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## BOUNDED STATE PROBLEMS AND THE CALCULUS OF VARIATIONS

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16. Abstract <p>An optimal control problem with bounded state variables is transformed into a Lagrange problem by means of differentiable mappings which take some Euclidean space onto the control and state regions. Whereas all such mappings lead to a Lagrange problem, it is shown that only those which are defined as acceptable pairs of transformations are suitable in the sense that solutions to the transformed Lagrange problem will lead to solutions to the original bounded state problem and vice versa. In particular, an acceptable pair of transformations is exhibited for the case when the control and state regions are right parallelepipeds.</p> <p>Finally, a description of the necessary conditions for the bounded state problem which were obtained by this method is given.</p>			
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## SUMMARY

An optimal control problem with bounded state variables is transformed into a Lagrange problem by means of differentiable mappings which take some Euclidean space onto the control and state regions. Whereas all such mappings lead to a Lagrange problem, it is shown that only those which are defined as acceptable pairs of transformations are suitable in the sense that solutions to the transformed Lagrange problem will lead to solutions to the original bounded state problem and vice versa. In particular, an acceptable pair of transformations is exhibited for the case when the control and state regions are right parallelepipeds.

Finally, a description of the necessary conditions for the bounded state problem which were obtained by this method is given.

## INTRODUCTION

Optimal control problems have been analyzed in a variety of ways by means of the Calculus of Variations. Perhaps the most well known technique, described in a paper by Berkovitz (1), involves adjoining additional variables to the system, commonly called slack variables, in order to transform inequality constraints into differential equation constraints. In the new problem all variables are unrestricted; hence, the classical theory can be applied. Necessary conditions for the Bolza problem are then translated into necessary conditions for optimal control. These conditions include the maximum principle of Pontryagin (20) which is seen to be a consequence of the Euler Lagrange equations and the Weierstrass necessary condition. In a paper (12) and in his book (13), Hestenes used a similar method to study optimal control problems. This method of slack variables has been widely used by a number of authors for some time. In fact, as early as 1937, F. A. Valentine (26) applied it to Lagrange problems with differential inequalities as added side conditions.

Later Berkovitz (2) applied this technique to problems with bounded state variables. He obtains essentially the results of Gamkrelidze in Chapter VI of (20) for the nonlinear problem with inequality constraints involving only state variables. Hestenes' results were extended by Russak (21) and Guinn (10) to include the bounded state case.

Another approach, described by Kalman (15), uses the Hamiltonian theory of the Calculus of Variations as it was developed by Caratheodory (5). Sagan also utilizes this method in his book (23) where the maximum principle is shown to follow from Caratheodory's lemma in a rather simple way. However, as is pointed out by the author, the usefulness of this technique is quite limited due to the

fact that unnecessary assumptions must be made regarding the differentiability of Hamilton's characteristic function and the existence of an admissible set of inception.

In this paper a technique described by Park in (18) and (19) and applied to a simple problem by this author in (11) is to be utilized to study properties of solutions to various general problems in Optimal Control Theory, in particular those with bounded state variables. By means of an appropriate transformation of variables, optimal control problems are converted into Lagrange problems of the Calculus of Variations. This is accomplished by using mappings satisfying certain properties which take some Euclidean space onto closed control and state regions.

Of course, the fundamental question must be considered as to under what conditions is the transformed problem equivalent to the original one. That is, one must know that solutions to the new problem lead to solutions of the old one and vice versa. This equivalence question is discussed in great generality by Park in (18); in this paper it will only be considered in relation to bounded state problems.

## ANALYSIS

### I. The Problem.

Let  $\Omega$  be a subset of  $R^m$ , Euclidean  $m$ -space, and  $\Gamma$  a subset of  $R^n$ . The sets  $\Omega$  and  $\Gamma$  will be referred to as the control region and state region respectively. Let  $x^0$  and  $x^1$  be points in  $\Gamma$ ;  $x^0$  will be referred to as the initial point and  $x^1$  the terminal point.

