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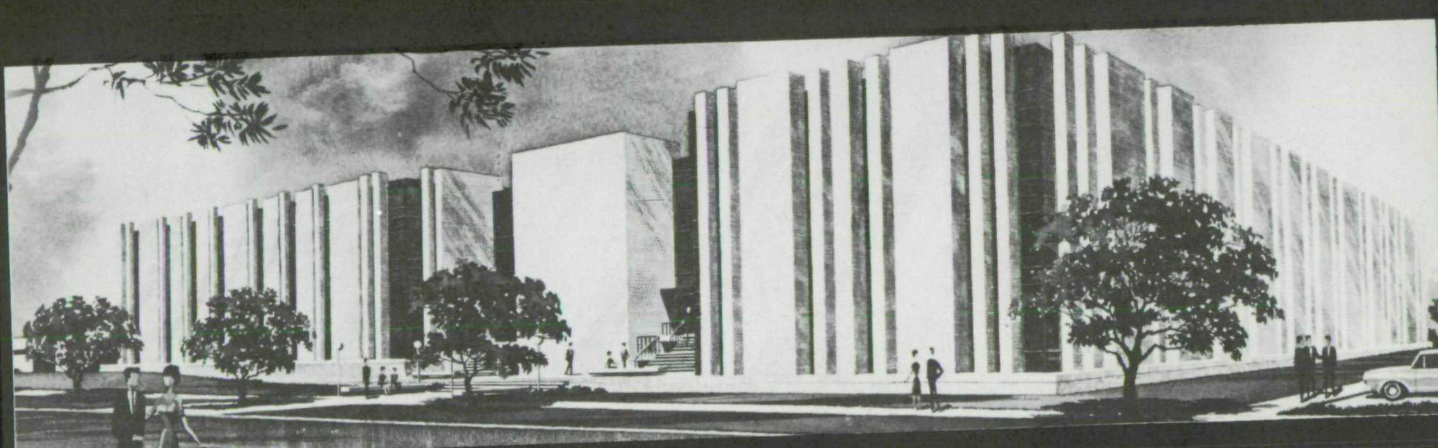

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SEQUENTIAL CONTROL CHART METHODOLOGY

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A modification of the "V" mask sequential control chart is proposed. In this modified scheme, a parabolic section is included in the mask to provide better performance when the process undergoes a large change in the mean from goal conditions. It is shown that the modified "V" mask can be implemented either in conventional graphic form, or in an algorithmic form suitable for a digital computer. Average run lengths are given for a typical range of circumstances. It also is shown that the conventional Shewhart chart is better than a sequential chart for the specific purpose of promptly detecting very large shifts of the mean from goal conditions.

Early Control Chart Methods

The earliest control charts were due to Shewhart (1931). He provided a method for controlling both the mean and the variability of a process. His basic idea is very simple. For the process mean a goal value is chosen. Limits at $\pm K$ units around this goal value

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are specified. These limits are chosen wide enough so that when the process is in control, it is unlikely that an observation occurs outside of these limits. These are called the control limits for the process. For controlling the mean, Shewhart recommended using control limits at ± 3 standard deviations of the sample mean from the goal value. These "three sigma" limits were found to work well for most industrial processes. When a process whose mean follows a normal distribution is in control, only about one reading in 400 will be outside of the 3 sigma (3σ) limits. Another way of saying this is to state that the Average Run Length (ARL) between false out-of-control signals when the process is in control is about 400. When an observation is outside these limits, the process is presumed to have gone out of control.

This method works well for detecting large shifts, but smaller shifts often go undetected. For example, there would be, on the average, about 44 samples taken before a persistent shift of 1 standard deviation (1σ) away from the goal would be detected.

For Shewhart charts, the limits are not always set at $\pm 3\sigma$. For example, when it is very inexpensive to check whether the process is out of control, it is often desirable to set limits tighter than $\pm 3\sigma$. Whenever a reading is out of these tighter control limits, the process is examined to see whether or not the process is really out of control. This scheme is recommended if frequent checking of an in-control process is much less expensive than the costs incurred by missing an out-of-control situation.

The main advantages of Shewhart control charts are their simplicity and their speed in detecting large deviations from goal. In Appendix A, we prove that when the mean is distributed normally Shewhart charts will detect large shifts in the mean faster than any other type of control chart. The primary disadvantage of Shewhart charts is that they do not detect moderate changes in the process quickly. The prime reason for this is that they use only the information from the last data point. To modify the Shewhart charts, additional criteria are often added to signal that the process is out of control. Examples of such modified criteria are:

1. 2 points in succession outside of 2σ limits;
2. K of the last N points outside 2σ limits;
3. 4 or 5 points in succession outside of 1σ limit;
4. A run of 8 or more points. This run might be a consistent upward trend or a downward trend, or it might simply be a run above or below the goal value.

With these modifications, the Shewhart chart begins to act very much like a sequential control chart. By the time these additional criteria are added to the Shewhart chart, it has lost its advantage of simplicity. The sequential methods that are discussed in this paper are easier to use than a Shewhart chart with multiple criteria.

The "V" Mask Sequential Control Procedure

The "V" mask scheme became popular after Barnard's (1959) article, though Page (1954) suggested essentially the same scheme earlier. Figure 1 illustrates a "V" mask. The successive sample numbers are plotted on the abscissa and the cumulative sums of deviations from goal are plotted on the ordinate. Early proponents of the "V" mask suggested that it should be drawn on a clear plastic sheet which is placed on a plot of the cumulative sums (cusums) of deviations from goal. The point on the "V" mask indicated by the arrow on Figure 1 is moved with every successive observation. The arrow is placed at the final cusum value. If any earlier reading lies outside the arms of the "V" mask, the process is taken to be out of control.

In practice, it may be tedious to construct a "V" mask and to move it with each successive observation. There are, however, computational methods that are easy to implement which are equivalent to the "V" mask. These are discussed in a later section of this paper.

A "V" mask is defined by two parameters; these are indicated by h and k in Figure 1. Other authors have defined the "V" mask in terms of the parameters d and θ where:

d = distance in sample units that the vertex of the "V" mask is ahead of the current cusum value;

θ = the angle between each of the arms of the "V" mask and the horizontal.

When one unit on the horizontal axis is equivalent to 2σ on the vertical axis, then:

$$k = 2\sigma \tan \theta;$$

$$h = 2\sigma d \tan \theta.$$

This scaling has been recommended for visual plotting. With this scaling, a 2σ shift in the mean gives a 45° trend on the cusum plot. Goldsmith and Whitfield (1961) used this convention for their graphs of ARL's. Note that the parameters d and θ are scale dependent while h and k can be conveniently defined as a multiple of the standard deviation of the measured variable. The h, k notation was used by Kemp (1961) who proved the computational method which we discuss later in this paper is equivalent to a "V" mask.

