UNIVERSITY OF CINCINNATI
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DYNAMIC CHARACTERISTICS OF A VARIABLE-MASS FLEXIBLE MISSILE

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Dynamics of a Two-Stage Variable-Mass
Flexible Rocket

by

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Abstract

The dynamic characteristics of a two-stage slender elastic body is investigated. The first stage, containing a solid-fuel rocket, posses variable mass while the second stage, envisioned as a flexible case, contains packaged instruments of constant mass. The mathematical formulation is in terms of vector equations of motion transformed by a variational principle into sets of scalar differential equations in terms of generalized coordinates. Solutions to the complete equations are obtained numerically by means of finite difference techniques. The problem has been programmed in the FORTRAN IV language and solved on an IBM 360/50 computer. Results for limited cases are presented showing the nature of the solutions.

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1. Introduction

During the past decade attempts have been made to describe the motion exhibited by a flexible missile throughout its flight phase. In the process, a large variety of mathematical models have been investigated. The treatment of the missile as a rigid-body of time dependent mass has been investigated by Grubin[1*] and Dryer[2] among others, and by Leitmann[3] and Meriam[4], who also consider the effects of a relative shift in the center of gravity of the body. The ballistic trajectories of spin-and fin-stabilized rigid bodies are treated in the book by Davis, Follin, and Blitzer[5].

Effort has also been devoted to the analysis of an elastic body subjected to longitudinal acceleration. Seide[6] has treated the effect of both a compressive and a tensile force on the frequencies and mode shapes of transverse vibration of a continuous slender body. Others, such as Beal[7], have been concerned with the problem of buckling instabilities of a uniform beam subjected to an end thrust as well as the change in the natural frequencies of such a system. These investigations regard the mass of the body as constant in time.

A series of reports by Miles, Young and Fowler[8] offers a comprehensive treatment of a wide range of subjects associated with the dynamics of missiles, including fuel sloshing. Again the mass variation is not accounted for.

Attempts have been made to consider simultaneously the mass variation and missile flexural elasticity by investigators such as Birnbaum[9] and Edelen[10]. Both were concerned with solid-fuel rockets and neither of them included the axial elasticity of the missile. Price[11], on the other hand, concerned himself with the internal flow in a solid-fuel rocket and ignored the vehicle motion.

A study by Meirovitch and Wesley[12,13] attempts to synthesize the problem of rocket dynamics by accounting for mass variation, rigid body translation and rotation, as well as axial and transverse elastic deformations. Later studies by Meirovitch[14] and

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* See publications listed in References.

Meirovitch and Bankovskis[15] consider a variable-mass two-stage rocket where the effects of discrete masses as well as axial and flexural elasticity are included. The work presented here gives a summary of the results obtained in References 14 and 15 as well as some numerical results not reported there.

This investigation is concerned with the dynamic characteristics of a two-stage, flexible missile. Of the two stages, only the first one posseses variable mass as it consists of a solid-fuel booster; the second stage consists of a flexible missile shell containing a certain number of instruments which are approximated by discrete masses attached to the missile casing by means of springs and dampers.

2. Mathematical Formulation

2.1 General Remarks on Systems of Changing Composition

Due to the mass decrease during powered flight, a rocket can be looked upon as a system of changing composition. For such a system the equations of motion can be derived by examining the change in the identity of matter within a given volume with time, where the shape of the control volume is assumed fixed. Since the system composition changes it is no longer proper to equate the time-derivative of the sum of momenta associated with the particles to the sum of the time derivatives, because the summation at different times is taken over different sets of particles.

Consider first a control volume fixed in space. This control volume may be a certain volume of the rocket. The procedure then is to write the force equation in the form $\underset{\sim}{F} = \dot{\underset{\sim}{p}}$*, where the rate of change of the momentum $\underset{\sim}{p}$ is derived by a limiting process consisting of determining $\underset{\sim}{p}$ at two different instants a time interval Δt apart, dividing the difference of the two values by Δt and letting $\Delta t \to 0$. In so doing we ensure that the same total mass is involved, although at one time it is entirely within the volume and at the other time part of the mass is outside as shown in Figure 1.

