AN INVESTIGATION OF FATIGUE FAILURE OF MATERIALS UNDER BROAD BAND RANDOM VIBRATIONS

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FATIGUE FAILURE OF MATERIALS UNDER BROAD BAND RANDOM VIBRATIONS

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Abstract

In the present paper the fatigue life of material under multi-factor influence of broad band random excitations has been investigated. Parameters which affect the fatigue life are postulated to be peak stress, variance of stress and the natural frequency of the system. Experimental data was processed by the hybrid computer. Based on the experimental results and regression analysis a best predicting model has been found. All values of the experimental fatigue lives are within the 95% confidence intervals of the predicting equation.

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INTRODUCTION

In the 50's random vibrations became an interesting subject due to technological development particularly in the aerospace industry. On the one hand vibration researchers were interested in the statistical and other properties of random vibrations for either excitation or response in both the time and frequency domains such as root mean square, standard deviation, distribution, spectral density, frequency response, etc. On the other hand material researchers were interested in the fatigue life of materials under random vibrations. In this respect the well known concept of cumulative damage [1] has been widely used, i.e. each cycle in random vibrations consumes a part of the material life based on the results of conventional constant amplitude sinusoidal fatigue tests.

One of the earliest attempts [2] in direct random fatigue experiment used a method called 'Block Form' test. In this method blocks of stress cycles, each at a particular amplitude of stress and a particular number of cycles are randomly numbered and used in the test.

A later attempt used a method called the 'Random Form' test [3]. A certain number of levels of load was used to simulate the maximum peaks of an assumed signal. These maximum values are called out at random to excite the test specimen for successive equal time intervals.

A third type of random fatigue test is called 'Analogus Random Process' testing. It involves the desired shape of spectral density applied to the specimen [4,5].

A fourth type of random testing is 'Acoustic Fatigue Testing' which involves exciting the specimen with sonic signals [6].

The purpose of our research program, "Fatigue under Random Vibrations" at the University of Wisconsin is to develop rational methods for fatigue tests, under the multifactor influence of various random excitations described statistically. This serves as an alternative to the unreliable method based on the concept of cumulative damage [7,8], the arbitrarily designed simulations involving no statistical properties such as block form and random form tests, and one parameter at a time tests such as the analogus random process test and the current acoustic fatigue test.

The program is being carried out in three directions according to excitation sources: namely narrow band, broad band and sonic excitations. "Fatigue Failure of Materials under Narrow band Random Vibrations-Part 1," has been reported [9]. Part 2 [10] will be reported at a later date. The results of the initial trial for fatigue tests with broad band random excitation is presented in this paper. Applying our experience with narrow and broad band random excitations we are planning to carry out studies on sonic fatigue which involves excitation with no direct contact between specimens and excitation sources.

PARAMETERS CONSIDERED

When a material is under random excitations the time to failure depends on several parameters of the random responses. These parameters can be described either in the amplitude domain, the frequency domain or in any combination of the two. In order to construct the most meaningful model, it is essential to include simultaneously all the significant parameters affecting the fatigue life. Initially, the parameters postulated in our tests were: mean stress, variance of stress, stess amplitude distribution, peak stress, and natural frequency of the test system.

In our tests the characteristics of the excitation (input) as well as the response (output) has zero mean. The response is described by specimen stress amplitude, and the amplitude distributions of the stresses are normal. Thus these two parameters are not in variation. The time to failure is then related only to the variance of stresses and peak stress of the specimen and the natural frequency of the test system.

The choice of related parameters in the amplitude and frequency domains is arbitrary. However, for convenience of experimental control the variance of stress amplitude was chosen instead of the spectral density in the frequency domain. The peak stress was used instead of the average of positive or negative stresses, both in amplitude domain.

It is well known that under random excitations of constant spectral density the test system response is extremely amplified in the neighborhood of its natural frequency - a resonant phenomenon. For high natural frequencies there will be more cycles per unit time and consequently more damaging effect. However, this effect will be reduced due to smaller amplitude responses at higher frequencies. In order to test its net effect the natural frequency of the system was considered as a parameter.

TEST EQUIPMENT

The major test equipment used was a 1200-lb MB-electromagnetic shaker with a sine-random signal generator. Recording instruments consisted of a Tektronics 'Q' unit and oscilloscope with camera, and a Precision Instrument FM tape recorder. In addition a GR spectrum analyzer was used.

The random output of the signal generator was fed into the MB amplifier producing a vertical motion of the shaker mass which strained the test specimen. As shown in Fig. 1, the shaker mass, test specimen, load cell and test fixture were connected in series. The load cell strain which was calibrated to specimen strain was amplified by the 'Q' unit, visually displayed by the oscilloscope and recorded by the FM tape recorder.

One load cell was used throughout the test. Four SR-4 wire resistance strain gages were used in a four element bridge configuration, for maximum sensitivity and temperature compensation. The maximum stress in the load cell, because of its large diameter and high elastic modulus relative to the specimen, was very low, thus reducing possible fatigue failure of the strain gages. The load cell calibration was checked before and after the fatigue testswere run to assure no changes in calibration during the test.

SPECIMENS

The specimens shown in Fig. 2 were machined from 1/2 inch diameter 60-61 T651 aluminum rod. A static tensile test of this material showed a yield point of approximately 36,000 psi and an ultimate strength of approximately 48,400 psi. A value of 36,000 psi was taken as the upper limit of specimen stress during the test to assure linearity of the system. The diameter of the testing length of the specimen was 0.200 inch. A large fillet radius of 3/8 in. was used to minimize stress concentrations at the sholders. Specimens of two different lengths were used so that the natural frequency of the system could be varied. The stability of these specimens under maximum possible values of compressive stress of 36,000 psi in the tests was assured by Eulers criterion for column buckling.

TEST SYSTEM

The vibration model of the test system is schematically represented in Fig. 3. The damping in the system, ignoring hysteresis effects, was produced mainly by rubber damper pads in the shaker. The spring constants for the test fixture, load cell and specimens were determined by static load-deflection tests to be 1.1×10^6 lb/in. for the fixture, 6.0×10^6 lb/in. for the load cell, 16.6×10^4 lb/in. for the long specimen, and 26.2×10^4 lb/in. for the short specimen. The combined spring constant for the support springs of the shaker mass was found by a free vibration test of the shaker to be 300 lb/in.