A Modified "V" Mask

The "V" mask sometimes takes too long to detect large changes in the process. Note that the first observation must be h+k units away from the goal value for it to be outside the arms of the "V" mask.

An anomaly of a Wald sequential likelihood ratio test for comparing a null hypothesis with a specific alternative hypothesis suggests an alternative shape for the mask. In the Wald procedure, an observation could cause the acceptance of an alternative hypothesis that was far from the null hypothesis value; while if the alternative hypothesis was closer to the null hypothesis value, it would not be accepted. For example compare the rejection region for a null hypothesis that the mean is at zero vs. the alternative hypothesis that the mean is at Δ for $\Delta = 1$ and $\Delta = 3$, and the observations are normally distributed with known standard deviation. Let $\alpha = \beta = .01$ and $\sigma = 1.0$.

To reject the hypothesis that a normally distributed mean is equal to zero, the sequential likelihood ratio test requires a single observation to be higher than

$$\frac{-\sigma^2 \ln \frac{\alpha}{1-\beta}}{\Delta} + \frac{\Delta}{2}$$

when $\Delta = 1$, this formula gives

$$\frac{1 \times 4.6}{1} + \frac{1}{2} = 5.10$$

while for $\Delta = 3$, this formula gives

$$\frac{1 \times 4.6}{3} + \frac{3}{2} = 3.03 \ .$$

Consider an observation at 3.5, when Δ is 3, the likelihood ratio indicates that a shift has occurred; while when $\Delta = 1$, no shift is detected.

This anomaly occurred because the observation at 3.5 was very unlikely both under the hypothesis that the mean was at zero and the alternative hypothesis that the mean was at $\Delta = 1$. With the alternative hypothesis that the mean was at $\Delta = 3$, the observation at 3.5 became much more likely for the alternative hypothesis. The hypothesis that the mean was at 0 was, therefore, rejected. This shows that the procedure is not uniform with respect to all alternatives, therefore we will change the alternative as more samples become available.

It seems reasonable to seek the smallest Δ that can be rejected at a given critical level for various sample numbers. For a given sample number n and given values of α and β, the value of the cusum needed for rejection of the null hypothesis is:

$$S_c = \frac{-\sigma^2 \ln \frac{\alpha}{1-\beta}}{\Delta} + \frac{n}{2} \Delta .$$

To find the smallest Δ that can be detected, for a given n, differentiate this cusum with respect to Δ, set it equal to zero, and solve, obtaining the following:

$$S_c = (-2\sigma^2 n \ln \alpha/1-\beta)^{1/2} = P\sqrt{n}$$

where

$$P = (-2\sigma^2 \ln \frac{\alpha}{1-\beta})^{1/2} .$$

The preceding indicates that a parabolic-shape mask would work better than a "V" mask for detecting large changes quickly. While both types of masks would eventually indicate an out of control situation since they will have finite run lengths, a parabolic mask would tend to detect very small changes after a process had been running near its goal for some time, more often than a "V" mask. Therefore, we wish to modify the purely parabolic mask. When the slope of the parabola is sufficiently small (this will be determined by the smallest deviations that we wish to detect), we no longer follow the parabolic curve. Rather we follow the "V" mask that is tangent to the parabolic curve.

Suggestions of different shapes for sequential masks are not new. Barnard (1959) noted that some of his colleagues suggested using a purely parabolic mask. However, this point was not pursued further in his paper.

The Construction of the Modified "V" Mask

The "V" mask is composed of a line with intercept h and slope k, and a second line which is the reflection of the first line. The formula for the upper line for a "V" mask is:

$$Y = h + (n)\ (k)$$

where Y is the distance of the upper arm from the centerline

h is the intercept

k is the slope

n is the distance from the last sample taken.

The formula for the upper half of a parabolic mask is

$$P\sqrt{n}$$

where P is a size constant.

To design a modified "V" mask, the values of P and k that give the desired operating characteristic (e.g., the desired ARL) are specified. The plots of ARL's (figures 3-7) are used in this step; this will be discussed later.

The modified "V" mask consists of a parabolic-shaped mask having parameter size constant P which is tangent to a "V" mask having slope k. The slope of the parabola is $\frac{P}{2\sqrt{n}}$. Where the parabola and the "V" mask meet, they must have the same slope. Therefore, they will meet at

$$n' = \frac{P^2}{4(k)^2} .$$

Choosing h so the heights will be equal at this point gives:

$$h = \frac{P^2}{4k} .$$

Note that if n' is ≤ 1.0, the parabolic section will not actually change the control limits for the "V" mask at any observed point; the two schemes are equivalent. This occurs when the slope of the "V" mask is greater or equal to one-half the P value.

As a typical example, consider a modified "V" mask with P = 3 and $k = \frac{1}{2}$. The parabola will meet the "V" mask at n' = 9 and h will be 4.5. Figure 2 illustrates this mask.

Specific needs will determine the values of P and k that will be used. Suppose it is desired to detect a departure of Δ units from the goal value and to have a specific Average Run Length when the process is in control. Calculations of Average Run Lengths for "V" masks and comparisons with likelihood ratio tests indicate that for a given ARL when the process is in control, the smallest ARL's (the quickest detections) when the process is running at Δ units from goal are obtained when k is approximately $\Delta/2$. As a rule of thumb, a k value of $\Delta/2$ is recommended. For $k = \Delta/2$, a P value giving the desired ARL when the process is in control can be chosen using figures 3-7 which plot the "Average Run Length" as a function of deviation of the mean from goal conditions for values of P and k that are usually met in practice.

Average Run Lengths for the Modified "V" Mask

The ARL's used in drawing figures 3-7 were first obtained by simulation on the 1108 Univac computer in DuPont's Engineering Department. An error in the computer program was discovered and the ARL's were rerun at Texas A&M University. Pseudo-random numbers were generated using a multiplicative congruence procedure. These were transformed to random normal deviates using the Box-Muller procedure (Muller 1959). A computer program used these generated random normal deviates, with the appropriate deviation from goal added, and the formulas for implementing a modified "V" mask, (Kemp 1961) to find run lengths for repeated trials using various parameter values and various deviations from goal conditions. For parameter values having a run length greater than 500, a "V" mask having approximately the same run length as the modified "V" mask being simulated was used as a control variable (Fieller and Hartley, 1954).