We now consider a differential system whose state at time  $t$  is characterized by a vector  $x(t) = (x_1(t), \dots, x_n(t))$  in  $\Gamma$  and whose value is determined or controlled at that time by a vector  $u(t) = (u_1(t), \dots, u_m(t))$  in  $\Omega$ , the so-called control vector. Let the system be defined by the differential equations  $\dot{x} = f(t, x, u)$  where  $f$  is assumed to be a continuously differentiable vector-valued function defined on  $R \times \Gamma \times \Omega$  where  $R$  is the real line. This is a non-autonomous system of  $n$  non-linear first order differential equations in the  $n + m$  unknowns  $x$  and  $u$ . Let an initial time  $t_0$  be given and designate the final time, which is variable, by  $t_1$ . In addition, let  $f_0(t, x, u)$  be a continuously differentiable real valued function defined on  $R \times \Gamma \times \Omega$ . We will refer to its integral over the interval  $[t_0, t_1]$  as the cost functional.

The problem to be considered is the following:

#### PROBLEM 1

Find a sectionally continuous control  $u(t)$  defined on  $[t_0, t_1]$  for some  $t_1 > t_0$  so that there exists  $x(t)$ , sectionally smooth, defined on  $[t_0, t_1]$  such that

(1)  $\dot{x}(t) = f(t, x(t), u(t))$  for all  $t \in [t_0, t_1]$  for which  $\dot{x}(t)$  is defined,

(2)  $x(t) \in \Gamma$  and  $u(t) \in \Omega$  for all  $t \in [t_0, t_1]$ ,

(3)  $x(t_0) = x^0$  and  $x(t_1) = x^1$ ,

(4)  $\int_{t_0}^{t_1} f_0(t, x(t), u(t)) dt$  is minimized.

This is an example of an optimal control problem with restricted state variables. Such a control  $u(t)$  satisfying (1), (2), and (3) is called an admissible control. A control for which (1), (2), (3), and (4) is satisfied is called an optimal control, and its corresponding  $x(t)$  is called an optimal trajectory.

## II. Review of Literature

This type of problem has been considered by a number of people. The papers by Berkovitz (2), Guinn (10) and Russak (21) have already been discussed in the introduction. They used the method of slack variables to translate necessary conditions for the Bolza problem into necessary conditions for the above problem.

Other approaches have been more direct and do not utilize the Calculus of Variations. Gamkrelidze (20) adjoined the total time derivative of the state constraint to the cost functional and treated the resulting problem in the same fashion as he and Pontryagin did for the unconstrained problem. His "regularity" assumption, which also arises in this paper but for different reasons, is made in order to insure that the control explicitly appears in the derivative of the constraint so that it directly influences his new cost functional.

In conjunction with numerical applications, Bryson et al. (4) developed techniques for avoiding this assumption in some cases by using higher order derivatives of the constraint. Results related to this are also demonstrated in a paper by Speyer and Bryson (24).

Dreyfus in (8) and (9) uses the method of dynamic programming to obtain results similar to those of Berkovitz. In fact, Berkovitz and Dreyfus compared their previous results in a joint paper (3) in 1965.



The penalty function approach is entirely different from all of the above. This method, first described by Chang in (6) and (7), involves the following. Instead of attempting a direct solution, an unconstrained problem is considered wherein the original cost functional is augmented by a non-negative penalty function which sharply increases the cost associated with trajectories which violate the state constraints. By using sequences of cost functionals involving more and more severe penalties it is to be expected in many cases that the desired solution to the original problem will be obtained as the limit of the solution of the approximate problem. This technique has been subsequently refined and further results obtained by Russell (22) and Jacobson et al. (14).

In (16) Khrustalev discusses the very difficult question of sufficiency for constrained state problems, and finally McIntire and Paiewonsky, in an expository paper written in 1964, survey the techniques known at that time.

### III. Transformation into a Lagrange Problem

We shall now state the method to be used herein to treat Problem 1 as a Lagrange problem in the Calculus of Variations.