The natural frequency of the system was varied by two methods. The first as noted before was to use two different specimen test lengths which provided two different spring constants. Because of machining difficulties, the limits on the specimen test lengths limited the natural frequency range. Therefore the natural frequency range was further increased by varying the shaker mass by bolting to it a series of steel rings. Using these two methods, five natural frequencies of 140, 170, 205, 250, and 300 cps were used in the test.

Response curves for these five systems are shown in Fig. 4. The bandwidth method for calculating the damping factor for each system was used on these curves. As shown in Table 1, the damping factor did not change appreciably around the value of 0.125. Therefore it was assumed that the damping factor was not a variable in the test.

The frequency input to the system was analyzed with the spectrum analyzer. As shown in Fig. 5, the spectral density of the signal generator output is approximately constant up to 10,000 cps and then rapidly falls off to zero at approximately 14,000 cps. An oscilloscope picture of the input to the system is shown in Fig. 6 which also shows the wide band characteristics of the input. The sweep speed was 100 cm/sec and the amplitude was 1 volt/cm.

The load cell output for the 140 cps system is shown in Fig. 7a. Since the sweep speed was 100 cm/sec it is easily seen that the predominant frequency component was approximately 140 cps. However, high frequency components are still visible. The 300 cps system response is shown in Fig. 7b. Once again, the predominant fundamental frequency was that of the system natural frequency with higher harmonics impressed on it. These two figures illustrate the filtering effect of the systems to the white noise input.

TEST PROCEDURE

After mounting the test specimen and load cell between the test fixture and shaker mass zero initial stress in each specimen was assured, as indicated by the 'zero balance' on the 'Q' unit, by adjusting the load cell position with respect to the test fixture. The five desired peak stresses of the specimens in each of the five natural frequency groups, were 26,000 psi to 34,000 psi with 2000 psi increments. However, because of the random nature of the response, these exact values were difficult to obtain. Therefore at the beginning of each run the shaker amplifier output was adjusted so that the corresponding desired maximum load cell strain was approximately obtained as observed on the oscilloscope. Once this adjustment was made it was unchanged throughout the run. Due to the nature of the input random signal of the test equipment the third factor, the variance of stress, was not directly controlled.

At about one-half of the assumed specimen fatigue life the 'Q' unit output of the load cell strain was recorded on the tape for a length of

100 ft. at 37.5 in/sec. When the specimen failed the elapsed time was recorded. Two typical failed specimens are shown in Fig. 8.

HYBRID COMPUTER DATA ANALYSIS

The recorded portion of the load cell strain was then analyzed by the University of Wisconsin's hybrid computer. The computer was programmed to sample and store one million ordinates of the analog signal at a rate of 5000 samples per second of analog time. The tape was played into the computer at 1/10 of the recording speed, therefore 50,000 samples per second of real time was taken by the computer. This reduced tape speed assured an almost continuous sampling process of the recorded signal. An ordered array of voltage values corresponding to load cell strain was printed showing the number of ordinates of the system response occuring within each 20 psi interval in the stress range of ±60000 psi.

From the array of stress values the probability density function for all specimens were drawn, two of which are shown in Fig. 9. Using the equations in Appendix 1 [11] the kurtosis and skewness of each response was calcuated from the array. Values for the first and last specimen of each natural frequency group are shown in Table 2. It is seen from the table that the kurtosis was very close to three and the skewness was very close to zero. This shows the amplitude distribution of this system to a white noise input was normal.

Peak stress was found in the array at the maximum address location in which at least one occurrence was stored. The computer was programmed to calculate the variance of stress from the array using standard statistical equations. Table 3 shows the data of each specimen for the five systems tested with corresponding values of peak stress and variance of stress.

MATHEMATICAL MODEL

It was desired to construct a mathematical model which relates the fatigue failure time to the peak stress, variance of stress and natural frequency of the system based on experimental data. This was done by regression analysis.

A four-variable model represented by the following equation

$$y = c_0 + c_1x_1 + c_2x_2 + c_3x_3 \tag{1}$$

was tried in which

y = function of fatigue time

x_1 = function of peak stress

x_2 = function of variance of stress

x_3 = function of natural frequency of the systems and

c_0, c_1, c_2 and c_3 are constants.

Using standard correlation equations on various trial models excellent correlations were found between logarithm of time and peak stress, logarithm of time and variance of stress, and peak stress and variance of stress. Poor correlation was found between frequency and other variables. However, later analysis shows the multiple correlation coefficient, which reflects the effect of all variables in the model, is greater than all the other correlation coefficients. This indicates that the frequency is an influential variable although it may be of lesser importance than either peak stress or variance of stress. It also shows that inclusion of frequency improves the model. Therefore the best model found is represented by

$$\log(T) = c_0 + c_1\,\sigma + c_2v + c_3\,\log(f) \tag{2}$$

where

T = fatigue time in sec.

σ = peak stress in psi

v = variance of stress $(psi)^2$

f = natural frequency of the system in cps.

Let the series of tests performed be represented by

$$\bar{y} = X\bar{c} \tag{3}$$

in which

$\bar{y}$ = Column matrix of y_i

X = Matrix of x

$\bar{c}$ = Column matrix of c_i

The column matrix $\bar{c}$ can be obtained as

$$\bar{c} = (X'X)^{-1} X'\bar{y} \tag{4}$$

in which

X' = Transpose of X

$(X'X)^{-1}$ = the inverse of $(X'X)$

Equation (4) was used to determine the coefficients c_0, c_1, c_2 and c_3 of Eq. (2) based on our 25 tests shown in Table 3.

RESULTS

(1) Predicating equation - With the numerical values of c_0, c_1, c_2 and c_3 evaluated from Eq. (4) the fatigue predicting equation based on our tests is

$$\log(T) = 6.9401 - 4.5613 \times 10^{-5}\, \sigma - 2.7964 \times 10^{-8}\, v - 0.12798 \log f \tag{5}$$

Predicted values of fatigue life under broad band random excitation were calculated by using Eq.(5) and compared with the fatigue data as shown in Table 4.