The "V" mask ARL values were obtained by solving the integral equation given by Page (1954). The solution was obtained by replacing the integral equation by a system of linear equations using Gaussian quadrature as the numerical integration scheme. The "V" mask ARL was double checked by recalculating the ARL following the iterative method suggested by Kemp (1958).

Table I is a table of the ARL's which were obtained. Note that occasionally a mask having larger parameter values had smaller ARL's than a mask having smaller parameter values. This is due to random error in the simulation method. The coefficient of variation of the computed ARL's is less than 5% for the entries in Table I. At least 40 Run Lengths were calculated at each deviation for each set of parameter values.

Table II illustrates a comparison between a "V" mask, a modified "V" mask, and a Shewhart control scheme. These three schemes were chosen to have nearly the same run lengths when there is no deviation from goal. For small deviations, on the order of 1σ, the run length for the cusum schemes is much lower than for the Shewhart scheme. For large deviations, greater than 3σ, the Shewhart scheme gives a somewhat smaller ARL than the "V" mask schemes.

The modified "V" mask has larger ARL's for very small deviations and smaller ARL's for large deviations. In this way, it performs better than the "V" mask. Following the rule of thumb that the k value is approximately half the deviation that we wish to detect, we

TABLE I

Average Run Lengths for Modified "V" Masks
These Values are Plotted in Figures 4-8

		Deviation from Goal (Multiple of σ)													Fig.
P	k	0.0	.25	.50	.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	n'	h	No.
2.0	.25	12.5	10.5	8.72	4.98	3.95	2.68	1.78	1.35	1.24	1.00	1.00	16.000	4.000	3
2.5	.25	32.2	24.6	14.2	8.32	5.69	2.94	2.39	1.72	1.41	1.05	1.02	25.000	6.250	3
3.0	.25	121.	57.7	24.2	12.7	8.19	4.72	3.02	1.99	1.58	1.18	1.00	36.000	9.000	3
3.5	.25	546.	125.	36.3	17.9	11.9	6.09	3.23	2.69	1.82	1.32	1.00	49.000	12.250	3
4.0	.25	2619.	248.	53.2	25.2	15.4	7.71	4.86	3.02	2.42	1.52	1.18	64.000	16.000	3
2.0	.50	13.2	12.9	7.88	5.98	3.62	2.11	1.66	1.36	1.05	1.00	1.02	4.000	2.000	4
2.5	.50	36.0	28.8	14.7	8.79	5.90	3.39	2.18	1.68	1.34	1.05	1.00	6.250	3.125	4
3.0	.50	132.	68.2	26.9	14.8	7.85	4.55	3.06	2.11	1.70	1.35	1.15	9.000	4.500	4
3.5	.50	561.	182.	47.5	19.9	10.7	6.27	3.46	2.69	1.94	1.27	1.05	12.250	6.125	4
4.0	.50	3150.	610.	71.6	23.7	13.2	7.48	4.78	3.40	2.44	1.63	1.10	16.000	8.000	4
2.0	.75	16.6	13.3	8.61	6.48	4.19	2.52	1.73	1.43	1.15	1.02	1.02	1.778	1.333	5
2.5	.75	44.0	33.3	16.7	9.42	6.51	3.48	2.27	1.68	1.36	1.02	1.02	2.778	2.083	5
3.0	.75	143.	87.6	36.1	15.6	8.92	4.41	2.85	1.80	1.52	1.18	1.00	4.000	3.000	5
3.5	.75	627.	259.	69.2	25.6	12.6	6.01	3.72	2.60	1.98	1.25	1.05	5.444	4.083	5
4.0	.75		1082.	159.	43.4	17.9	7.18	4.91	3.14	2.28	1.58	1.17	7.111	5.333	5
2.0	1.00	17.3	13.8	11.4	7.09	4.88	2.60	1.91	1.43	1.20	1.02	1.00	1.000	1.000	6
2.5	1.00	56.5	40.7	20.4	11.5	7.72	3.62	2.35	1.66	1.27	1.12	1.00	1.563	1.563	6
3.0	1.00	179.	102.	46.3	22.0	11.2	4.54	2.88	2.04	1.64	1.18	1.05	2.250	2.250	6
3.5	1.00	779.	368.	116.	37.0	16.7	6.67	3.42	2.47	1.93	1.41	1.10	3.062	3.062	6
4.0	1.00		1715.	304.	71.4	25.4	7.93	4.43	2.90	2.44	1.40	1.20	4.000	4.000	6
2.0	1.50	32.0	25.6	17.9	10.5	7.25	3.45	2.06	1.60	1.21	1.05	1.00	.444	.667	7
2.5	1.50	80.3	63.5	37.3	18.4	10.7	4.97	2.86	1.74	1.35	1.12	1.02	.694	1.042	7
3.0	1.50	264.	177.	88.4	43.4	21.1	6.93	3.42	2.40	1.58	1.12	1.05	1.000	1.500	7
3.5	1.50	1347.	776.	298.	98.1	42.0	10.4	4.32	2.80	1.95	1.27	1.05	1.361	2.042	7
4.0	1.50			1000.	222.	82.0	14.2	5.96	3.44	2.54	1.53	1.15	1.778	2.667	7

TABLE II

An ARL Comparison for Three Control Schemes

Control Scheme	Deviation from Goal						
	0	$\sigma/2$	σ	2σ	3σ	4σ	5σ
Shewhart	320	137	39.5	5.89	1.92	1.17	1.02
Modified "V" Mask	320	54.2	10.6	3.37	1.73	1.29	1.08
"V" Mask	319	42.3	9.23	3.50	2.09	1.59	1.21

Shewhart Control Limits at $\pm 2.96\sigma$

Modified "V" Mask k = .7 p = 3.31
"V" Mask k = .7 h = 3.5

(The Shewhart and "V" mask numbers are from Goldsmith and Whitfield (1961)).

see that these schemes are designed to detect deviations of about 1.4σ. In the range between 1σ and 2σ, both "V" masks have significantly smaller ARL's than the Shewhart chart; to detect deviations in this range, both "V" masks are superior to the Shewhart chart.

Note that a Shewhart chart is a special case of both the "V" mask and the modified "V" mask. A "V" mask with h = 0 and k = 3σ is equivalent to a modified "V" mask with P = 0 and k = 3σ. Both are equivalent to a Shewhart chart with control limits at $\pm 3\sigma$. A "V" mask or modified "V" mask designed to detect large deviations quickly is very similar to a Shewhart chart; and if only very large deviations are to be detected, a Shewhart chart is best.