Consider the functions  $\psi$  and  $\phi$  defined so that  $\psi: R^k \rightarrow \Omega$  and  $\phi: R^\ell \rightarrow \Gamma$  where  $k$  and  $\ell$  are natural numbers. Assume that both  $\psi$  and  $\phi$  are onto and continuously differentiable. Then if we let  $x = \phi(y)$  and  $u = \psi(\dot{z})$  and restate our original problem in terms of the new variables  $y = (y_1, \dots, y_\ell)$  and  $\dot{z} = (\dot{z}_1, \dots, \dot{z}_k)$  we obtain the following:

#### PROBLEM 2

Find a sectionally continuous  $\dot{z}(t)$  defined on  $[t_0, t_1]$  for some  $t_1 > t_0$  so that there exists  $y(t)$ , sectionally smooth, defined on  $[t_0, t_1]$  such that

$$(1)' \quad \frac{\partial \phi}{\partial y}(y(t)) \dot{y}(t) = f(t, \phi(y(t)), \psi(\dot{z}(t))) \text{ for all } t \in [t_0, t_1] \text{ for}$$

which  $\dot{y}(t)$  is defined,

(2)'  $y(t) \in R^l$  and  $\dot{z}(t) \in R^k$  for all  $t \in [t_0, t_1]$ ,

(3)'  $\phi(y(t_0)) = x^0$  and  $\phi(y(t_1)) = x^1$ ,

(4)'  $\int_{t_0}^{t_1} f_0(t, \phi(y(t)), \psi(\dot{z}(t))) dt$  is minimized.

In (1)'  $\frac{\partial \phi}{\partial y}$  is used to denote the  $n \times l$  matrix of partial derivatives of the components of  $\phi$  with respect to the components of the vector  $y$ . Notice also that this new problem is unconstrained in the sense that the new variables  $y$  and  $\dot{z}$  are allowed to take on any values in  $R^l$  and  $R^k$  respectively.

The variable  $\dot{z}$  which takes the place of the control  $u$  is introduced as a derivative so that the new problem will satisfy the hypotheses for the necessary conditions to the Lagrange problem which require that solutions, in this case  $y(t)$  and  $z(t)$ , be sectionally smooth. That is, a sectionally smooth solution of Problem 2,  $y(t)$  and  $z(t)$ , will lead to a sectionally continuous control  $u(t) = \psi(\dot{z}(t))$  and a sectionally smooth trajectory  $x(t) = \phi(y(t))$  as its counterpart in Problem 1.

We may now view Problem 2 as a Lagrange problem with unknowns  $y$  and  $z$ , constraining differential equations given in (1)', boundary conditions (3)' and with (4)' giving the functional to be minimized. Hence all the well known classical theory associated with this problem may be applied in order to find solutions. A full discussion of this theory is contained in Chapter 6 of (23).

However, before we proceed farther along these lines, we must establish under what conditions is Problem 2 actually equivalent to Problem 1, in the sense that a solution to Problem 2 leads to a solution of Problem 1 and vice versa. That is, we must be able to translate necessary and sufficient conditions for  $y(t)$  and  $z(t)$  in Problem 2 into necessary and sufficient conditions for  $u(t) = \psi(\dot{z}(t))$  and  $x(t) = \phi(y(t))$  as optimal controls and optimal trajectories in Problem 1. The next two theorems provide an answer to this question.

DEFINITION 1

The mappings  $\psi$  and  $\phi$  shall be called an acceptable pair of transformations provided the following are true:

- (1)  $\psi: R^k \rightarrow \Omega$ ,  $\phi: R^l \rightarrow \Gamma$  for some  $k$  and  $l$  and both are onto and continuously differentiable;
- (2) for any sectionally smooth  $x(t)$  defined on some interval  $[t_0, t_1]$  such that  $x(t) \in \Gamma$  for all  $t$ , there exists a sectionally smooth  $y(t)$  defined on  $[t_0, t_1]$  such that  $y(t) \in E^l$  and  $\phi(y(t)) = x(t)$  for all  $t \in [t_0, t_1]$ ;
- (3) for any sectionally continuous  $u(t)$  defined on some interval  $[t_0, t_1]$  such that  $u(t) \in \Omega$  for all  $t$ , there exists a sectionally continuous  $\dot{z}(t)$  defined on  $[t_0, t_1]$  such that  $\psi(\dot{z}(t)) = u(t)$  for all  $t \in [t_0, t_1]$ .

