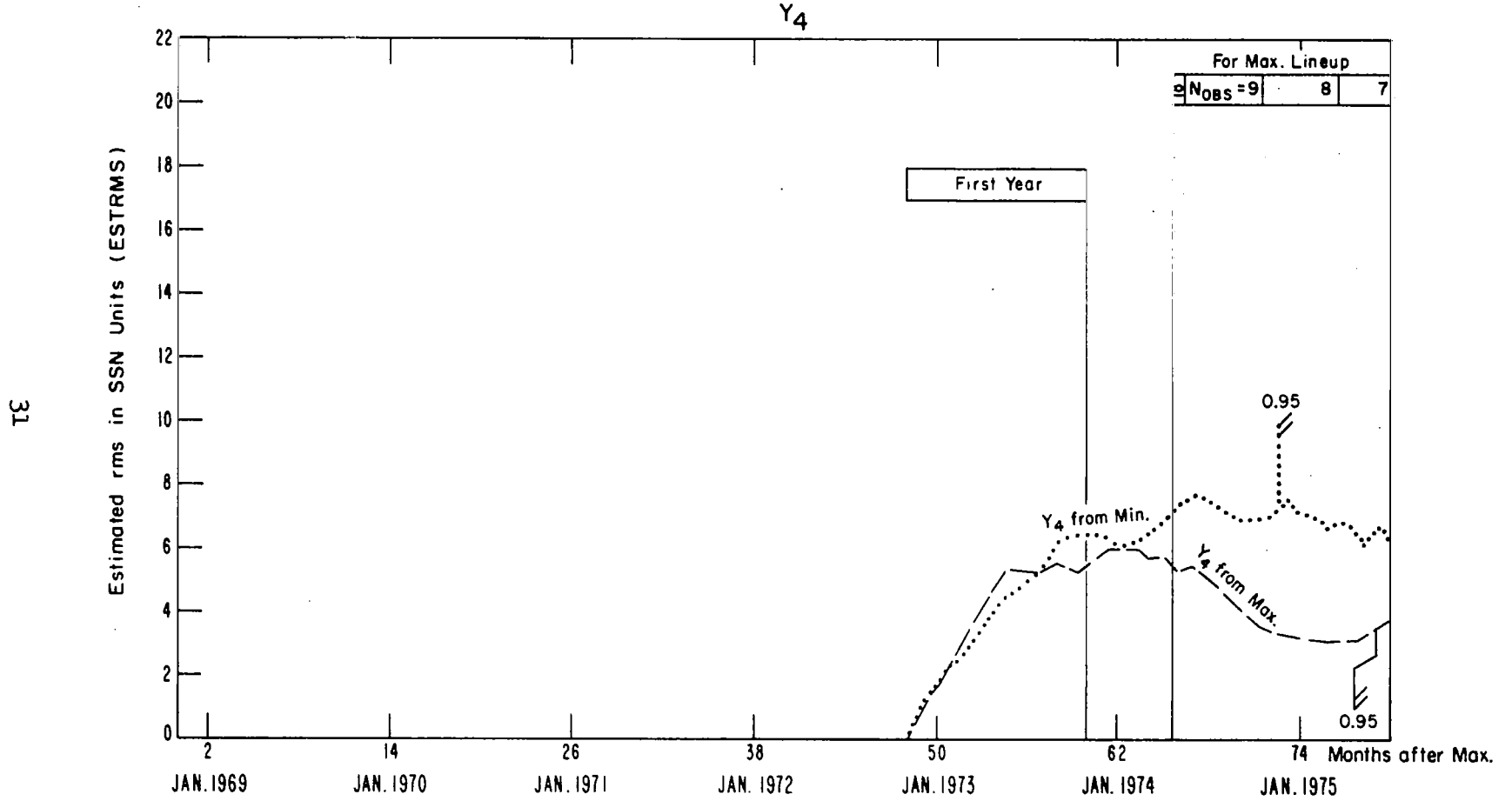


Figure 4.7

COMPARISON OF MAX. AND MIN. LINEUPS

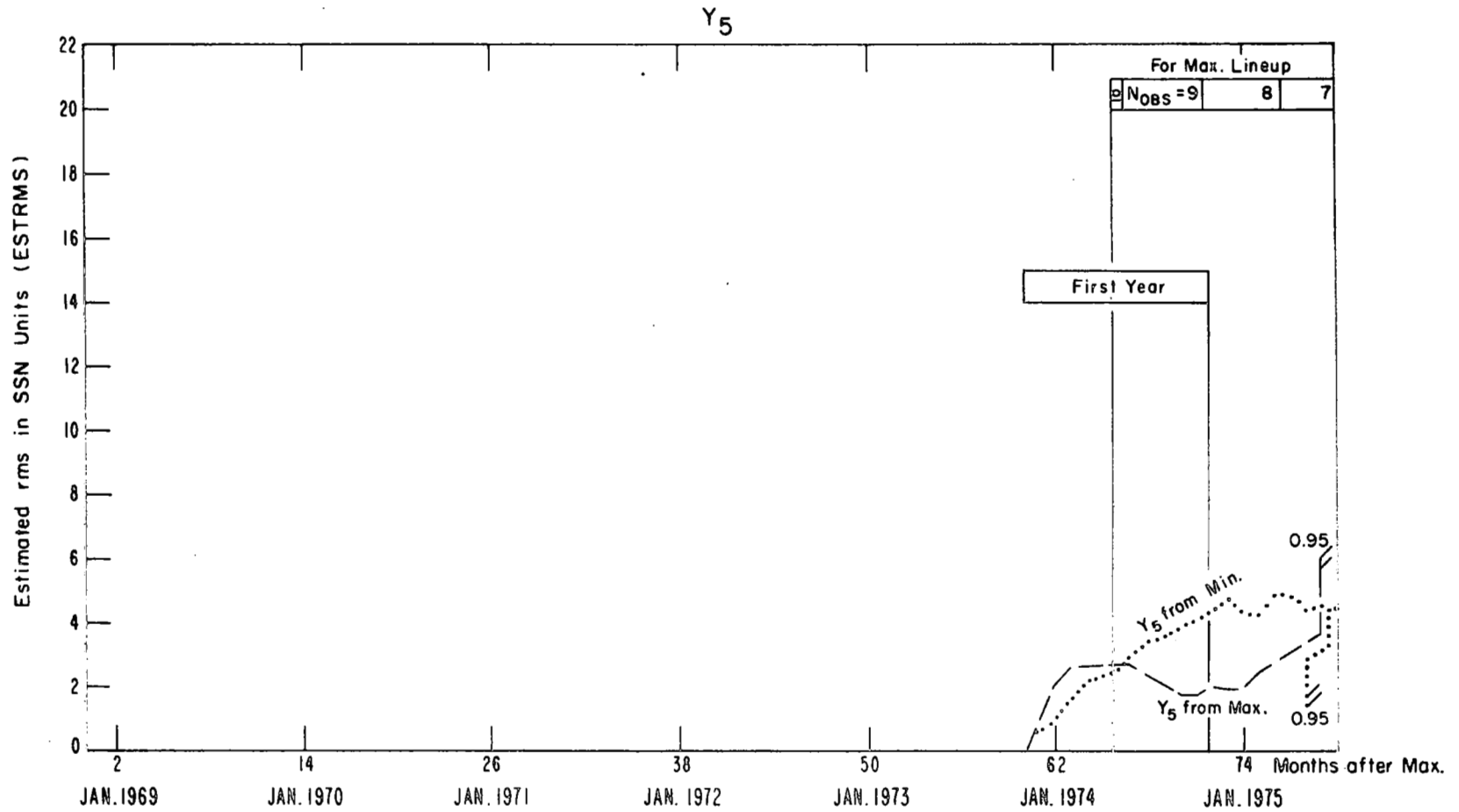


Figure 4.8

An examination of all of the figures 4.3 through 4.8 shows a consistency that appears indicative of a possible improved forecasting method, but it is not fully satisfying. For the first $1\frac{1}{2}$ years after the maximum, forecasts prepared from the maximum lineup are distinctly better than the others. From $1\frac{1}{2}$ to $3\frac{1}{2}$ years after the maximum, the contrary holds: forecasts prepared from the minimum lineup are the better. Then from $3\frac{1}{2}$ years to the end of the cycle the situation reverses again, and the forecasts prepared from the maximum lineup are always at least as good as the others and frequently better.

This may perhaps be a valid conclusion, but it is uncomfortably close to falling into the statistical trap of using different analyses to get good fits to short portions of a long data set. It is well known that if enough different functions are fitted to a long set of data, one function may fit the data well for a short time, another function for a different short time, and so on. It could be argued that such is not the case with this result because of the continuity of the conclusions over a sizable portion of the cycle, and because it comes not from a search among many different formulas but rather from a single *a priori* hypothesis: that behavior shortly after the maximum should be influenced more by the time from maximum than the time from minimum some 4 years or so farther away.

It could even be argued that the sudden drop in the standard deviation of the minimum lineup data that occurs $6\frac{1}{2}$ years into the cycle might well be only a statistical fluctuation resulting from the small number of cycles available for analysis, thus leading to a possibly false preference for the minimum lineup formulas at that time. However, we prefer to stay away from that trap and these arguments. The approach that has been used in this analysis is not to look

for a formula that will be effective for only a short time, but instead to look for a formula that will be generally better than the former method during all of the falling portion of the cycle, and never be noticeably any worse. Thus the results of figure 4.2 are interpreted as not showing an acceptable improvement over the updated McNish-Lincoln method shown in figure 4.1.

The most obvious single linear predictor to choose is the latest observation of the SSN, and that is what was done in figures 4.1 and 4.2. During the falling portion of the cycle there are two other obvious possible predictors, the value of the maximum and the risetime from the minimum to the maximum. The former of these coincides with curve Y0 of figure 4.2, and so has already been done. The latter is shown in figure 4.9, where it is plotted together with the Y0 curves from figure 4.3. It can be seen that the confidence of this being a significant predictor lasts longer than it does for either of the other curves, but there are only a few months where the resulting estimated error is any better than for the other curves; most of the time it is much worse.

FORECAST FROM RISETIME
(Max. Lineup)

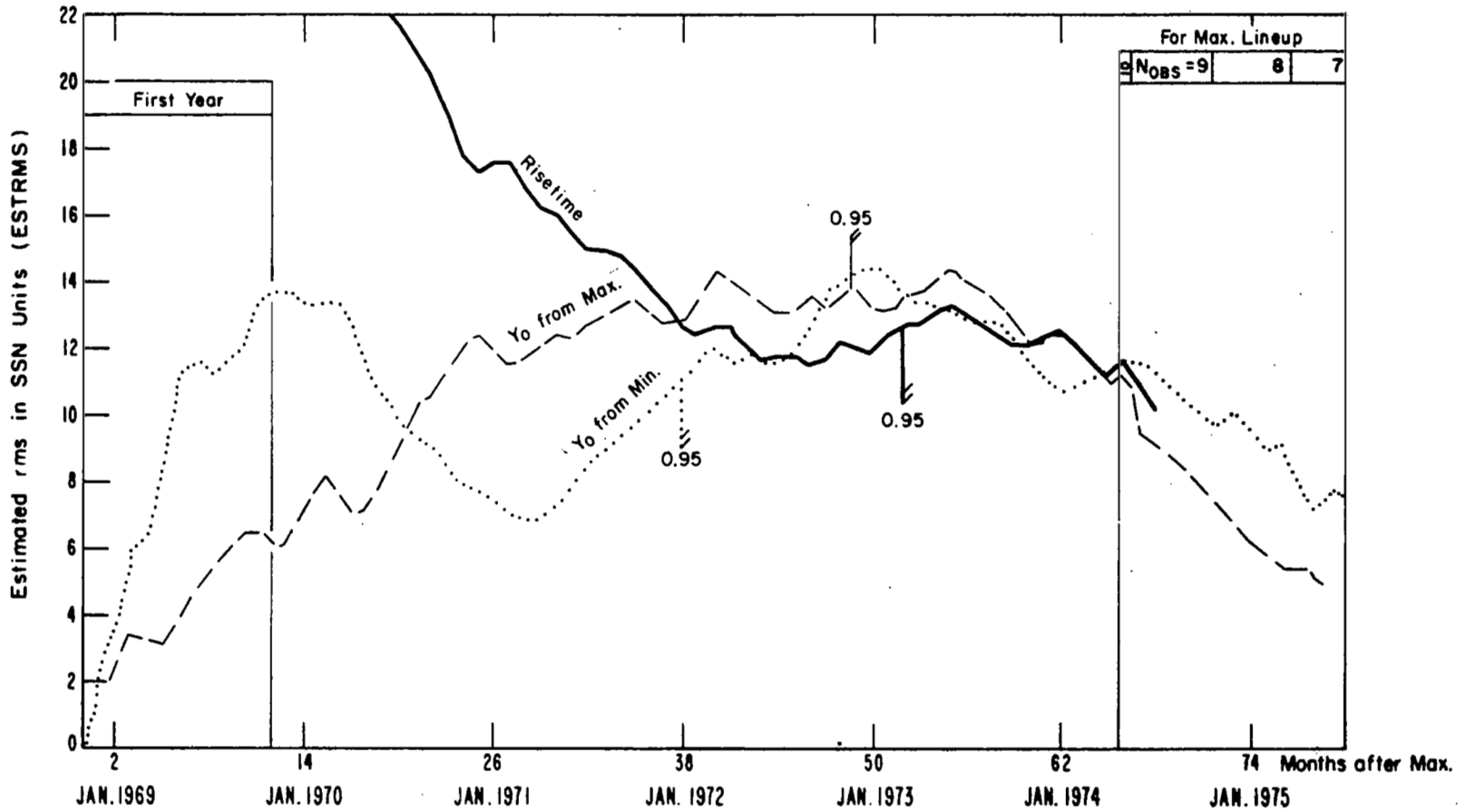


Figure 4.9

5. MULTIPLE LINEAR PREDICTORS

In selecting a single predictor for a forecast formula, the most obvious choice is to use the latest available observation of the phenomenon to be predicted. In selecting additional predictors, an appropriate choice is to use one or more earlier independent observations. The smoothed annual sunspot number, S_k , is a weighted average of observations taken during 13 months. Thus, two S_k a year apart are nearly independent and make a convenient choice to use. Using this choice, the formula used is

$$\hat{S}_j = b_0 + b_1 S_{k-12} + b_2 S_k, \quad (F3)$$

with the data lined up with cycle minima coinciding. The results from this formula are shown in figures 5.1 through 5.3. In the figures, the estimated rms error of this formula is shown superimposed on the curves from figure 4.3, to permit direct comparison with the results from formulas (F1) and (F2). In figure 5.1, it can be seen that (F3) has a slight advantage after times corresponding to March 1972, but this is so small as to be of doubtful significance. For all other times, there is almost no difference visible from (F1).

Figure 5.2 shows results from the same formula applied to data a year later, and this time there is no visible advantage at all from including the second predictor. The same is also true in figure 5.3, which shows the results for data still another year later.

When the same formula is applied to the data lined up with their maxima coinciding, the result is equally negative, as shown in figure 5.4. The second predictor makes essentially no change from the first predictor. Thus this formula is considered unsuccessful.