(2) 95% confidence intervals - Due to experimental error the estimated fatigue life is subject to deviation from the experimental value. Therefore we seek estimation within an interval known as a confidence interval. In other words the probability that our value falls within an upper and lower range is $1-\alpha$ where α is taken as .05. This is known as a 95% confidence interval and was calculated from the expression [12]

$$\hat{y} \pm t(n-p-1,\ 1-\frac{\alpha}{2}).\ s\sqrt{\frac{1}{g} + X_0'\ CX_0}$$

in which

$\hat{y}$ = estimated failure time

t = student t-statistic

n = number of tests

p = number of variables

s^2 = estimated variance

g = number of observations

X_0 = column matrix of x

C = $(X'X)^{-1}$

X_0' = transpose of X_0

The 95% confidence intervals are shown in Table 5. It is seen that all the test results are within these intervals. However, it should be noted these intervals are large because there is only one observation ($g = 1$) for each test available for estimation. A reduced interval would be realized for a greater number of observations ($g > 1$) for each test.

(3) Correlation coefficients - The zero order (R_{ij}), partial ($R_{ij \cdot k}$ and $R_{ij \cdot k\ell}$) and multiple ($R_{i \cdot jk\ell}$) correlation coefficients corresponding to Eq. (5) are shown in Table 6. Equations to calculate these coefficients [11] are shown in Appendix 2.

The correlation coefficients shown in Table 5 show an improvement in the percentage of explained variation by the inclusion of all variables in the model. Using the expressions in Appendix 3, $R_{13.2}$ shows a 46% improvement by including x_3 vs. $x_1 = f(x_2)$ only. $R_{12.3}$ shows a 15% improvement by including x_2 vs. $x_1 = f(x_3)$ only. The improvement shown by including x_4 is low. A 9.8% improvement is shown by including x_4 vs. $x_1 = f(x_2)$ only and a 4% improvement is shown by including x_4 vs. $x_1 = f(x_3)$ only.

Coefficient $R_{12.34}$ shows a 12% improvement by including x_2 vs. $x_1 = f(x_3, x_4)$ only. $R_{13.24}$ shows 41% improvement by including x_3 and $R_{14.23}$ shows a .05% improvement by including x_4. However $R_{1.234}$ shows that 92% variation is explained by using all the variables which is greater than any percentage when not all variables were included. This means the model is best when all variables are included.

(4) Response planes - The response planes for peak stress, variance of stress and frequency based on Eq. (5) are shown in Fig. 10. These planes give the overall view of the fatigue life within the test ranges of the variables.

CONCLUSION

The present investigation is a part of the research program in fatigue failure of materials under random vibrations. The purpose of the program is to explore the important statistical variables in random signals, to build the best models for predicting equations of failure time, and to establish the procedures of experiment design and data processing.

In the present analysis three conjectured parameters which affect the fatigue life under random vibrations were considered simultaneously. A best model was found based on zero order correlation coefficients and a further study of correlation coefficients indicates that all variables conjectured should be included. The fact that all experimental results are within the 95% intervals indicates that the model is good and experimental error is within the prescribed limits.

The overall results shown in this paper are very promising. Continuing research effort is in progress and will be reported.

ACKNOWLEDGEMENT

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Table 1. Equivalent Viscous Damping Factor

System Natural Frequency cps	Damping Factor
140	.131
170	.128
205	.116
250	.136
300	.125

Table 2. Skewness and Kurtosis

Specimen No.	Coefficient of Skewness	Coefficient of Kurtosis
1	.0040	3.017
5	.0014	2.995
6	.0017	3.016
10	.0046	2.966
11	.0011	3.182
15	.0041	2.923
16	.0025	3.165
20	.0005	2.996
21	.0015	3.092
25	.0030	2.976

Table 3. Test Data

Specimen No.	System Natural Frequency cps	Peak Stress psi	Variance of Stress $(psi)^2 x 10^{-7}$
1	140	25,728	4.2210
2	140	27,794	4.9884
3	140	29,860	5.8091
4	140	30,026	6.7225
5	140	34,366	7.9936
6	170	26,451	4.3764
7	170	27,402	4.8748
8	170	30,481	6.2480
9	170	32,857	6.7123
10	170	35,895	8.1577
11	205	26,265	4.1059
12	205	27,980	4.6840
13	205	30,626	5.4536
14	205	32,237	6.5965
15	205	33,643	8.2498
16	250	26,286	4.5759
17	250	28,848	4.5938
18	250	31,617	5.3752
19	250	32,651	5.8075
20	250	33,374	7.0658
21	300	26,100	3.7183
22	300	26,079	4.5903
23	300	30,088	5.4489
24	300	33,023	6.1367
25	300	34,924	7.7050

Table 4. Test Results

Specimen No.	Observed Failure Time Sec.	Predicted Failure Time Sec.
1	42,240	20,485
2	9,085	10,059
3	6,343	4,773
4	2,839	2,605
5	956	728
6	20,852	16,757
7	5,080	11,001
8	3,410	3,288
9	1,250	1,899
10	775	544
11	21,052	19,858
12	6,686	11,429
13	3,970	5,274
14	1,354	2,133
15	430	635
16	18,110	14,273
17	11,460	10,780
18	4,330	4,873
19	3,412	3,309
20	1,570	1,364
21	30,630	24,700
22	11,390	14,120
23	7,028	5,331
24	2,700	2,515
25	1,090	750

Table 5. 95% Confidence Intervals

Specimen No.	Observed log(T)	Predicted log(T)	95% Confidence Interval in log(T)	
1	4.6257	4.3114	3.9737	4.6760
2	3.9583	4.0026	3.6532	4.3597
3	3.8023	3.6788	3.3331	4.0333
4	3.4532	3.4158	3.0590	3.7830
5	2.9805	2.8624	2.5058	3.2305
6	4.3191	4.2242	3.8783	4.5748
7	3.7059	4.0714	3.7000	4.3881
8	3.4830	3.5169	3.1801	3.8609
9	3.0810	3.2787	2.9354	3.6294
10	2.8893	2.7359	2.3795	3.1016
11	4.3233	4.2979	3.9522	4.6456
12	3.8252	4.0580	3.7185	4.4002
13	3.5988	3.7221	3.3798	4.0678
14	3.1316	3.3290	2.9926	3.6704
15	2.6335	2.8025	2.4248	3.1878
16	4.2570	4.1545	3.7959	4.5135
17	4.0592	4.0326	3.6850	4.3802
18	3.6365	3.6878	3.3287	4.0476
19	3.5330	3.5197	3.1596	3.8811
20	3.1959	3.1348	2.7891	3.4839
21	4.4861	4.3927	4.0307	4.7514
22	4.0565	4.1498	3.7694	4.5285
23	3.8468	3.7268	3.3775	4.0751
24	3.4314	3.4006	3.0424	3.7582
25	3.0374	2.8753	2.5091	3.2433