THEOREM 1

If  $\psi$  and  $\phi$  are an acceptable pair of transformations and  $\dot{z}(t)$ ,  $y(t)$  defined on  $[t_0, t_1]$  yield a solution to Problem 2 then  $u(t) = \psi(\dot{z}(t))$  and  $x(t) = \phi(y(t))$  yield a solution to Problem 1.

Proof:

We shall verify that  $x(t)$  and  $u(t)$  satisfy (1), (2), (3) and (4) of Problem 1. Since  $\dot{x}(t) = \phi_y(y(t))\dot{y}(t) = f(t, \phi(y(t)), \psi(\dot{z}(t))) = f(t, x(t), u(t))$  we see that (1) is satisfied. Clearly (2) follows from (1) of Definition 1, and  $x(t_0) = \phi(y(t_0)) = x^0$ ,  $x(t_1) = \phi(y(t_1)) = x^1$ , so (3) is satisfied. Now suppose (4) were not true, then there would exist  $\bar{x}(t)$ ,  $\bar{u}(t)$  and  $\bar{t}_1$  satisfying (1), (2) and (3) such that

$$\int_{t_0}^{\bar{t}_1} f_0(t, \bar{x}(t), \bar{u}(t)) dt < \int_{t_0}^{t_1} f_0(t, x(t), u(t)) dt.$$

Now applying (2) and (3) of Definition 1 there exists  $\bar{y}(t)$  and  $\dot{\bar{z}}(t)$  such that  $\bar{x}(t) = \phi(\bar{y}(t))$  and  $\bar{u}(t) = \psi(\dot{\bar{z}}(t))$  for all  $t \in [t_0, \bar{t}_1]$ . Then we have

$\phi_y(\bar{y}(t)) \dot{\bar{y}}(t) = \dot{\bar{x}}(t) = f(t, \bar{x}(t), \bar{u}(t)) = f(t, \phi(\bar{y}(t)), \psi(\dot{\bar{z}}(t)))$  for all  $t$  and moreover  $x^0 = \bar{x}(t_0) = \phi(\bar{y}(t_0))$ ,  $x^1 = \bar{x}(t_1) = \phi(\bar{y}(t_1))$ . However,

$$\int_{t_0}^{\bar{t}_1} f_0(t, \phi(\bar{y}(t)), \psi(\dot{\bar{z}}(t))) dt = \int_{t_0}^{\bar{t}_1} f_0(t, \bar{x}(t), \bar{u}(t)) dt < \int_{t_0}^{t_1} f_0(t, x(t), u(t)) dt = \int_{t_0}^{t_1} f_0(t, \phi(y(t)), \psi(\dot{z}(t))) dt$$

which contradicts the fact that  $y(t)$  and  $\dot{z}(t)$  yield a solution for Problem 2.

Thus (4) must be true and hence  $x(t)$  and  $u(t)$  yield a solution to Problem 1.

The next theorem is the converse of Theorem 1.

#### THEOREM 2

If  $\psi$  and  $\phi$  are an acceptable pair of transformations and  $u(t)$ ,  $x(t)$  defined on  $[t_0, t_1]$  yield a solution to Problem 1 then any sectionally smooth  $y(t)$  and  $z(t)$  such that  $x(t) = \phi(y(t))$  and  $u(t) = \psi(\dot{z}(t))$  for  $t \in [t_0, t_1]$  yield a solution to Problem 2.

