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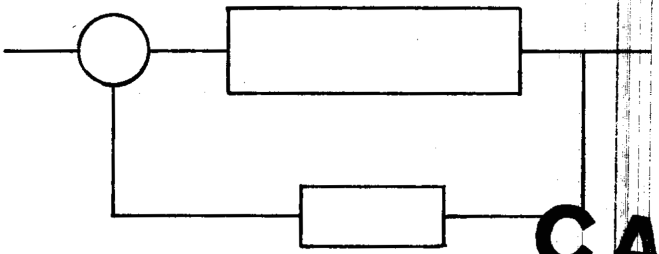

Bounded State Space

by

EARL D. EYMAN

(Principal Investigator)

March 1, 1972

Prepared for: NASA Scientific and Technical Information Center
P.O. Box 33
College Park, Maryland 20740

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FINAL REPORT

BOUNDED STATE SPACE

EARL D. EYMAN
PRINCIPAL INVESTIGATOR

March 1, 1972

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ELECTRICAL ENGINEERING DEPARTMENT
THE UNIVERSITY OF IOWA
IOWA CITY, IOWA 52240

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13. ABSTRACT

This investigation is divided functionally into three different areas: (1) study of bounded state space, (2) nonlinear smoothing theory, and (3) system identification.

(1) Study of Bounded State Space: Necessary and sufficient conditions for an optimal control are obtained for a bounded state space optimal control problem. The difficulty of determining the so called "jump conditions" is eliminated; however, the problem of determining the points where the response either enters or leaves the boundary still remains unsolved.

(2) Nonlinear Smoothing Theory: Nonlinear fixed-interval, fixed-point and fixed-lag smoothing of a random signal generated by a stochastic differential equation are investigated. Results on the asymptotic stability of a linear constant-parameter fixed-interval smoothing filter are obtained.

(3) System Identification: A particular stochastic modelling problem is solved. An Ito stochastic integral equation is used to mathematically model a black box having multiple inputs and multiple outputs. A new method for identifying system parameters is presented.

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Nonlinear Smoothing Theory						
Stability of Smoothing						
System Identification						
Stochastic Modelling						

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Chapter I

SUMMARY OF REPORT

This report contains four chapters. The first chapter gives an overall summary of the report, the remaining three chapters contain research work performed by individual investigators in the areas of (1) bounded state space control theory, (2) nonlinear smoothing theory, and (3) stochastic system modelling and identification. The following summaries give the results obtained in each area.

(1) Bounded State Space Control Theory

The most powerful tool in the study of the ordinary optimal control problem is the Pontryagin's maximum principle. Even in the bounded state space problem, a maximum principle is available; however, in this case it is not as useful as in the ordinary case because of two main reasons: The first is that the adjoint solution is not necessarily continuous, and hence it is necessary to find the jump condition at the points where discontinuities occur; however, there are no known methods available for determining the jump conditions. The second is that no methods are available for determining the jump points. It is the first difficulty with which this study is concerned. We show that at least in linear systems the adjoint solution can be continuous, thereby eliminating the difficulty. Furthermore in certain cases, as shown in the example in the report, this enables one to determine the times at which the response trajectory enters or leaves the boundary of the state constraint set and the number of switchings.

(2) Nonlinear Smoothing Theory

The problems of fixed-interval, fixed-point, and fixed-lag nonlinear smoothing are considered. Stochastic differential equations satisfied by the fixed-interval, fixed-point, and fixed-lag smoothing probability density functions are derived. Dynamical equations are developed for the minimum-variance fixed-interval, fixed-point, and fixed-lag smoothed estimates and also for their corresponding covariance matrices. By utilizing the nonlinear results obtained in this paper, it is shown that, not only the problems of fixed-interval, fixed-point, and fixed-lag linear smoothing with observations contaminated by Gauss-Markov (correlated) noise can immediately be solved, but also much insight of the general linear and nonlinear smoothing problems is obtained.

The stability properties associated with a constant-parameter fixed-interval linear smoothing filter are also investigated. It is shown that the fixed-interval smoothing filter is exponentially asymptotically stable. It is noted that the fixed-interval smoothing filter is an important filter for data smoothing purposes.

(3) Stochastic Modeling and Identification

A particular stochastic modeling problem is solved and a method is presented for generating a random process having a specified power spectral density matrix using "available" laboratory white noise.

An Ito stochastic integral equation is used to mathematically model a black box having multiple inputs and multiple outputs, where, when the black box has no inputs, the outputs have an ergodic correlation function matrix. The stochastic integral equation model is derived from the standpoint of measure-theoretic probability theory. Three methods of spectral factorization are demonstrated in the process of obtaining all the matrix parameters in the stochastic integral equation model. A numerical example is worked to illustrate the theory of modeling a black box having only outputs.

A new method of obtaining a realization corresponding to a given transfer function matrix is obtained as part of the particular stochastic modeling problem. In obtaining the new method of realizing a transfer function matrix, a method is given for putting a linear constant coefficient differential equation with multiple differentiated inputs into standard state variable form.

Chapter II

STUDY OF BOUNDED STATE SPACE

STUDY OF BOUNDED STATE SPACE

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INTRODUCTION

The most powerful tool in the study of the ordinary optimal control problem is the Pontryagin's maximum principle. Even in the bounded state space problem, a maximum principle is available; however, in this case it is not as useful as in the ordinary case because of two main reasons: The first is that the adjoint solution is not necessarily continuous, and hence it is necessary to find the jump condition at the points where discontinuities occur; however, there are no known methods available for determining the jump conditions. The second is that no methods are available for determining the jump points. It is the first difficulty with which this study is concerned. We show that at least in linear systems the adjoint solution can be continuous, thereby eliminating the difficulty. Furthermore in certain cases, as shown in the example below, this enables one to determine the times at which the response trajectory enters or leaves the boundary of the state constraint set and the number of switchings.

MAXIMUM PRINCIPLE

Consider the linear system

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$$

where $x \varepsilon R^n$ is an (nxl) state vector, $u \varepsilon R^m$ is an (mxl) control vector, and A(t) and B(t) are continuous matrices of compatible dimensions. Let Ω be a given convex compact control restraint set in R^m, and let S be a given closed convex state space constraint set in R^n. An admissible control u(t) on $[t_o, t_1]$ is a measurable function on $[t_o, t_1]$ with values in Ω, i.e., $u(t)\ \varepsilon \Omega$, such that the corresponding response x(t) is in S for all t on $[t_o, t_1]$, i.e., $x(t) \varepsilon S$, $t \varepsilon [t_o, t_1]$. Let G be a given closed convex target set in S, and let C(u) be the cost functional which is defined by

$$C(u) = g\ (x(t_1)\ ) + \int_{t_o}^{t_1} \{f(x(t),t) + h(u(t),t\}\, dt$$

where g(x) is a C^1 convex function in x, f(x,t) is a C^1 convex function in x for each t, and h(u,t) is a C^o convex function of u for each t on $[t_o, t_1]$.

The problem is to find an admissible control function u(t) on a given interval $[t_o, t_1]$ which steers the corresponding response x(t) from a given initial point $x(t_o) = x_o$ to the target set G at t_1, i.e., $x(t_1) \varepsilon G$, and minimizes the cost functional C(u). From the definition of admissible control it follows that x(t) is in S for all $t \varepsilon [t_o, t_1]$.