Table 6. Correlation Coefficients

Zero order R_{ij}	Partial $R_{ij\cdot k}$	Partial $R_{ij\cdot k\ell}$	Multiple $R_{i\cdot jk\ell}$
$R_{12} = -.924$	$R_{12.3} = -.391$	$R_{12.34} = -.345$	$R_{1.234} = .960$
$R_{13} = -.952$	$R_{12.4} = -.931$	$R_{13.24} = -.637$	
$R_{14} = .089$	$R_{13.2} = -.681$	$R_{14.23} = -.022$	
	$R_{13.4} = -.954$		
	$R_{14.2} = .313$		
	$R_{14.3} = -.198$		

Appendix 1 Kurtosis and Skewness [11]:

Moment M

$$M_3 = M_3' - 3M_1' M_2' + 2 (M_1')^2 \quad (A1)$$

$$M_4 = M_4' - 4M_1' M_3' + 6(M_1')^2 M_2' - 3(M_1')^4 \quad (A2)$$

where

$$M_r' = \frac{\Sigma X^2}{N} \quad (A3)$$

N = number of tests

Kurtosis K and Skewness S

$$K = \frac{M_4}{v^2} \quad (A4)$$

$$S = \frac{(M_3)^2}{v^3} \quad (A5)$$

where

v = variance

Appendix 2 Equations of correlation coefficients [11]:

$$R_{ij} = \frac{N\Sigma X_i X_j - (\Sigma X_i)(\Sigma X_j)}{[(N\Sigma X_i^2 - (\Sigma X_i)^2)(N\Sigma X_j^2 - (\Sigma X_j)^2]^{1/2}} \tag{A6}$$

$$R_{ij\cdot k} = \frac{R_{ij} - (R_{ik})(R_{jk})}{[(1-R_{ik}^2)(1-R_{jk}^2)]^{1/2}} \tag{A7}$$

$$R_{ij\cdot k\ell} = \frac{R_{ij\cdot\ell} - R_{ik\cdot\ell}\,(R_{kj\cdot\ell})}{[(1-R_{ik\cdot\ell}^2)(1-R_{kj\cdot\ell}^2)]^{1/2}} \tag{A8}$$

$$R_{i\cdot jk\ell} = [1 - (1-R_{ij}^2)(1 - R_{ik\cdot j}^2)(1 - R_{i\ell\cdot jk}^2)]^{1/2} \tag{A9}$$

Appendix 3   Expressions for percentage improvement [11]:

Percent improvement by including X_k vs. $X_i = f(X_j)$ only is

$$(R_{ik\cdot j})^2 \cdot 100$$

Percent improvement by including X_ℓ vs. $X_i = f(X_j, X_k)$ only is

$$(R_{i\ell\cdot jk})^2 \cdot 100$$

Percent explained variation when all variables are included is

$$(R_{i\cdot jk\ell})^2 \cdot 100$$

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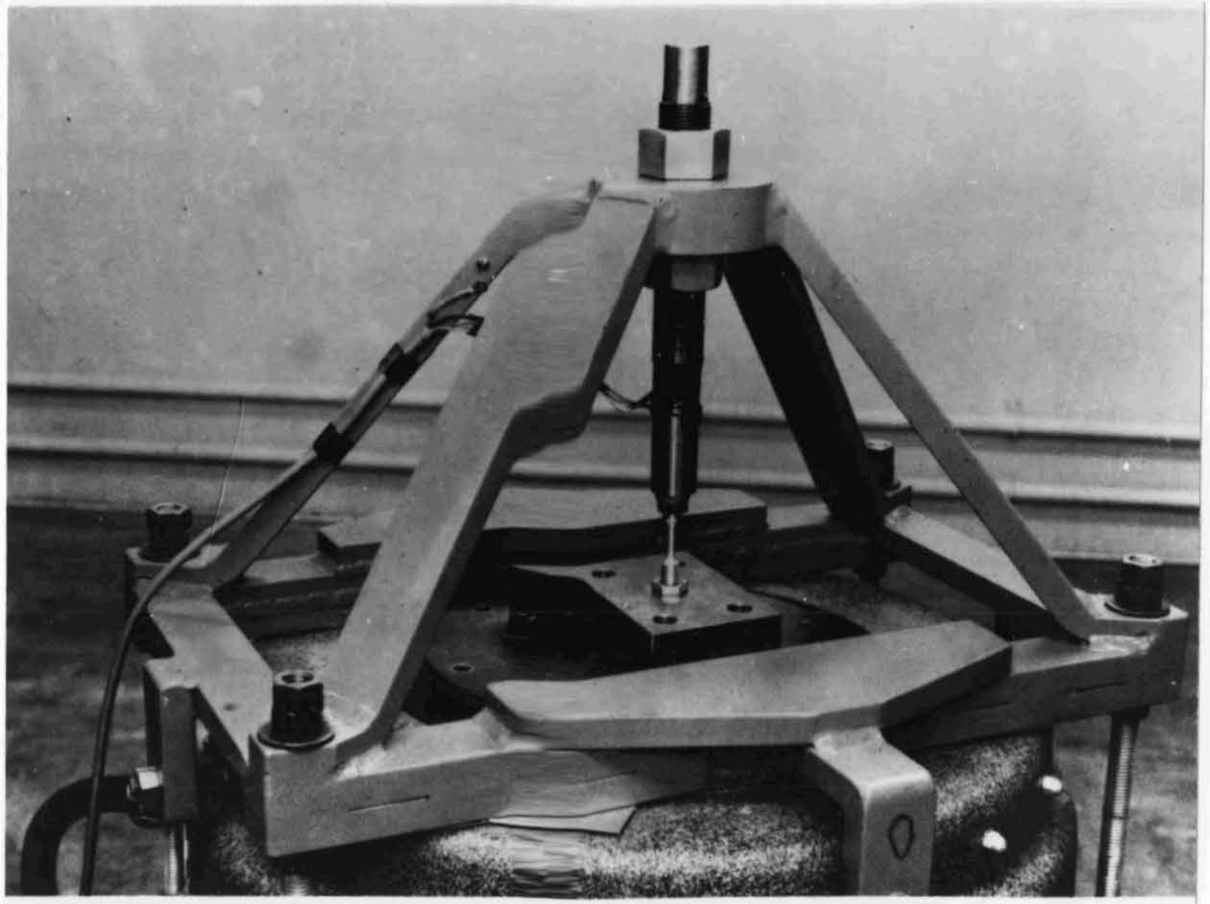