Proof:

We know that at least one such  $y(t)$  and  $\dot{z}(t)$  exist by (2) and (3) of Definition 1. We must therefore show that such a  $y(t)$  and  $\dot{z}(t)$  satisfy (1)', (2)', (3)' and (4)' of Problem 2. Notice that  $\phi_y(y(t))\dot{y}(t) = \dot{x}(t) = f(t, x(t), u(t)) = f(t, \phi(y(t)), \psi(\dot{z}(t)))$ ; thus we have that (1)' is true. Moreover (2)' is trivially satisfied and since  $\phi(y(t_0)) = x(t_0) = x^0$  and  $\phi(y(t_1)) = x(t_1) = x^1$ , (3)' is also. Now if  $\bar{y}(t)$  and  $\dot{\bar{z}}(t)$  defined on  $[t_0, \bar{t}_1]$  satisfies (1)', (2)', (3)' and

$$\int_{t_0}^{\bar{t}_1} f_0(t, \phi(\bar{y}(t)), \psi(\dot{\bar{z}}(t))) dt < \int_{t_0}^{t_1} f_0(t, \phi(y(t)), \psi(\dot{z}(t))) dt$$

then it is easily seen that  $\bar{x}(t) = \phi(\bar{y}(t))$  and  $\bar{u}(t) = \psi(\dot{\bar{z}}(t))$  will contradict the optimality of  $x(t)$  and  $u(t)$ . Thus  $y(t)$  and  $\dot{z}(t)$  must yield a solution to Problem 2.

The following corollary follows immediately from Theorems 1 and 2.

#### COROLLARY 1

If  $\psi$  and  $\phi$  are an acceptable pair of transformations then any necessary or sufficient condition for  $y(t)$  and  $\dot{z}(t)$  to be a solution of Problem 2 yields a necessary or sufficient condition for  $x(t)$  and  $u(t)$  to be a solution of Problem 1 when the condition is restated in terms of  $x(t) = \phi(y(t))$  and  $u(t) = \psi(\dot{z}(t))$ .

A much more general discussion of the equivalence of minimization problems is contained in (18), particularly in regard to unconstrained problems.

#### IV. The Control and State Regions as Right Parallelepipeds

Consider Problem 1 with

$$\Omega = \{u \in \mathbb{R}^m : a_i \leq u_i \leq b_i, i = 1, \dots, m\} \text{ and}$$

$$\Gamma = \{x \in \mathbb{R}^n : c_i \leq x_i \leq d_i, i = 1, \dots, n\}$$

where  $a_i, b_i, c_i$  and  $d_i$  are real numbers such that  $a_i < b_i$  and  $c_i < d_i$  for each  $i$ .

Then  $\Omega$  and  $\Gamma$  defined in this way constitute right parallelepipeds. Now define

$\psi: \mathbb{R}^m \rightarrow \Omega$  by  $\psi = (\psi_1, \dots, \psi_m)$  where

$$u_i = \psi_i(\dot{z}) = \frac{1}{2} ((b_i - a_i) \sin \dot{z}_i + (b_i + a_i))$$

for  $i = 1, \dots, m$ . Also let  $\phi: \mathbb{R}^n \rightarrow \Gamma$  be given by  $\phi = (\phi_1, \dots, \phi_n)$  where

$$x_i = \phi_i(y) = \frac{1}{2} ((d_i - c_i) \sin y_i + (d_i + c_i))$$

for  $i = 1, \dots, n$ .

#### THEOREM 3

For this choice of  $\Omega$ ,  $\Gamma$ ,  $\psi$  and  $\phi$ , the mappings  $\psi$  and  $\phi$  constitute an acceptable pair of transformations.

Proof:

Since  $-1 \leq \sin \dot{z}_i \leq 1$  for all real numbers  $\dot{z}_i$ , we have

$$\frac{(a_i - b_i)}{2} \leq \frac{(b_i - a_i)}{2} \sin \dot{z}_i \leq \frac{(b_i - a_i)}{2}, \text{ and therefore}$$

$$\begin{aligned} a_i &= \frac{(a_i - b_i)}{2} + \frac{(b_i + a_i)}{2} \leq \frac{(b_i - a_i)}{2} \sin \dot{z}_i + \frac{(b_i + a_i)}{2} = \psi_i(\dot{z}) \\ &\leq \frac{(b_i - a_i)}{2} + \frac{(b_i + a_i)}{2} = b_i \end{aligned}$$

for  $i = 1, \dots, n$ .