FORECAST $\hat{S} = b_0 + b_1 S_{y-1} + b_2 S_{y0}$
 FROM MIN. LINEUP

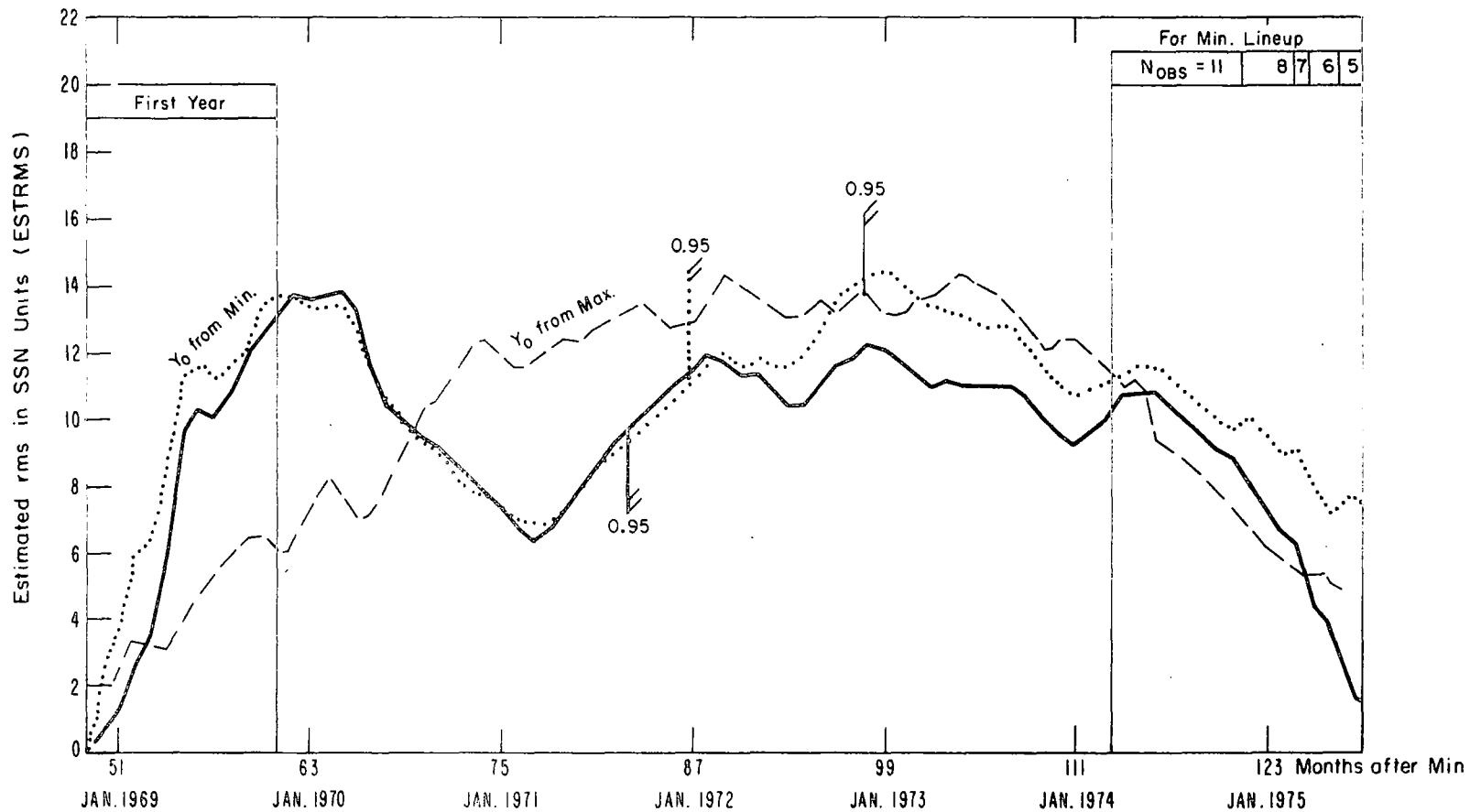


Figure 5.1

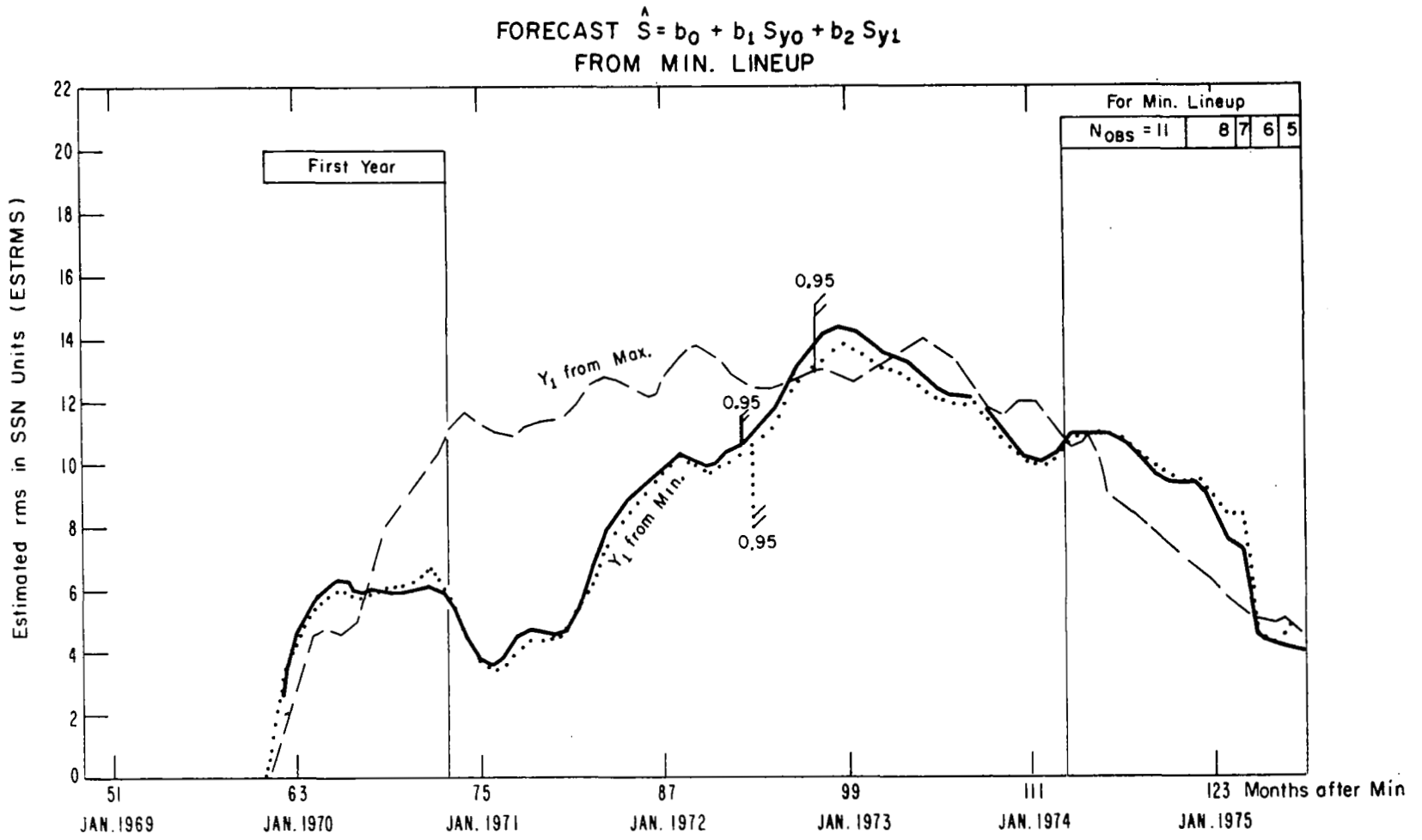


Figure 5.2

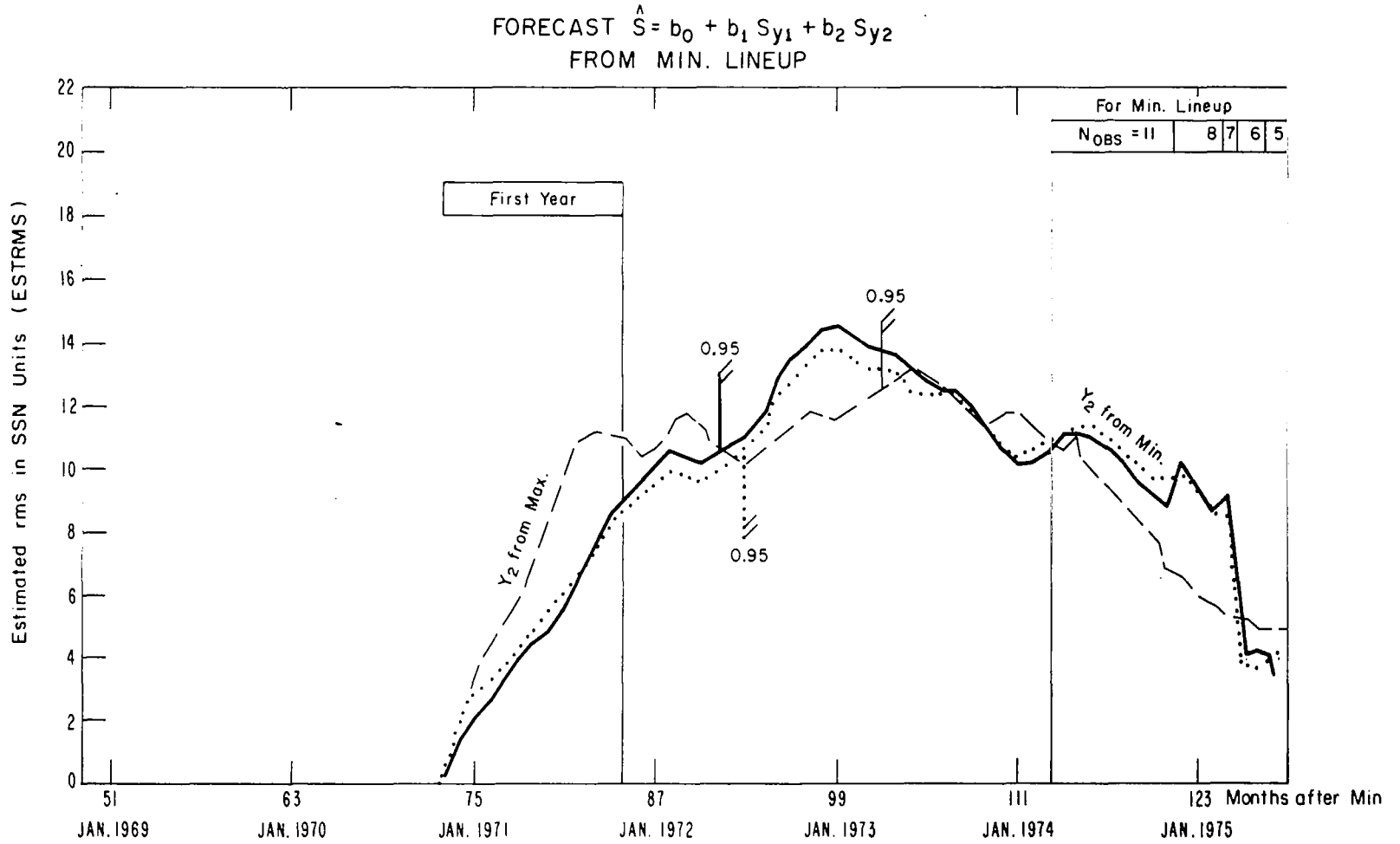


Figure 5.3

$$\text{FORECAST } \hat{S} = b_0 + b_1 S_{y0} + b_2 S_{y1}$$

FROM MAX. LINEUP

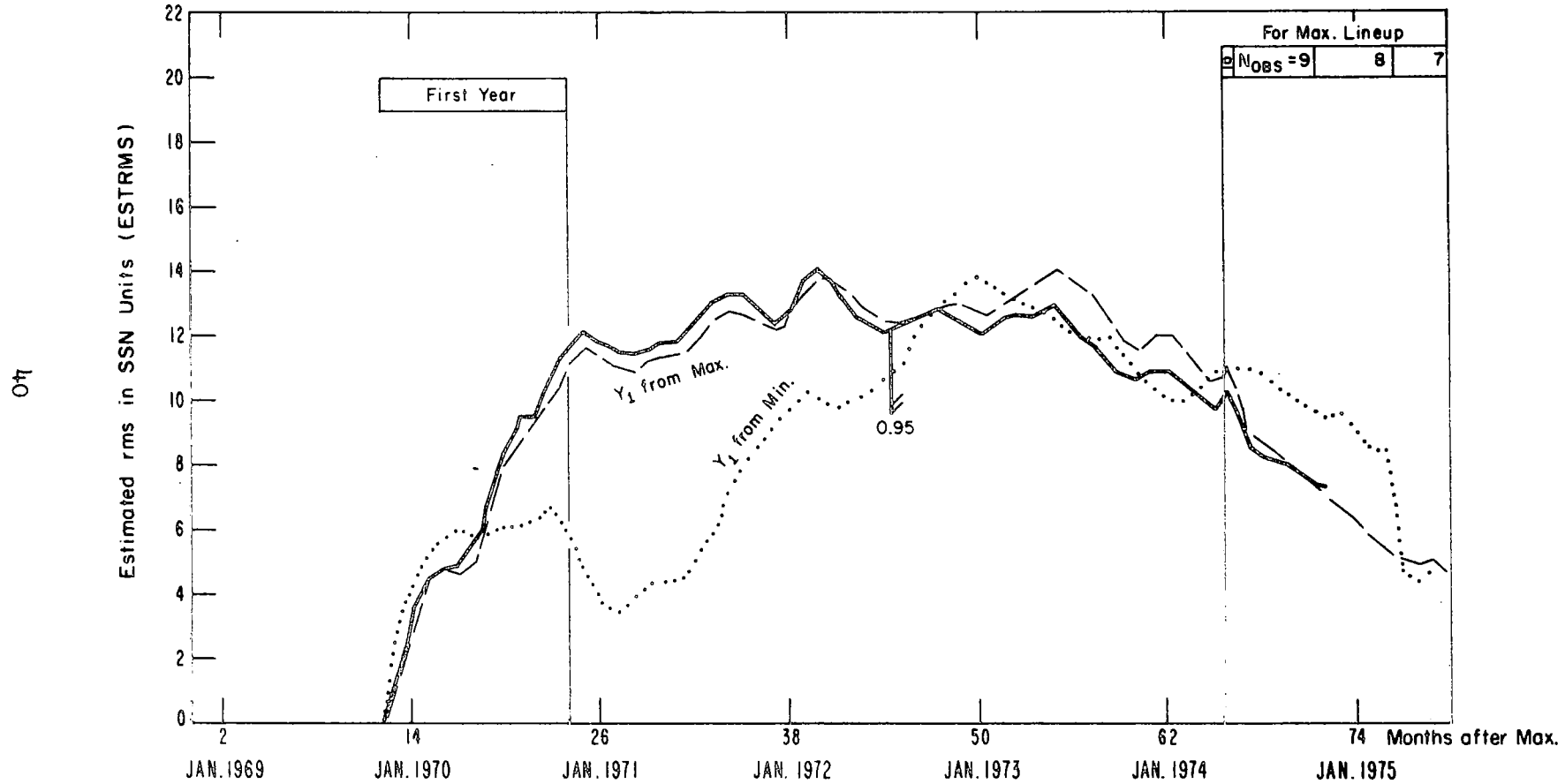


Figure 5.4

FORECAST $\hat{S} = b_0 + b_1 \text{RISETIME} + b_2 S_{y_0}$
 FROM MAX. LINEUP

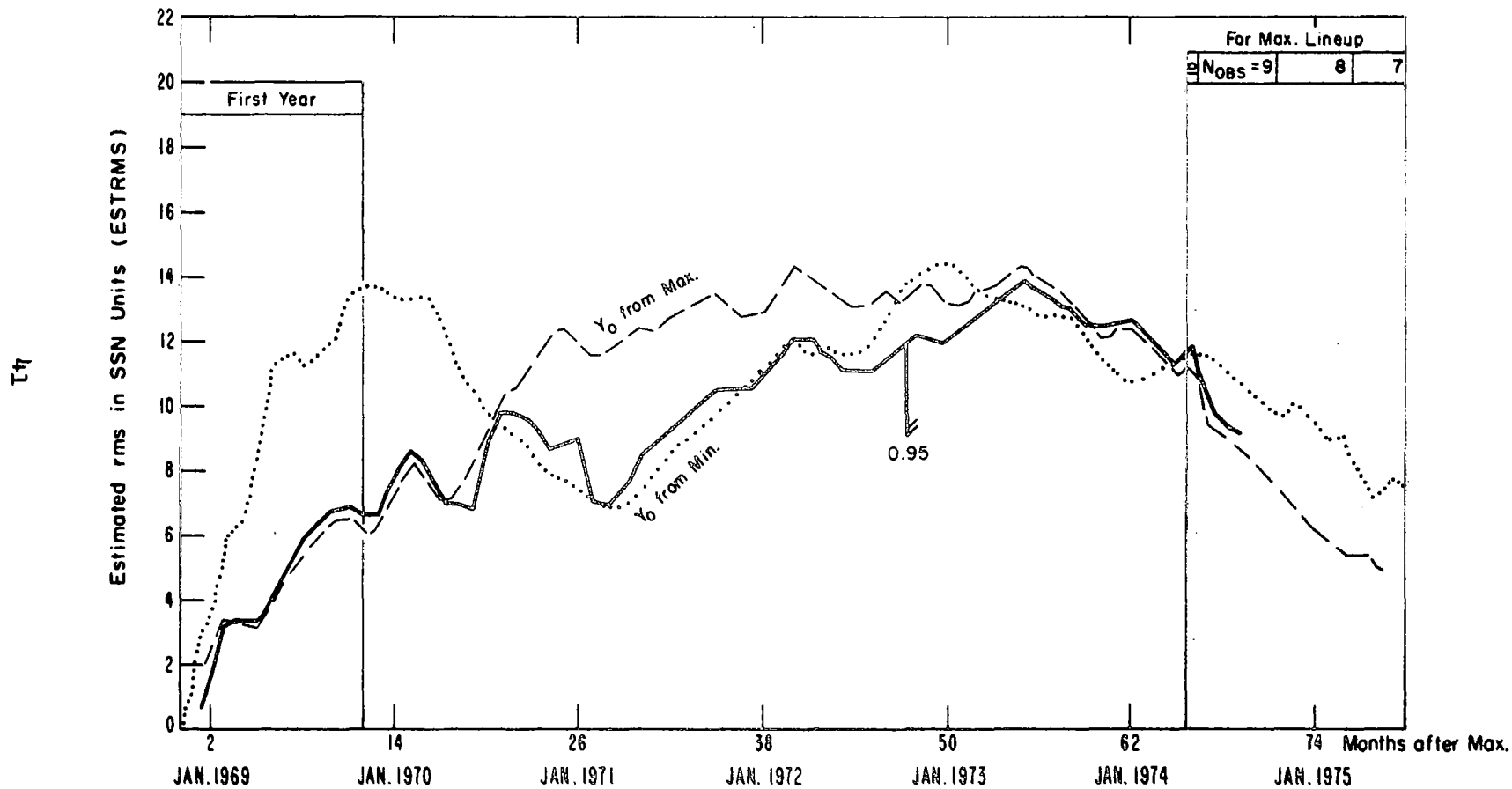


Figure 5.5

FORECAST $\hat{S} = b_0 + b_1 \text{RISETIME} + b_2 S_{y_1}$
 FROM MAX. LINEUP

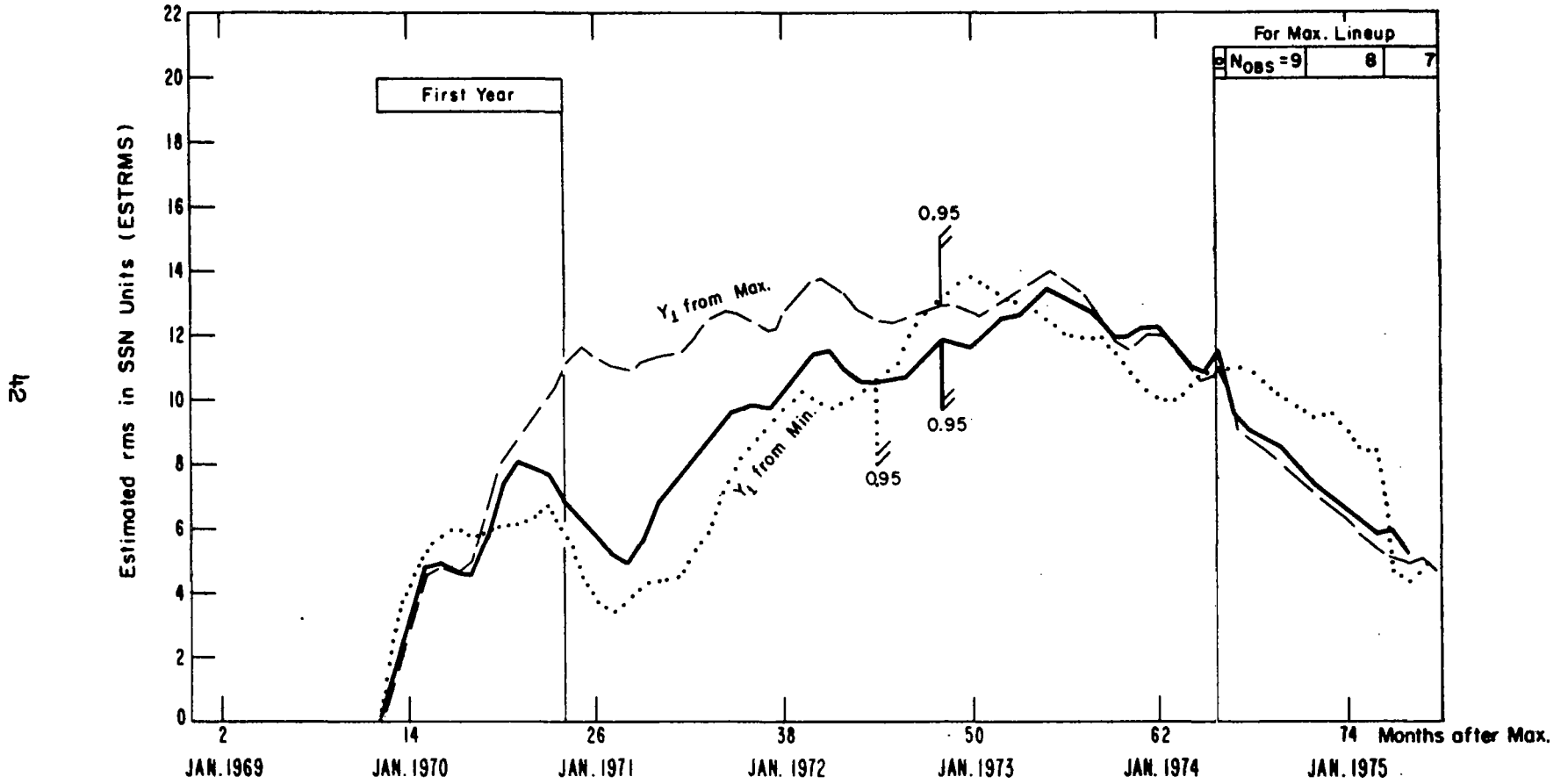


Figure 5.6

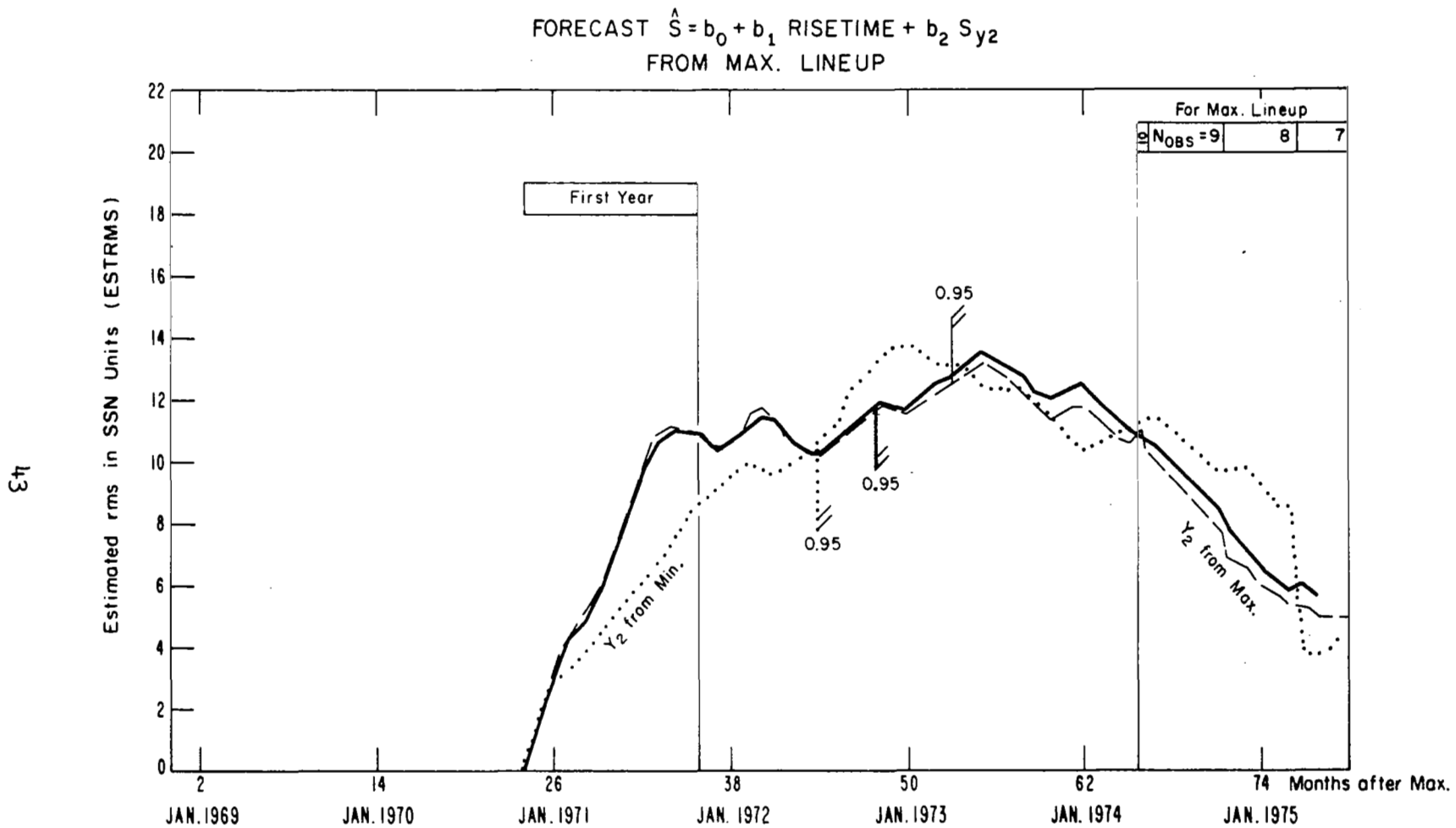


Figure 5.7

When the second predictor is chosen as the risetime of the cycle, the results are much better. The formula here is

$$\hat{S}_j = b_0 + b_1 R + b_2 S_k \quad (F4)$$

where R is the risetime of the cycle in months. This formula is inappropriate if applied to the early portion of data lined up with their minima coinciding. The longest risetime in the data base is 63 months, so prior to 63 months after the minimum this formula would presume prescience of a value for the risetime that is not yet completed. After that time it would be possible to use this formula for the minimum lineup data, but its calculations did not show enough improvement to be worth studying further.

When this formula is applied to the maximum lineup data, however, it gives very good improvements, as can be seen from figures 5.5 through 5.7. Here again the new results are superimposed on the graphs from figures 4.3 through 4.5, so direct comparison is possible with the former formulas. In figure 5.5 it can be seen that the new formula is at all times as good as the better of the two previous formulas. For the first $1\frac{1}{2}$ years it follows the maximum lineup curve, and thereafter it follows the minimum lineup curve. In figure 5.6 the new formula is almost as good as the minimum lineup curve. In figure 5.7, however, the new formula follows the poorer of the two other curves, that from the maximum lineup. This is not too surprising, since the influence of the risetime on the prediction could be expected to become small when reaching a time 3 or more years into the falling part of the cycle.

Averaged over all of the times shown in figures 5.5 through 5.7, this formula gives a decided advantage over either formula (F1) or (F2) which were derived from a single observation. The poorer performance in figure 5.7 is outweighed by the much

better performance shown in figures 5.5 and 5.6. Whereas formulas (F1) and (F2) had advantages one over the other at different parts of the cycle, this new formula represents a uniform forecasting method whose overall performance is better than either of the others. Thus this formula does satisfy the criteria that were established and can be considered to be an improvement over the previous formulas.

6. NONLINEAR PREDICTORS

When nonlinear predictors are admitted to the forecast formula, it makes possible a very large choice of predictors. It must be remembered that the regression calculations being used are based on a linear sum of the predictors, but the form of the individual predictors is completely arbitrary. They may be linear, nonlinear, or transcendental functions of any number of the available variables. There is the computational limitation that none of them may be linearly dependent on the others with respect to the set of observations used in the calculation, but this is easily avoided and is checked in the calculations. Other than that, the only limitation is one of reasonableness of the complexity used.

The use of nonlinear combinations of the variables is more than a mere blind search. For instance, it is not unreasonable *a priori* to suspect that regression relationships which may be appropriate for high parts of the cycle differ from those that would be appropriate for low parts of the cycle. One way of testing this suspicion would be to separate the data into groups having different average values, and then to see if the regression relationships differ when developed on the different groups. A more convenient process for computer solution is to include in the regression equation not only terms linear in the observations but also terms of higher order, like the square or the cube. If there is indeed any significant difference in the regression relationship depending on the amplitude of the observation, better results can be expected from the equations including the higher powers than from the equations not including them. If the relationship is a strong one, a search can then be made for the power that best represents it. On the other hand, if the dependence on amplitude is a weak one it will make little difference which power is included in the equation.

Thus trials were made of equations that involved observations made at a single time, together with the square of these observations. Then trials were made of equations involving observations made at two different times, together with the squares and cross products of these observations. Most of these equations showed more or less advantage for a restricted portion of the cycle, but they did not meet the criterion we had established of giving general improvement throughout the falling portion of the cycle, without ever being significantly worse than the former equations. For instance, equations tried included

$$\hat{S}_j = b_0 + b_1 S_k + b_2 S_k^2 . \quad (F5)$$

When this was applied to the data from minimum lineup of the cycles, it gave no visible improvement over formula (F1), which does not include the term in S_k^2 . Thus, the supposition of dependence of the regression relationship on amplitude was not confirmed. On the other hand, when this formula was tried on the data from the maximum lineup of the cycles, some improvement was evident shortly after cycle maximum and also near the end of the cycle, but not as much as from formula (F4). But this gave encouragement to the inclusion of the square term in other formulas being tried.

Turning to formulas that involve two observations at different times, we tried

$$\hat{S}_j = b_0 + b_1 S_k + b_2 S_{k-12} S_k . \quad (F6)$$

For the minimum lineup of the cycles, this gave no visible improvement. For the maximum lineup, it gave some improvement during the later phases of the cycle decay, but was distinctly worse during the earlier phases.

Other formulas that were tried without satisfactory improvement included

$$\hat{S}_j = b_0 + b_1 S_{k-12} + b_2 S_k + b_3 S_k^2 \quad (F7)$$

$$\hat{S}_j = b_0 + b_1 S_{k-12} + b_2 S_k + b_3 S_{k-12} S_k \quad (F8)$$

$$\hat{S}_j = b_0 + b_1 S_{k-12} S_k + b_2 S_k + b_3 S_k^2 \quad (F9)$$

$$\hat{S}_j = b_0 + b_1 S_{k-12} + b_2 S_k + b_3 S_{k-12}^2 + b_4 S_k^2 \quad (F10)$$

$$\hat{S}_j = b_0 + b_1 S_{k-12} + b_2 S_k + b_3 S_{k-12} S_k + b_4 S_k^2 \quad (F11)$$

Formulas (F7) through (F11) all involved only observations that were not specifically related to the cycle shape, and thus are equally well applicable to the data derived from the lineup of cycle minimum or maximum. In addition, by restricting attention to the maximum lineup, predictors such as the risetime R can be included, and were already found to be useful in (F4). Thus we tried

$$\hat{S}_j = b_0 + b_1 R + b_2 S_k + b_3 R S_k \quad (F12)$$

$$\hat{S}_j = b_0 + b_1 R + b_2 S_k + b_3 S_{k-12} S_k \quad (F13)$$

$$\hat{S}_j = b_0 + b_1 R + b_2 S_k + b_3 S_k^2 \quad (F14)$$

$$\hat{S}_j = b_0 + b_1 R + b_2 S_k + b_3 S_k^2 + b_4 R^2 \quad (F15)$$

The first two of these, (F12) and (F13), did not give satisfactory improvement; however, (F14), which is the same as (F4) but with the addition of a term in S_k^2 , was found to be an improvement on (F4) and on all previous formulas tried. Formula (F15),

which is the same as (F14) but with the addition of a term in R^2 , gave slightly less improvement than (F14) did, showing that the additional term in R^2 contributed nothing but noise to the forecast.

Thus formula (F14) was chosen as the final formula. Its error comparisons are shown in figures 6.1 through 6.4, where it is compared with the formulas using a single predictor both from the maximum lineup and from the minimum lineup of the data. It can be seen that this formula (F14) gives an estimated error of its forecast that is nearly as small as the smaller of the other formulas, whichever that may be. For example, in figure 6.1 the single-term formula from the maximum lineup is much better than that from the minimum lineup for the first year and half. The new formula agrees with this better estimate. Then, after the first year and a half, the single-term formulas switch places, that from the minimum lineup becoming the better. But the new formula (F14) continues to follow the better of the two, which now is a different one from that which it was following previously. Thus we have a single formula that gives as low an error estimate as would have been obtainable from the earlier formulas only by using each of them for only a selected portion of the cycle. In figures 6.2 through 6.4 the pattern continues, with the new formula being essentially as good as the better of the two former ones --- sometimes slightly better and sometimes slightly worse, but not significantly so. Thus this formula satisfies the criterion that was established in searching for an improved means of preparing the forecasts.