Fig. I Ex[redacted]eriment Set-up

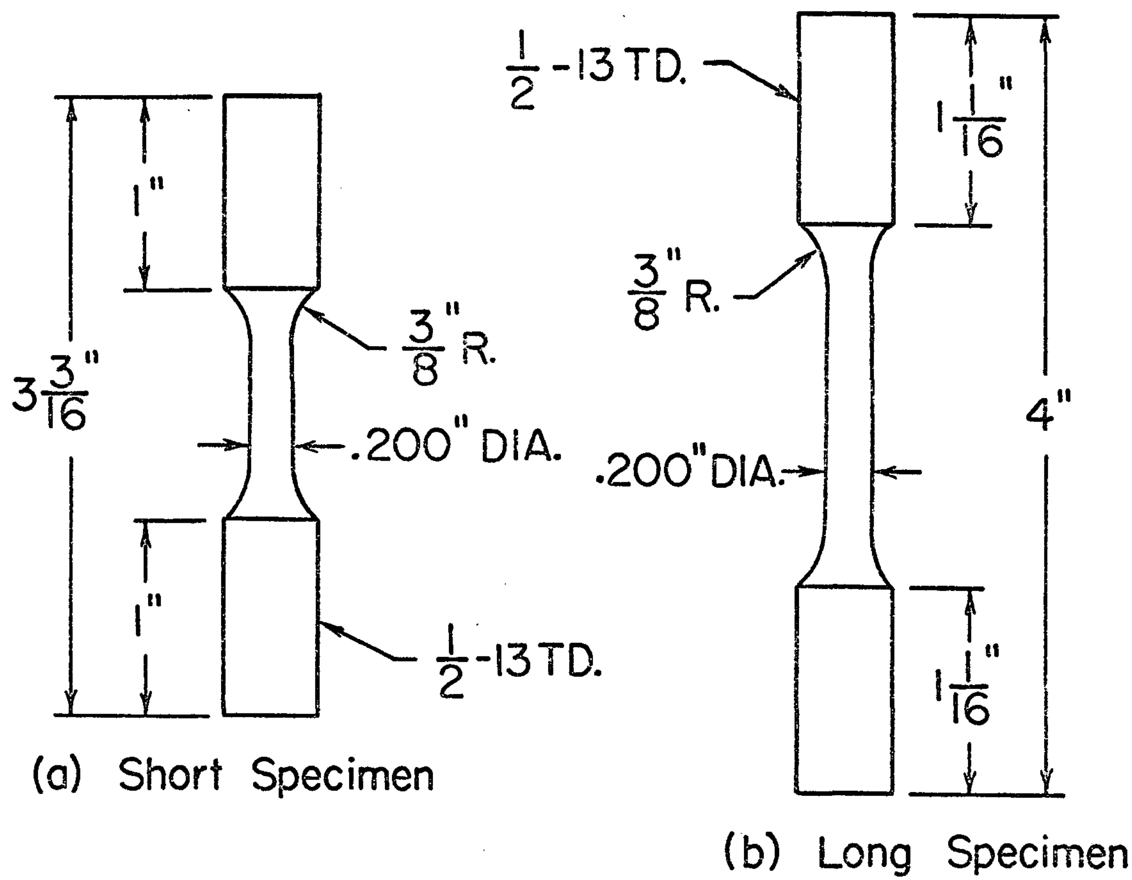


Fig. 2 Test Specimens

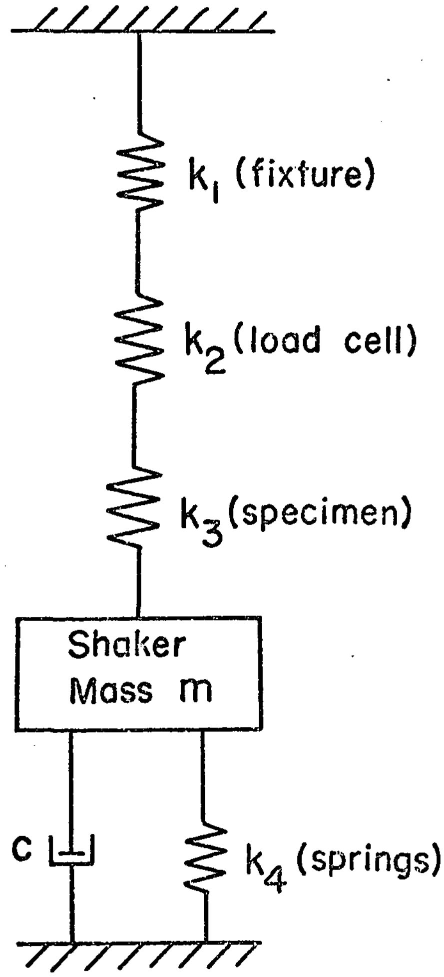


Fig. 3 Vibration Model of the Test System

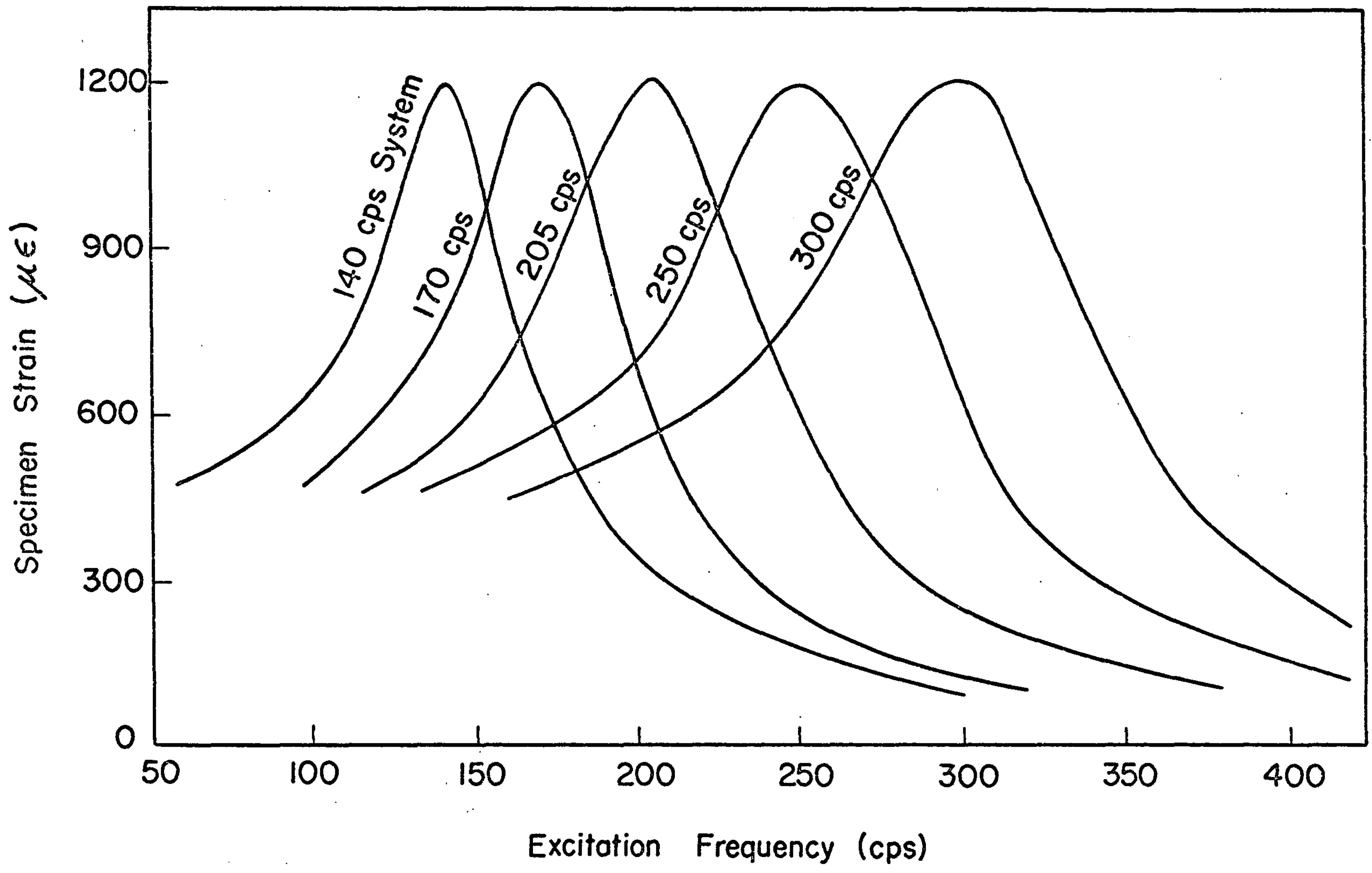