Define the function $f^o(x)$ by

$$f^o(x) = \begin{cases} 0 & \text{if} \quad x \in S \\ d(x,S) & \text{if} \quad x \notin S \end{cases}$$

where $d(x,S) = \min_{y \in S} |x-y|$, that is, $d(x,S)$ is the usual distance between the point x and the set S in the eucliden space R^n. Obviously $f^o(x)$ is a continuous convex function of x in R^n, and $\frac{\partial f^o}{\partial x}$ exists and is continuous everywhere in R^n except on the boundary S of the set S. If x is an interior point of S, then $\frac{\partial f^o}{\partial x} = 0$, for $f^o(x) = 0$ for all $x \in S$. Let

$$C_o(u) = \int_{t_o}^{t_1} f^o(x(t)) \, dt.$$

Then, since $f^o(x) \geq 0$, $C_o(u) \geq 0$ for all measurable control function $u(t)$ on $[t_o, t_1]$. Here $x(t)$ is the response of the system corresponding to the control $u(t)$. Furthermore, since $f^o(x) = 0$ if and only if $x \in S$, $C_o(u) = 0$ if and only if $x(t) \in S$ almost everywhere on $[t_o, t_1]$. Since $x(t)$ is a solution of the linear differential equation it is absolutely continuous. Therefore, $C_o(u) = 0$ if and only if $x(t) \in S$ for all $t \in [t_o, t_1]$, that is, $u(t)$ is an admissible control on $[t_o, t_1]$. Since $C_o(u) \geq 0$, $u(t)$ is an admissible control if and only if $u(t)$ minimizes the functional $C_o(u)$ and the minimum is zero. Let $\Omega(t_o, t_1)$ be the set of all measurable functions $u(t)$ on $[t_o, t_1]$ with their values in Ω, that is, $u(t) \in \Omega$ on $[t_o t_1]$.

We can now rephrase our problem as follows: Find a control function $u^*(t)$ in $\Omega(t_o t_1)$ which steers the corresponding response

$x^*(t)$ from the given initial point x_o at t_o to the target set G at t_1 and satisfies the following conditions:

(1) $C_o(u^*) \leq C_o(u)$ for all measurable functions u(t) in Ω which steers the corresponding response endpoint $x_u(t_1)$ to G.

(2) $C(u^*) \leq C(u)$ for all measurable functions u(t) in Ω such that $x_u(t_1) \varepsilon G$ and $C_o(u) = C_o(u^*)$.

If $C_o(u^*)$ is not zero then of course an optimal control does not exist. In other words, an optimal control always satisfies the conditions above.

This problem has been studied by Chyung in [1]. In the paper it was assumed that $\frac{\partial f^o}{\partial x}$ exists and is continuous everywhere in R^n. In the present case this condition is not satisfied, for $\frac{\partial f^o}{\partial x}$ does not exist on ∂S. Therefore the results in [1] cannot be applied directly to the present problem. However, if we let $\frac{\partial f^o}{\partial x}(x) = n(x)$ on the boundary ∂S of the state constraint set S then it can be shown that the results obtained in [1] is still valid to cover the present case. Here n(x) is a vector function of x which is defined on ∂S such that it is continuous at x if the boundary hypersurface ∂S is smooth at x and is exterior normal to the convex set S at x on ∂S. An example of such a function n(x) is the unit exterior normal vector to S at $x \varepsilon \partial S$. The normal vector always exists, for S is convex. If the boundary hypersurface ∂S is not smooth at x, then it may happen that there is more than one supporting hyperplane to S at x, and hence there is more than one exterior normal vector to S at x with different direction, i.e., there is more than one exterior normal unit vector to S at x. In this case, any one of the normal vectors may be chosen for $n(x)$. If the set S is defined by $s(x) \leq 0$, i.e., $S = \{x \varepsilon R^n | \; s(x) \leq 0\}$, the boundary ∂S is defined by $s(x) = 0$, and grad $s(x)$ exists and grad $s(x) \neq 0$ on ∂S; then, obviously, $n(x)$ may be chosen as $n(x) = \text{grad } s(x)$ on ∂S, for grad $s(x)$ is always exterior normal to the set S at $x \varepsilon \partial S$.

Since $n(x)$ is an exterior normal to S on ∂S at x if the response $x(t)$ lies on the boundary ∂S on an interval $[t_2, t_3)$ with positive measure it is clear that $n(x(t)) \dot{x}(t) = 0$ on $[t_2, t_3)$ whenever $\dot{x}$ exists. Combining this result with the results in [1], we then obtain the following.

First, let us consider the case when $g(x) = 0$, that is, the cost functional is given by

$$C(u) = \int_{t_o}^{t_1} \{ f + h\} \, dt.$$

Let an admissible control $u^*(t)$ with corresponding response $x^*(t)$ be an optimal control on $[t_o, t_1]$. Let $I \subset [t_o, t_1]$ be the set of time t at which $x^*(t)$ lies on the boundary ∂S of the state constraint set S, i.e., $I = \{t | t \varepsilon [t_o, t_1], \; x^*(t) \varepsilon \partial S\}$. If there exists a function $n(x)$ such that $n(x^*(t))$ is integrable on I, then with respect to this particular function $n(x)$, there exists a nontrivial continuous (1xn) vector solution $p(t)$ of the equation

$$\dot{x}^*(t) = A(t) \, x^*(t) + B(t) \, u^*(t), \quad x^*(t_o) = x_o$$

$$\dot{p}(t) = -p(t) \, A(t) - p_o \frac{\partial f^o}{\partial x} (x^*(t)) - p_1 \frac{\partial f}{\partial x} (x^*(t))$$

with the endpoint conditions either $x^*(t_1) \epsilon G$, p_o = constant <0, p_1 = constant ≤ 0, $p(t_1) = 0$ or $x^*(t_1) \epsilon \partial G$, p_o = constant ≤ 0, p_1 = constant ≤ 0, $p(t_1)$ is interior normal to G at $x^*(t_1)$, such that

$$p_1 h(u^*(t),t) + p(t)\ B(t)\ u^*(t) = \max_{u \epsilon \Omega} \{p_1 h(u,t) + p(t)B(t)u\}$$

almost everywhere on $[t_o, t_1]$.

If $g(x) \neq 0$, that is, $C(u) = g(x(t_1)) + \int_{t_o}^{t_1} \{f + h\}dt$, but $G = R^n$ (free endpoint), then the above result is still valid except that the endpoint conditions should be replaced by simpler conditions

$$p(t_1) = -\text{grad}\ g(x^*(t_1)),\ p_o \leq 0,\ p_1 < 0.$$

Conversely if a measurable control $u(t)$ in Ω on $[t_o, t_1]$ satisfies the above maximal condition and $p_o \neq 0$, $p_1 \neq 0$, then it is an optimal control.

CONCLUSIONS

We have derived necessary conditions and sufficient conditions for an optimal control. It is also shown that the adjoint solution is a continuous function. This eliminates the difficulty of determining the so called "jump conditions". However, the second difficulty, the determination of the points where the response either enters or leaves the boundary ∂S, still remains unsolved.