The coefficients for this equation are given in tables 6.1 through 6.6. In table 6.1, it can be seen by examining the size of the coefficients that the value of the observation S_k has a strong influence on the early forecasts; this influence gradually decreases up to a j of 24 months after the observation and then increases again. At the same time,

FORECAST MADE FROM DATA AT TIME OF MAXIMUM

$$\hat{S} = b_0 + b_1 \text{RISE TIME} + b_2 S_{y_0} + b_3 S_{y_0}^2$$

FROM MAX. LINEUP

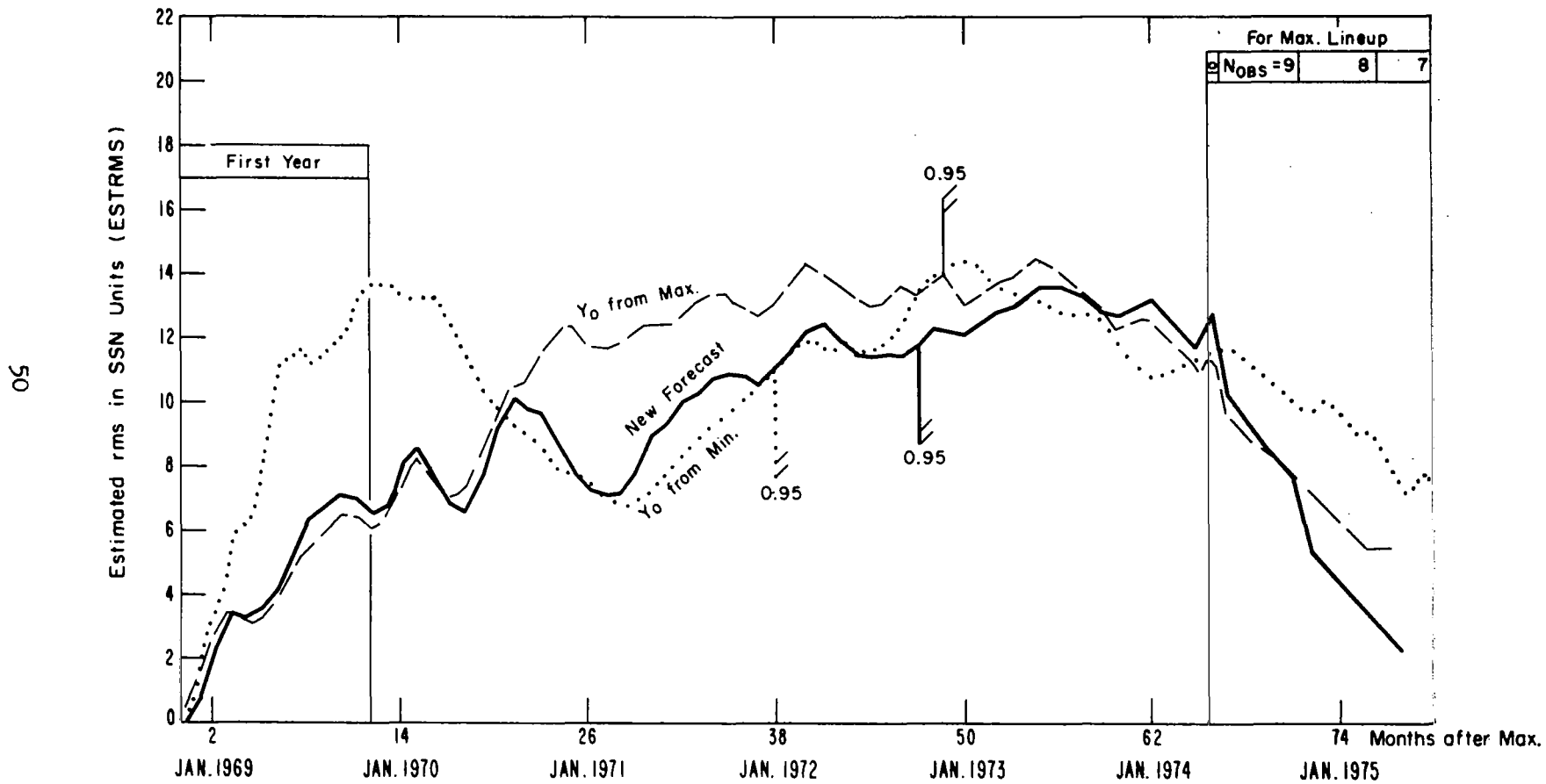


Figure 6.1

FORECAST MADE FROM DATA 1 YEAR AFTER MAXIMUM

$$\hat{S} = b_0 + b_1 \text{RISETIME} + b_2 S_{y_1} + b_3 S_{y_1}^2$$

FROM MAX. LINEUP

51

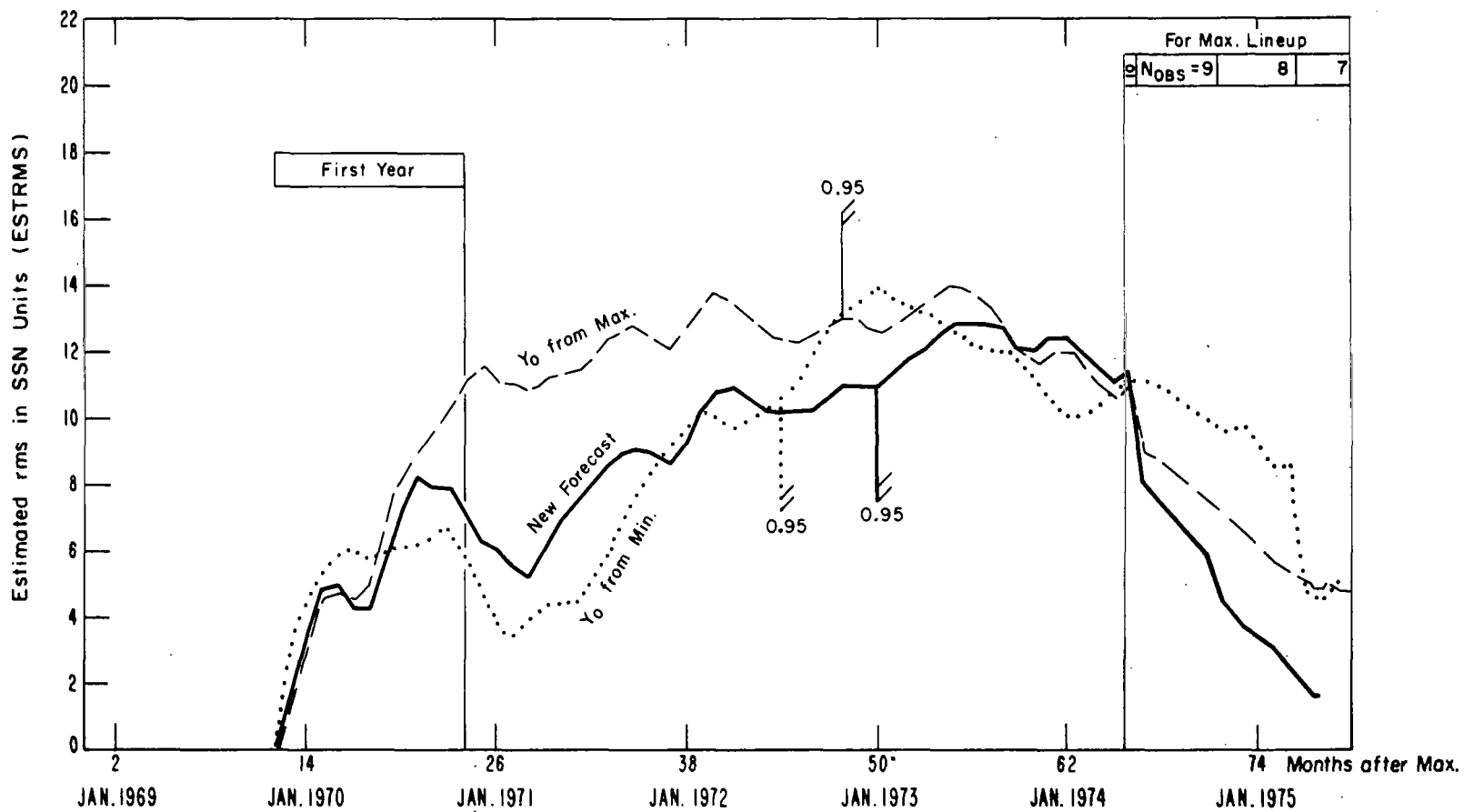


Figure 6.2

FORECAST MADE FROM DATA 2 YEARS AFTER MAXIMUM
 $\hat{S} = b_0 + b_1 \text{RISETIME} + b_2 S y_2 + b_3 S y_2^2$
 FROM MAX. LINEUP

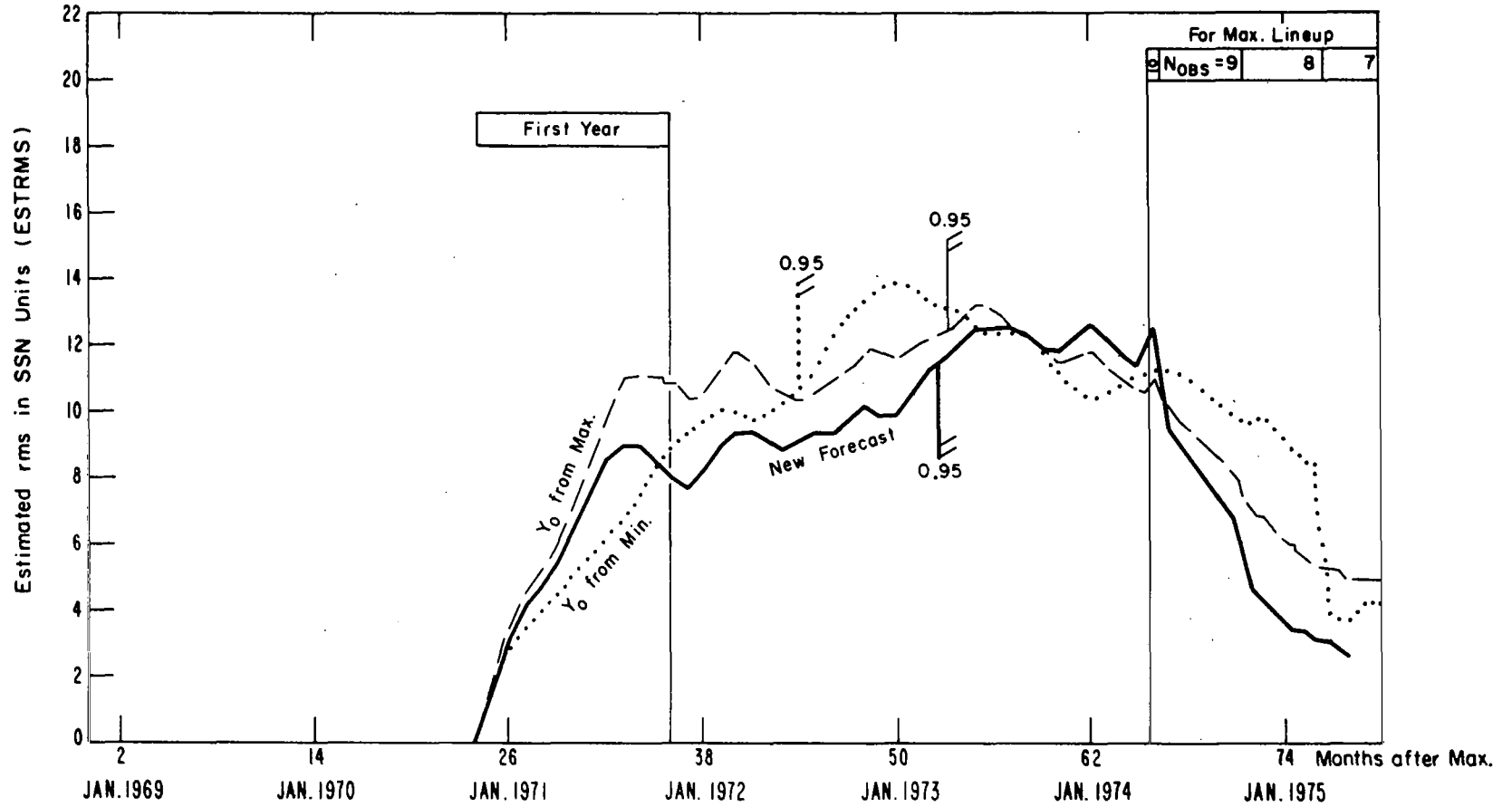


Figure 6.3

FORECAST MADE FROM DATA 3 YEARS AFTER MAXIMUM

$$\hat{S} = b_0 + b_1 \text{RISETIME} + b_2 S_{y_3} + b_3 S_{y_3}^2$$

FROM MAX. LINEUP

53

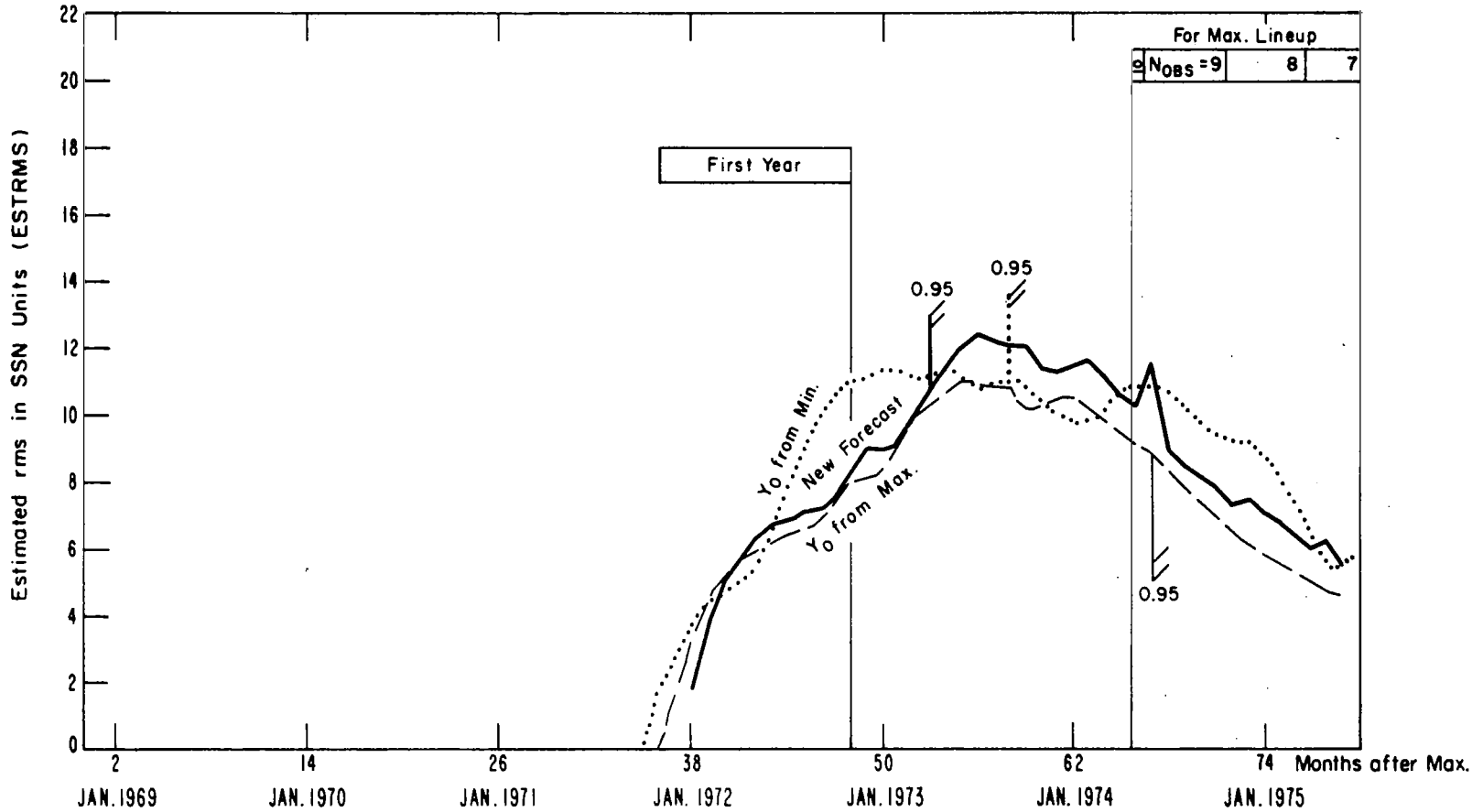


Figure 6.4

the observed value of the risetime has a small influence on early forecasts, with its influence increasing up to a j of 27 months and then gradually decreasing again. Thus, these two predictors have differing importance at different times of the cycle. It also may be noted that in this table b_1 is always negative, which implies that the future forecast will be depressed for cycles that take longer times to reach the same maximum.

In all of these tables, the portion enclosed in a box of dashed lines is where the test of correlation coefficient indicates a confidence of less than 0.95 that the regression relationship is meaningful. The tables are continued into this region because there may be some slight practical advantage to using the regression relationship there rather than just the cycle mean, but the weak confidence must be kept in mind.

Tables 6.1 through 6.6 were derived from a data set where all 12 of the developmental cycles were included, whether or not an individual cycle had passed its minimum. This is the practical situation that would face the forecaster near the end of a given cycle. However, note that the confidence test cuts off the forecast either before the time of minimum of the mean cycle or not significantly after it. Even in table 6.6 the confidence disappears after 15 months from the observation --- just at the time of the minimum of the mean cycle. As will be seen in more detail later, there is no visible hope for forecasting the subsequent rise of the following cycle any better than by using the mean cycle to represent it. The tables show how the various coefficients all become irregular when the confidence limit is approached.

The coefficients in tables 6.1 through 6.6 also show that the influence of the risetime becomes less and less as the cycle progresses. This could well be expected, and in the later

stages of a cycle it makes little difference whether this term is or is not included in the regression. The term is essential, however, for the success of this formula during the first few years after the maximum, so it is here left in to keep the forecasting formula uniform throughout its region of application.

Table 6.1. Coefficients for Formula
 $\hat{S}_j = b_0 + b_1 \text{risetime} + b_2 S_k + b_3 S_k^2$

j in months after max	For k = time of cycle maximum				S _j for mean cycle (116)
	b ₀	b ₁	b ₂	b ₃	
3	6.2	-0.178	1.034	-0.00048	110
6	1.5	-0.048	0.922	-0.00005	105
9	-1.7	-0.007	0.891	0.00006	102
12	16.7	-0.077	0.548	0.00122	95
15	44.0	-0.177	0.240	0.00188	91
18	68.9	-0.570	0.163	0.00170	85
21	83.9	-0.831	0.091	0.00162	77
24	115.9	-1.437	0.064	0.00143	72
27	102.3	-1.475	0.226	0.00086	67
30	89.7	-1.382	0.321	0.00025	61
33	75.6	-1.302	0.471	-0.00056	56
36	48.8	-1.062	0.658	-0.00148	50
39	54.5	-1.142	0.628	-0.00175	44
42	54.0	-1.053	0.530	-0.00147	41
45	42.1	-0.931	0.578	-0.00166	38
48	27.9	-0.812	0.703	-0.00231	34
51	18.2	-0.674	0.701	-0.00229	32
54	10.2	-0.644	0.775	-0.00266	28
57	16.5	-0.599	0.575	-0.00196	24
60	40.6	-0.335	0.487	-0.00152	21
63	27.9	-0.324	0.435	-0.00126	18

Table 6.2. Coefficients for Formula

$$\hat{S}_j = b_0 + b_1 \text{risetime} + b_2 S_k + b_3 S_k^2$$

j in months after max	For k = 12 months after cycle maximum				S _j for mean cycle
	b ₀	b ₁	b ₂	b ₃	
15	25.7	-0.099	0.638	0.00097	91
18	60.8	-0.556	0.403	0.00135	85
21	77.6	-0.820	0.266	0.00152	77
24	107.8	-1.410	0.245	0.00121	72
27	89.3	-1.409	0.502	0.00011	67
30	67.2	-1.231	0.704	-0.00108	61
33	49.1	-1.106	0.910	-0.00231	56
36	24.4	-0.851	1.071	-0.00331	50
39	31.8	-0.927	0.977	-0.00336	44
42	30.8	-0.838	0.863	-0.00295	41
45	21.2	-0.727	0.886	-0.00307	38
48	5.9	-0.577	1.019	-0.00387	34
51	-1.1	-0.455	0.978	-0.00368	32
54	-7.4	-0.421	1.019	-0.00396	28
57	-1.7	-0.388	0.802	-0.00306	24
60	-11.5	-0.161	0.690	-0.00245	21
63	-13.0	-0.158	0.644	-0.00218	18

Table 6.3. Coefficients for Formula

$$\hat{S}_j = b_0 + b_1 \text{ risetime} + b_2 S_k + b_3 S_k^2$$

j in months after max	For k = 24 months after cycle maximum				S _j for mean cycle
	b ₀	b ₁	b ₂	b ₃	
27	-22.3	-0.025	1.518	-0.00328	67
30	-41.4	0.034	1.910	-0.00642	61
33	-62.5	0.012	2.546	-0.01132	56
36	-84.8	0.202	2.789	-0.01332	50
39	-58.5	-0.076	2.444	-0.01216	44
42	-56.5	0.005	2.209	-0.01079	41
45	-54.4	0.022	2.069	-0.01011	38
48	-65.2	0.063	2.304	-0.01211	34
51	-73.6	0.193	2.279	-0.01196	32
54	-65.2	0.102	2.128	-0.01133	28
57	-53.4	0.084	1.749	-0.00919	24
60	-63.7	0.330	1.589	-0.00804	21
63	-50.5	0.239	1.275	-0.00611	18
66	-34.8	0.163	0.896	-0.00395	15
69	-37.8	0.187	0.955	-0.00461	14
72	-36.5	0.182	0.931	-0.00477	12
75	-55.0	0.420	1.112	-0.00603	11
78	-87.4	0.811	1.436	-0.00778	12
83	-180.1	1.961	2.321	-0.01224	15

Table 6.4. Coefficients for Formula

$$\hat{S}_j = b_0 + b_1 \text{risetime} + b_2 S_k + b_3 S_k^2$$

j in months after max	For k = 36 months after cycle maximum				S _j for mean cycle
	b ₀	b ₁	b ₂	b ₃	
39	14.0	-0.167	0.660	0.00195	44
42	87.5	-0.125	0.787	-0.00032	41
45	11.5	-0.115	0.525	0.00207	38
48	-16.7	-0.014	0.870	-0.00234	34
51	-9.5	0.039	1.087	-0.00530	32
54	2.3	-0.061	0.653	-0.00130	28
57	0.1	-0.026	0.552	-0.00084	24
60	-14.5	0.215	0.524	-0.00044	21
63	-1.9	0.154	-0.131	0.00670	18
66	1.2	0.102	-0.195	0.00663	15
69	1.6	0.076	-0.083	0.00449	14
72	1.6	0.069	-0.044	0.00335	12
75	-13.0	0.247	0.293	-0.00089	11

Table 6.5. Coefficients for Formula

$$\hat{S}_j = b_0 + b_1 \text{risetime} + b_2 S_k + b_3 S_k^2$$

j in months after max	For k = 48 months after cycle maximum				S _j for mean cycle
	b ₀	b ₁	b ₂	b ₃	
51	-15.4	0.192	1.191	-0.00258	32
54	15.9	0.017	0.459	0.00712	28
57	-48.0	0.101	0.383	0.00758	28
60	-21.5	0.342	0.538	0.00502	21
63	-16.5	0.104	-0.102	0.01309	18
66	-19.2	0.081	-0.045	0.01064	15
69	3.4	0.031	-0.151	0.01023	14
72	7.5	-0.012	-0.207	0.00909	12
75	-4.3	0.176	0.050	0.00379	11
78	-13.9	0.346	0.315	-0.00159	12
81	-29.0	0.604	0.731	-0.00932	13
84	-43.6	0.953	0.980	-0.01539	17

Table 6.6. Coefficients for Formula

$$\hat{S}_j = b_0 + b_1 \text{ risetime} + b_2 S_k + b_3 S_k^2$$

j in months after max	For k = 60 months after cycle maximum				S _j for mean cycle
	b ₀	b ₁	b ₂	b ₃	
63	8.1	-0.135	0.583	0.00763	18
66	6.2	-0.117	0.537	0.00615	15
69	11.6	-0.158	0.223	0.00921	14
72	14.4	-0.171	0.058	0.00897	12
75	12.4	-0.066	-0.157	0.00954	11
78	20.4	-0.072	-0.641	0.01497	12
81	24.3	0.004	-1.014	0.01748	13
84	24.7	0.215	-1.487	0.02093	17
87	25.7	0.433	-1.987	0.02512	20

7. COMPARISON WITH FORMER CYCLES

As was discussed in section 2, with only 12 cycles in the data base we felt it best to use all 12 of them in the developmental sample. This leaves no independent sample on which the results can be verified. Instead, in this study dependence was placed on using the statistically determined estimated error of the forecast as a criterion of goodness of fit. Still, it gives a certain intellectual satisfaction to see how well the resulting formula fits some of the historical data. Accordingly, some comparisons were made between the historical data and forecasts made from the final formula.

Initially these comparisons were made for cycle 19 for two reasons. One reason was that cycle 19 is the most recent of the historical ones. Thus, if the formula development here undertaken had been done some 11 years earlier, it could not have included data from cycle 19. The formula would have been selected on the basis of data extending up through cycle 18, and then would be applied in real time to cycle 19 as it occurred. The second reason was that cycle 19 was an outstanding one, having a much higher maximum than any previous one --- even higher than any of the first seven that were rejected by the analysis of section 2. Thus it is of particular interest to see how well the new formula would have worked on this outstanding cycle.