Fig. 4 Response Curves for Constant Maximum Strain

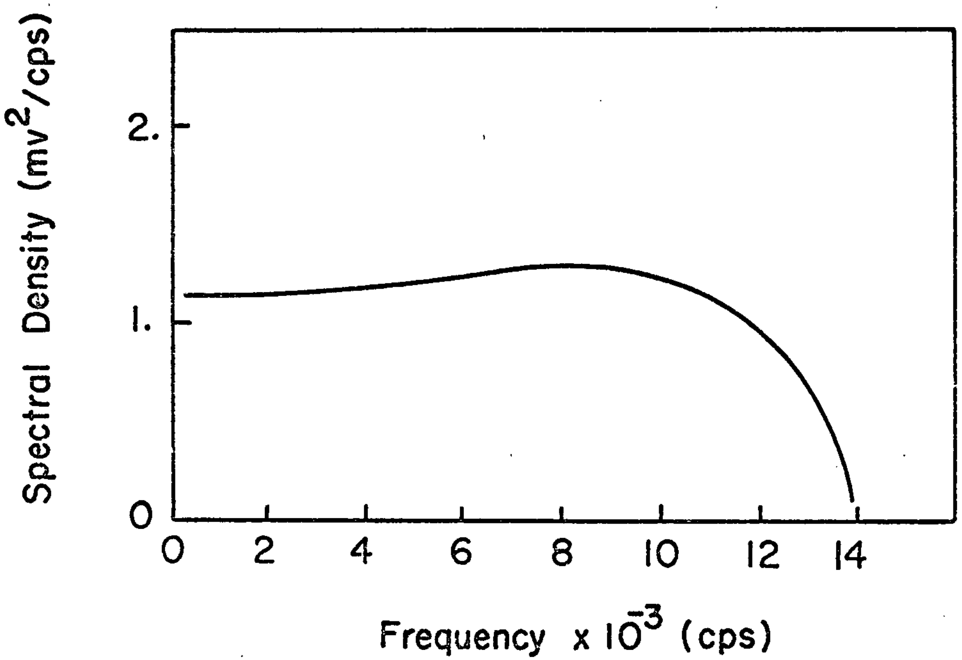


Fig. 5 Input Spectral Density

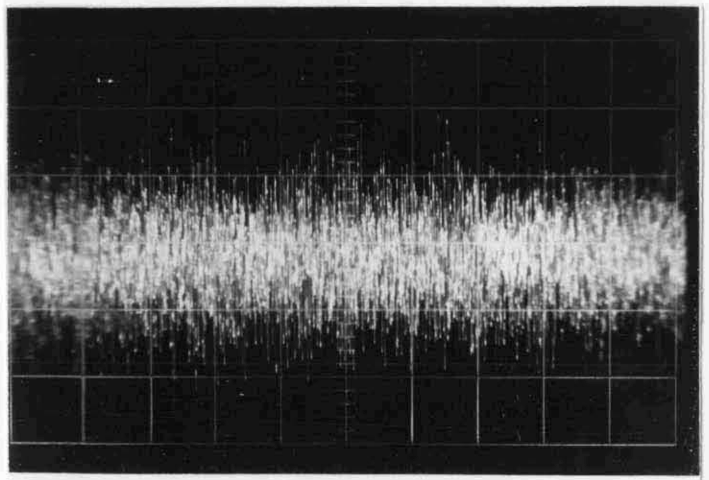
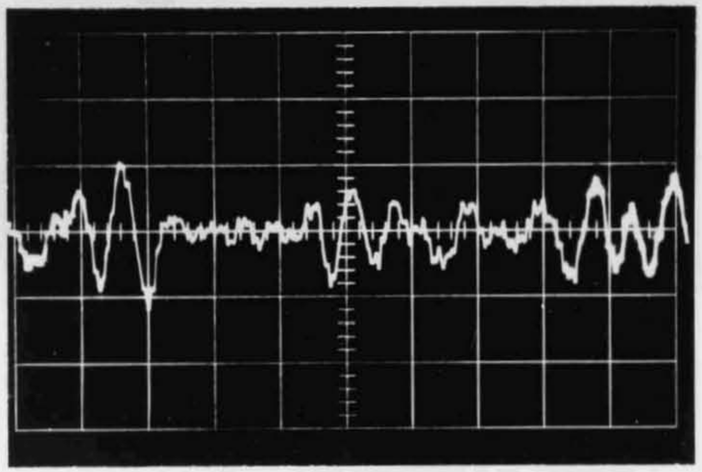
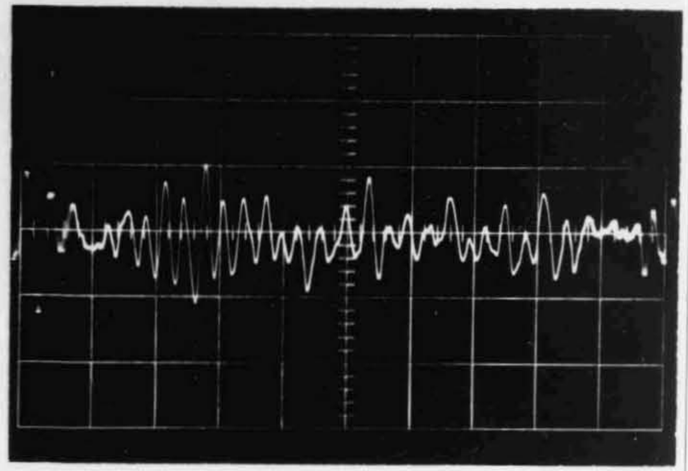