EXAMPLE

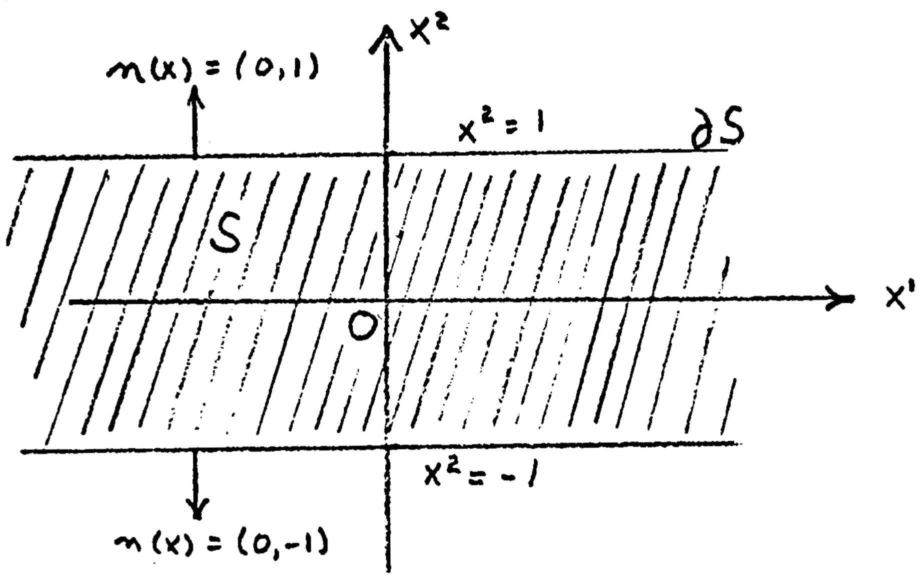
Consider the well-known, and the simplest, time optimal control problem of the system $\ddot{x}=u$. The problem is to find a control which steers the response x(t) of the system $\ddot{x}=u$ from $x(0)=-2, \dot{x}(0)=0$ to the origin $x=0$, $\dot{x}=0$ in minimum time with the restrictions $|u|\leq 1$ and $|\dot{x}|\leq 1$. Using vector notation,

$$\begin{pmatrix}\dot{x}^1\\ \dot{x}^2\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x^1\\ x^2\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix} \qquad , \begin{pmatrix}x^1(0)\\ x^2(0)\end{pmatrix} = \begin{pmatrix}-2\\ 0\end{pmatrix}$$

and the cost functional is

$$C(u) = \int_0^T 1\, dt \; , \quad \text{that is,} \quad f=1, \quad h=0,$$

where T is the minimum time.



Let n(x) be the exterior normal unit vector to S at $x\varepsilon\partial S$, that is,

$$n(x) = \begin{cases} (0,1) & \text{if } x^2 = 1 \\ (0,-1) & \text{if } x^2 = -1 \end{cases}$$

Then

$$\frac{\partial f^\circ}{\partial x}(x) = \begin{cases} (0,0) & \text{if} \quad |x^2|<1 \\ (0,1) & \text{if} \quad x^2 = +1 \\ (0,-1) & \text{if} \quad x^2 = -1 \end{cases}$$

and if $x(t)$ is on the boundary ∂S, that is, $x^2(t) = \pm 1$, then

$$n(x(t))\, \dot{x}(t) = (0,\pm 1) \cdot [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)] = \pm u(t) = 0.$$

Therefore, if $x(t)$ is on ∂S then the optimal control $u(t)=0$, that is, $u(t)=0$ if $x^2(t)=\pm 1$, except the moment when $x(t)$ leaves the boundary ∂S. Let $p(t) = (p^1(t),\ p^2(t)\ )$. Then, rememberi g $f=1$,

$$(\dot{p}^1, \dot{p}^2) = (p^1, p^2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - p_0 \frac{\partial f^\circ}{\partial x}, \qquad p_0 \leq 0$$

Thus,

$$\dot{p}^1 = 0$$

$$\dot{p}^2 = \begin{cases} -p^1 & \text{if} \quad |x^2|<1 \\ -p^1 - p_0 & \text{if} \quad x^2 = +1 \\ -p^1 + p_0 & \text{if} \quad x^2 = -1 \end{cases}$$

Therefore,

$$p^1 = \text{constant}$$

$$p^2(t) = -p^1 t + C_o \qquad \text{if } |x^2(t)| < 1$$

$$p^2(t) = -(p^1 + p_o)t + C_1 \qquad \text{if } x^2(t) = 1$$

$$p^2(t) = (-p^1 + p_o)t + C_2 \qquad \text{if } x^2(t) = -1$$

Notice that p_o is a fixed constant throughout the entire interval [0,T].

From the maximal condition, an optimal control must satisfy the condition, remembering h = 0,

$$p(t)\ B(t)\ u(t) = \max_{|u| \le 1} p(t)\ B(t)\ u = \max_{|u| \le 1} p^2(t)u.$$

Therefore

$$u(t) = \text{sgn}\ p^2(t)$$

and u(t) = 1 or -1 unless $p^2 = 0$. On the other hand u(t)=0 if x(t) is on ∂S, that is, $x^2(t) = \pm 1$. Thus $p^2(t) = 0$ when $x^2(t) = \pm 1$, i.e., $x(t) \varepsilon \partial S$.

Now $x^2(0) = 0$. Therefore the response x(t) starts from an inner point of S. Now suppose the response reaches the boundary of the state constraint set at t_1, that is, $x^2(t_1) = \pm 1$, and $|x^2(t)| < 1$ for $t < t_1$. Assume (arbitrarily) that $x^2(t_1) = 1$, it stays on the boundary until $t_2 \ge t_1$, that is, $x^2(t) = 1$ for $t_1 \le t < t_2$, and then the response leaves the boundary at t_2, that is, $|x^2(t)| < 1$, for $t > t_2$. Then on $[t_1, t_2)$

$$p^2(t) = -(p^1 + p_o)\ t + C_1 = 0$$

and so $p^1 + p_o = 0$, $C_1 = 0$, or $p_o = -p^1$. Since $p_o \le 0$, $p^1 \ge 0$. On $[0, t_1]$,

$$p^2(t) = -p^1 t + C_o.$$

Since $p^2(t)$ is continuous,

$$-p^1 t_1 + C_o = -(p^1+p_o)t_1+C_1 = 0.$$

But then, since $p^1 \geq 0$, unless $p_o=p^1=0$,

$$p^2(t) = -p^1 t + C_o > 0$$

on $[0,t_1]$, and so $u(t)=1$ on $[0,t_1]$. Therefore t_1 is the first possible switching time (from 1 to 0). For $t > t_1$, again since

$$p^2(t_2)=-p^1 t_2+C_o=0$$

and $p^1 > 0$ (unless $p_o=0$),

$$p^2(t) = -p^1 t + C_o < 0$$

and so $u(t) = -1$. Since $u(t) = \text{sgn } p^2 (t)$, the only time a switching can occur is at the time when $p^2(t) = 0$. But $p^2(t)<0$ for all $t>t_2$. Thus once the response leaves the boundary ∂S there can be no more switchings. Hence the maximum number of switching is two, and it occurs in the order of $1 \to 0 \to -1$.

If $x^2(t_1) = -1$ instead of +1, then we obtain the same result except that the switching sequence is now $-1 \to 0 \to 1$.

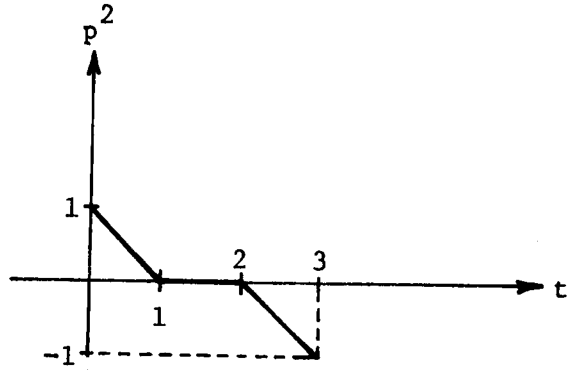
Therefore an optimal control can have at most two switchings, and the sequence is either $1 \to 0 \to -1$ or $-1 \to 0 \to 1$. This important and extremely useful result is due to that fact that $p(t)$ is continuous and p_o is constant on the whole interval $[0,T]$. This result is obtained for the first time in this report, and it cannot be obtained by any other previous work.