First, the forecasts for cycle 19 were made using the coefficients from table 6.1. The results are plotted in figure 7.1, which shows the forecast, its 90% prediction interval, and the actual observations for that cycle. The results are unexpectedly good --- in fact they seem to be too good for such an unusual cycle. But it must be remembered that not only was the formula itself selected from tests that included cycle 19 as one of the total of 12 cycles analyzed, but the coefficients entering into it were also derived from data

OBSERVATIONS vs FORECAST FOR CYCLE 19
 FORMULA DERIVED FROM CYCLES 8-19

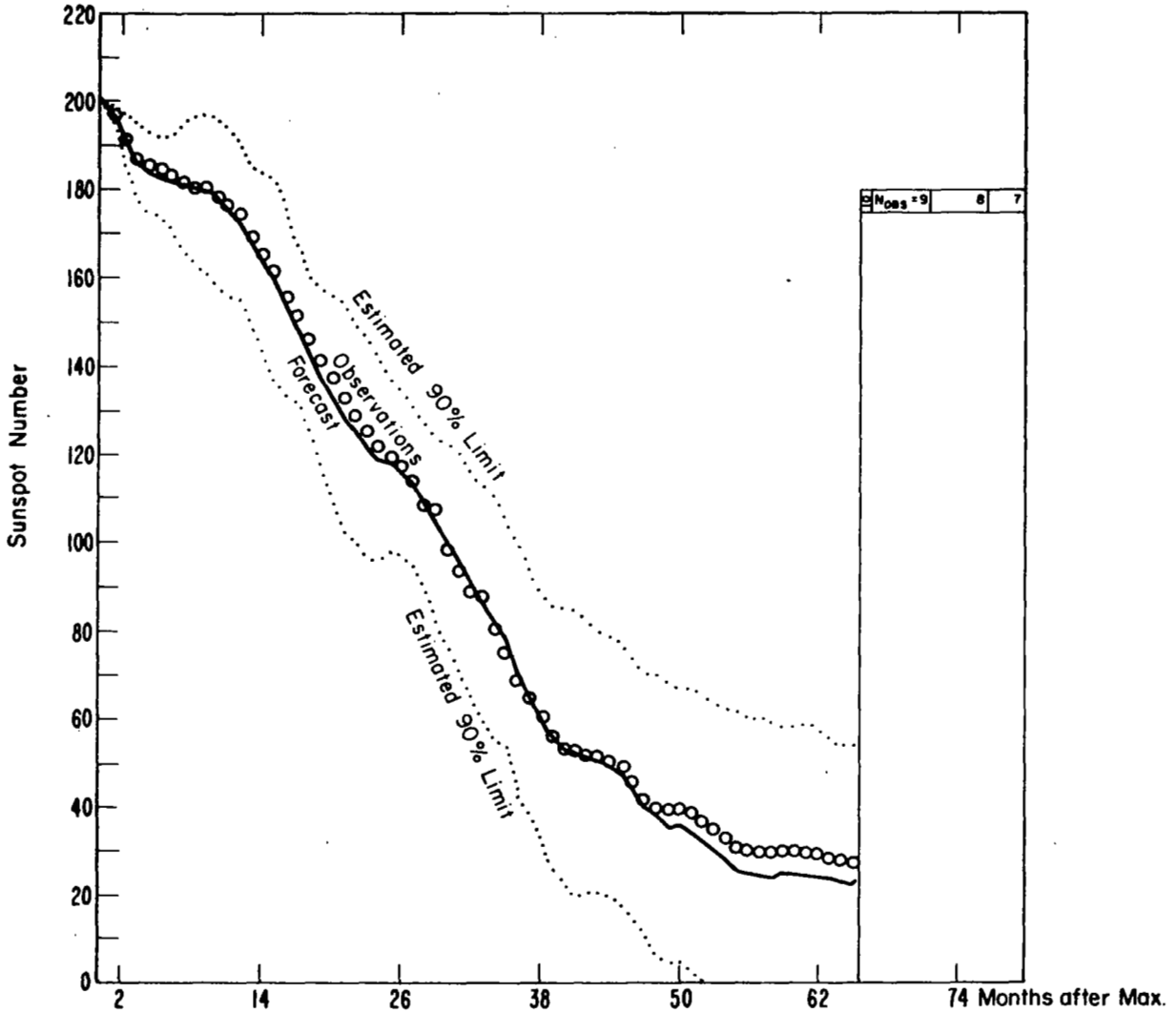


Figure 7.1

including cycle 19. Neither of these two features should be included in a statistically valid verification. At least the second can be eliminated by recomputing the coefficients using only cycles 8 through 18, and not using cycle 19. This was done, with the rather startling results shown in figure 7.2. About the only thing good that can be said about this figure is that the observations do indeed lie within the 90% prediction interval, but that appears to have occurred only because the 90% prediction interval has expanded so violently.

A careful analysis was made of why the comparison is so poor in figure 7.2 when it was so good in figure 7.1. The reason seems to lie in the uniqueness of cycle 19. Its maximum is more than a third higher than any of the other cycles used in the data base. Cycle 19 had a maximum of 201, while the next highest cycle was number 18, with a maximum of 148. Thus when data from only cycles 8 through 18 are used in developing the regression relationships, forecasting for cycle 19 represents a considerable extrapolation beyond the range of data used for development.

The new formula, having as it does four adjustable parameters, has greater sensitivity to variations from one cycle to another than did earlier formulas with only two adjustable parameters. Thus fluctuations can be expected to be especially large when the formula is used in an extrapolation process rather than in the interpolation process for which it is designed. The mean data for cycles 8 through 18 show a slightly depressed region near 21 months after the maximum, and a slightly elevated region near 33 months after the maximum. When the formula is extrapolated for cycle 19, these become the large fluctuations seen in figure 7.2.

OBSERVATIONS vs FORECAST FOR CYCLE 19
 FORMULA DERIVED FROM CYCLES 8-18

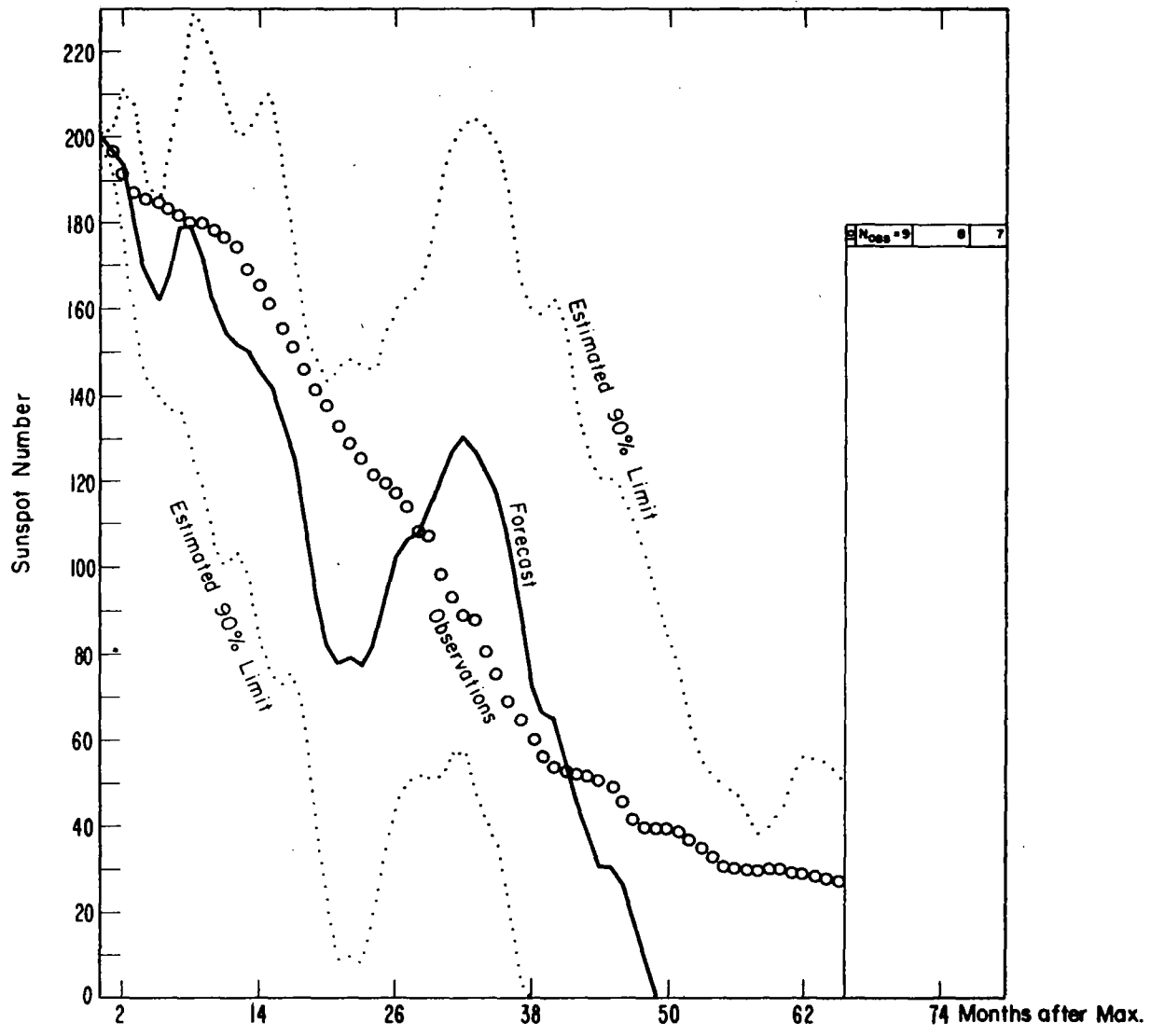


Figure 7.2

The same analysis shows why the forecast of figure 7.1 fits so well the observations of cycle 19. With cycle 19 so far away from all of the other cycles in the data base, it requires only a small adjustment of the parameters to fit it very closely. Thus the very good fit shown in figure 7.1 should not be taken as evidence of success of the formula --- to an unusually large extent it was fitted to that curve.

As another comparison, the forecast was made for cycle 15, with the regression relationships derived from cycles 8 through 14 and 16 through 19, so they did not include the cycle being forecast. Cycle 15 was chosen because it is near the middle of the distribution function of cycle maxima, and also because its maximum was the closest of any in the data base to the current cycle 20. The results of this comparison are shown in figure 7.3 and are much more what one would expect for such a comparison.

It must be remembered that the comparisons shown in this section are for forecasts made with the latest available data being that of the maximum of the cycle. Thus the forecasts shown cover a period up to more than 5 years after the latest data.

OBSERVATIONS VS FORECAST FOR CYCLE 15
 FORMULA DERIVED FROM CYCLES 8-14, 16-19

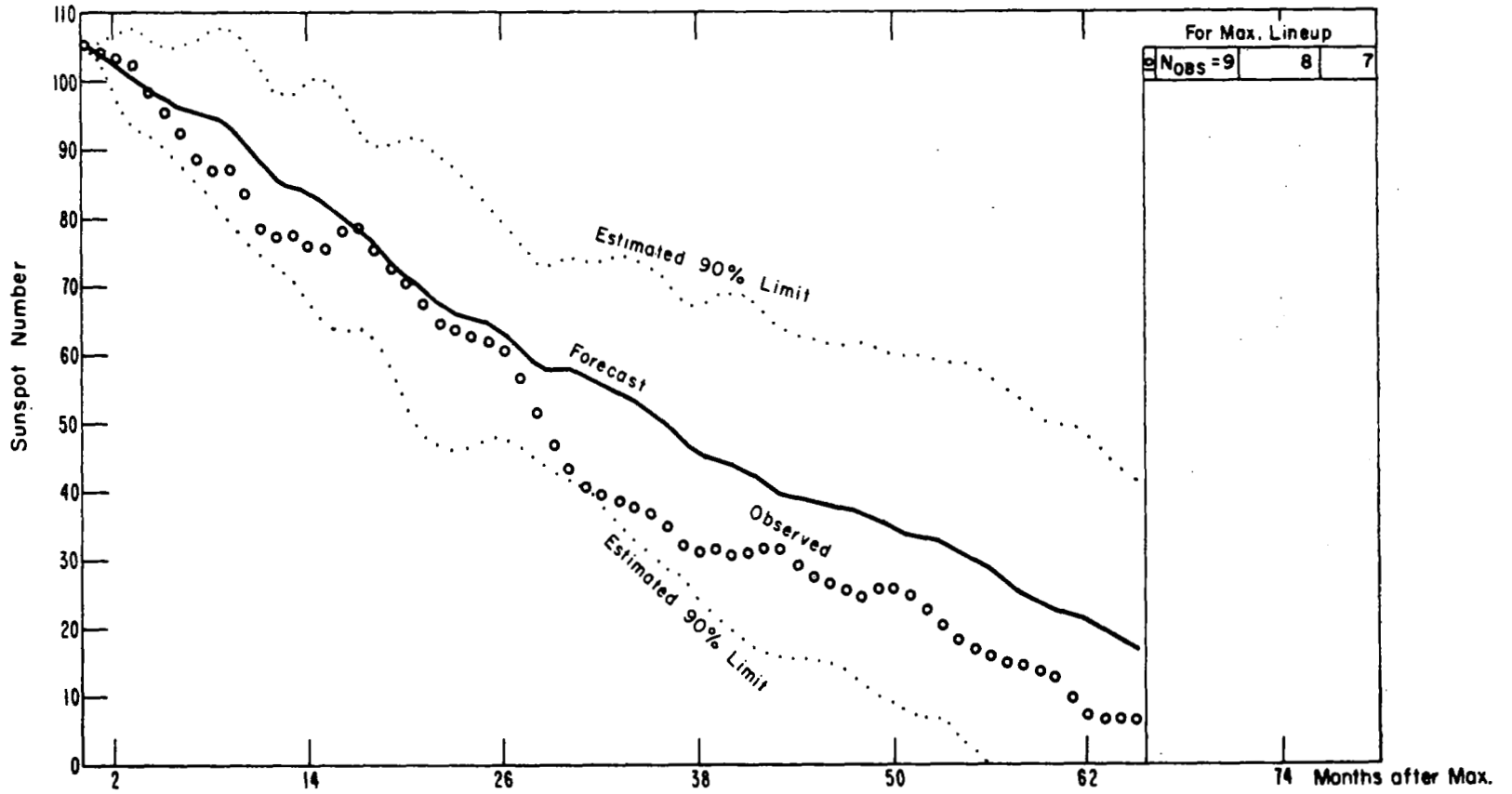


Figure 7.3

8. PREDICTION OF THE CURRENT CYCLE

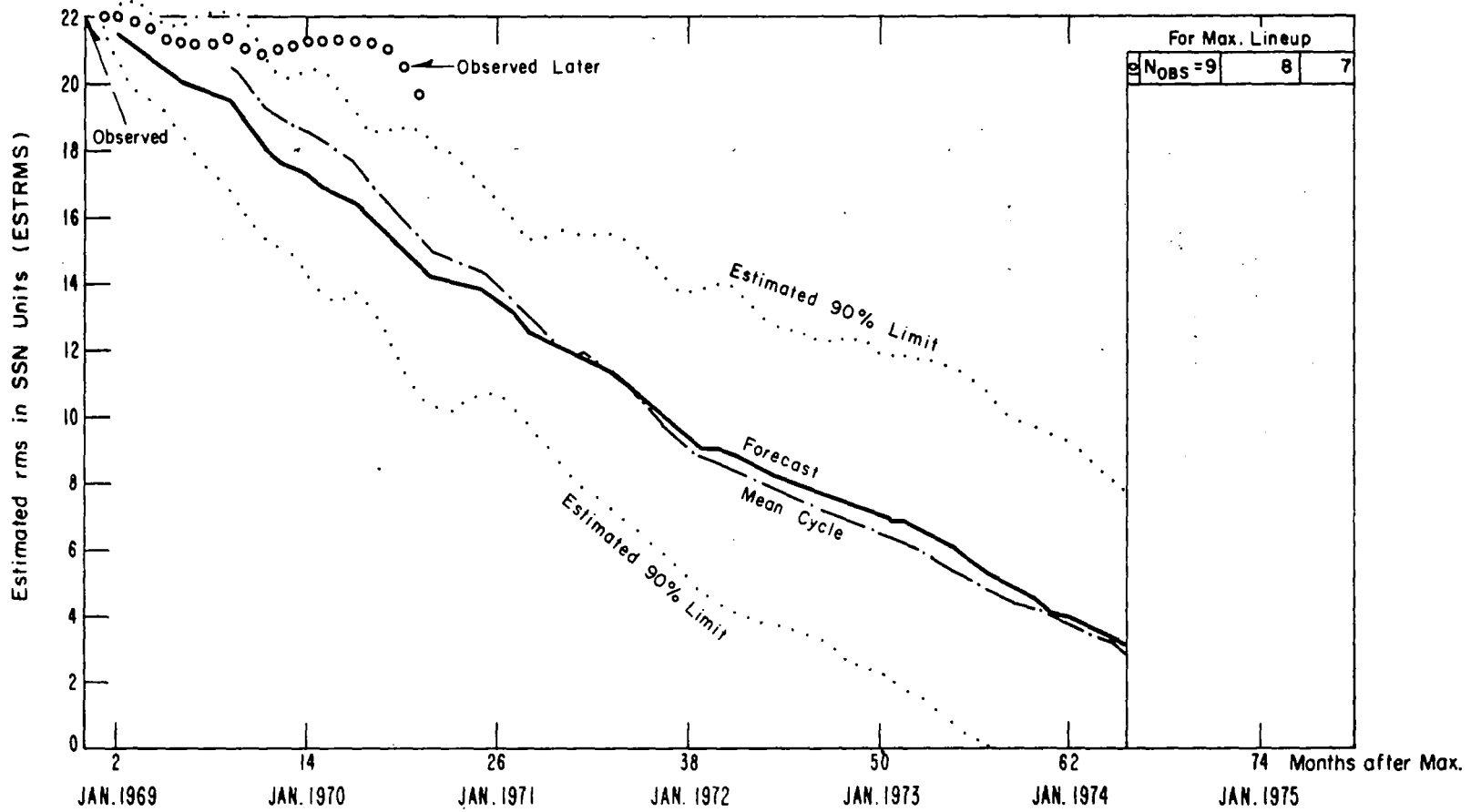
Figure 8.1 shows the forecast that would have been made for the present cycle (number 20) using the new formula and data at the time of cycle maximum (November 1968). Shown in the figure are the forecast, the 90% limits on the forecast estimated from the regression calculations, and the values for the mean of the cycles 8 through 19. Note that the forecast would indicate this cycle to be very near to the mean cycle. Also shown on this figure are the values for the smoothed SSN that have been observed since November 1968. It can be seen that the sunspot number has decreased much more slowly than would have been predicted for this cycle.

Examination of the cycles in the data base shows that once more the sunspot cycles are making their statistical study difficult by producing extreme values at the end of the series. Not only was the cycle 18 higher than any cycle preceding it in the data base, but cycle 19 was in its turn higher still, and so gained the same distinction in its turn. Now we are having in cycle 20 the one that is falling the slowest from its maximum. When the smoothed sunspot number at 19 months after maximum is expressed as a percentage of the maximum value, the numbers for previous cycles run from 62% to 86%. But for the present cycle it is 95%, well above all of the previous cycles.

Figure 8.2 shows the same forecast made using a McNish-Lincoln type of forecast from a single predictor. It misses the recent observations almost as much, but its confidence limits were so much wider that they covered the miss.

Figures 8.3 and 8.4 repeat these last two figures, but this time using the latest data available, the smoothed sunspot number for August 1970. It can be seen that now that the sunspot number has stayed high for so long, both prediction methods now forecast that it will remain well above the mean cycle for the rest of this cycle.