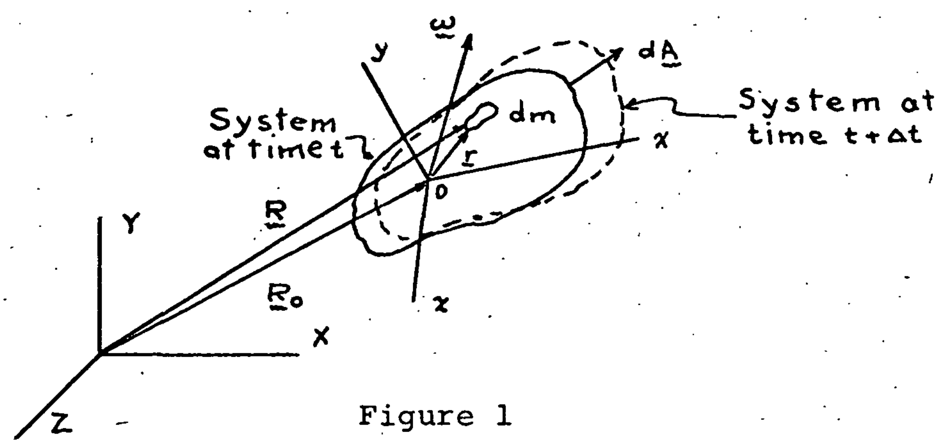

Figure 1

Using the above analysis the resulting force equation for an inertial control volume has the form (Reference 15, page 96)

$$\underset{\sim}{F} = \underset{\sim}{F}_S + \underset{\sim}{F}_B = \int_M \underset{\sim}{a}\, dM = \frac{\partial}{\partial t}\int_{CV} \underset{\sim}{v}\, dM + \int_{CS} \underset{\sim}{v}(\rho \underset{\sim}{v} \cdot d\underset{\sim}{A}) \tag{2.1}$$

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* A wavy line under a symbol represents a vector quantity.

where $\underset{\sim}{F}_S$ and $\underset{\sim}{F}_B$ are the resultant surface and body forces, respectively, $\underset{\sim}{a}$ is the acceleration of the mass elements dM relative to the control volume, $\underset{\sim}{v}$ is the velocity of that element, ρ is the mass density, and $d\underset{\sim}{A}$ is a vector normal to the control surface at any given point whose magnitude is equal to the area element dA of the control surface. If the control volume is fixed with respect to a set of body axes, x,y,z translating and rotating relative to the inertial system X,Y,Z, then the force equation becomes

$$\underset{\sim}{F}_S + \underset{\sim}{F}_B = \int_M [\underset{\sim}{a}_0 + \dot{\underset{\sim}{v}} + 2\underset{\sim}{\omega} \times \underset{\sim}{v} + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})]\, dM$$

$$= \int_M [\underset{\sim}{a}_0 + 2\underset{\sim}{\omega} \times \underset{\sim}{v} + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})]\, dM + \frac{\partial}{\partial t}\int_{CV} \underset{\sim}{v}\, dM + \int_{CS} \underset{\sim}{v}(\rho \underset{\sim}{v} \cdot d\underset{\sim}{A}) \qquad (2.2)$$

where $\underset{\sim}{a}_0$ is the acceleration of the origin 0 of the body axes x,y,z, $\underset{\sim}{\omega}$ is the angular velocity vector of these axes relative to the inertial space, and $\underset{\sim}{r}$ is the position of the mass element dM relative to the body axes. Introducing the notation

$$\begin{aligned} \underset{\sim}{F}_C &= -2\underset{\sim}{\omega} \times \int_{CV} \underset{\sim}{v}\, dM \\ \underset{\sim}{F}_U &= -\frac{\partial}{\partial t}\int_{CV} \underset{\sim}{v}\, dM \\ \underset{\sim}{F}_R &= -\int_{CS} \underset{\sim}{v}(\rho \underset{\sim}{v} \cdot d\underset{\sim}{A}) \end{aligned} \qquad (2.3)$$

where $\underset{\sim}{F}_C$, $\underset{\sim}{F}_U$, and $\underset{\sim}{F}_R$ are a set of equivalent forces referred to as the Coriolis force, the force due to the flow unsteadiness, and the reactive force, respectively, we can write Eq. (2.2) in the form

$$\underset{\sim}{F}_S + \underset{\sim}{F}_B + \underset{\sim}{F}_C + \underset{\sim}{F}_U + \underset{\sim}{F}_R = \int_M [\underset{\sim}{a}_0 + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})]\, dM \qquad (2.4)$$

where the expression enclosed by square brackets in the integrand is recognized as the rigid body accleration of the mass element.