Implementing the "V" Mask or the Modified "V" Mask

When the data from the process comes slowly, a graphical procedure is adequate. Simply drawing the mask on a clear transparency and placing it on a cusum plot is not difficult or time-consuming. When many processes are examined simultaneously or when the data arrives rapidly, graphical procedures are not adequate. A computational procedure is much more efficient. The following method has not received much publicity considering its usefulness.

Using figures 3-7 choose the desired values of P and k (or for a "V" mask choose values of h and k). The computational method equivalent to the "V" mask or the "V" mask section of the modified "V" mask takes three columns to implement. The first column records the individual readings. The second column calculates:

$$S_{H(i)} = \max[0,\ x_i - (\text{Goal} + k) + S_{H(i-1)}] \ .$$

The third column calculates:

$$S_{L(i)} = \max[0,\ (\text{Goal} - k) - x_i + S_{L(i-1)}]$$

where

x_i - is the individual reading

k - is the slope of the "V" mask

Goal - is the goal value

$\max[a,b]$ - is the maximum of a and b

$S_{H(0)} = S_{L(0)} = 0$.

Both the second and third columns cumulate deviations greater than k units away from the goal value, with the cumulation starting anew (being reset to zero) whenever it becomes negative.

Whenever either of columns 2 or 3 becomes greater than h, the process is considered to be out of control. The observations would then be outside the arms of the "V" mask.

For the parabolic section of the modified "V" mask calculate:

$$\left| \sum_{j=i-n+1}^{i} (x_j - \text{Goal}) \right|$$

for n = 1, 2, ... up to the maximum integer less than n'. If any of these values are greater than P/n, an out-of-control situation is indicated by the parabolic section of the mask.

Table III illustrates the computational procedure for a "V" mask. It is helpful to define two more columns, column 4 and column 5, that record the number of successive readings that the cusum has been greater than 0. With these, it is possible to obtain an estimate of the process average. An easily-calculated estimate of the process average is:

$$\text{Goal} \pm \frac{N \times k + \max\,[S_{H(i)}, S_{L(i)}]}{N}$$

where the (+) is used with $S_{H(i)}$, the (-) is used with $S_{L(i)}$, and N is obtained from column 4 or 5 as appropriate. When both cusums are 0(N = 0), the goal is used as the estimate of process average.

TABLE III

The Computational Form of the "V" Mask

Column	1 Individual Reading x_i	2 S_H	3 S_L	4 N_{HIGH}	5 N_{LOW}	Reading No.
	102	0	0	0	0	1
	101	0	0	0	0	2
	104	1	0	1	0	3
	98	0	0	0	0	4
	96	0	1	0	1	5
	91	0	7	0	2	6
	95	0	9	0	3	7
	94	0	12	0	4	8
	101	0	8	0	5	9
	93	0	12	0	6	10
	93	0	16*	0	7	11

x_i = Individual Reading Values

* = Out-of-Control Point

100 = Goal Value

3 = k = Slope of "V" Mask (allowable slack in the process)

14 = h = Intercept of the "V" Mask

N_{HIGH}, N_{LOW} = No. of Readings the Cusum has been Positive

Estimated Process Average at Out-of-Control Point $= \text{Goal} - \dfrac{N \times k + \max[S_{H(i)}, S_{L(i)}]}{N} = 100 - \dfrac{7 \times 3 + 16}{7} = 94\frac{5}{7}$.

Acknowledgements

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APPENDIX

When Shewhart Charts are Best

A Shewhart Chart having a given average run length when the process is running at goal conditions will have its single-observation control limits closer to the goal value than any other control scheme having the same average run length. For detecting large shifts in the mean from goal conditions, the single-observation control limits are the important ones. This section compares the average run length of control schemes having different single-observation control limits. It proves that for detecting large shifts in the mean of a normal population the control scheme that has its single-observation control limit closest to goal has the shortest average run length. Thus the Shewhart Chart is optimal for detecting large shifts in the mean.

Theorem: Given two control schemes for controlling the mean of a normal population with single-observation control limits respectively at $\pm$ Z and $\pm$ (Z + δ) away from the goal, there exists a deviation Δ such that for all deviations $\Delta' \geq \Delta$, the average run length of the control scheme with single-observation control limits at $\pm$ Z is smaller than the average run length of the control scheme with single-observation control limits at $\pm$ (Z + δ), [regardless of what other control criteria may apply to later observations].

Proof (in two steps)

Step 1: We prove that if the population mean makes a shift of $\Delta > Z$, the probability contained in the interval from Z to Z + δ is greater than the probability contained in the interval from $-\infty$ to Z. Formally, we prove (after a translation that places the mean at zero, thereby placing the goal at $-\Delta$ and making $Z < 0$) that:

$$\int_{Z}^{Z+\delta} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt > \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt = F(Z) .$$

The left-hand side is greater than

$$\left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right)\right) \delta .$$

Mills ratio R(Z) is the ratio of the tail area to the bounding ordinate (Kendall and Stuart (1963))

$$R(Z) = F(Z) / \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right)$$

where

$$R(Z) = \frac{1}{Z} - \frac{1}{Z^3} + \frac{1.3}{Z^5} - \dots (-1)^j \frac{1 \cdot 3 \cdot 5 \dots (2j-1)}{Z^{2j+1}} \dots .$$

In this series, the remainder is less in absolute value than the last term taken into account. We need only consider the first term to obtain:

$$\int^{Z+\delta} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt > \left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right)\right)\delta$$

$$> \frac{1}{Z}\left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right)\right)$$

$$> R(Z)\left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right)\right) = F(Z) .$$

The inequality holds when:

$$\delta > \frac{1}{Z} .$$

Thus, for all Δ greater than $Z + \frac{1}{\delta}$, the probability contained in the interval from Z to Z + δ is greater than the probability contained in the interval from $-\infty$ to Z.

Step 2: We show that the condition of step 1 is sufficient to prove the theorem.

When previous conditions hold, there is a probability, say Q, of not detecting a shift with a single observation with the tighter single-observation limits and a probability > 2Q with the looser single-observation control limits. If the scheme with looser limits always detects an out-of-control situation on the second observation, the best it can do, its ARL is slightly greater than:

$$1(1-2Q) + 2(2Q) = 1 + 2Q .$$

While the ARL for the scheme with tighter limits is no greater than:

$$1(1-Q) + 2(1-Q)Q + 3(1-Q)Q^2 + \ldots + n(1-Q)Q^{n-1} + \ldots$$

$$= 1 + Q + Q^2 + Q^3 + \ldots + Q^n + \ldots \quad .$$

This is a geometric series. Its sum is $1/(1-Q)$.

$$1 + 2Q > \frac{1}{1-Q} \quad \text{when} \quad Q < 1/2 \; .$$

Since Z and $Z + \delta$ are on the same side of the goal value, Q is $\leq 1/2$. Q.E.D.