Fig. 6 Randon Input Signal



(a) 140 cps System



(b) 300 cps System

Fig. 7 System Responses

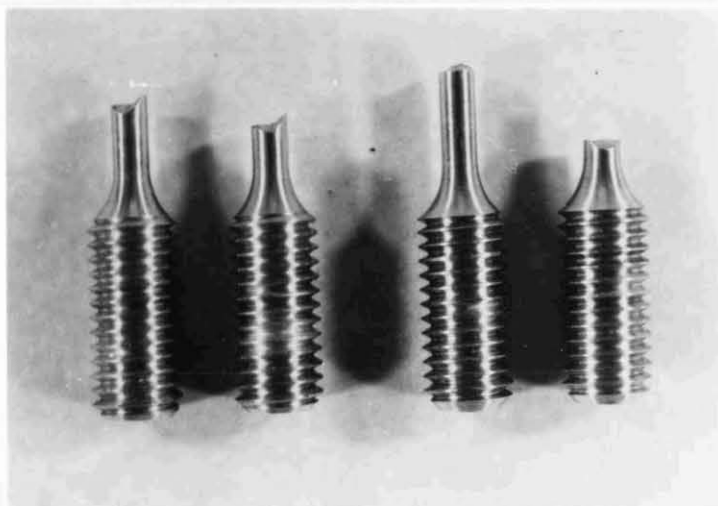


Fig. 8 Failed Fatigue Specimens