FORECAST FOR CURRENT CYCLE
MADE FROM DATA AT MAX. (NOV. 1968)
NEW FORMULA



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Figure 8.1

FORECAST FOR CURRENT CYCLE
 MADE FROM DATA AT MAX. (NOV. 1968)
 MCNISH-LINCOLN FORMULA - NEW DATA BASE

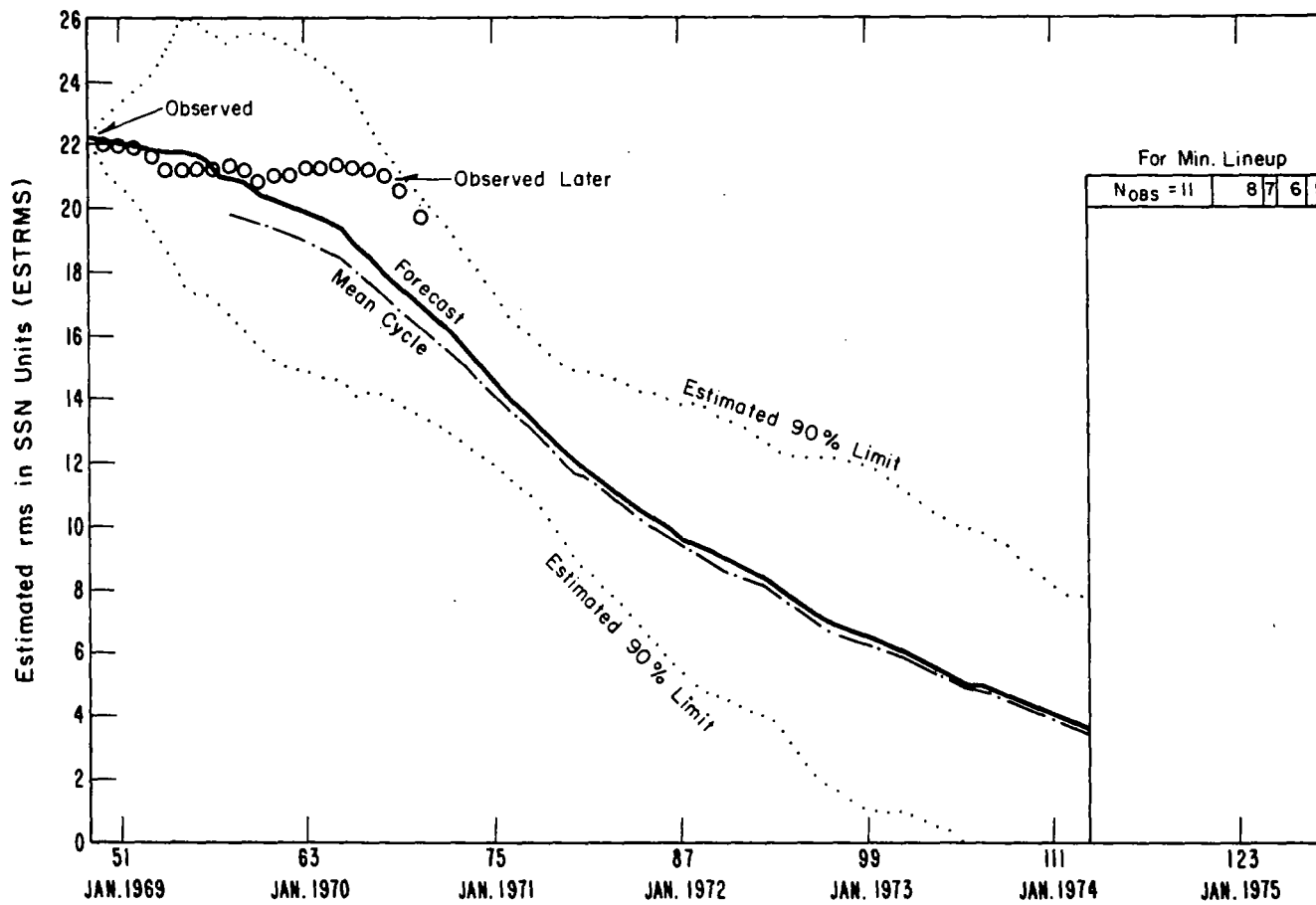


Figure 8.2

LATEST FORECAST OF THE PRESENT CYCLE
NEW FORMULA

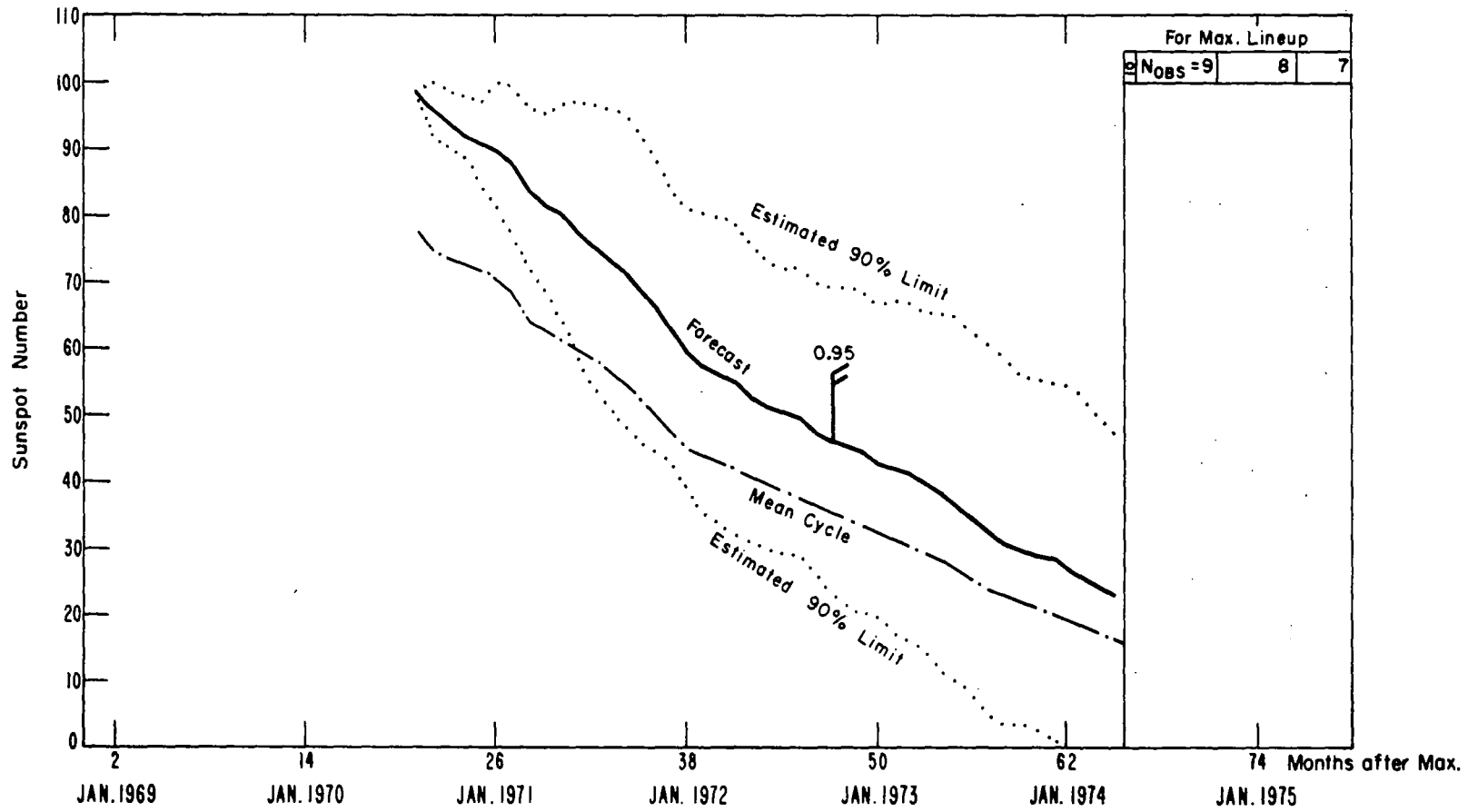
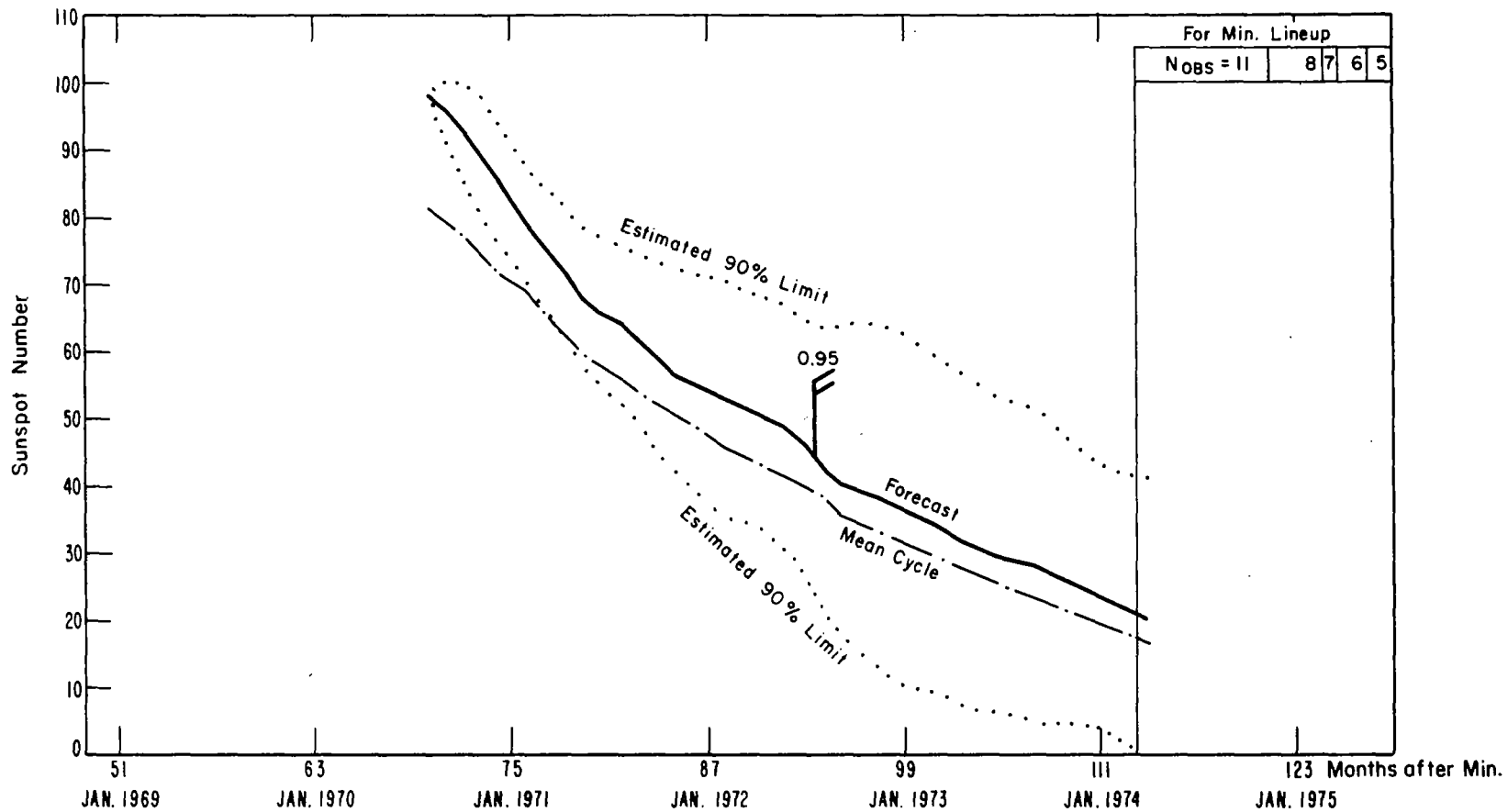


Figure 8.3

LATEST FORECAST OF THE PRESENT CYCLE
 MC NISH-LINCOLN METHOD



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Figure 8.4

9. PREDICTION PAST THE CYCLE MINIMUM

It has long been known that the correlation of sunspot numbers is very poor when the numbers lie on opposite sides of the minimum separating two cycles. The cycles have many characteristics of independent eruptions, with little relationship from one to another. Because of the magnetic characteristics of sunspot groups, a 22-year cycle has been proposed (Vitinskii, 1962) in which cycles having even numbers in the Zurich classification can be used to predict some of the characteristics of the following odd-numbered cycle. These relationships were examined for the data base used in this study, and some correlations were found which exceeded the 0.95 confidence limit. However, these lasted for a period of only some 3 years, and when allowance was made for the relatively high year-to-year autocorrelation of the sunspot numbers it was felt that there was insufficient duration in this relationship to be sure that it was not a chance occurrence.

The analyses reported in the preceding sections were carried out for durations no greater than the shortest cycle in the data base. Beyond that time, a straightforward application of the regression calculations would combine data from long cycles that are still decreasing with data past the end of shorter cycles, where the next following cycle is already increasing. Instead of this, more precise estimates could be made for the rising portion of the following cycle if it were possible to get a good estimate of the duration of the current cycle. Then beyond that time the dependent data set could once more be lined up with cycle minima coinciding and so give a more uniform data set for the regression analysis.

Although the cycle risetime is well known to be related to the cycle maximum, this is of no practical advantage at this time in the current cycle. Regressions were calculated for the cycle falltime and for the total time of the cycle

as functions of the maximum and the risetime. Unfortunately, these turned out to have correlation coefficients far below significance, and so this method of predicting cannot be used. Instead, it was necessary to use the straightforward method of continuing the data base on into the future whether or not it combines data from long and short cycles.

Doing this for the latest forecast gives the result shown in figure 9.1. Once the estimated rms error approaches the standard deviation of the raw data, at about 50 months after the cycle maximum, it then remains close except for a short time near 80 months after the cycle maximum. The flags in the figure show the boundaries where 0.95 confidence of significance to the regression are passed, and it can be seen that after 48 months there is just a short period near 80 months where the 0.95 confidence is exceeded.

This is more clearly shown in figure 9.2, where the correlation coefficient from the regression is plotted and the limit of 0.95 confidence is shown. Since the peak near 80 months is so narrow and isolated, we once more reject its significance. This is confirmed in figure 9.3, which shows the correlation coefficient for a forecast prepared $3\frac{1}{2}$ years later, 5 years after cycle maximum. The apparent significance near 80 months in figure 9.2 is no longer there. Even for a forecast so close to that time as that shown in figure 9.3, the apparent significance has vanished. Thus the new formula does not give any significant regression past the cycle minimum, and all that can be done with it for that time is to use the mean and standard deviation of the data base as an estimate of conditions then.

The question comes up whether the former McNish-Lincoln type of formula gives any better results past the cycle minimum. Figures 9.4 through 9.6, which parallel figures 9.1 through 9.3, show that this is not the case. In figure 9.4

ESTIMATED rms ERROR FOR LATEST FORECAST
NEW FORMULA

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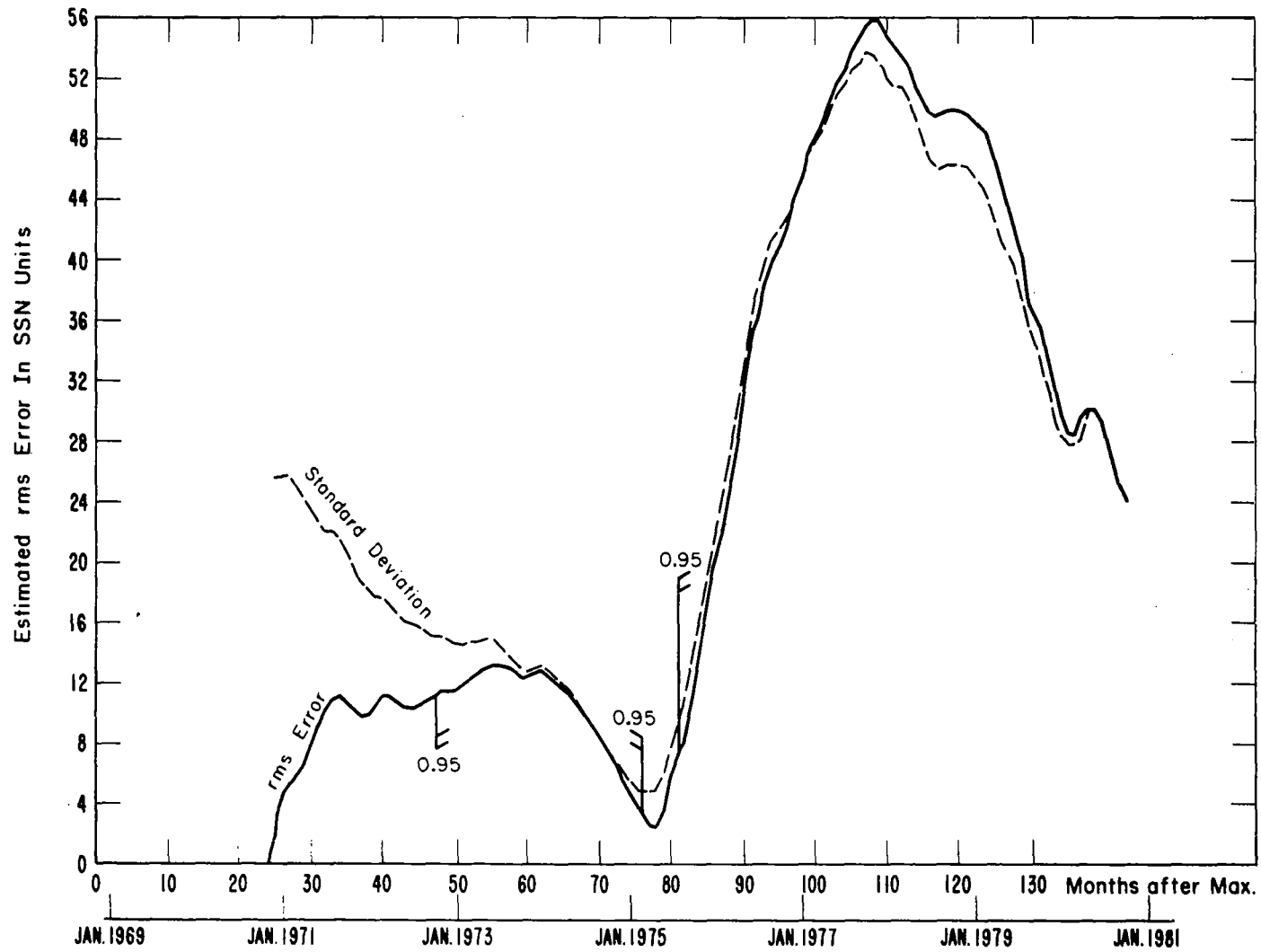


Figure 9.1

MULTIPLE CORRELATION COEFFICIENT FOR LATEST FORECAST
NEW FORMULA

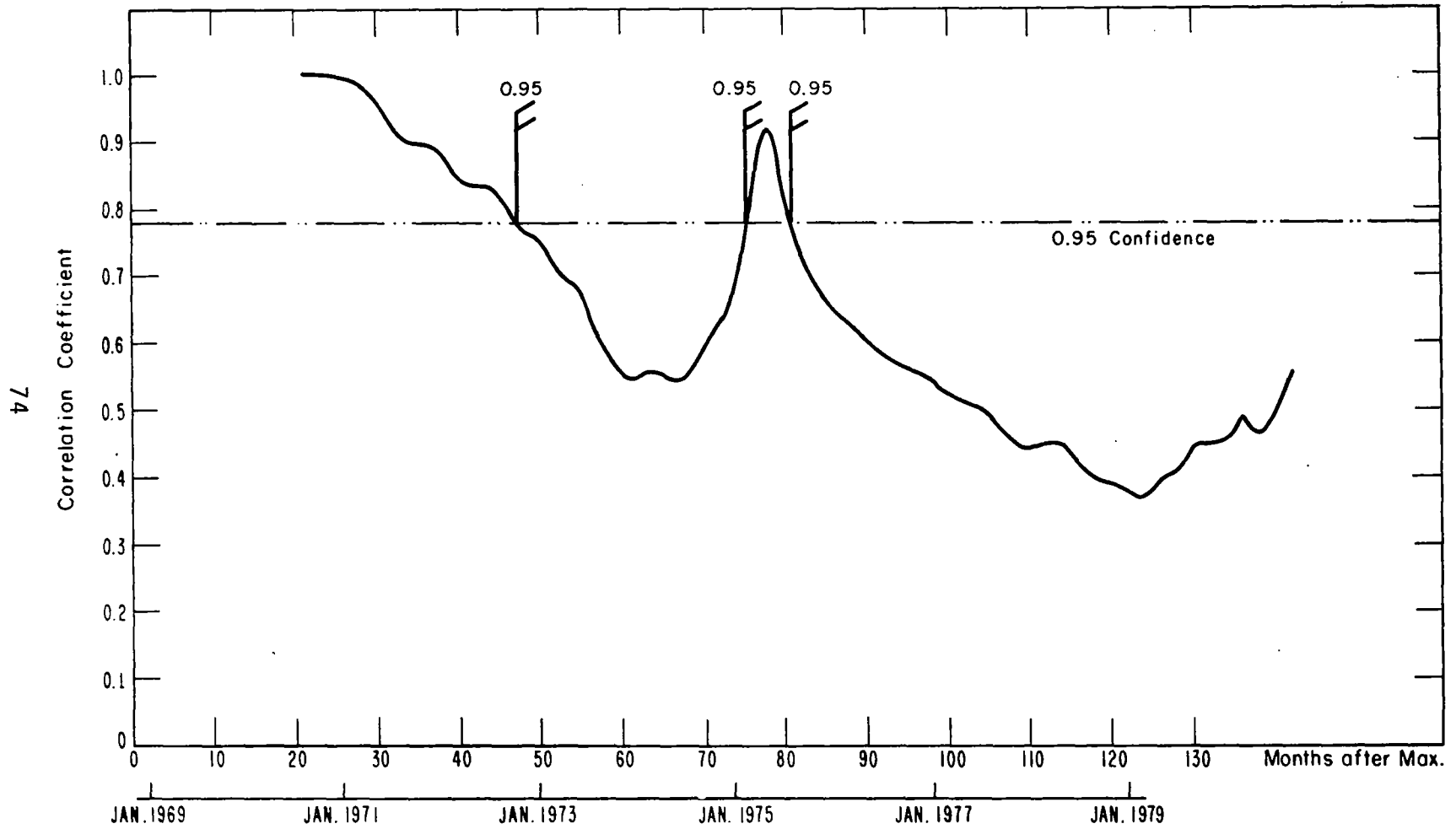


Figure 9.2

MULTIPLE CORRELATION COEFFICIENT
FOR FORECAST 5 YEARS AFTER MAXIMUM
NEW FORMULA

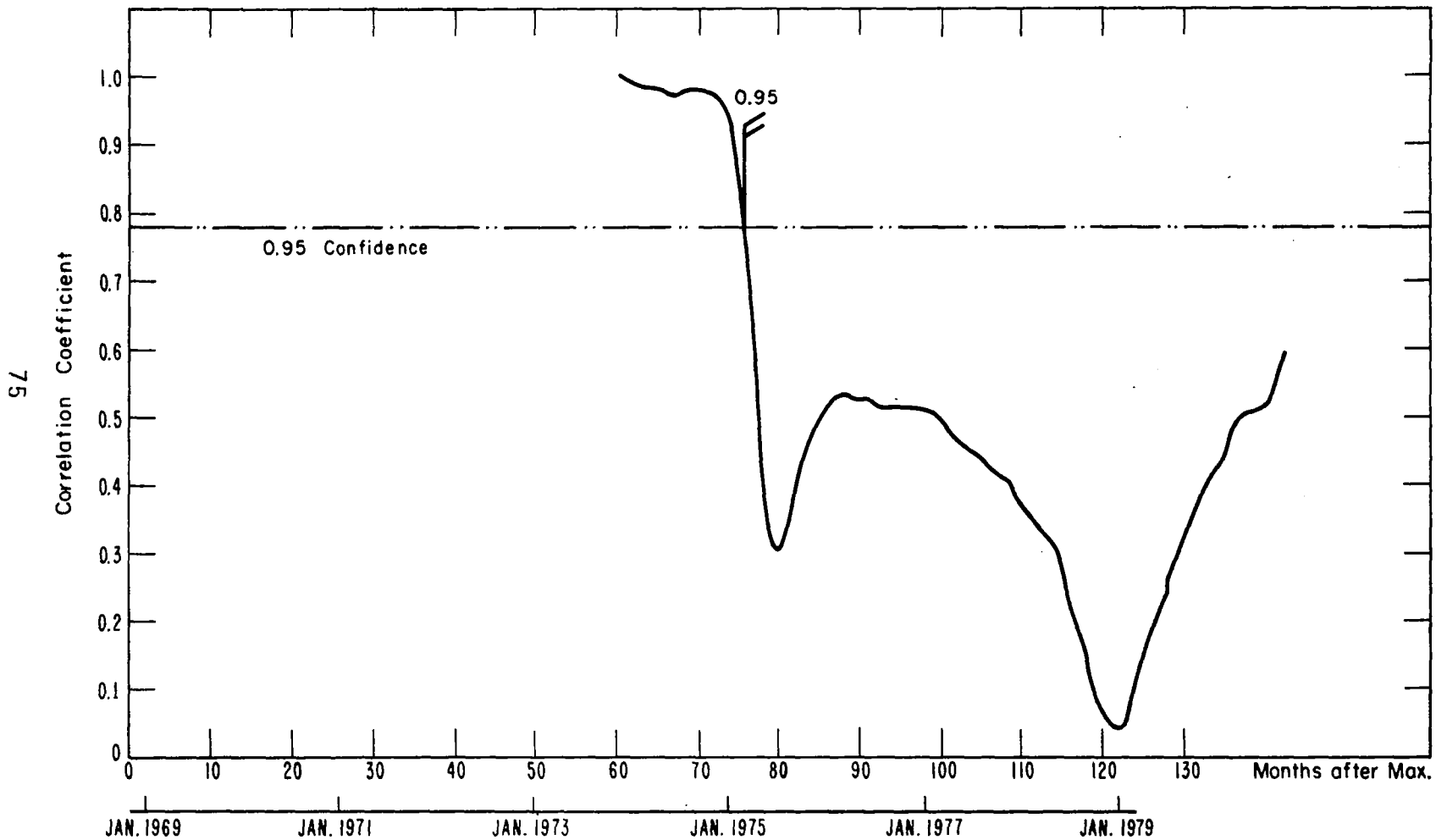
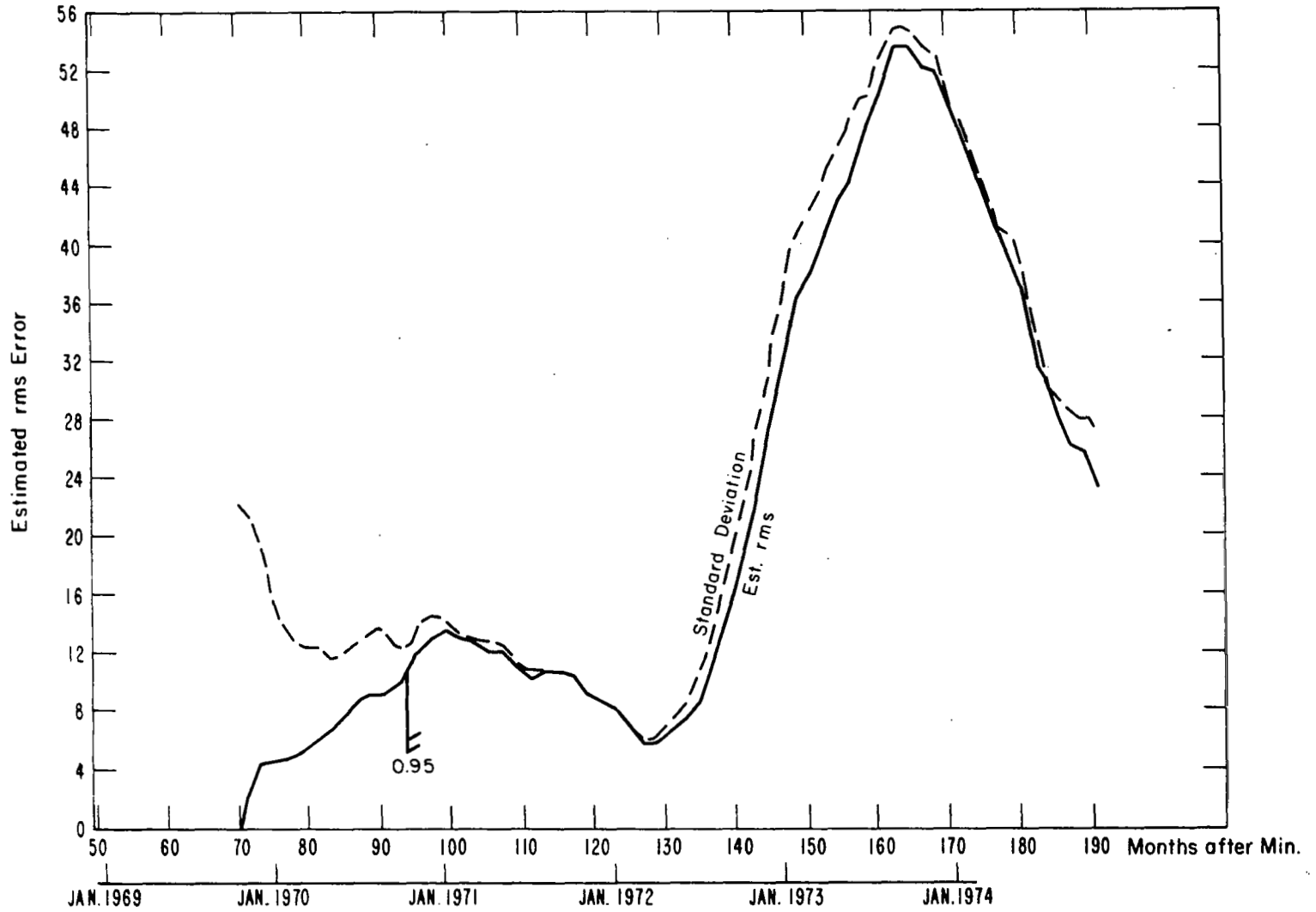