2.2 Equations of Motion for Case Element and Discrete Masses.

To derive the equations of motion of a flexible, solid-fuel missile it will prove convenient to work with a vehicle element of unit length comprising the missile casing, the unburned fuel, and the hot gases flowing relative to the first two, as shown in Figure 2. We shall assume that every point of the casing and unburned fuel element has the same motion and the same is true for the corresponding gas element. Hence, denoting quantities relating to the casing and unburned fuel by the subscript c and the quantities pertaining to the gas flow by f, we can use the analogy with Eq. (2.2), and write for the element of Figure 2b

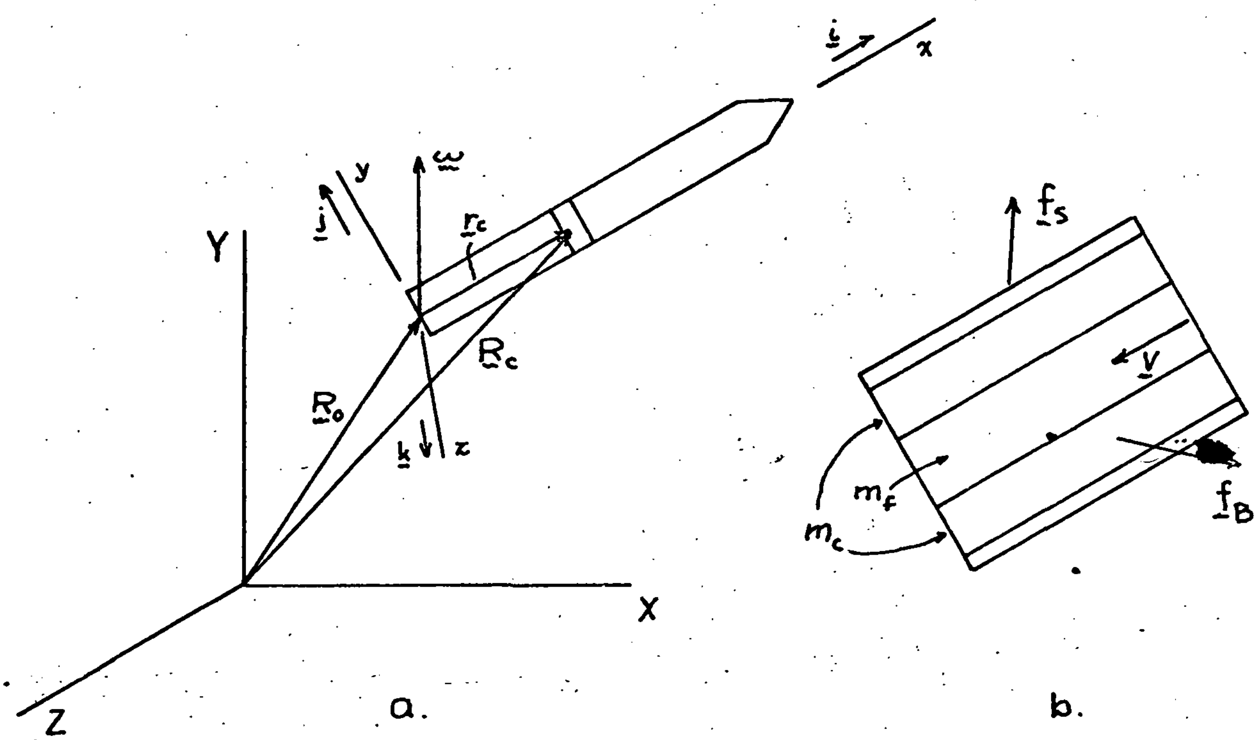


Figure 2

$$\underset{\sim}{f}_s + \underset{\sim}{f}_B = m_c[\underset{\sim}{a}_o + \dot{\underset{\sim}{v}}_c + 2\underset{\sim}{\omega} \times \underset{\sim}{v}_c + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_c + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_c)]$$