The proof depends on the speed with which the tails approach the axis. The proof extends immediately to any distribution having a finite range, i.e. for any "real" distribution.

Note that this proof indicates the optimum sampling plan for processes which make only large shifts in the mean when they go out of control. The optimum plan is to take a single sample as often as possible.

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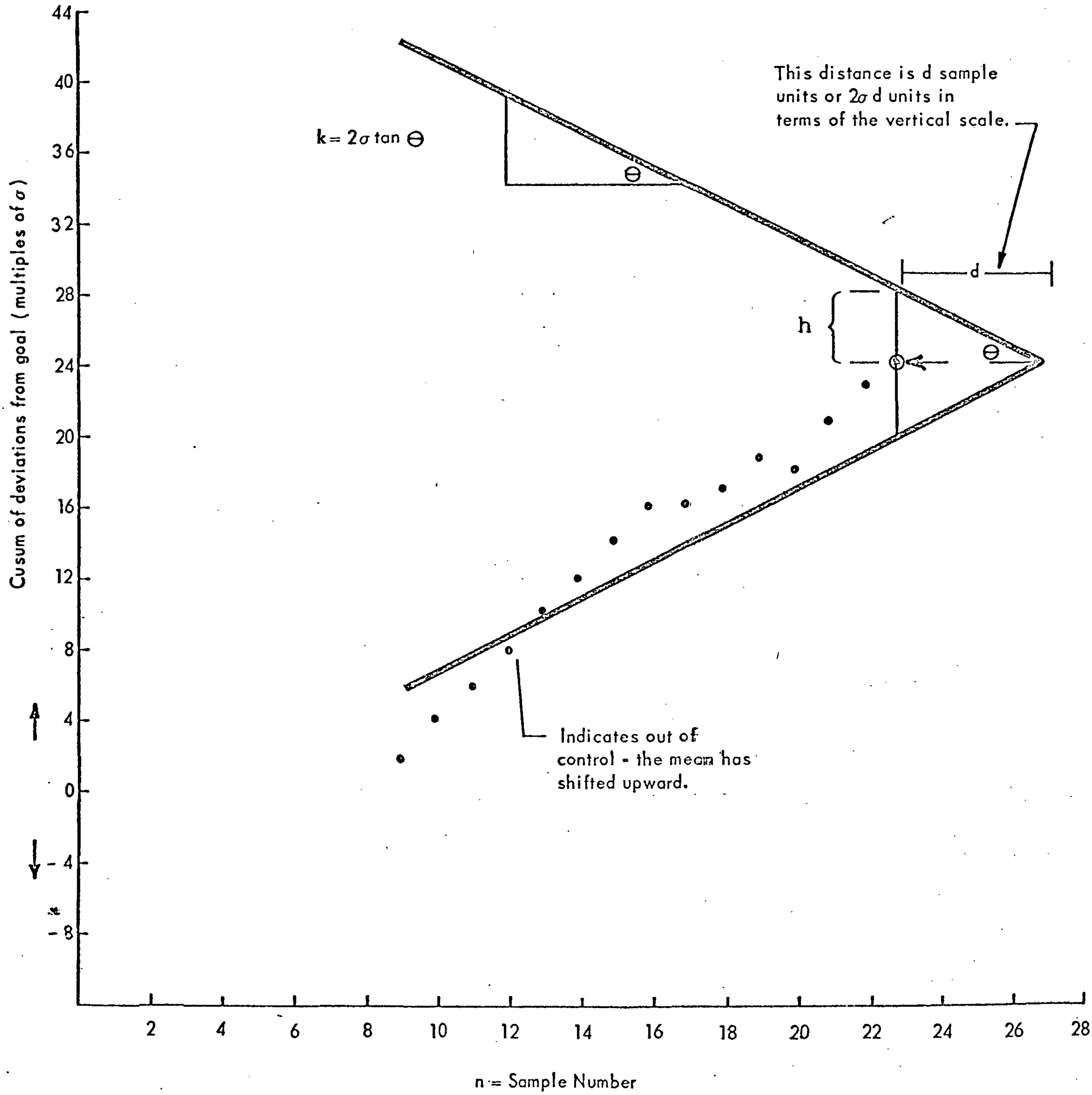
FIGURE 1