Figure 9.3

ESTIMATED rms ERROR FOR LATEST FORECAST
MCNISH-LINCOLN FORMULA



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Figure 9.4

CORRELATION COEFFICIENT FOR LATEST FORECAST
MCNISH - LINCOLN FORMULA

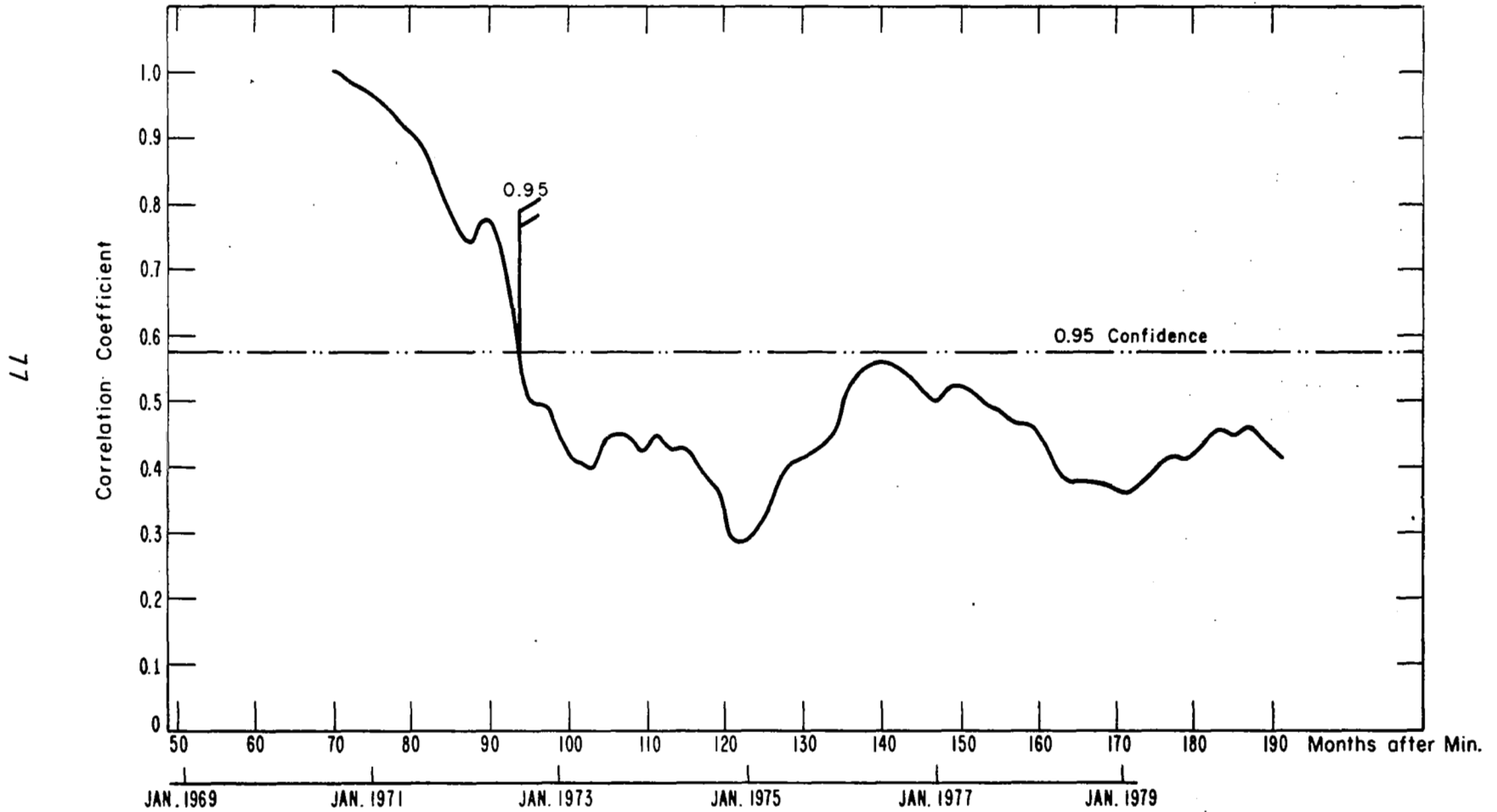


Figure 9.5

CORRELATION COEFFICIENT
FOR FORECAST 5 YEARS AFTER MAXIMUM
MCNISH - LINCOLN FORMULA

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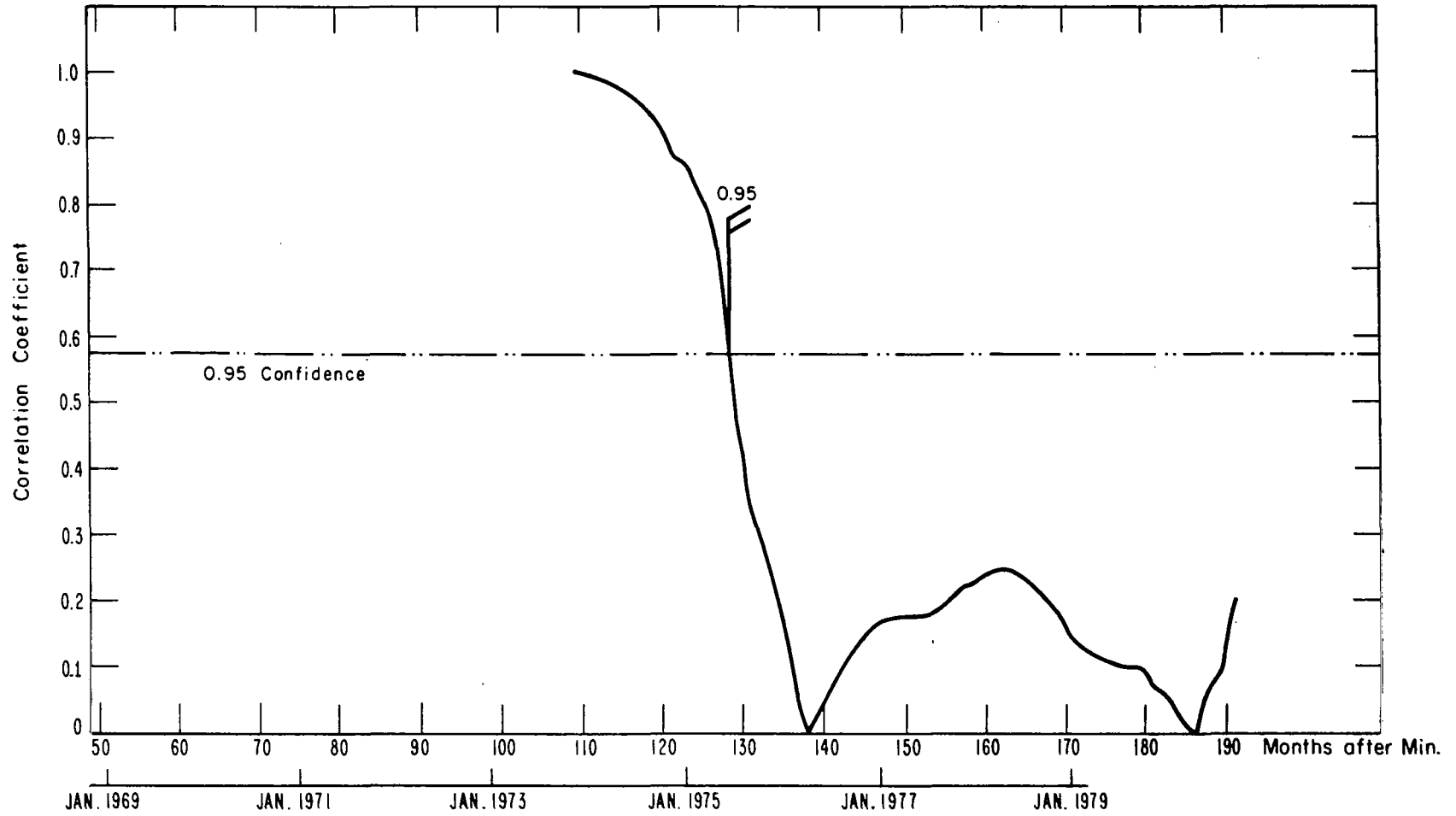


Figure 9.6

there is a small reduction in the estimated rms error from the standard deviation of the data, but figure 9.5 shows that this does not have significance at the 0.95 level. As a matter of fact, for forecasts prepared a little later on, the correlation rises briefly above the 0.95 confidence level, but so briefly that it is felt its significance should be rejected. Figure 9.6 shows that even this small effect has vanished for forecasts prepared from data much closer to the end of the cycle. Both the new formula and the McNish-Lincoln one lose their confidence at very nearly the same time --- 4 months later in one case and 3 months earlier in another, but neither extending beyond the cycle minimum.

When we reach the time of the cycle minimum, some 4 to 4½ years from now, it will become possible to prepare meaningful forecasts thereafter, but no method has been found in these studies for doing it from present data. All that has been found to be meaningful so far in the future is to use the characteristics of the mean cycle. Thus the final long-term forecast from present data is that shown in figure 9.7, where the new formula was used out to its 0.95 confidence limit, the characteristics of the mean cycle were used thereafter, and a brief subjective transition region was used between the two.

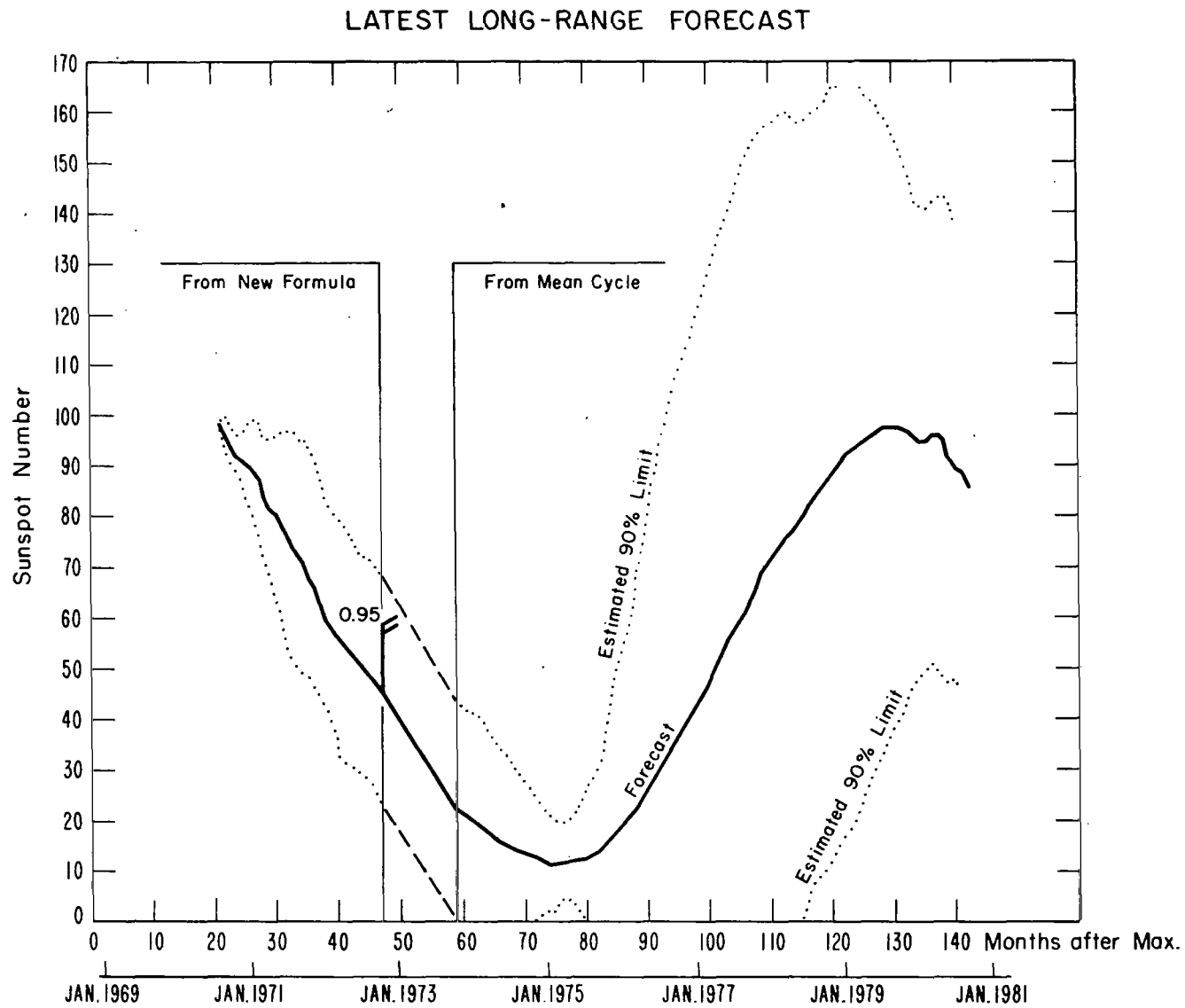


Figure 9.7

10. COMPARISON WITH NASA-MARSHALL FORMULA

The prediction program that was provided from NASA-Marshall differed in only one respect from the formula plotted in figure 4.1. It was the same in using the cycles lined up with their minima coinciding, in using the latest observation as the single predictor, and in using this single predictor to forecast all future time. It differed only in using all 19 cycles as its data base rather than just cycles 8 through 19.

When the estimated rms error of the forecasts is calculated as in figure 4.1 but based on all 19 cycles, it differs little from what is shown in the figure. At 61 months after sunspot minimum it is about 1.5 sunspot numbers lower than the curve of figure 4.1; at 77 months it is about two sunspot numbers higher.

Since the differences in estimated rms error are so small between the two sets of forecasts, a comparison between the new forecast and the NASA-Marshall formula can be based on figures 6.1 through 6.4. The comparison is between the curves labeled "New Forecast" and those labeled "Y0 from Min." The small differences between using all 19 cycles and only cycles 8-19, which were noted above, produce minor differences in interpretation in the period 29 to 39 months after maximum, but these differences favor the new formula.

The conclusions reached in comparing the NASA-Marshall formula with the new formula are that there is an obvious advantage of the new formula during the first couple of years after cycle maximum. Unfortunately, this result will provide no practical advantage until the time of maximum of the next cycle many years from now. Less obvious, but still significant, is the advantage that for the formulas shown in figures 6.1 through 6.3 there is a significantly longer time during which confidence is greater than 0.95 that the regression equation is meaningful.

It is felt that another, more subtle, advantage of the present analysis and forecast is clearer recognition of the time beyond which use of the regression relationship is not justified. We recommend that beyond that time the characteristics of the mean cycle should be used for the forecast, rather than placing too much dependence on a formula that has not been shown to have significance.

11. CONCLUSIONS AND RECOMMENDATIONS

Although the Zurich listing of sunspot data extends back in time for over 19 solar cycles, the first seven of these cycles were derived from historical data examined only in retrospect after the sunspot number was defined. The analysis reported in section 2 showed that there is significant difference between the statistics of these seven cycles and that of the remaining cycles. Thus it was concluded that only the remaining 12 cycles should be included in the data base to be analyzed for prediction purposes.

Because we are now in the falling part of the twentieth cycle, and will be for several more years, emphasis was placed on prediction methods that would be particularly applicable to that part of the cycle. By using information about the rise-time of the cycle, together with nonlinear predictors, we found the new formula (F14) of section 6 to give better error estimates than former formulas derived from the work of McNish and Lincoln.

Neither the former formulas nor the new one was found to give any significant correlation past the minimum of one cycle and on into the rising portion of the next cycle. Thus, for predictions extending that far, nothing better has been found than to use the characteristics of the mean cycle.

Since the current study included so wide a variety of multiple predictors and nonlinear predictors, it is felt that there is little more to be gained from further statistical analyses of the present data series. Extending the data series significantly will require many more years of observations. Thus we recommend that any further work in this area should now be based on a study of the physics of the phenomena rather than just the statistics of the sunspot cycles.

12. REFERENCES

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SOLAR ACTIVITY PREDICTION

PART II

Relationship Between Ottawa 10.7 cm Solar Radio

Noise Flux and Zurich Sunspot Number

Frank Stewart and Margo Leftin

A statistical analysis is conducted to describe the relationship between moving average 10.7 cm solar flux and moving average Zurich sunspot numbers. Attention is given to the autocorrelation of the observations and confidence intervals for the derived relationships are presented. The accuracy of a predicted value of 10.7 cm solar flux from a predicted sunspot number is discussed.

I. INTRODUCTION

The solution of many practical scientific problems requires an estimate of the sun's activity. Although there are several indices of solar activity available, most of them do not have a long enough series of observations for prediction purposes. The Zurich sunspot numbers have been the most important index of solar activity. The 10.7 cm solar radio noise flux, since it is measured objectively, would be preferable to sunspot numbers for predicting solar activity. However, this time series is too short for present prediction techniques. Therefore, since it is well known that the two indices are highly correlated, it was decided to derive a relationship between 10.7 cm solar radio flux and Zurich sunspot numbers.

Zurich sunspot numbers (relative sunspot numbers) were introduced in 1848 by Rudolf Wolf and expressed as

$$R = k (10 \cdot g + f)$$

where f is the total number of spots, g is the number of sunspot groups, and k is the factor assigned to a particular observer (Waldmeier 1961).

Wolf, in his determination of the sunspot numbers, used a factor of $k=1$. The method for estimating sunspot numbers was changed in 1882, and to preserve homogeneity with Wolf's original observations, a new factor of $k=.60$ was established for Zurich and has remained constant since then.

Because of the subjectivity involved in the determination of g and f , it is difficult to assess the accuracy of the Zurich sunspot numbers as an index of solar activity. The earlier observations are probably less reliable due to fewer observatories and the limitations of the observing instruments. Since, at present, sunspot numbers from many observatories with a considerable range of K factors are used to compute the final Zurich relative sunspot number, there are still some reservations as to the objectivity of this index.

Observations of the intensities of solar radio emissions at 10.7 cm (2800 MHz) are made at the Algonquin Radio Observatory of the National Research Council near Ottawa, Canada. The equipment used is a four-foot-diameter parabolic reflector and an associated radiometer of the Dicke comparison type. The solar flux in watts per square meter per Hz bandwidth is measured daily in addition to a continuous monitor to detect any enhanced radiation or burst activity. The relative errors over a long period of time are estimated to be about $\pm 2\%$ (Covington, 1969).

2. ASSUMPTIONS

Sunspot number data have been accumulated for almost three centuries and provide the longest time series of solar observations. On the other hand, observations of 10.7 cm solar flux did not begin until 1947; therefore, this study is restricted to the data for the period November 1947 to November 1968.

Little consideration was given to the relationship between 10.7 cm solar radio flux and Zurich sunspot numbers for periods less than one half a solar cycle (1 cycle \approx 11 years). As seen in figures 1 and 2, many variations with oscillations of less than one-half cycle can be observed in the monthly observations. To filter out these short-term variations it was decided to smooth the data over 13 months. This average is defined as follows:

$$R_{12,j} = \frac{R_{j-6} + R_{j+6} + 2 \sum_{i=-5}^5 R_{j+i}}{24}$$

where R_i is the observed monthly sunspot number for month i , and $R_{12,j}$ is the 12-month smoothed mean Zurich sunspot number for month j . The 12-month smoothed mean Ottawa 10.7 cm solar radio flux ($\varphi_{12,i}$) is calculated in a similar manner. The plots of $R_{12,j}$ and $\varphi_{12,i}$ may be seen in figures 1 and 2, respectively.

In multiple normal regression theory it is assumed that y , the dependent variable, is normally distributed about an expected value η with variance σ^2 and that all the observations are independent. It is also necessary that η be a simple linear function of the independent variables X_1, X_2, \dots, X_n . However, there is no requirement that X_1, X_2, \dots, X_n be independent in the statistical sense.

The assumptions that φ_{12} is normally distributed about some expected value η and that η is a linear function of the independent variable R_{12} were made a priori. The assumption of independent observations, however, cannot be postulated in this application. According to our definitions of R_{12} and φ_{12} , there is certainly time dependence in both sets of observations due to the averaging over a 13-month period, i. e. R_{12} and φ_{12} are autocorrelated for at least 12 months.

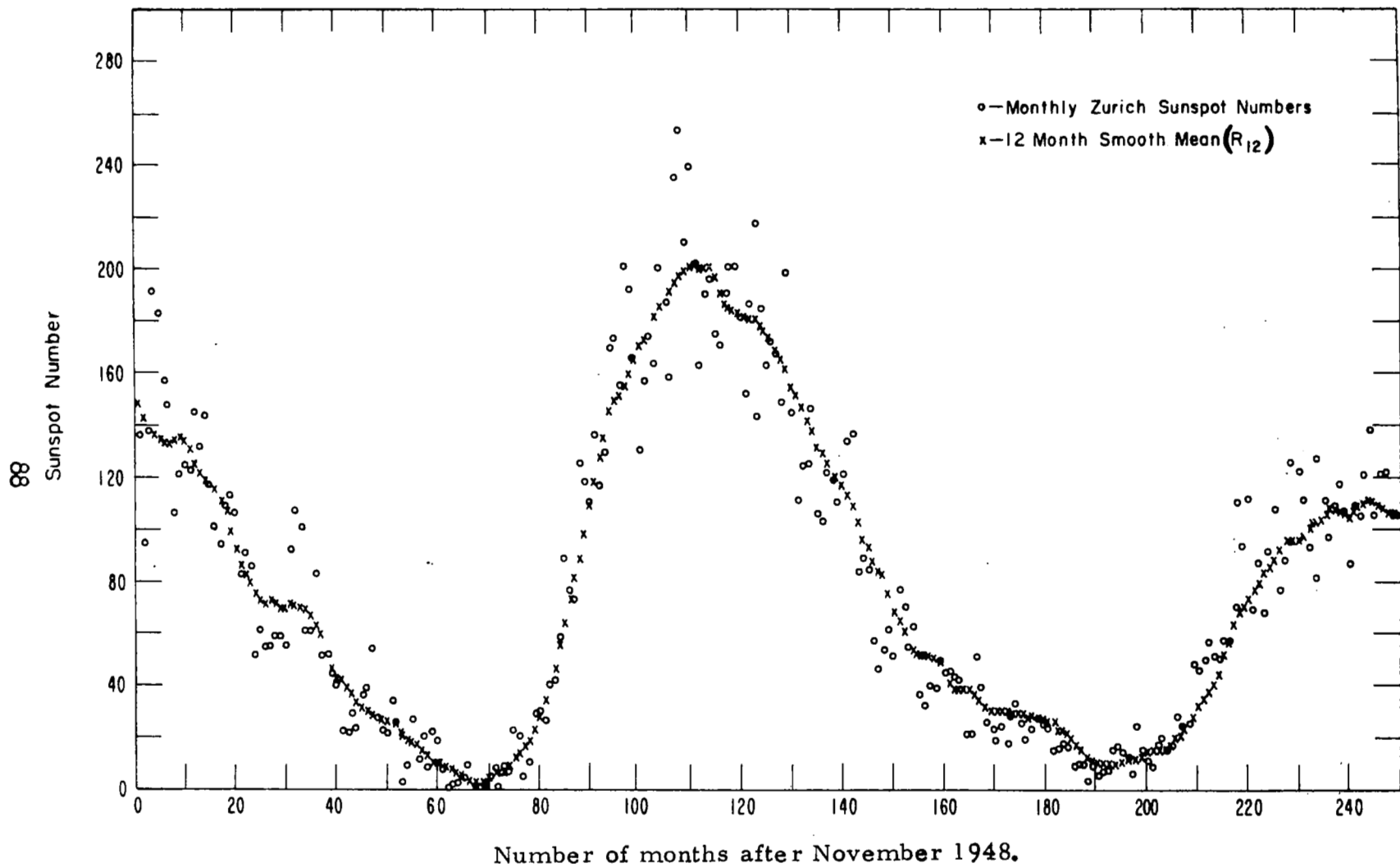


Figure 1. Monthly and Smoothed Mean Zurich Sunspot Numbers

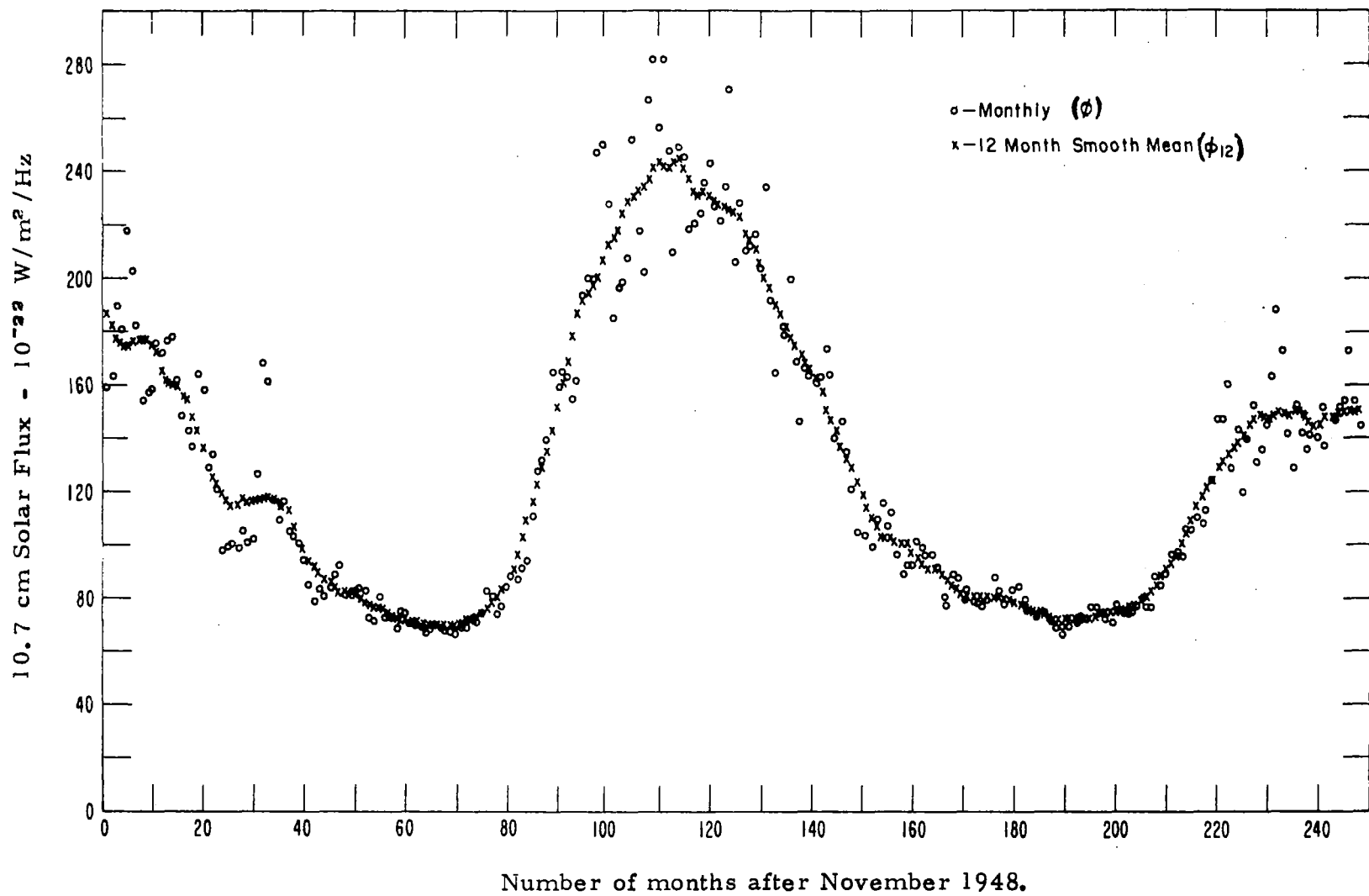


Figure 2. Monthly and Smoothed Mean 10.7-cm Solar Flux.