Once the above fact is known the remaining derivation is easy. The result is given below:

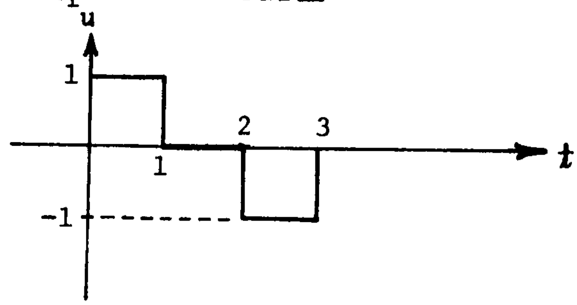
$P_o = -1,\ p^1 = 1,\ t_1=1,\ t_2=2,\ T=3,$

$p^1(t) = 1$

$$p^2(t) = \begin{cases} -t+1 & 0 \le t \le 1 \\ 0 & 1 \le t \le 2 \\ -t+2 & 2 \le t \le 3 \end{cases}$$

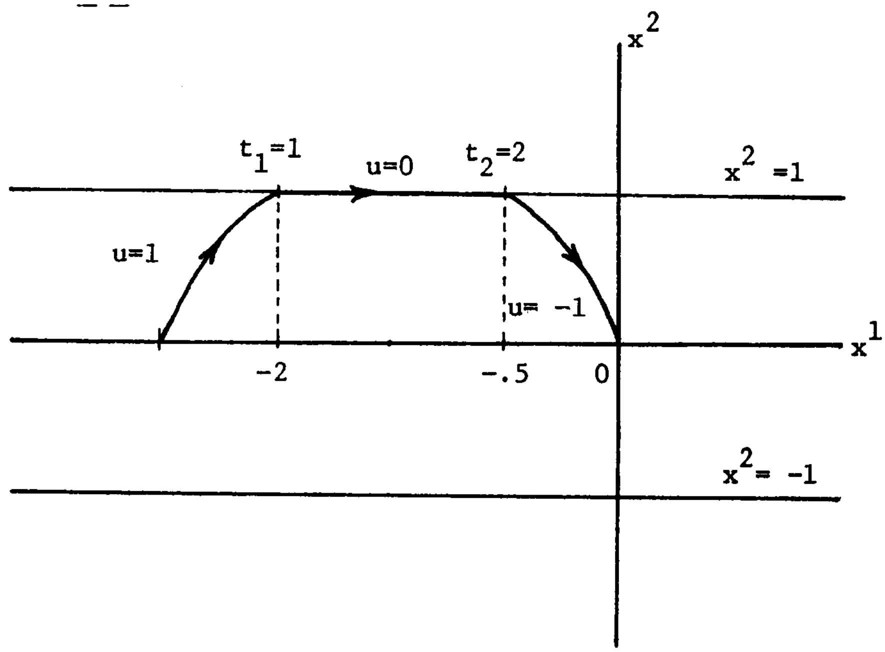


$$u(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ -1 & 2 \le t \le 3 \end{cases}$$



$$x^1(t) = \begin{cases} \frac{1}{2}t^2 - 2, & 0 \le t \le 1 \\ t - \frac{5}{2}, & 1 \le t \le 2 \\ -\frac{1}{2}t^2 + 3t - \frac{9}{2}, & 2 \le t \le 3 \end{cases}$$

$$x^2(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \\ 3-t & 2 \le t \le 3 \end{cases}$$



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Chapter III

NONLINEAR SMOOTHING THEORY

(a) NONLINEAR SMOOTHING THEORY, WITH APPLICATIONS TO CORRELATED NOISE PROCESSES*

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ABSTRACT

The problems of fixed-interval, fixed-point, and fixed-lag nonlinear smoothing are considered. Stochastic differential equations satisfied by the fixed-interval, fixed-point, and fixed-lag smoothing probability density functions are derived. Dynamical equations are developed for the minimum-variance fixed-interval, fixed-point, and fixed-lag smoothed estimates and also for their corresponding covariance matrices. By utilizing the nonlinear results obtained in this paper, it is shown that, not only the problems of fixed-interval, fixed-point, and fixed-lag linear smoothing with observations contaminated by Gauss-Markov (correlated) noise can immediately be solved, but also much insight of the general linear and nonlinear smoothing problems is obtained.

I. INTRODUCTION:

The theory of linear and nonlinear filtering [1]-[3], linear and nonlinear prediction [1], [4]-[5], linear and nonlinear smoothing [4], [6]-[7] for stochastic processes with observations contaminated by Gaussian white-noise disturbances is well established. The generalization of the filtering theory to processes where the observations contain Gauss-Markov (correlated) noise is an important problem, and was apparently first considered by Bryson and Johansen [9] for the linear continuous-time systems. Since the pioneering work of Bryson and Johansen, various results in linear smoothing for correlated noise have been obtained [3], [10].

In this paper, the problem of linear and nonlinear smoothing for stochastic processes with observations contaminated by Gauss-Markov noise will be considered. The following three smoothing problems will be solved, namely (a) fixed-interval smoothing, (b) fixed-point smoothing, and (c) fixed-lag smoothing.

II. PROBLEM STATEMENT:

Let a dynamical system be described by the following nonlinear Ito's stochastic differential equation

$$dx(t)=f(x(t),t)dt+G(t)d\xi(t),x(0)=x_o. \tag{1.a}$$

The noisy observations on x(t) are obtained via a nonlinear channel,

$$dz(t) = a(z(t),t)dt + h(x(t),t)dt + d\eta(t),\ z(0) = z_o \tag{1.b}$$

In eq. (1), $x(t)$ is the n-dimensional state vector, $z(t)$ is the m-dimensional output vector, and $G(t)$ is an nxr matrix; $f(x,t)$, $a(z,t)$, and $h(x,t)$ are vector valued functions defined for their arguments; the initial condition $x(0)$ is a Gaussian random variable (vector) with mean $\bar{x}(0)$ and covariance $P(0)$, and is assumed to be independent of $\xi(t)$ and $\eta(t)$ for all

*Paper presented at the Second Annual Pittsburgh Conference on Modeling and Simulation, University of Pittsburgh, March 29-30, 1971.

$t \geq 0$; $\xi(t)$ and $\eta(t)$ are, respectively, r-vector and m-vector zero-mean Wiener processes with covariance matrices $Q(t)$, $R(t)$, and $C(t)$ satisfying

$$E[\ \xi(t)\xi^T(t)] = \int_o^t Q(t)d\tau,$$
$$E[\eta(t)\eta^T(t)] = \int_o^t R(t)d\tau \qquad (2)$$
$$E[\xi(t)\eta^T(t)] = \int_o^t C(t)d\tau.$$

It will be shown in Section IV that the cross-covariance matrix $C(t)$, which is often assumed to be zero, is of great importance in solving the smoothing problems when the observations contain Gauss-Markov (correlated) noise.