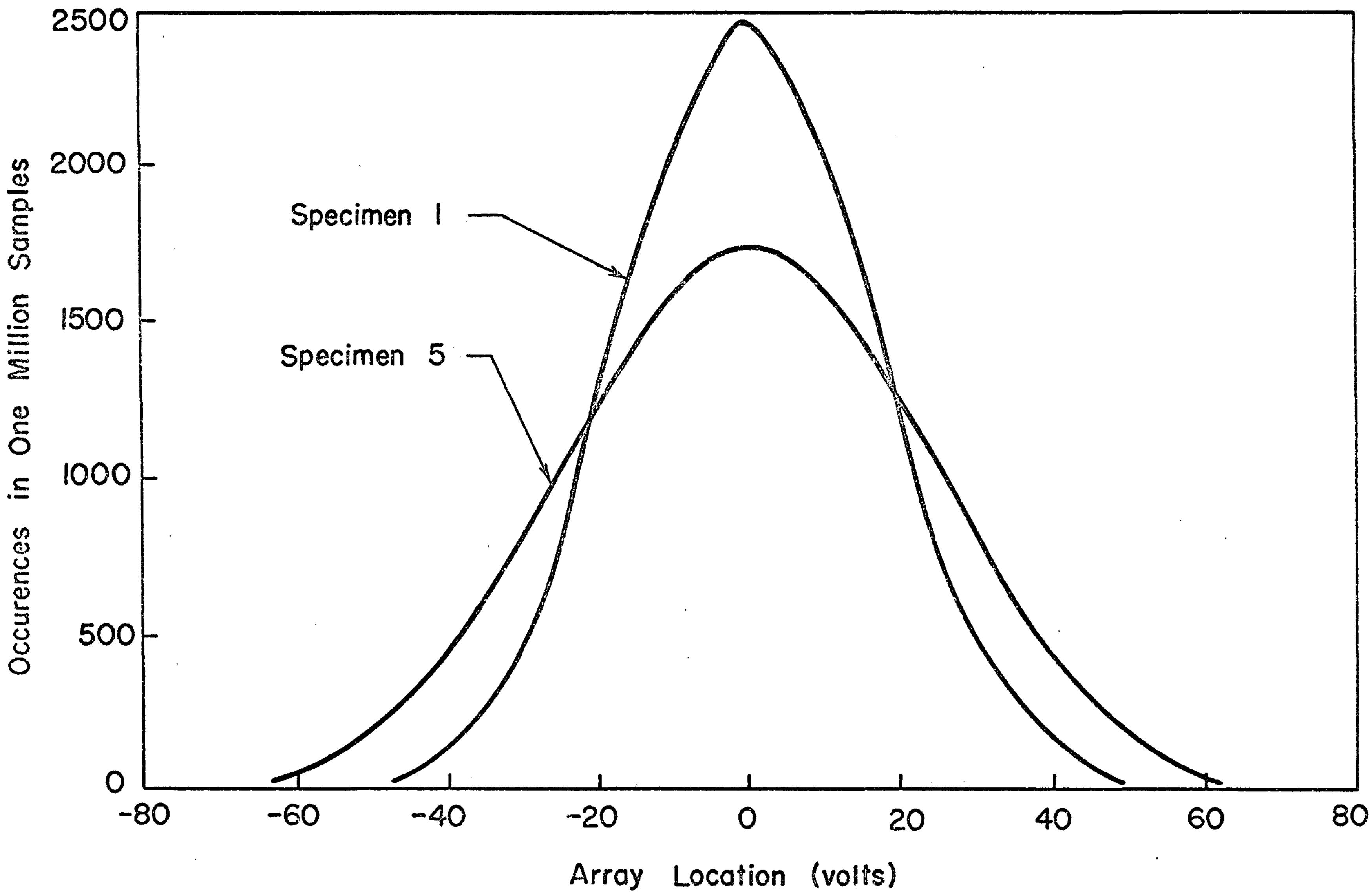


Fig. 9 Probability Density for 140 cps System

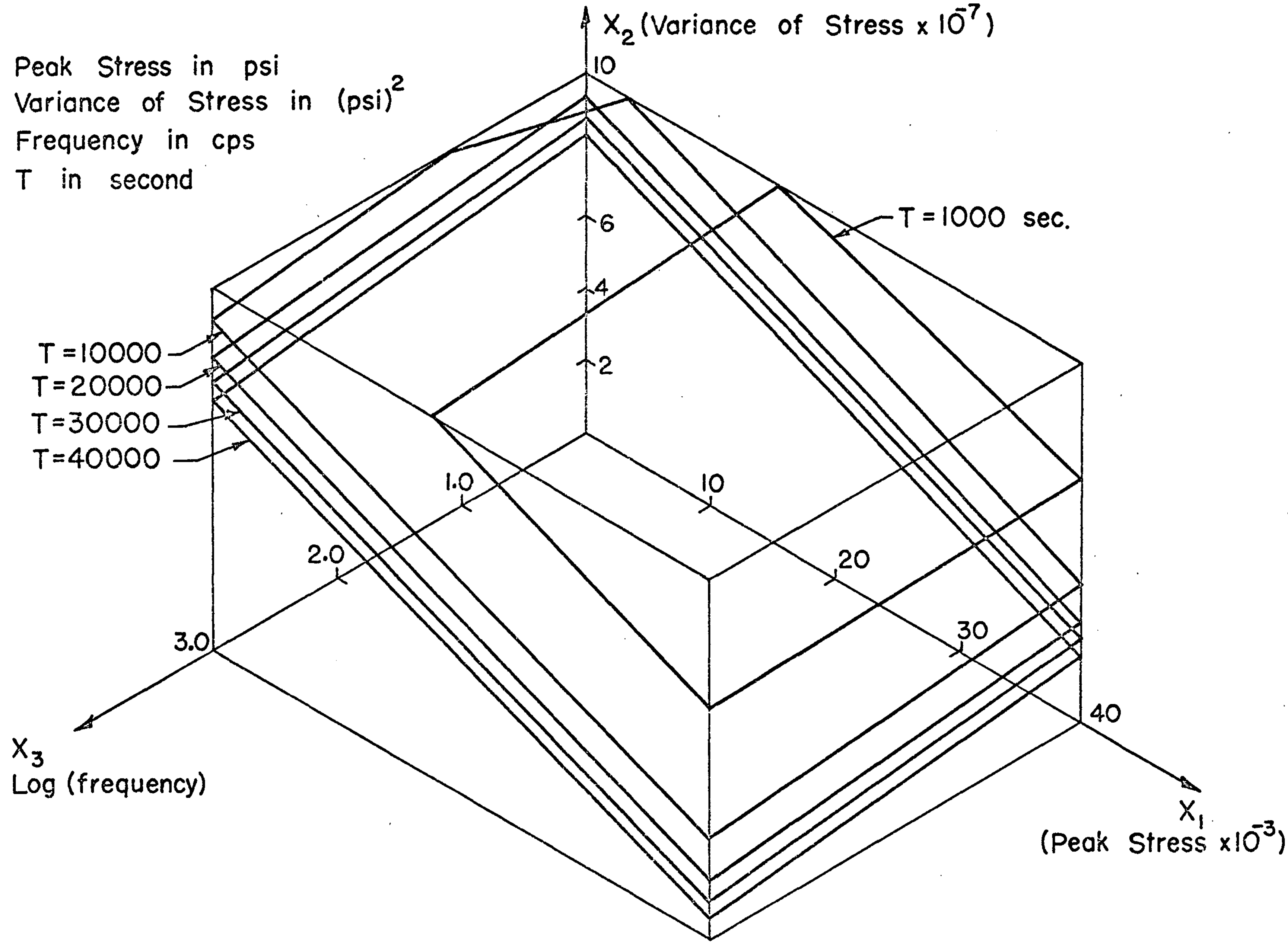


Fig. 10 Response Planes