Thus we see that  $\psi(\dot{z}) \in \Omega$  for all  $\dot{z} \in \mathbb{R}^m$ . Similarly we can show that  $\phi(y) \in \Gamma$  for all  $y \in \mathbb{R}^n$ . Moreover  $\psi$  and  $\phi$  are clearly continuously differentiable. Now let  $u(t)$  be a sectionally continuous function defined on  $[t_0, t_1]$  such that  $u(t) \in \Omega$  for all  $t \in [t_0, t_1]$ . Define  $\dot{z}(t)$  by

$$\dot{z}_i(t) = \sin^{-1} \left[ \frac{2u_i(t) - (b_i + a_i)}{(b_i - a_i)} \right].$$

We first notice that each  $\dot{z}_i(t)$  is well defined for all  $t \in [t_0, t_1]$ . This is true since  $u(t) \in \Omega$  means that  $a_i \leq u_i(t) \leq b_i$  for  $i = 1, \dots, m$  and hence it follows that  $a_i - b_i = 2a_i - (b_i + a_i) \leq 2u_i(t) - (b_i + a_i) \leq 2b_i - (b_i + a_i) = b_i - a_i$ .

Dividing this inequality by  $b_i - a_i$  we obtain

$$-1 \leq \frac{2u_i(t) - (b_i + a_i)}{(b_i - a_i)} \leq 1.$$

Hence the  $\sin^{-1}$  of the middle term in the above inequality is well defined.

Note that this argument yields that  $\psi$  is onto since  $u(t)$  may in general be any point of  $\Omega$  and clearly if  $\dot{z}$  is defined in the above fashion for any  $u \in \Omega$  we have  $\psi(\dot{z}) = u$ . Moreover, since  $\sin^{-1}$  is a sectionally continuous function, we see that the  $\dot{z}(t)$  which we have defined will be sectionally continuous also.

Therefore  $\psi$  satisfies condition (3) of Definition 1. If  $x(t)$  is a sectionally smooth function defined on  $[t_0, t_1]$  such that  $x(t) \in \Gamma$  for all  $t \in [t_0, t_1]$ , we can satisfy condition (2) of Definition 1 by defining

$$y_i(t) = \sin^{-1} \left[ \frac{2x_i(t) - (d_i + c_i)}{(d_i - c_i)} \right] + 2\pi j(i,t)$$

for  $i = 1, \dots, n$  and  $t \in [t_0, t_1]$  where  $j(i,t)$  is an integer chosen for each  $i$  and  $t$  to insure that  $y_i(t)$  is not just sectionally continuous but also sectionally smooth. Since  $\sin(2\pi j) = 0$  for all integers  $j$ , this term does not affect the relationship  $\phi(y(t)) = x(t)$ . Thus the theorem is proved and we have shown that  $\psi$  and  $\phi$  are an acceptable pair of transformations.

We now have that for this particular choice of  $\Omega$ ,  $\Gamma$ ,  $\psi$  and  $\phi$  that Theorems 1 and 2 apply, and hence, in this case, Problem 1 is equivalent to Problem 2.

In the case that some of the components of  $u$  or  $x$  are to be unrestricted, while the remainder are constrained between maximum and minimum values as previously, one simply defines  $\psi_i(\dot{z}) = u_i$  or  $\phi_i(y) = x_i$  for those particular ones and defines the rest of the components of  $\psi$  and  $\phi$  as is done previously with the sine function. Clearly, in this case, the results of Theorem 3 also follow. For numerical applications the previous formulation is more desirable as in effect it covers both cases. When a component of  $u$  or  $x$  is to be unrestricted, one simply inputs to the system maximum and minimum values of the variable which are exceedingly large and exceedingly small respectively, thus effectively eliminating the constraint.