III. SOLUTIONS OF THE NONLINEAR SMOOTHING PROBLEM

We wish to obtain the equations of evolution of the smoothing probability density function $p(x,t|Z(s))$, the smoothed estimate $\hat{x}(t|s)$ and the smoothed covariance matrix $P(t|s)$ of $x(t)$ defined by, for $t<s$,

$$p(x,t|Z(s) \triangleq \frac{\partial^n P(x(t)\leq x|Z(s))}{\partial x_1\ \partial x_2\ \ldots\ \partial x_n} \qquad (3)$$

$$\hat{x}(t|s) \triangleq \int_{R^n} xp(x,t|Z(s))dx \qquad (4)$$

$$P(t|s) \triangleq \int_{R^n} [x-\hat{x}(t|s)][x-\hat{x}(t|s)]^T p(x,t|Z(s))dx \qquad (5)$$

where $Z(s) \triangleq \{z(t),\ 0\leq\tau\leq s\}$.

For s fixed, implying that the observation interval $[o,s]$ is fixed, $\hat{x}(t|s)$ is a fixed-interval smoothed estimate of $x(t)$ given $Z(s)$; for t fixed, $x(t)$ is an unknown constant random variable and $\hat{x}(t|s)$ is a fixed-point smoothed estimate of $x(t)$ conditioned on a growing record of observed data $Z(s)$; and for $s=t+\lambda$, where $\lambda>0$ is fixed implying that the signal $x(t)$ lags the observation $z(s)$ by a constant λ units of time, $\hat{x}(t|t+\lambda)$ is a fixed-lag smoothed estimate of $x(t)$ given $Z(s)$. This classification for the smoothed estimates is due to Meditch [4].

The main results in this section are summarized in the following theorems.

Theorem 1 (Fixed-Interval Smoothing):

The fixed-interval smoothing density $p(x,t|Z(s))$ satisfies, with respect to t for s fixed ($t<s$),

$$d_t p(x,t|Z(s)) = \frac{p(x,t|Z(s))}{p(x,t|Z(t))} Lp(x,t|Z(t))dt - p(x,t|Z(t))L^*[\frac{p(x,t|Z(s))}{p(x,t|Z(t))}]dt-$$
$$-(\frac{\partial}{\partial x})^T\{G(t)C(t)R^{-1}(t)[dz(t)-a(z(t),t)dt-h(x,t)dt]p(x,t|Z(s))\}-$$
$$-\frac{dt}{2}\, tr[G(t)C(t)R^{-1}(t)C^T(t)G^T(t)(\frac{\partial}{\partial x})(\frac{\partial}{\partial x})^T p(x,t|Z(s))] \qquad (6)$$

$$p(x,t|Z(s))|_{t=s} = p(x,s|Z(s)),$$

where

$$Lp = -\left(\frac{\partial}{\partial x}\right)^T [f(x,t)p] + \frac{1}{2}\mathrm{tr}[G(t)\bar{Q}(t)G^T(t)\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)^T p] \quad (7)$$

$$L^* p = f^T(x,t)\frac{\partial p}{\partial x} + \frac{1}{2}\mathrm{tr}[G(t)\bar{Q}(t)G^T(t)\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)^T p] \quad (8)$$

$$\bar{Q}(t) = Q(t) - C(t)R^{-1}(t)C^T(t). \quad (9)$$

Theorem 2 (Fixed-Point Smoothing):

The fixed-point smoothing density $p(x,t|Z(s))$ satisfies, with respect to s for t fixed $(t<s)$,

$$d_s p(x,t|Z(s)) = p(x,t|Z(s))[\bar{h}(s|s) - \hat{h}(s|s)]^T R^{-1}(s)[dz(s) - a(z(s),s)ds - \hat{h}(s|s)ds]$$

$$p(x,t|Z(s))\big|_{s=t} = p(x,t|Z(t)), \quad (10)$$

where

$$\hat{h}(s|s) \triangleq \int_{R^n} h(y,s)p(y,s|Z(s))dy \quad (11)$$

$$\bar{h}(s|s) = \int_{R^n} h(y,s)p(y,s|x(t)=x;Z(s))dy. \quad (12)$$

Theorem 3 (Fixed-lag Smoothing):

The fixed-lag smoothing density $p(x,t|Z(s))$ satisfies with respect to t for $s=t+\lambda$, where $\lambda>0$ is fixed,

$$d_t p(x,t|Z(s)) = \frac{p(x,t|Z(s))}{p(x,t|Z(t))} Lp(x,t|Z(t))dt - p(x,t|Z(t))L^*\left[\frac{p(x,t|Z(s))}{p(x,t|Z(t))}\right]dt -$$

$$-\left(\frac{\partial}{\partial x}\right)^T \{G(t)C(t)R^{-1}(t)[dz(t) - a(z(t),t)dt - h(x,t)dt]p(x,t|Z(s))\} -$$

$$-\frac{dt}{2}\mathrm{tr}[G(t)C(t)R^{-1}(t)C^T(t)G^T(t)\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)^T p(x,t|Z(s))] +$$

$$+p(x,t|Z(s))[\bar{h}(s|s) - \hat{h}(s|s)]^T R^{-1}(s)[dz(s) - a(z(s),s)ds - \hat{h}(s|s)ds]$$

$$p(x,t|Z(s))\Big|_{\substack{t=o\\ s=\lambda}} = p(x_o,0|Z(\lambda)), \quad (13)$$

where $\hat{h}(s|s)$ and $\bar{h}(s|s)$ are as defined in eqs.(11) and (12).

The equations of evolution of the smoothed estimates $\hat{x}(t|s)$ and the smoothed covariance matrices $P(t|s)$ are omitted due to the lack of space. These equations can be obtained by utilizing eqs. (6)-(13), together with an application of Ito's Lemma.

IV. APPLICATIONS TO CORRELATED NOISE PROCESSES

We shall consider linear systems described by the following vector stochastic differential equation,

$$dx(t) = F(t)x(t)dt + G(t)d\xi(t)$$

$$x(0) = x_o, \quad x\epsilon R^n, \quad \xi\epsilon R^r, \quad (14)$$

with noisy observations obtained via a linear channel,

$$z(t) = M(t)\, x(t) + v(t), \quad (z,v)\epsilon R^m \quad (15)$$

where $v(t)$ is the Gauss-Markov (correlated) noise satisfying,

$$dv(t) = A(t)v(t) + B(t)\, d\alpha(t)$$

$$v(0) = v_o, \quad v\epsilon R^m, \quad \alpha\epsilon R^p, \quad (16)$$

$$E[\alpha(t)\alpha^T(t)] = \int_0^t N(t)d\tau$$
$$E[\xi(t)\alpha^T(t)] = \int_0^t S(t)d\tau. \quad (17)$$

Equations (14)-(17) constitute the problem of linear smoothing for correlated noise. The various smoothed estimates $\hat{x}(t|s)$ and covariance matrices $P(t|s)$ will be obtained by using the results presented in Theorems 1-3 in Section III.

Now by applying Ito's Lemma to eq.(15) and utilizing eqs.(14) and (16), we obtain the following stochastic differential equation satisfied by $z(t)$

$$dz(t) = A(t)z(t)dt + H(t)x(t)dt + d\eta(t) \quad (18)$$
$$z(0) = M(0)x(0) + v(0),$$

where

$$H(t) \triangleq \dot{M}(t) + M(t)F(t) - A(t)M(t), \quad (19)$$

$$d\eta(t) \triangleq M(t)G(t)d\xi(t) + B(t)\,d\alpha(t), \quad (20)$$

$$E[\eta(t)\eta^T(t)] = \int_0^t R(\tau)d\tau, \quad (21)$$

$$E[\xi(t)\eta^T(t)] = \int_0^t C(\tau)d\tau, \quad (22)$$

$$R(t) \triangleq M(t)G(t)Q(t)G^T(t)M^T(t) + B(t)N(t)B^T(t) + M(t)G(t)S(t)B^T(t) + B(t)S^T(t)G^T(t)M^T(t), \quad (23)$$

$$C(t) \triangleq Q(t)G^T(t)M^T(t) + S(t)B^T(t). \quad (24)$$

Utilizing eqs.(14) and (18), and Theorems 1-3, we can now summarize the main results for linear smoothing with observations contaminated by Gauss-Markov (correlated) noise in the following theorems.