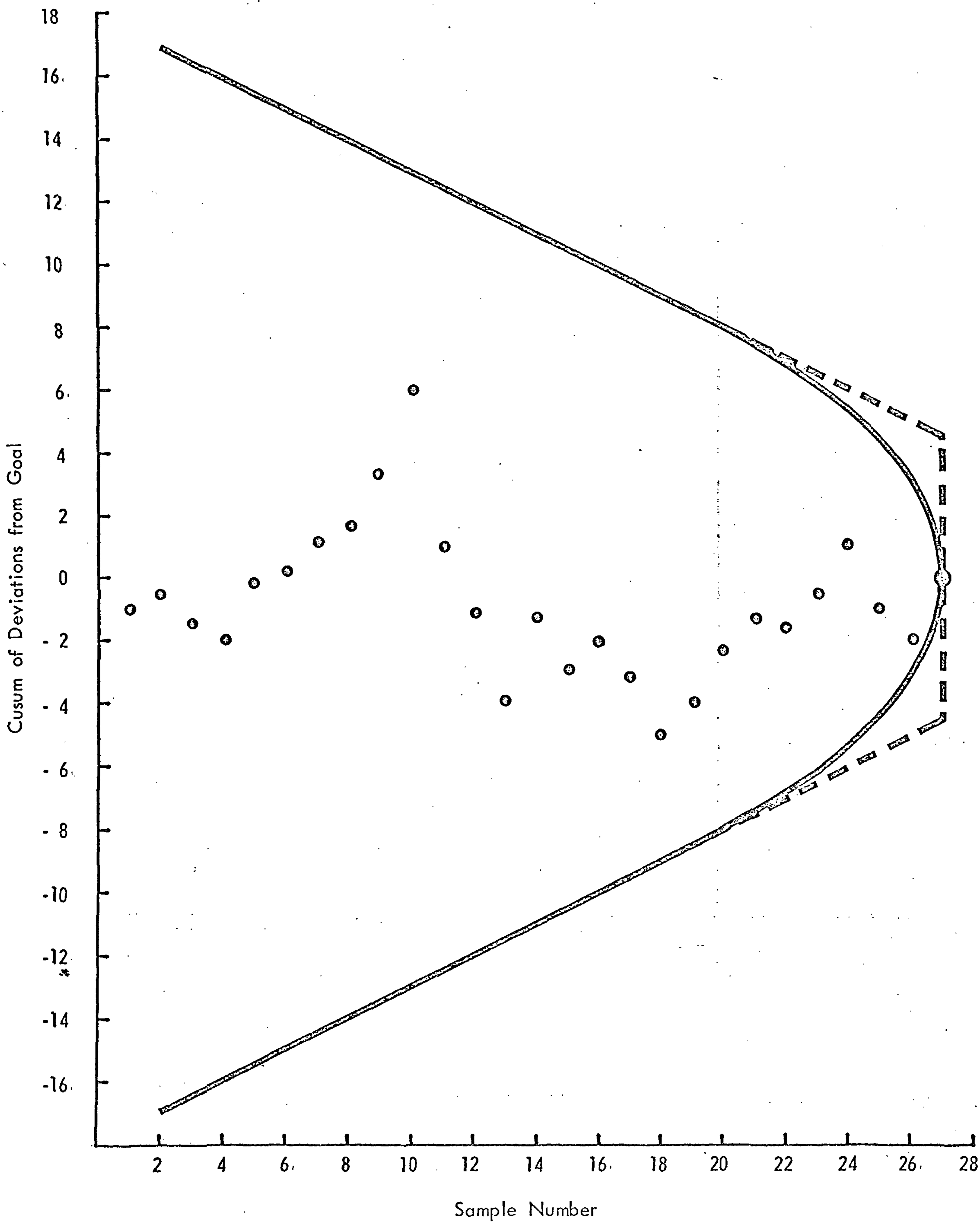
A "V" MASK CUSUM CHART



Scale 1 sample unit (on the abscissa) = 2σ (on the ordinate)

FIGURE 2

A MODIFIED "V" MASK CUSUM CHART

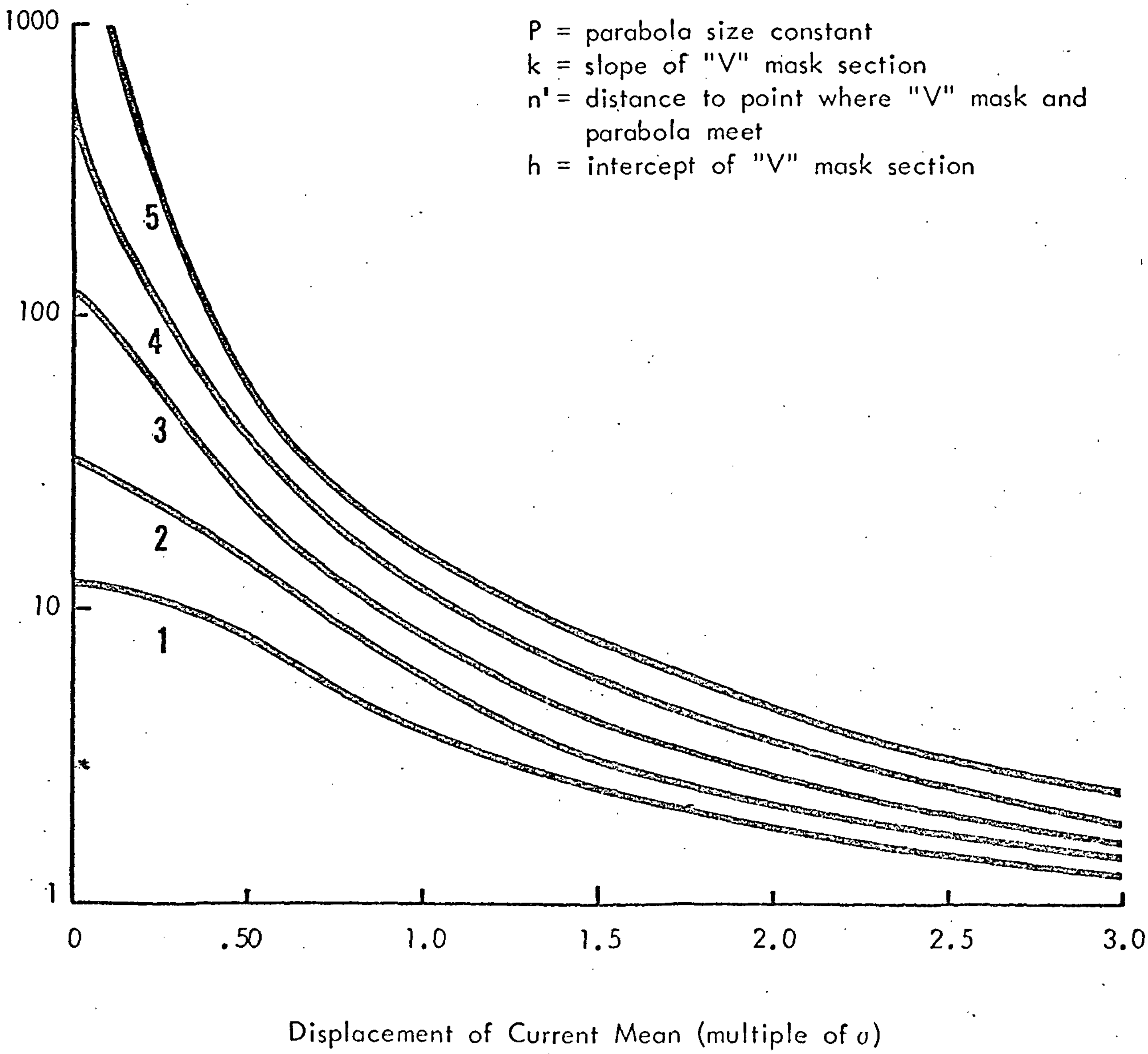


AVERAGE RUN LENGTHS FOR
MODIFIED "V" MASKS

FIGURE 3

CURVE	P	k	n'	h
1	2.0	.25	16.0	4.0
2	2.5	.25	25.0	6.25
3	3.0	.25	36.0	9.0
4	3.5	.25	49.0	12.25
5	4.0	.25	64.0	16.0

P = parabola size constant
k = slope of "V" mask section
n' = distance to point where "V" mask and parabola meet
h = intercept of "V" mask section