In the case where  $\Omega$  is the unit  $m$ -cube and  $\Gamma$  is the unit  $n$ -cube we set  $a_i = -1$ ,  $b_i = 1$  for  $i = 1, \dots, m$  and  $c_i = -1$ ,  $d_i = 1$  for  $i = 1, \dots, n$ . Then  $\psi$  and  $\phi$  reduce to  $\psi(\dot{z}) = u = \sin \dot{z} = (\sin \dot{z}_1, \dots, \sin \dot{z}_m)$  and  $\phi(y) = x = \sin y = (\sin y_1, \dots, \sin y_n)$ .

Examples of acceptable choices of  $\psi$  and  $\phi$  corresponding to more general control and state regions are contained in (18).

## V. Results.

Let us now consider Problem 1 with  $\Omega = \{u \in R^m: |u_i| \leq 1, i = 1, \dots, m\}$  and  $\Gamma = \{x \in R^n: |x_i| \leq 1, i = 1, \dots, n\}$ . We then set  $\phi(y) = x = \sin y$  and  $\psi(\dot{z}) = u = \sin \dot{z}$ . By Theorem 3 this choice of  $\psi$  and  $\phi$  constitute an acceptable pair of transformations. Therefore, by Theorems 1 and 2, Problem 2 may be viewed as a Lagrange Problem and necessary conditions for solutions to it may be transformed back into necessary conditions for solutions to Problem 1, the bounded state problem. I shall now summarize some of the results obtained by this approach.

Assume that  $x(t)$  and  $u(t)$  defined on  $[t_0, t_1]$  yield a solution to Problem 1. At any time  $t \in [t_0, t_1]$ , let  $\bar{x}(t)$  be the subvector of  $x(t)$  consisting of its components with absolute value one and let  $\tilde{x}(t)$  consist of all other components. Subdivide the vectors  $f$  and  $\lambda$ , the Lagrange multipliers, according to this same rule. Similarly divide  $u(t)$  into  $\bar{u}(t)$  consisting of all  $u_i(t)$  with absolute value one and  $\tilde{u}(t)$  containing all  $u_i(t)$  for which  $|u_i(t)| < 1$ .

We assume that the regularity hypothesis holds for Problem 2, that is, that the matrix of partials of the constraining differential equations with respect to the variables involved has maximum row rank. This yields the following theorem.

### THEOREM 4

For all  $t \in [t_0, t_1]$ ,  $\bar{f}_{\bar{u}}(t, x(t), u(t))$  has maximum row rank. As a corollary to this theorem we obtain that the total number of coordinates of  $x(t)$  and  $u(t)$  with absolute value one cannot exceed the number of control variables in the system or equivalently that the number of variables with absolute value less than one must exceed the number of state variables.

Let the function  $H$ , called the Hamiltonian, be defined by

$$H(t, x, u, \lambda) = \lambda_0 f_0(t, x, u) + \lambda \cdot f(t, x, u).$$



Then the Euler Lagrange Equations and Transversality Conditions for Problem 2 imply the following theorem.

THEOREM 5

There exists a vector function  $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))$  defined on  $[t_0, t_1]$  which is continuous except possibly at the points where  $u(t)$  is not continuous and a constant  $\lambda_0 \leq 0$  with  $(\lambda_0, \lambda_1(t), \dots, \lambda_n(t)) \neq 0$  for all  $t \in [t_0, t_1]$  such that for  $i = 1, \dots, n$  either

$$|x_i(t)| = 1 \text{ or } \dot{\lambda}_i(t) + H_{x_i}(t, x(t), u(t), \lambda(t)) = 0$$

and for  $j = 1, \dots, m$  either

$$|u_j(t)| = 1 \text{ or } H_{u_j}(t, x(t), u(t), \lambda(t)) = 0$$

on every smooth arc of  $x(t)$ .