Theorem 4 Fixed-Interval Smoothing:

The fixed-interval smoothing density $p(x,t|Z(s))$, fixed-interval smoothed estimate $\hat{x}(t|s)$, and fixed-interval smoothed covariance $P(t|s)$ of $x(t)$ given $Z(s)$ satisfy, respectively, with respect to t for s fixed ($t<s$),

$$d_t p(x,t|Z(s)) = -\left(\frac{\partial}{\partial x}\right)^T\{[\bar{F}(t)+G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)]x\,p(x,t|Z(s))\}dt - \frac{dt}{2}\mathrm{tr}[G(t)Q(t)G^T(t)\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)^T p(x,t|Z(s))] - p(x,t|Z(s))[x-\hat{x}(t|s)]^T P^{-1}(t|s)G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)\hat{x}(t|t)dt - \left[\frac{\partial p(x,t|Z(s))}{\partial x}\right]^T G(t)C(t)R^{-1}(t)[dz(t)-A(t)z(t)dt], \quad (25)$$

$$d_t\hat{x}(t|s) = \bar{F}(t)\,\hat{x}(t|s)dt + G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)\,[\hat{x}(t|s)-\hat{x}(t|t)]\,dt + G(t)C(t)R^{-1}(t)\,[dz(t)-A(t)z(t)dt], \quad (26)$$

$$\frac{dP(t|s)}{dt} = [\bar{F}(t) + G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)]\,P(t|s) + P(t|s)\,[\bar{F}(t) + G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)]^T - G(t)\bar{Q}(t)G^T(t), \quad (27)$$

where

$$\bar{F}(t) \triangleq F(t) - G(t)C(t)R^{-1}(t)\,H(t) \quad (28)$$

$\bar{Q}(t)$ is as given by eq.(9), and the boundary conditions for (25), (26) and (27) are, respectively, the filtered density $p(x,s|Z(s))$, filtered estimate $\hat{x}(s|s)$, and filtered covariance $P(s|s)$:

$$
\begin{aligned}
p(x,t|z(s))|_{t=s} &= p(x,s|Z(s)) \\
\hat{x}(t|s)|_{t=s} &= \hat{x}(s|s) \\
P(t|s)|_{t=s} &= P(s|s).
\end{aligned}
\tag{29}
$$

It should be noted that eqs.(26)and(27) have been obtained in a recent paper by Fujita and Fukao [11]. However, our approach here is different from that in [11].

Theorem 5 (Fixed-Point Smoothing):

The fixed-point smoothing density $p(x,t|Z(s))$, fixed-point smoothed estimate $\hat{x}(t|s)$, and fixed-point smoothed covariance $P(t|s)$ of $x(t)$ given $Z(s)$ satisfy, respectively, with respect to s for t fixed $(t<s)$,

$$
\begin{aligned}
d_s p(x,t|Z(s)) &= p(x,t|Z(s))\,[\bar{x}(s|s)-\hat{x}(s|s)]^T H^T(s) R^{-1}(s)\cdot \\
&\quad \cdot [dz(s)-A(s)z(s)ds-H(s)\hat{x}(s|s)ds] \\
p(x,t|Z(s))|_{s=t} &= p(x,t|Z(t)),
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
d_s\hat{x}(t|s) &= P(t|t)\Psi^T(s,t)H^T(s)R^{-1}(s)[dz(s)-A(s)z(s)ds-H(s)\hat{x}(s|s)ds] \\
\hat{x}(t|s)\,|_{s=t} &= \hat{x}(t|t),
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\frac{dP(t|s)}{ds} &= -P(t|t)\Psi^T(s,t)H^T(s)R^{-1}(s)H(s)\Psi(s,t)P(t|t) \\
P(t|s)\,|_{s=t} &= P(t|t),
\end{aligned}
\tag{32}
$$

where
$$
\bar{x}(s|s) = \int_{R^n} y p(y,s|x(t) = x;\ Z(s))dy,
\tag{33}
$$

and $\Psi(s,t)$ is the transition matrix associated with

$[F(s) - P(s|s)H^T(s)R^{-1}(s)H(s)]$.

Theorem 6 (Fixed-Lag Smoothing):

The fixed-lag smoothing density $p(x,t|Z(s))$, fixed-lag smoothed estimate $\hat{x}(t|s)$, and fixed-lag smoothed covariance $P(t|s)$ of $x(t)$ given $Z(s)$ satisfy, respectively, with respect to t for $s=t+\lambda$, where $\lambda>0$ is fixed,

$$
\begin{aligned}
d_t p(x,t|Z(s)) &= -\left(\frac{\partial}{\partial x}\right)^T \{[\bar{F}(t)+G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)]x p(x,t|Z(s))dt- \\
&\quad -\frac{dt}{2}\,\mathrm{tr}\,[G(t)Q(t)G^T(t)\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)^T p(x,t|Z(s))]- \\
&\quad -p(x,t|Z(s))[x-\hat{x}(t|s)]^T P^{-1}(t|s)G(t)\bar{Q}(t)G^T(t)G^T(t)P^{-1}(t|t)\hat{x}(t|t)dt- \\
&\quad -\left[\frac{\partial p(x,t|Z(s))}{\partial x}\right]^T G(t)C(t)R^{-1}(t)\,[dz(t)-A(t)z(t)dt]+ \\
&\quad +p(x,t|Z(s))[\bar{x}(s|s)-\hat{x}(s|s)]^T H^T(s)R^{-1}(s)[dz(s)-A(s)z(s)ds-H(s)\hat{x}(s|s)ds] \\
p(x,t|Z(s))\Big|_{\substack{t=0\\ s=\lambda}} &= p(x,0|Z(\lambda)),
\end{aligned}
\tag{34}
$$

$$d_t\hat{x}(t|s)=\bar{F}(t)\hat{x}(t|s)dt + G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)\ [\hat{x}(t|s)-\hat{x}(t|t)]dt +$$
$$+G(t)C(t)R^{-1}(t)[dz(t)-A(t)z(t)dt] +$$
$$+P(t|t)\Psi^T(s,t)H^T(s)R^{-1}(s)[dz(s)-A(s)z(s)ds-H(s)\hat{x}(s|s)ds] \tag{35}$$
$$\hat{x}(t|s)\Big|_{\substack{t=0\\ s=\lambda}} = \hat{x}(0|\lambda),$$

$$\frac{dP(t|s)}{dt} = [\bar{F}(t) + G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)]\ P(t|s) +$$
$$+P(t|s)[\bar{F}(t)+G(t)\bar{Q}(t)G^T(t)P^{-1}(t|t)]-G(t)\bar{Q}(t)G^T(t)$$
$$-P(t|t)\Psi^T(s,t)H^T(s)R^{-1}(s)H(s)\Psi(s,t)P(t|t)$$
$$P(t|s)\Big|_{\substack{t=0\\ s=\lambda}}=P(0|\lambda), \tag{36}$$

where $\bar{F}(t)$ and $\bar{Q}(t)$ are as given in eqs.(28) and (9), and $\Psi(s,t)$ is the transition matrix associated with $[F(s)-P(s|s)\ H^T(s)R^{-1}(s)H(s)]$.