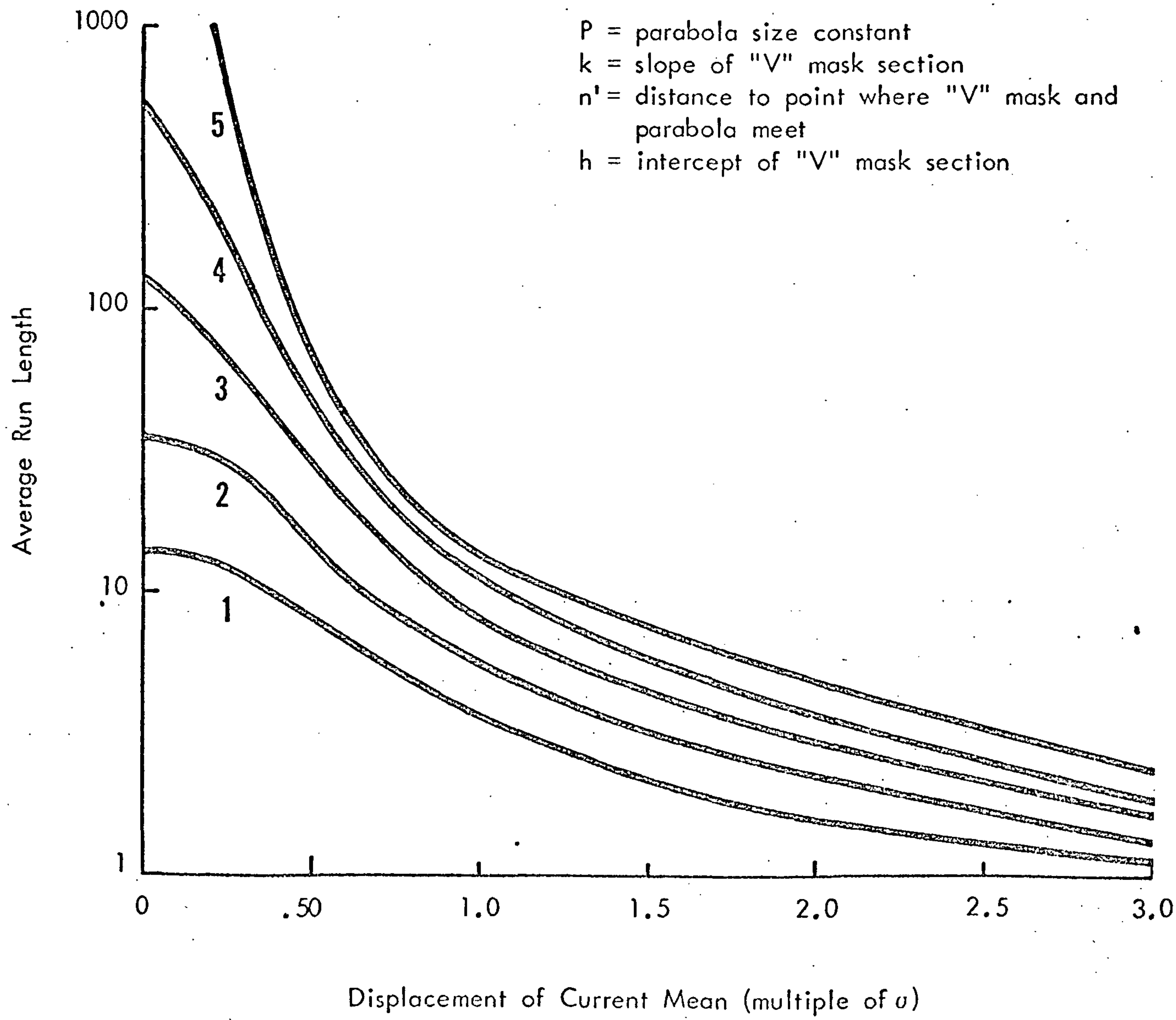


AVERAGE RUN LENGTHS FOR
MODIFIED "V" MASKS

FIGURE 4

CURVE	P	k	n'	h
1	2.0	.50	4.0	2.0
2	2.5	.50	6.25	3.12
3	3.0	.50	9.0	4.5
4	3.5	.50	12.25	6.12
5	4.0	.50	16.0	8.0

P = parabola size constant
k = slope of "V" mask section
n' = distance to point where "V" mask and parabola meet
h = intercept of "V" mask section