The Weierstrass-Erdmann Corner Conditions also yield an analogous theorem for the bounded state case

THEOREM 6

For each  $i = 1, \dots, n$ ,  $\lambda_i(t)$  is continuous except possibly at corners of  $x(t)$  where  $|x_i(t)| = 1$ . The function  $H(t, x(t), u(t), \lambda(t))$  is continuous for all  $t \in [t_0, t_1]$ .

In addition by applying the time transversality conditions, we find that the Hamiltonian function evaluated along the solution has the value 0 for  $t = t_0$  and  $t = t_1$ . Moreover, by taking the total time derivative of the Hamiltonian along the solution we obtain:

THEOREM 7

If the function  $u(t)$  is differentiable except at its points of discontinuity then  $H(t, x(t), u(t), \lambda(t))$  is differentiable except at these points and

$$\frac{d}{dt} H(t, x(t), u(t), \lambda(t)) = H_t(t, x(t), u(t), \lambda(t)).$$

The Legendre-Clebsch Condition also yields a generalization for the bounded state case. Its corollary is perhaps a more usable form.

THEOREM 8

Let  $t \in [t_0, t_1]$  and  $\gamma \in R^m$ . Decompose  $\gamma$  into two subvectors  $\bar{\gamma}$  consisting of those components of  $\gamma$  corresponding to coordinates of  $u(t)$  for which  $|u_j(t)| = 1$ , and  $\tilde{\gamma}$  consisting of those coordinates of  $u(t)$  for which  $|u_j(t)| < 1$ . Decompose the vector  $u$  similarly. Then for all  $\bar{\gamma}$  and for all  $\tilde{\gamma}$  satisfying the equations  $f_{\tilde{u}} \tilde{\gamma} = 0$ ,

$$\bar{\gamma} \cdot (H_{\tilde{u}} * \tilde{u} * \bar{\gamma}) - \tilde{\gamma} \cdot H_{\tilde{u}\tilde{u}} \tilde{\gamma} \geq 0 \text{ at time } t,$$

where the  $*$  product of two vectors represents the vector the components of which are the products of corresponding components of the two vectors.

COROLLARY 1

Let  $t \in [t_0, t_1]$  and  $\bar{u}, \tilde{u}, \bar{\gamma}$  and  $\tilde{\gamma}$  be defined as in Theorem 8. Then each component of the vector  $H_{\tilde{u}} * \tilde{u}$  is non-negative, and for all  $\tilde{\gamma}$  such that  $f_{\tilde{u}} \tilde{\gamma} = 0$ ,  $\tilde{\gamma} \cdot H_{\tilde{u}\tilde{u}} \tilde{\gamma} \leq 0$ .

Finally we obtain a generalization of Pontryagin's Maximum Principle as a consequence of the Weierstrass Necessary Condition.

THEOREM 9

Let  $t \in [t_0, t_1]$ . Then for all  $v \in R^m$  such that  $|v_i| \leq 1$  for  $i = 1, \dots, m$  and  $\bar{f}(t, x(t), v) = 0$  where  $\bar{f}$  consists of all coordinates of  $f$  corresponding to components of  $x(t)$  for which  $|x_k(t)| = 1$ , the following is satisfied:

$$H(t, x(t), u(t), \lambda(t)) \geq H(t, x(t), v, \lambda(t)).$$

Therefore one can see that quite a comprehensive set of necessary conditions for the bounded state problem may be obtained by using this transformation approach.

## CONCLUDING REMARKS

A method has been described whereby an optimal control problem with bounded state variables may be transformed into an equivalent Lagrange problem. This was accomplished by means of differentiable mappings which take some Euclidean space onto the closed and bounded control and state regions. Whereas all such mappings lead to a Lagrange problem, it has been shown that only those which were defined as acceptable pairs of transformations are suitable in the sense that solutions to the Lagrange problem lead to solutions to the bounded state problem and vice versa. In particular, an acceptable pair of transformations was exhibited for the case when the control and state regions are right parallelepipeds.

In the last section some of the necessary conditions for the bounded state problem which were developed using this transformation approach were described.

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