An autocorrelation analysis was performed and the autocorrelation coefficients for lags up to 36 months are listed in table 1. Both indices have a high positive correlation for lags up to 12 months and are substantially correlated for time lags greater than 12 months. The amount of autocorrelation remaining in lags greater than 13 months cannot be attributed to the smoothing process. In a rigorous statistical analysis of the data, the autocorrelation should be taken into account.

However, for this analysis, the additional effort required to properly account for all of the autocorrelation was not feasible. Methods of approximation are available which were considered more practical in this instance.

The effective number of observations must be reduced by at least a factor of 13 to account for the autocorrelation introduced by the averaging process. One might argue that the number of degrees of freedom should be reduced by a factor of 31, but this would result in a conclusion that is overly pessimistic. Since it was difficult to determine the added effect of autocorrelation for lags greater than 12 and further, since the effect of a reduction of 13 is already overwhelming, it was decided that further reduction in the number of degrees of freedom could not be justified. This could be accomplished also by using only every thirteenth value in the analysis; however, the data excluded may contain valid information. Therefore, all the available data were used in the analysis and the significance of the relationships and their parameters were tested using the total number of observations divided by 13 as the effective sample size.

From a visual examination of a plot of R_{12} versus ϕ_{12} , as shown in figure 3, a linear fit to the data would appear to satisfy the relationship adequately; certainly no equation greater than a cubic would be necessary. Thus the investigation was restricted to a relationship of the following type:

Table 1. Autocorrelation Coefficients ϕ_{12}

Lag	Coefficient	Lag	Coefficient
1	0.995	19	0.530
2	0.987	20	0.489
3	0.977	21	0.447
4	0.964	22	0.405
5	0.949	23	0.362
6	0.932	24	0.319
7	0.912	25	0.276
8	0.891	26	0.233
9	0.868	27	0.190
10	0.841	28	0.147
11	0.814	29	0.104
12	0.783	30	0.062
13	0.751	31	0.020
14	0.717	32	-0.020
15	0.683	33	-0.060
16	0.647	34	-0.100
17	0.609	35	-0.137
18	0.571	36	-0.174

Autocorrelation Coefficients for R_{12}

Lag	Coefficient	Lag	Coefficient
1	0.997	19	0.537
2	0.990	20	0.496
3	0.980	21	0.455
4	0.967	22	0.412
5	0.952	23	0.369
6	0.934	24	0.327
7	0.914	25	0.284
8	0.892	26	0.241
9	0.868	27	0.198
10	0.842	28	0.155
11	0.814	29	0.112
12	0.784	30	0.070
13	0.753	31	0.028
14	0.720	32	-0.013
15	0.687	33	-0.053
16	0.651	34	-0.093
17	0.615	35	-0.132
18	0.577	36	-0.170

$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3$$

The individual rising and declining phases of the solar cycle were analyzed to determine whether there was a consistent relationship among them. The divisions of the data are shown in table 2.

Table 2

Phases of the Solar Cycle

Sample A = Nov 1947 to April 1954	(declining part of cycle 18)
Sample B = May 1954 to March 1958	(rising part of cycle 19)
Sample C = April 1958 to Oct 1964	(declining part of cycle 19)
Sample D = Nov 1964 to Nov 1968	(rising part of cycle 20)

3. ANALYSIS AND STATISTICAL DISCUSSION

The numerical results of the analysis outlined in the previous section are displayed in tables 3 through 6, respectively. The "number of observations" is the actual number of observations in the sample. The "effective sample size" is the number of "independent" observations after correcting for autocorrelation as indicated previously. In the matrix of simple correlation coefficients, the correlation is presented between the first-column symbols and the first-row symbols. Under the heading of "Statistics of the fit," the b_i are the estimates of the β_i coefficients and s_{b_i} are the estimated standard error of the partial regression coefficient b_i . The residual variance for each equation is tabulated under its respective heading and gives an indication of the relative significance of the three equations. To test the significance of the last coefficient in each equation a student t test was used where

$$t_i = \frac{b_i}{s_{b_i}}$$

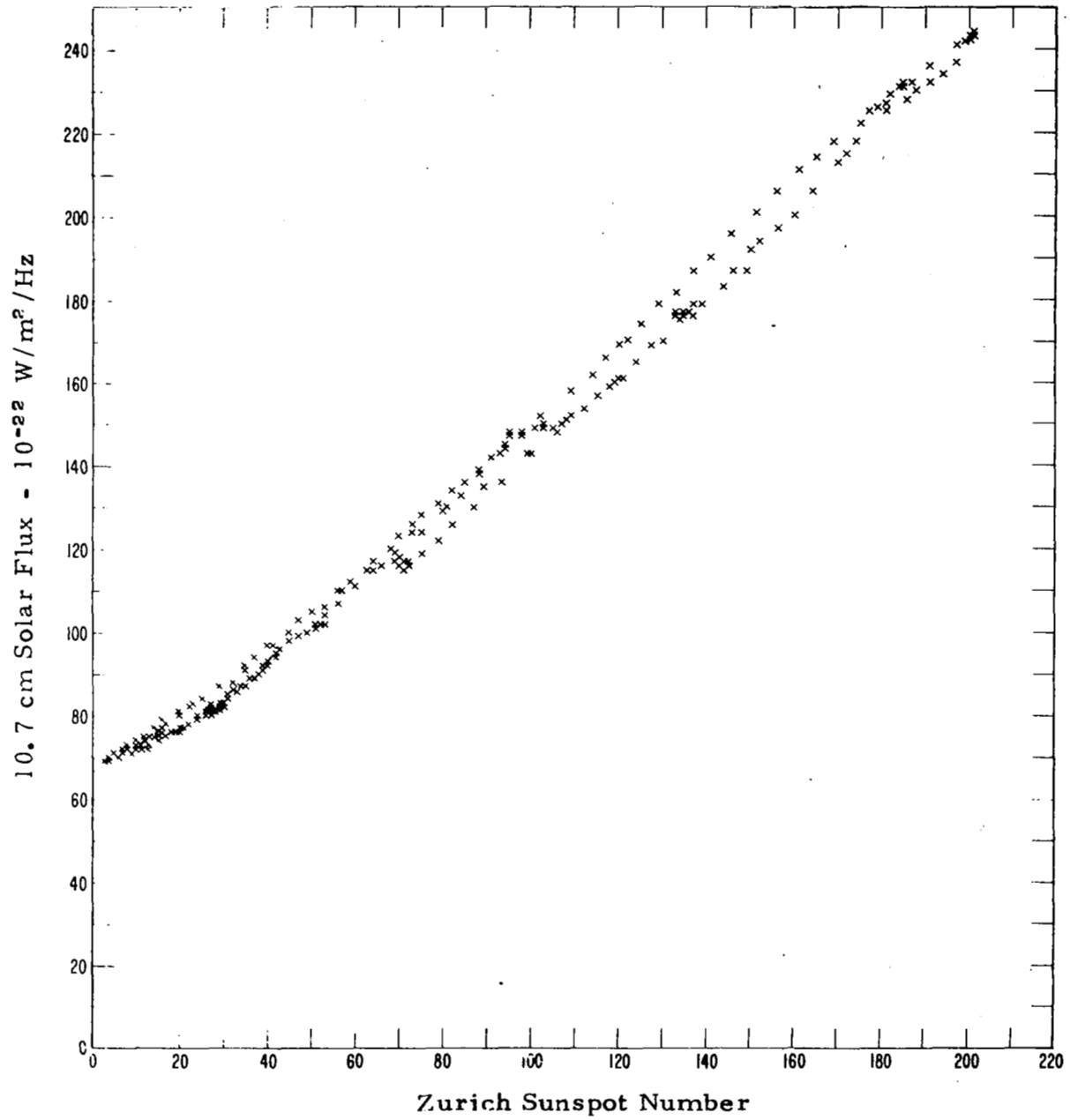


Figure 3. Smoothed Mean 10.7-cm Solar Flux versus Smoothed Mean Zurich Sunspot Number

Table 3

Sample A (November 1947 to April 1954)

Number of Observations = 78

Effective Sample Size = 6

Matrix of Simple Correlation Coefficients

	R_{12}	R_{12}^2	R_{12}^3	ϵ_{12}
R_{12}	1.0	0.975	0.939	0.998
R_{12}^2		1.0	0.991	0.984
R_{12}^3			1.0	0.953

Statistics of the Fit

Equation	<u>First Degree</u>		<u>Second Degree</u>		<u>Third Degree</u>	
Term	b_i	s_{b_i}	b_i	s_{b_i}	b_i	s_{b_i}
Constant	6.01207×10^1	(--)	6.43838×10^1	(--)	6.51713×10^1	(--)
R_{12}	8.47499×10^{-1}	(6.46×10^{-3})	6.64028×10^{-1}	(2.00×10^{-2})	5.97388×10^{-1}	(5.41×10^{-2})
R_{12}^2			1.17764×10^{-3}	(1.25×10^{-4})	2.31749×10^{-3}	(8.70×10^{-4})
R_{12}^3					-5.06868×10^{-6}	(3.83×10^{-8})
Residual Variance	7.96		3.70		3.67	

Table 4

Sample B (May 1954 to March 1958)

Number of Observations = 47

Effective Sample Size = 4

Matrix of Simple Correlation Coefficients

	R_{12}	R_{12}^2	R_{12}^3	ϕ_{12}
R_{12}	1.0	0.976	0.937	0.999
R_{12}^2		1.0	0.990	0.985
R_{12}^3			1.0	0.952

Statistics of the Fit

Equation	<u>First Degree</u>		<u>Second Degree</u>		<u>Third Degree</u>	
Term	b_i	s_{b_i}	b_i	s_{b_i}	b_i	s_{b_i}
Constant	6.24378×10^1	(--)	6.68630×10^1	(--)	6.67558×10^1	(--)
R_{12}	8.81635×10^{-1}	(6.48×10^{-3})	6.97585×10^{-1}	(9.78×10^{-3})	7.06194×10^{-1}	(2.53×10^{-2})
R_{12}^2			9.00918×10^{-4}	(4.67×10^{-5})	7.93299×10^{-4}	(2.94×10^{-4})
R_{12}^3					3.43814×10^{-7}	(9.28×10^{-7})
Residual Variance	10.9		1.17		1.20	

Table 5

Sample C (April 1958 to October 1964)

Number of Observations 79

Effective Sample Size = 6

Matrix of Simple Correlation Coefficients

	R_{12}	R_{12}^2	R_{12}^3	φ_{12}
R_{12}	1.0	0.977	0.938	0.999
R_{12}^2		1.0	0.989	0.981
R_{12}^3			1.0	0.943

8

Statistics of the Fit

Equation	<u>First Degree</u>		<u>Second Degree</u>		<u>Third Degree</u>	
Term	b_i	s_{b_i}	b_i	s_{b_i}	b_i	s_{b_i}
Constant	5.61003×10^1	(--)	5.89609×10^1	(--)	6.52597×10^1	(--)
R_{12}	9.45009×10^{-1}	(4.72×10^{-3})	8.45517×10^{-1}	(1.90×10^{-2})	4.90703×10^{-1}	(2.35×10^{-2})
R_{12}^2			5.05898×10^{-4}	(9.46×10^{-5})	4.97246×10^{-3}	(2.77×10^{-4})
R_{12}^3					-1.50232×10^{-5}	(9.20×10^{-7})
Residual Variance	6.58		4.84		1.08	

Table 6

Sample D (November 1964 to November 1968)

Number of Observations = 49

Effective Sample Size = 4

Matrix of Simple Correlation Coefficients

	R_{12}	R_{12}^2	R_{12}^3	r_{12}
R_{12}	1.0	0.983	0.951	0.996
R_{12}^2		1.0	0.991	0.974
R_{12}^3			1.0	0.935

Statistics of the Fit

Equation	<u>First Degree</u>		<u>Second Degree</u>		<u>Third Degree</u>	
Term	b_i	s_{b_i}	b_i	s_{b_i}	b_i	s_{b_i}
Constant	6.40477×10^1	(--)	6.15553×10^1	(--)	7.10189×10^1	(--)
R_{12}	8.24179×10^{-1}	(1.03×10^{-2})	9.60207×10^{-1}	(5.31×10^{-2})	1.42629×10^{-1}	(1.11×10^{-2})
R_{12}^2			-1.14068×10^{-3}	(4.38×10^{-4})	1.50889×10^{-2}	(2.11×10^{-3})
R_{12}^3					-8.92102×10^{-5}	(1.15×10^{-5})
Residual Variance		7.11		6.33		2.76

Table 7 displays the t values and their number of degrees of freedom ν . In samples B and D the effective sample size is too small even to consider the third degree equation while samples A and C have t values of 1.3 and 16.3 respectively.

Table 7. t Values for Testing the Significance of the Last Coefficient

Equation Sample	First degree		Second degree		Third degree	
	t	ν	t	ν	t	ν
A	131.2	(4)	9.4	(3)	1.3	(2)
B	136.0	(2)	19.3	(1)	---	(0)
C	200.2	(4)	5.3	(3)	16.3	(2)
D	80.0	(2)	2.6	(1)	---	(0)

With the exception of sample C, the null hypothesis that $\beta_3 = 0$ in the third degree equation, is satisfied at the 5% level of significance. The same hypothesis applied to the last coefficient of the second degree equation is rejected in all cases except D. Although there is still substantial autocorrelation remaining after reducing the sample size by a factor of 13, it was decided that the second degree equation adequately represented the data in samples A, B, C, and D.

A further test was necessary to determine whether a single second degree equation could be applied to all the samples. The hypothesis that all the second degree equations are from the same population was tested by the formula,

$$t_i = \frac{(b_{1i} - b_{2i})}{\sqrt{s^2_{b_{1i}} - s^2_{b_{2i}}}} \quad (\text{Bennett, Franklin, 1954})$$

which approximates the t -distribution with ν degrees of freedom where ν is given by

$$\frac{1}{v} = \frac{1}{v_1} \left(\frac{s^2_{b1i}}{s^2_{b1i} + s^2_{b2i}} \right)^2 + \frac{1}{v_2} \left(\frac{s^2_{b2i}}{s^2_{b1i} + s^2_{b2i}} \right)^2,$$

and $v_1 = (n_1 - 3)$, $v_2 = (n_2 - 3)$ for a second degree equation.

Table 8 displays the t value for each b_1 and b_2 coefficient for the four second degree equations. The number of degrees of freedom associated with each t value is tabulated in parentheses.

Table 8. t Values for the b_1 and b_2 Coefficients of the Second Degree Equation

	<u>b_1</u>		
	A	B	C
D	5.0(1)	4.9(1)	2.0(1)
C	6.6(6)	6.3(6)	
B	1.5(4)		

	<u>b_2</u>		
	A	B	C
D	5.0(1)	4.6(1)	3.7(1)
C	4.3(6)	3.7(4)	
B	2.1(4)		

Comparing the t values in table 8 with a set of standard t values, at the 5% significance level we conclude that the b_1 and b_2 coefficients for samples A, B, and D are from the same population. Both coefficients for sample C appear to be significantly different and therefore, it is not conclusive that they are from the same population. The statistics would seem to indicate that there is no systematic relationship between the four half-cycles. In fact, a different equation would be necessary to represent Sample C.

For practical considerations this is not very useful since the relationship cannot be determined until after the observations are made. To obtain a useful relationship for predicting solar flux, it was decided to combine all the available data and form one large sample. This was justified by the small number of effective independent observations in the samples A, B, C, and D and the fact that there is significant auto-correlation still remaining. Regression equations, up to the third degree, were fitted to the new sample. The results of these analyses are shown in table 9. The effective sample size in this case was 23. For the third degree equation there were 19 degrees of freedom and the t value at the 5% significance level was 2.093. From table 9 the calculated t value for b_3 was 2.86 and thus the pooled data is not consistent with the null hypothesis that $\beta_3=0$. However, the second order equation with $t=13.53$, is considerably more significant than the third order equation. Furthermore, the reduction in the residual variation due to the third degree is so small that it was concluded that the second order equation sufficiently represented the independent variations in the data. To illustrate this, the 95% confidence intervals (based on the effective sample size) for the ordinate to the true regression line were constructed and plotted for samples A, B, C, D and the pooled sample. The plots of the data, the regression lines, and the 95% confidence bands are presented in figures 4 through 8. The confidence limits in figure 7 seem rather extreme at first glance; the explanation for this is that the effective number of independent observations is only 4 and the large variance due to the restricted range of that particular sample. The confidence limits in figure 8 are narrower than figures 4 through 7 and can be explained by the larger effective number of independent observations. The confidence intervals plotted in figures 4 through 8 are limits for the true mean η and not for an individual predicted Y.

Table 9

Pooled Sample (November 1947 to November 1968)

Number of Observations = 253

Effective Sample Size = 23

Matrix of Simple Correlation Coefficients

	R_{12}	R_{12}^2	R_{12}^3	ϕ_{12}
R_{12}	1.0	0.964	0.902	0.997
R_{12}^2		1.0	0.983	0.975
R_{12}^3			1.0	0.920

Statistics of the Fit

Equation	<u>First Degree</u>		<u>Second Degree</u>		<u>Third Degree</u>	
Term	b_i	s_{b_i}	b_i	s_{b_i}	b_i	s_{b_i}
Constant	5.90588×10^1	(--)	6.37451×10^1	(--)	6.51066×10^1	(--)
R_{12}	8.95393×10^{-1}	(4.46×10^{-3})	7.28015×10^{-1}	(1.28×10^{-2})	6.41014×10^{-1}	(3.28×10^{-2})
R_{12}^2			8.90443×10^{-4}	(6.58×10^{-5})	2.00138×10^{-3}	(3.92×10^{-4})
R_{12}^3					-3.72201×10^{-6}	(1.30×10^{-5})
Residual Variance		17.13		9.93		9.65

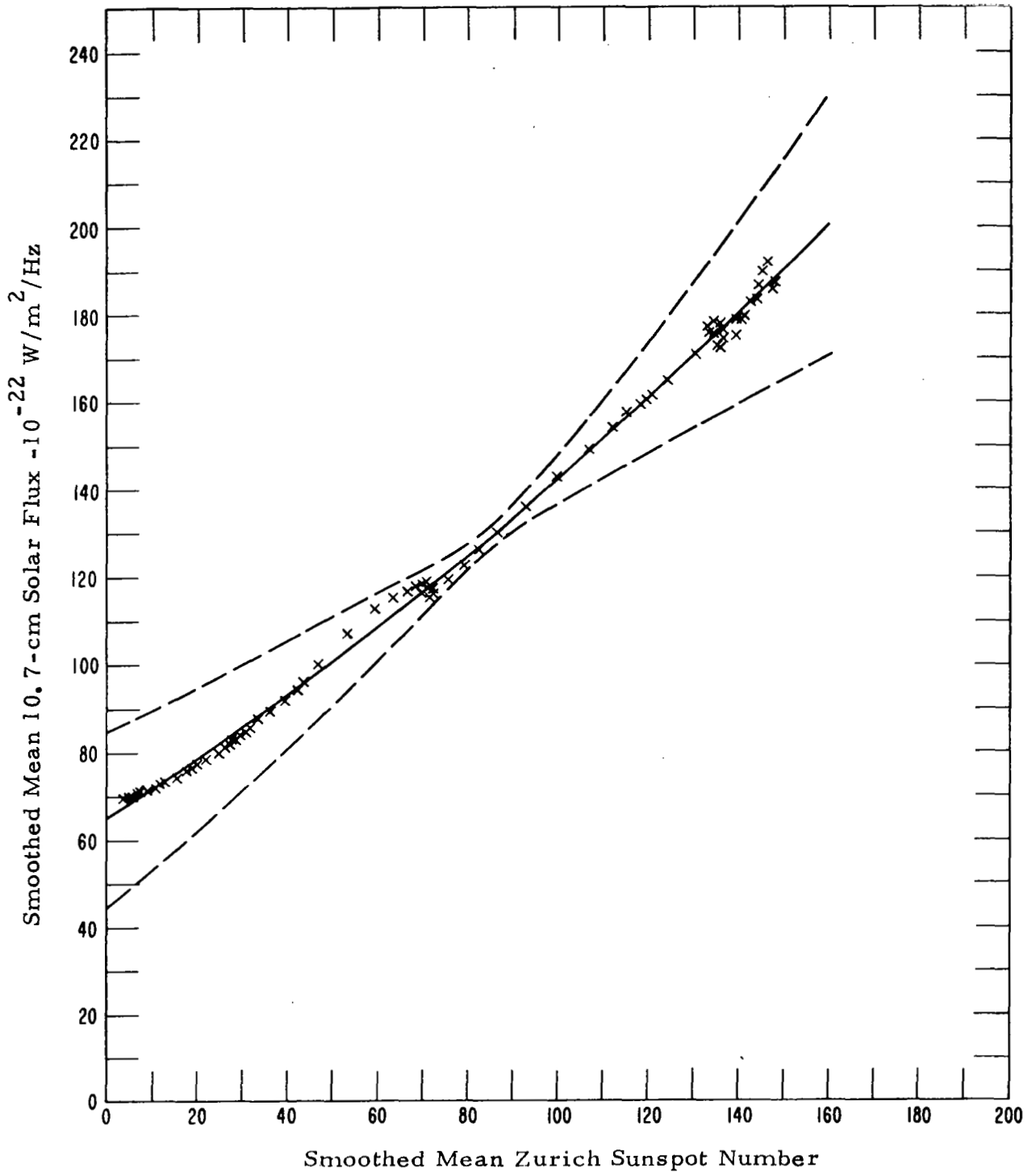


Figure 4. Data and Fitted 2nd Degree Equation with 95% Confidence Intervals November 1947 to April 1954