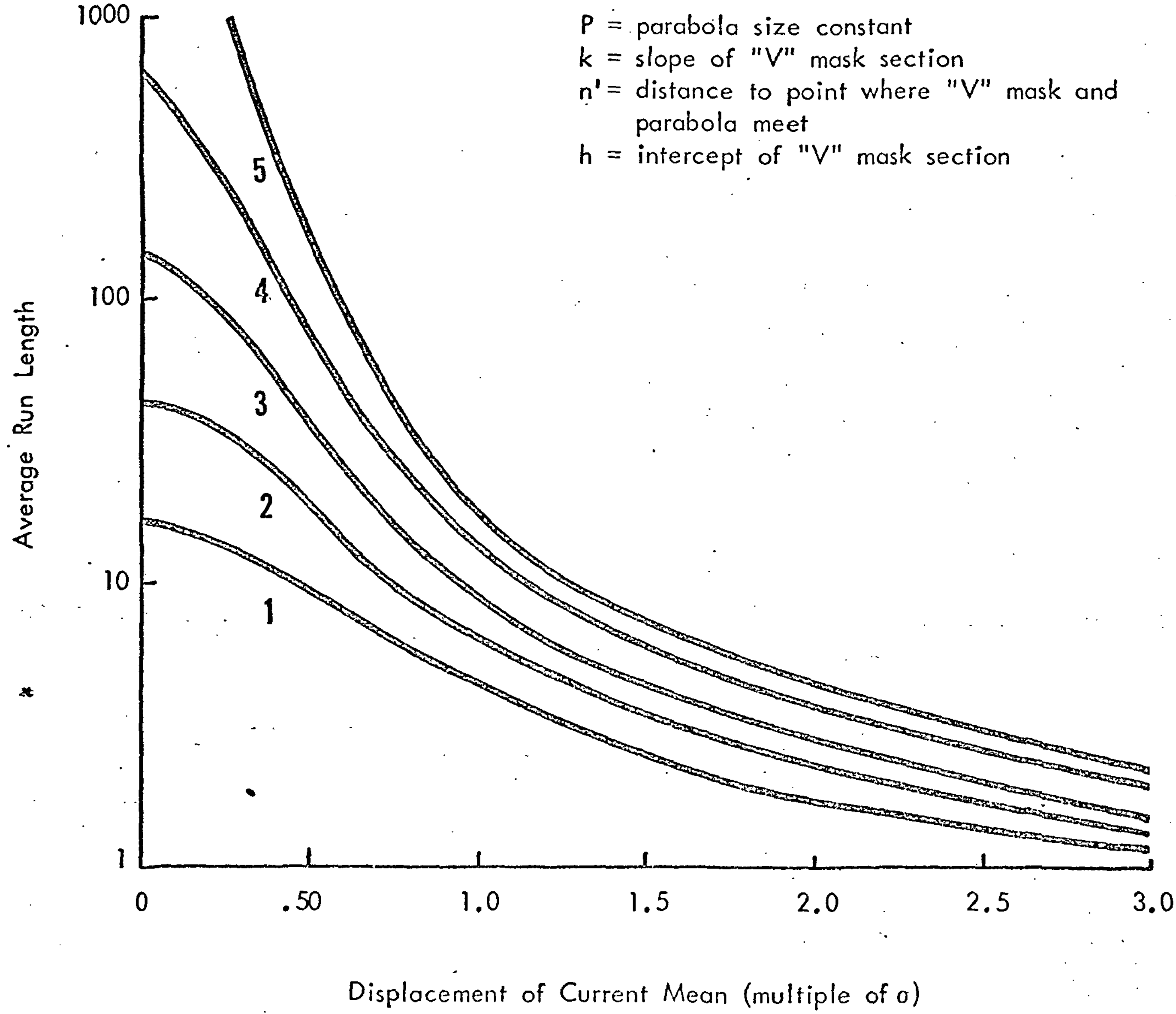


AVERAGE RUN LENGTHS FOR
MODIFIED "V" MASKS

FIGURE 5

CURVE	P	k	n'	h
1	2.0	.75	1.77	1.33
2	2.5	.75	2.77	2.08
3	3.0	.75	4.0	3.0
4	3.5	.75	5.44	4.08
5	4.0	.75	7.11	5.33

P = parabola size constant
k = slope of "V" mask section
n' = distance to point where "V" mask and parabola meet
h = intercept of "V" mask section

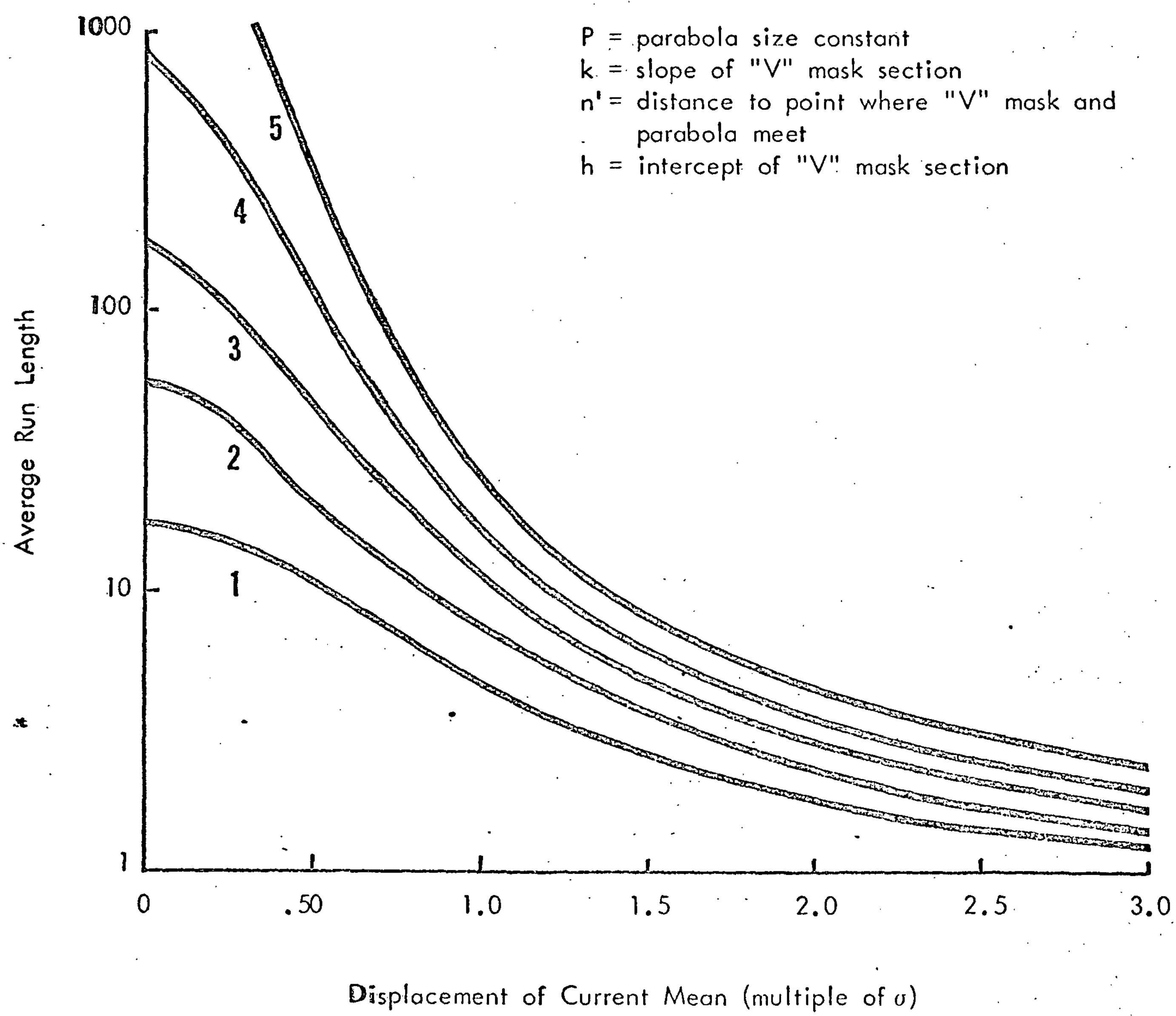


AVERAGE RUN LENGTHS FOR
MODIFIED "V" MASKS

FIGURE 6

CURVE	P	k	n'	h
1	2.0	1.00	1.00	1.00
2	2.5	1.00	1.56	1.56
3	3.0	1.00	2.25	2.25
4	3.5	1.00	3.06	3.06
5	4.0	1.00	4.0	4.0

P = parabola size constant
k = slope of "V" mask section
n' = distance to point where "V" mask and parabola meet
h = intercept of "V" mask section



AVERAGE RUN LENGTHS FOR
MODIFIED "V" MASKS

FIGURE 7

CURVE	P	k	n'	h
1	2.0	1.50	.44	.67
2	2.5	1.50	.69	1.04
3	3.0	1.50	1.00	1.50
4	3.5	1.50	1.77	2.66
5	4.0	1.50	1.77	2.66