REMARKS:

It is obvious from eq.(22) and Theorem 4 and 6 that the nxm matrix C(t) plays an important role in linear fixed-interval and fixed-lag smoothing when the observations are contaminated by Gauss-Markov (correlated) noise. The matrix C(t) was a cross-covariance matrix in eq.(2) and Theorems 1 and 3; however, it is not entirely a cross-covariance matrix in eq.(22). The results in Theorems 1 and 3 may not be used to solve the problems of linear fixed-interval and fixed-lag smoothing for correlated noise if C(t) had been assumed zero in eq.(2).

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(b) ON THE STABILITY OF FIXED-INTERVAL LINEAR SMOOTHING, WITH APPLICATION TO COLORED NOISE

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ABSTRACT

The stability properties associated with a constant-parameter fixed-interval linear smoothing filter are investigated. It is shown that the fixed-interval smoothing filter is exponentially asymptotically stable. It is noted that the fixed-interval smoothing filter is an important filter for data smoothing purposes.

I. INTRODUCTION

The stability properties associated with the Kalman-Bucy filter are well known[1]. The stability properties associated with the constant-parameter fixed-interval linear smoothing filter will be presented here. The stability of a filter is of utmost importance because without it, the estimate generated by the filter is useless. It is noted that the fixed-interval smoothing filter is an important filter for data smoothing purposes.

The linear time-invariant system is assumed to be modelled by the following differential equation

$$\dot{x}(t) = Fx(t) + G\xi(t)$$
$$x(0) = x_o \sim N[\bar{x}_o, P_x(0)], \tag{1.a}$$

with noisy observations via a linear channel,

$$\dot{z}(t) = Az(t) + Hx(t) + \eta(t)$$
$$z(0) = z_o \sim N[\bar{z}_o, P_z(0)]. \tag{1.b}$$

In (1), x, z, ξ, and η are n, m, r, and m vectors, respectively; F, G, A, and H are constant matrices with appropriate dimensions; $N[\bar{y}, P_y]$ denotes a normal distribution with mean $\bar{y}$ and constant covariance P_y; and $\xi(t)$ and $\eta(t)$ are zero-mean Gaussian white-noise sources with covariances

$$E[\xi(t)\xi^T(\tau)] = Q\ \delta(t-\tau)$$
$$E[\eta(t)\eta^T(\tau)] = R\ \delta(t-\tau) \tag{2}$$
$$E[\xi(t)\eta^T(\tau)] = C\ \delta(t-\tau).$$

The initial conditions x(0) and z(0) are assumed to be independent of $\xi(t)$ and $\eta(t)$ for all $t \geq 0$. It is also assumed that[1] $Q \geq 0$, $R > 0$, and $(Q - CR^{-1}C^T) \geq 0$.

II. STABILITY OF FIXED-INTERVAL SMOOTHING

Given the observations $Z(T) \triangleq \{z(\tau),\ 0 \leq \tau \leq T\}$ on the fixed interval [0,T], it can be shown [2] [3] that the fixed-interval smoothed estimate $\hat{x}(t|T) = E[x(t)|Z(T)]$ of x(t) satisfies with respect to t, for $t < T$,

$$\frac{d}{dt}\hat{x}(t|T) = \hat{F}\hat{x}(t|T) + \hat{G}\hat{G}^T\Pi^{-1}[\hat{x}(t|T) - \hat{x}(t|t)] + \\ + GCR^{-1}[\dot{z}(t) - Az(t)] \tag{3}$$

$$\hat{x}(t|T)|_{t=T} = \hat{x}(T|T),$$

where

$$\hat{F} \triangleq F - CR^{-1}H$$

$$\hat{G}\hat{G}^T \triangleq G(Q - CR^{-1}C^T)G^T \geq 0.$$

The estimate $\hat{x}(t|t)$ is the Kalman-Bucy filtered estimate of x(t) given $Z(t) \triangleq \{z(\tau),\ 0 \leq \tau \leq t\}$ [1]; Π is the constant covariance matrix associated with $\hat{x}(t|t)$, and Π^{-1} satisfies,

$$\Pi^{-1}\hat{F} + \hat{F}^T\Pi^{-1} + \Pi^{-1}\hat{G}\hat{G}^T\Pi^{-1} - H^TR^{-1}H = 0. \tag{4}$$

1 A>B ($A \geq B$) means A-B is positive definite (positive semidefinite).

As is evident from the terminal condition $\hat{x}(T|T)$, equation (3) is to be integrated backward in time from $t = T$ to $t = 0$. It is convenient to set $s = T - t$, so that (3) may be written as, in terms of the backward-time variable $\hat{x}_B(s)$,

$$\dot{\hat{x}}_B(s) = -(\hat{F} + \hat{G}\hat{G}^T\Pi^{-1})\ \hat{x}_B(s) + \hat{G}\hat{G}^T\Pi^{-1}\hat{x}(T-s|T-s) - GCR^{-1}\ [\dot{z}(T-s) - Az(T-s)] \tag{5}$$

$$\hat{x}_B(0) = \hat{x}(T|T).$$

The stability result on (5), and therefore on (3), is given in the following theorem.

Theorem: Suppose the pair $(\hat{F},\hat{G})$ is completely controllable, and the pair $(\hat{F},\hat{H})$ is completely observable, where $\hat{H}^T\hat{H} \triangleq H^TR^{-1}H$. Then the fixed-interval smoothing filter (5) is exponentially asymptotically stable, i.e., there exist positive constants k_1 and k_2 such that $||\Phi_D(t_2,t_1)|| \le k_1 \cdot \exp[-(t_2-t_1)k_2]$, where $\Phi_D(t_2,t_1)$ is the transition matrix associated with $D \triangleq \hat{F} + \hat{G}\hat{G}^T\Pi^{-1}$.

Remark: The complete controllability and complete observability assumptions also guarantee the exponential asymptotic stability of the constant-parameter Kalman-Bucy filter, and the position definiteness of Π and Π^{-1} [1].

Proof of Theorem: Consider the homogenous part of (5),

$$\dot{y}(s) = -(F + \hat{G}\hat{G}^T\Pi^{-1})\ y(s), \tag{6}$$

and the Lyapunov function,

$$V(y) = y^T(s)\Pi^{-1}y(s). \tag{7}$$

From (4), (6) and (7), it follows that,

$$\dot{V}(y) = -y^T(s)[\Pi^{-1}\hat{G}\hat{G}^T\Pi^{-1} + \hat{H}^T\hat{H}]\ y(s) \tag{8}$$

so that $\dot{V}(y)$ is nonpositive, but not necessarily negative definite. However, if $\dot{V}(y)$ does not vanish identically along any nonzero trajectory determined by (6), then exponential asymptotic stability follows [4].

Now assume that $\dot{V}(y) \equiv 0$ but $y(0) \neq 0$. From (8), $\hat{G}^T\Pi^{-1}y(s) \equiv 0$ and $\hat{H}y(s) \equiv 0$, and so from (6),

$$\dot{y}(s) = -\hat{F}y(s). \tag{9}$$

Equation (9) yields $y(s) = \Phi_{\hat{F}}(0,s)\ y(0)$, so that

$$\hat{H}y(s) = \hat{H}\Phi_{\hat{F}}(0,s)\ y(0) \equiv 0. \tag{10}$$

Since $(\hat{F},\hat{H})$ is completely observable, the n columns of $\hat{H}\Phi_{\hat{F}}(0,\cdot)$ are linearly independent on $[0,T]$; therefore (10) implies that $y(0) = 0$ which contradicts the assumption that $y(0) \neq 0$. Hence (6) and therefore (5) are exponentially asymptotically stable.