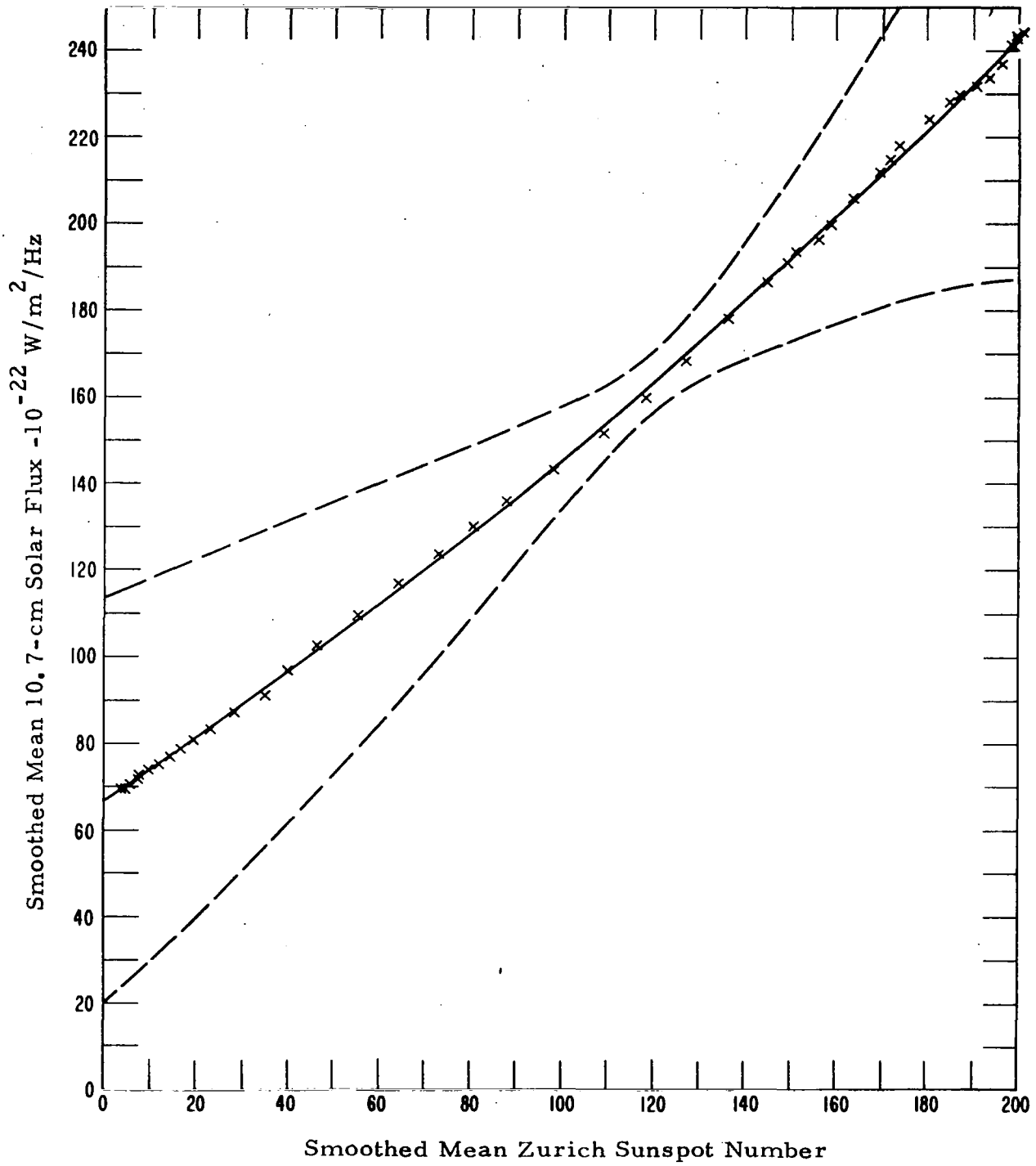


Figure 5. Data and Fitted 2nd Degree Equation with 95% Confidence Intervals May 1954 to March 1958

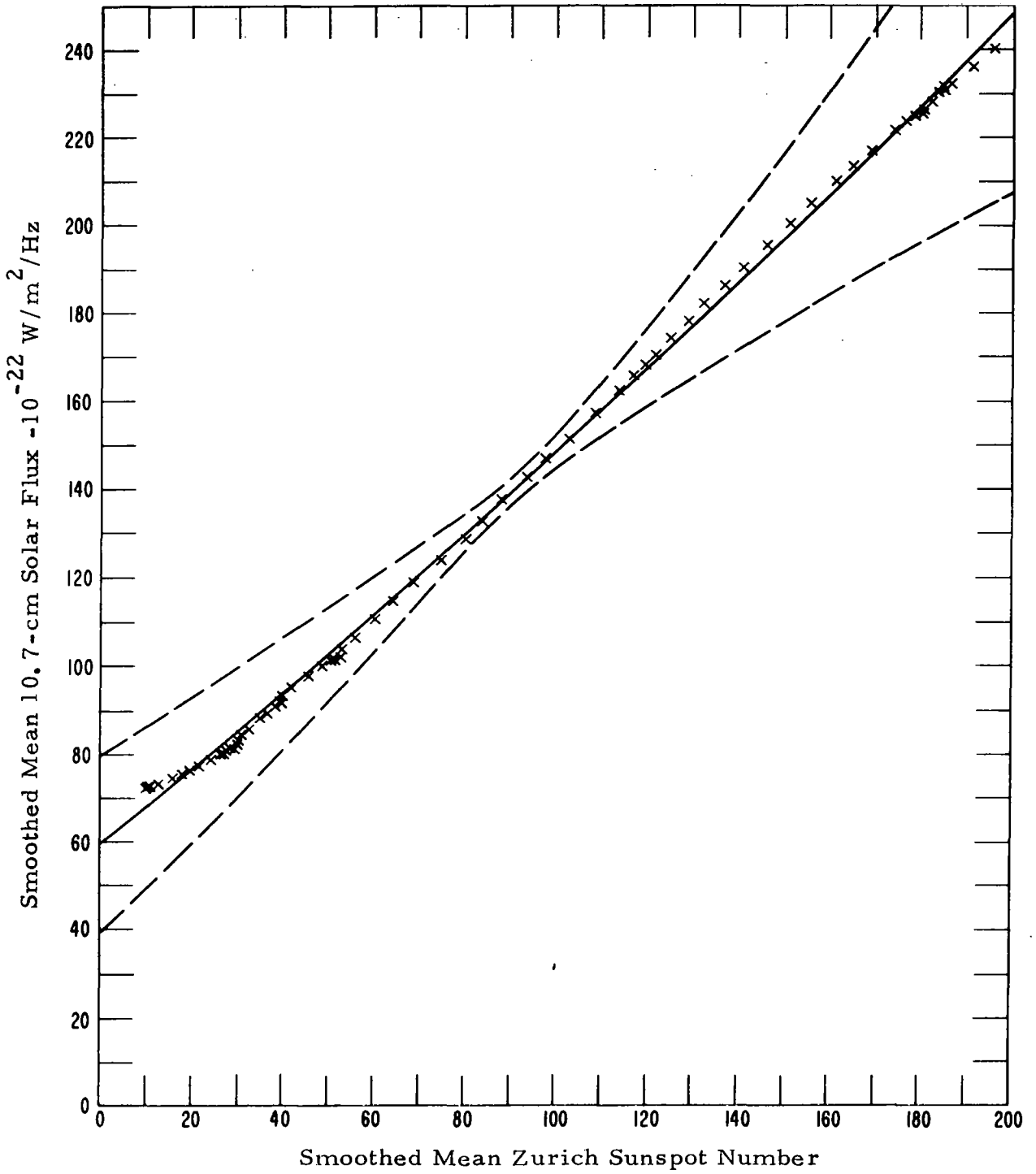


Figure 6. Data and Fitted 2nd Degree Equation with 95% Confidence Intervals April 1958 to October 1964

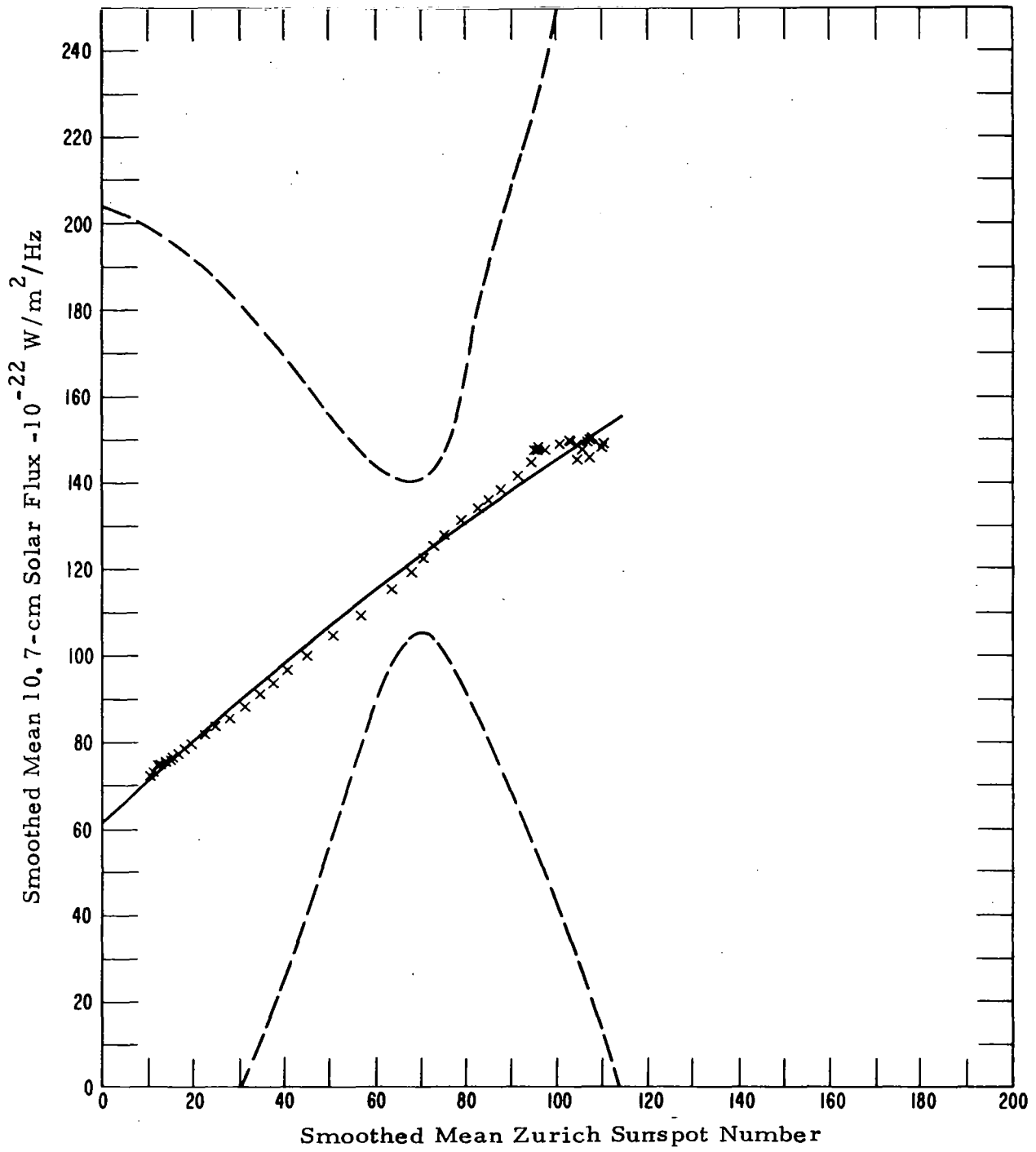


Figure 7. Data and Fitted 2nd Degree Equation with 95% Confidence Intervals, November 1964 to November 1968

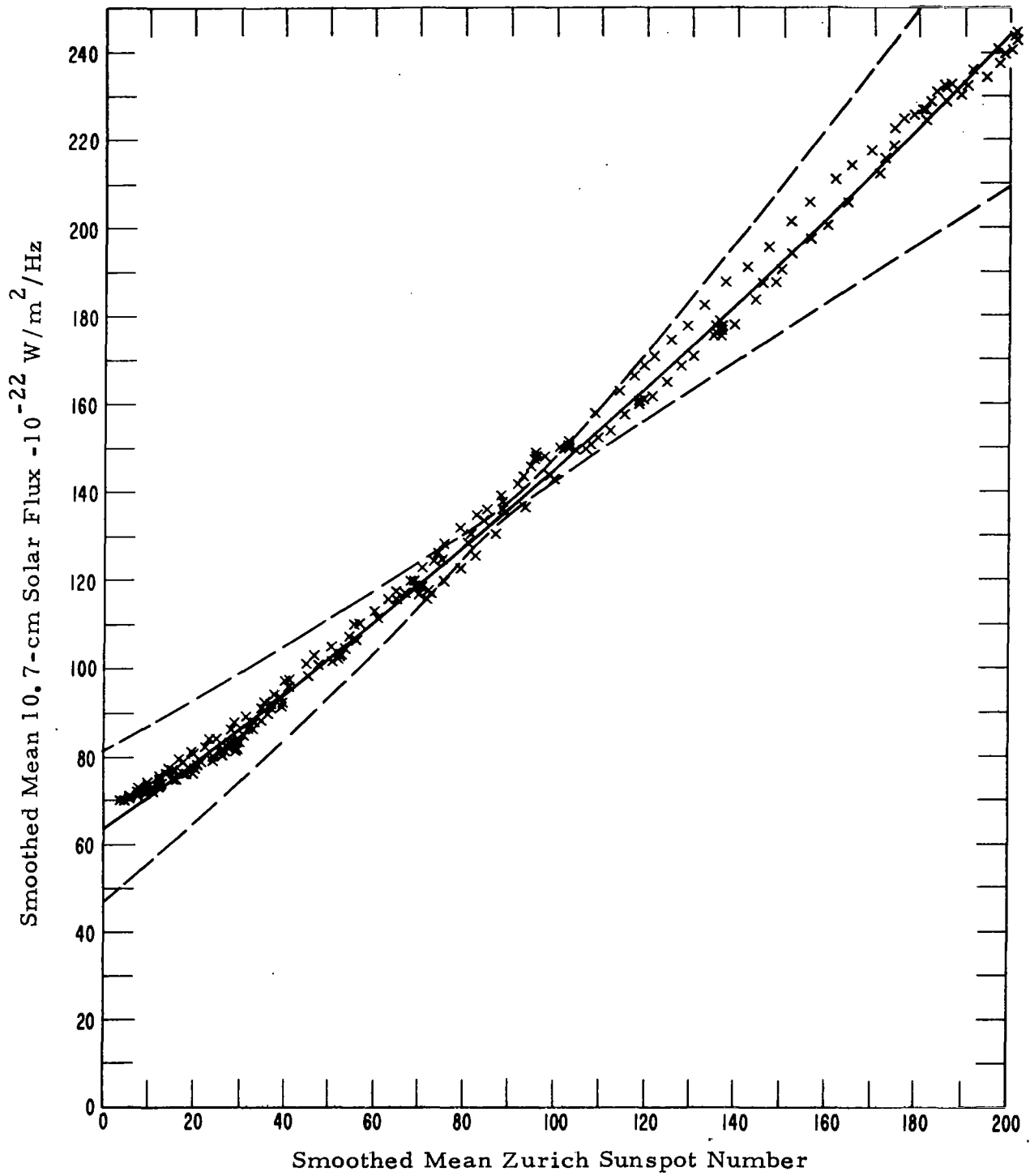


Figure 8. Data and Fitted 2nd Degree Equation with 95% Confidence Intervals, November 1947 to November 1968

Using the second degree relationship derived from the pooled sample as the best approximation to the true relationship between φ_{12} and R_{12} , we have the following equation:

$$\varphi_{12} = 63.7451 + 0.728015 R_{12} + 0.000890443 (R_{12})^2 \quad (1)$$

This second degree equation was compared with an expression described in a NASA report (T. J. Richards, 1965) which consists of the following three straight line segments

$$\begin{aligned} \varphi_{12} &= 68.0 + 0.60 R_{12}, & 0 < R_{12} \leq 30 \\ \varphi_{12} &= 60.0 + 0.825 R_{12}, & 30 < R_{12} \leq 70 \\ \varphi_{12} &= 50.0 + 0.967 R_{12}, & 70 < R_{12} \leq 200 \dots \end{aligned} \quad (2)$$

The curves in figure 9 show that there is little difference between values of φ_{12} derived from (1) and (2) for any given values of R_{12} from 0 to 200.

4. PREDICTION RELIABILITY

The discussion up to this point has been concerned with establishing a relationship between φ_{12} and R_{12} . The validity of using this relationship for predicting φ_{12} from a predicted value of R_{12} must be considered. For a predicted value Y , derived from a predicted value X , we have the following:

$$\text{Var}[Y] = E[(Y - \mu_Y)^2] = E[\text{Var}(Y|X)] + \text{Var}[E(Y|X)], \quad (3)$$

where μ_Y is the mean value of Y . This relationship is derived in Parzen (1962), page 55, equation (2.25). For a second degree polynomial relationship such as (1), we have for the second part of (3)

$$\text{Var}[\mu_Y|X] = \text{Var}[\alpha_0 + \alpha_1 X + \alpha_2 X^2] = \alpha_1^2 \text{Var}[X] + \alpha_2^2 \text{Var}[X^2] + 2\alpha_1\alpha_2 \text{Cov}[X, X^2].$$

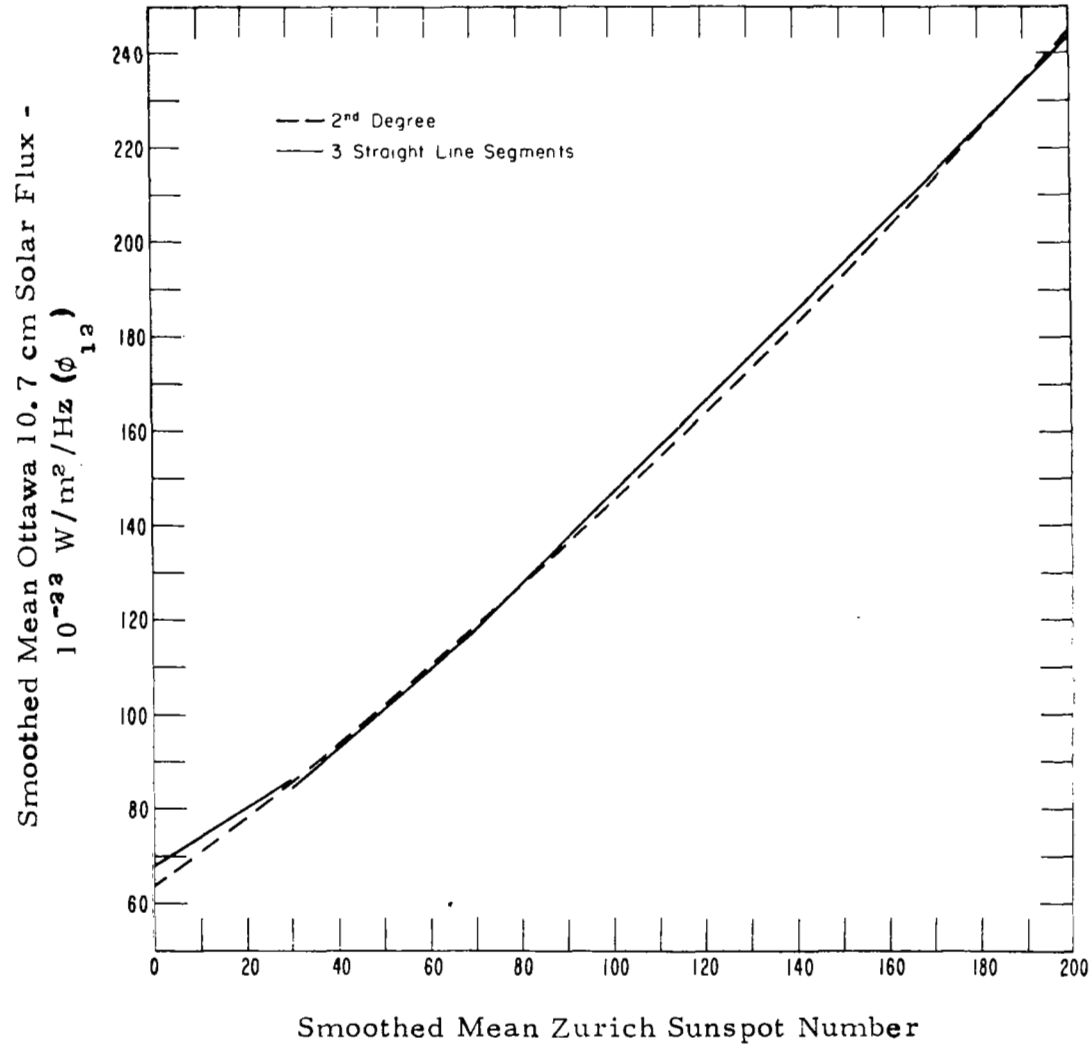


Figure 9. Comparison of the Second Degree and Three Straight Line Segment Expressions for the Relationship between Smoothed Means of Ottawa 10.7-cm Solar Radio Flux and of Zurich Sunspot Number.

If normality of X is assumed, we have

$$\text{Var}[\mu_Y | X] = \mu_2[\alpha_1^2 + \alpha_2^2 (2\mu_2 + 4\mu^2) + 4\alpha_1\alpha_2\mu] \quad (4)$$

where

$$\mu = E[X] \text{ and } \mu_2 = \text{Var}[X].$$

For the first part of (3) we have

$$E[\text{var}(Y | X)] \approx \text{Var}[Y | X = x],$$

where

$$\text{Var}[Y | X = x] = \sigma^2_{Y|X} \left\{ 1 + \frac{1}{M} + \frac{(X - \bar{X})^2 \sum_{i=1}^M (x_i^2 - \bar{x}^2)^2 - 2(X - \bar{X})(X^2 - \bar{x}^2) \sum_{i=1}^M (x_i - \bar{x})(x_i^2 - \bar{x}^2) + (X^2 - \bar{x}^2)^2 \sum_{i=1}^M (x_i - \bar{x})^2}{\sum_{i=1}^M (x_i - \bar{x})^2 \sum_{i=1}^M (x_i^2 - \bar{x}^2)^2 - \left\{ \sum_{i=1}^M (x_i - \bar{x})(x_i^2 - \bar{x}^2) \right\}^2} \right\} \quad (5) \text{ (Hald 1952)}$$

and M is the number of independent observations used to establish the relationship between Y and X.

To illustrate the magnitude of the variance of a predicted value of φ_{12} the following approximations were made:

$$E[X] = E[R_{12}] \approx \text{predicted sunspot number } (\hat{R}_{12})$$

$$\text{Var}[X] = \text{Var}[\hat{R}_{12}] \approx \text{estimated variance of the predicted value of } \hat{R}_{12}$$

$$\sigma^2_{Y|X} \approx \text{estimated variance about regression equation (1)}$$

$$\text{Sample variance of } \varphi_{12} = 2.75 \times 10^3$$

$$\text{Sample variance of } R_{12} = 3.41 \times 10^3$$

The predicted \hat{R}_{12} for April 1970 based on observed R_{12} for October 1969 was 97.5 and the approximate variance of the predicted number was 37.6. The above values were based on the McNish-Lincoln method using cycles 8 through 19 (Stewart and Ostrow, 1970). Substituting these values into (4) we obtained the following:

$$\begin{aligned} \text{Var}[\mu_Y | X] &= 37.6 [(72.8015 \times 10^{-2})^2 + (89.0443 \times 10^{-5})^2 (2 \times 37.6 + 4 \times 97.5^2) \\ &\quad + 4(72.8015 \times 10^{-2})(89.0443 \times 10^{-5})(97.5)] \approx 30.6 \end{aligned}$$

With $X = 97.5$, $M = 23$ and $\sigma^2_{Y|X} = 9.93$, we find that

$$\text{Var}[Y | X = x] = 9.93 [1.1193] = 11.11 .$$

Therefore, the variance of ϕ_{12} is 41.7. Thus we see that the variance of the predicted sunspot number in this example is 0.9% of the total variance of R_{12} while the predicted 10.7 cm solar flux in the example has a variance which is 1.5% of the total variance of ϕ_{12} . As one would expect using this sort of scheme, the predicted value of ϕ_{12} has more relative uncertainty associated with it than the predicted value of R_{12} . This also demonstrates that the error bounds associated with a predicted value \hat{R}_{12} cannot be used to describe the reliability of a value of $\hat{\phi}_{12}$ predicted from that particular \hat{R}_{12} . It is of considerable interest that most of the variance in predicting $\hat{\phi}_{12}$ arises from the uncertainty in the predicted sunspot number rather than from the variability of ϕ_{12} for a known sunspot number.

5. CONCLUSIONS

Using observed data from November 1947 to November 1968 it was determined that a second degree equation best described the relationship between φ_{12} and R_{12} . This relationship was selected in preference to the others investigated on the basis of the statistical evidence and the highly positive correlation of the observations. No systematic relationship for the rising and declining phases of the solar cycle could be determined due to the limited sample size and the small number of effective independent observations. Further research should be conducted to eliminate all of the autocorrelation. It is conceivable that the additional statistics would add further support to these conclusions and might present evidence that only a linear relationship is justified.

This second degree relationship can be used to predict φ_{12} from a predicted value of R_{12} . Since most of the variance associated with the predicted value of φ_{12} results from the prediction of R_{12} , the variance of the predicted value of φ_{12} is only slightly greater than the variance of the predicted R_{12} . In addition, it should be noted that the variance of the predicted φ_{12} must be greater than that of the predicted value of R_{12} .

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