III. APPLICATION TO COLORED OBSERVATION NOISE

The result of the stability theorem in the previous section can be applied to fixed-interval smoothing where the observations contain colored or time-correlated noise processes.

Consider (1a) with observations given by

$$z(t) = Mx(t) + v(t). \tag{11}$$

The process $\{v(t)\}$ is the colored or time-correlated noise determined by,

$$\begin{aligned} \dot{v}(t) &= Av(t) + B\alpha(t) \\ v(0) &= v_o \sim N[\bar{v}_o, P_v(0)], \end{aligned} \tag{12}$$

where $\alpha(t)$ is a k-vector zero-mean Gaussian white-noise source with covariance $E[\alpha(t)\alpha^T(\tau)] = R_1\delta(t-\tau)$ and $E[\xi(t)\alpha^T(\tau)] = C_1\delta(t-\tau)$. It is assumed that $v(0)$ and $\alpha(t)$ are independent for all $t \geq 0$. It can be shown that the correlation matrix $K(t_2,t_1) \triangleq E[v(t_2)v^T(t_1)]$ is given by, for all $t_2 \geq t_1$,

$$\begin{aligned} K(t_2,t_1) &= \Phi_A(t_2,t_1)\, K(t_1,t_1) \\ &= \Phi_A(t_2,t_1)\, [\Phi_A(t_1,0)\, K(0,0)\Phi_A^T(t_1,0) + \\ &\quad \int_0^{t_1} \Phi_A(t_1,\tau)BR_1B^T\Phi_A^T(t_1,\tau)\, d\tau], \end{aligned} \tag{13}$$

where $K(0,0) = P_v(0) - \bar{v}_o\bar{v}_o^T$. A block diagram of the system is shown in Fig. 1.

Now from (1a), (11) and (12), it follows that,

$$\dot{z}(t) = Az(t) + Hx(t) + \eta(t), \tag{14}$$

where

$$H \triangleq MF - AM \tag{15}$$

$$\eta(t) \triangleq MG\xi(t) + B\alpha(t) \tag{16}$$

$$\begin{aligned} E[\eta(t)\eta^T(\tau)] &= [MGQG^TM^T + BR_1B^T + MGC_1B^T + BC_1^TG^TM^T]\ \delta(t-\tau) \\ &\triangleq R\delta(t-\tau) \end{aligned} \tag{17}$$

$$\begin{aligned} E[\xi(t)\eta^T(\tau)] &= [QG^TM^T + C_1B^T]\ \delta(t-\tau) \\ &\triangleq C\ \delta(t-\tau) \end{aligned} \tag{18}$$

As before, it is assumed that $R > 0$ and $(Q - CR^{-1}C^T) \geq 0$.

Equations (1a) and (14) constitute a standard form for the smoothing problem. The fixed-interval smoothing filter equation in the present case has exactly the same form as (3) or (5), i.e.,

$$\begin{aligned} \dot{\hat{x}}_B(s) = &-(\hat{F} + \hat{G}\hat{G}^T\Pi^{-1})\ \hat{x}_B(s) + \hat{G}\hat{G}^T\Pi^{-1}\hat{x}(T-s|T-s) - \\ &GCR^{-1}[\dot{z}(T-s) - Az(T-s)], \end{aligned} \tag{19}$$

where the H, R, and C matrices are, however, now given by (15), (17), and (18), respectively.

By the stability theorem in the previous section, it follows that if $(\hat{F},\hat{G})$ is completely controllable and $(\hat{F},\hat{H})$ is completely observable, where $\hat{F} \triangleq F - CR^{-1}H$, $\hat{G}\hat{G}^T \triangleq G(Q - CR^{-1}C^T)G^T$, and $\hat{H}^T\hat{H} \triangleq H^TR^{-1}H$, then the fixed-interval smoothing filter (19) is exponentially asymptotically stable. It should be noted that in (5), the matrix C is a cross-covariance matrix between the system noise $\xi(t)$ and the observation noise $\eta(t)$; however, in (19) C is not just a cross-covariance matrix as is evident from (18). Hence, the matrix C is of theoretical significance as the results in the previous section can be applied to the fixed-interval smoothing problem in which the observations contain colored noise only when $C \neq 0$ in (3) or (5).

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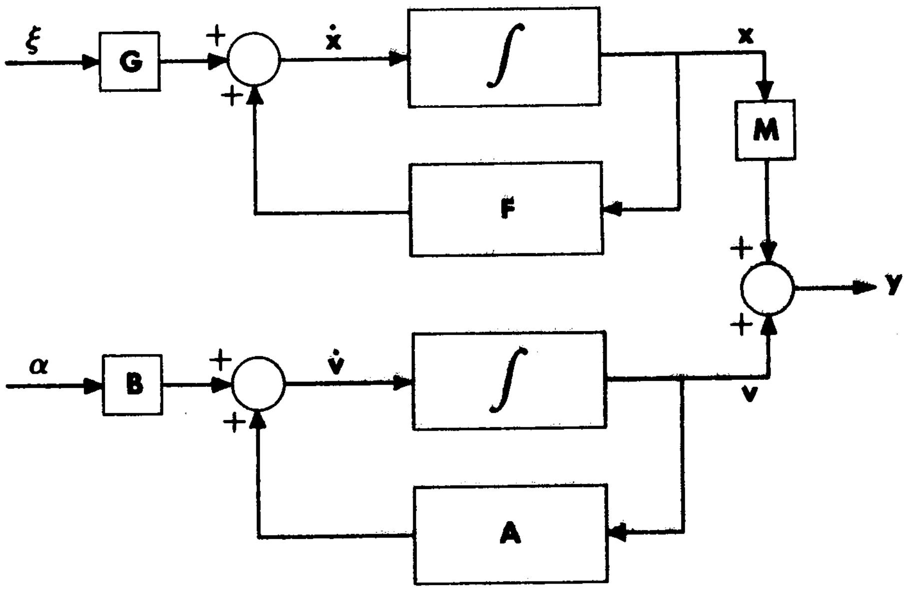

Figure 1

Chapter IV

STOCHASTIC MODELLING AND IDENTIFICATION

STOCHASTIC MODELING AND IDENTIFICATION

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ABSTRACT

A particular stochastic modeling problem is solved and a method is presented for generating a random process having a specified power spectral density matrix using "available" laboratory white noise W(t)[1]. ("Available" means that between W(t) and W(t+τ) there is a fixed correlation for all t; fixed correlations would be encountered in using the congruence method of generating pseudo-random numbers by computer.)

An Ito stochastic integral equation is used to mathematically model a black box having multiple inputs and multiple outputs, where, when the black box has no inputs, the outputs have an ergodic correlation function matrix. The stochastic integral equation model is derived from the standpoint of measure-theoretic probability theory. Three methods of spectral factorization are demonstrated in the process of obtaining all the matrix parameters in the stochastic integral equation model. A numerical example is worked to illustrate the theory of modeling a black box having only outputs.

A new method for obtaining a realization corresponding to a given transfer function matrix is obtained as part of the particular stochastic

modeling problem. In obtaining the new method of realizing a transfer function matrix, a method is given for putting a linear constant coefficient differential equation with multiple differentiated inputs into standard state variable form.

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