$$+ m_f[\underset{\sim}{a}_o + \dot{\underset{\sim}{v}}_f + 2\underset{\sim}{\omega} \times \underset{\sim}{v}_f + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_f + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_f)] \tag{2.5}$$

where m_c and m_f are the corresponding mass elements. But the mass centers of the case and gas elements coincide, so that $\underline{r}_f = \underline{r}_c$. Moreover, denoting by $\underline{v}$ the velocity of flow relative to the case, we have $\underline{v}_f = \underline{v}_c + \underline{v}$, from which it follows that

$$\underline{f}_S + \underline{f}_B = m[\underline{a}_0 + \dot{\underline{v}}_c + 2\underline{\omega} \times \underline{v}_c + \dot{\underline{\omega}} \times \underline{r}_c + \underline{\omega} \times (\underline{\omega} \times \underline{r}_c)] + \dot{\underline{v}} m_f + 2\underline{\omega} \times \underline{v} m_f \tag{2.6}$$

where $m = m_c + m_f$ is the combined mass. By analogy with the preceding results, Eq. (2.6) can be written in the form

$$\underline{f}_S + \underline{f}_B + \underline{f}_C + \underline{f}_U + \underline{f}_R = m\ddot{\underline{R}}_c \tag{2.7}$$

in which

$$\begin{aligned} \underline{f}_C &= -2\underline{\omega} \times \underline{v} m_f \\ \underline{f}_U &= -\frac{\partial}{\partial t}(m_f \underline{v}) \\ \underline{f}_R &= -\frac{\partial}{\partial x}(m_f v \underline{v}) \end{aligned} \tag{2.8}$$

are the corresponding equivalent distributed forces, where the latter assumes one-dimensional flow. Moreover

$$\ddot{\underline{R}}_c = \underline{a}_0 + \dot{\underline{v}}_c + 2\underline{\omega} \times \underline{v}_c + \dot{\underline{\omega}} \times \underline{r}_c + \underline{\omega} \times (\underline{\omega} \times \underline{r}_c) \tag{2.9}$$

is the absolute acceleration of the case element.

It is assumed that the gas flow takes place along the missile longitudinal axis

$$\underline{v}(x,t) = -v(x,t)\,\underline{i} \tag{2.10}$$

with the possible exception of the nozzle, which implies that the flow is not affected by any transverse elastic deformations.

Equation (2.7) is valid for a rocket element in the first stage, namely in the interval $0 \le x \le L_1$, inside which there is mass flow present. The validity of the equation can be extended to the entire length of the missile, including the segment $L_1 \le x \le L$, by writing it in the form

$$\underset{\sim}{f}_s + \underset{\sim}{f}_B + (\underset{\sim}{f}_c + \underset{\sim}{f}_v + \underset{\sim}{f}_R)[h(x) - h(x - L_1)] = m\ddot{\underset{\sim}{R}}_c \tag{2.11}$$

where $h(x - x_0)$ is a spatial unit step function applied at $x = x_0$. We recall that m is time-dependent in the interval $0 \leq x \leq L_1$ and constant for $L_1 \leq x \leq L$.

Newton's second law for the discrete masses M_i, contained in the second stage, yields simply

$$\underset{\sim}{F}_{Si} + \underset{\sim}{F}_{Bi} = M_i \ddot{\underset{\sim}{R}}_i \tag{2.12}$$

where $\underset{\sim}{F}_{Si}$ anf $\underset{\sim}{F}_{Bi}$ are surface and body forces, respectively, and $\ddot{\underset{\sim}{R}}_i$ is the acceleration of mass M_i.

The gas flow must satisfy the continuity equation

$$m_f v = -\int_x^{L_1} \dot{m}\, d\xi \tag{2.13}$$

where $\dot{m}$ is the rate of mass decrease per unit length due to burning and ξ is a dummy variable of integration.

3. Differential Equations in Terms of Generalized Coordinates

3.1 Coordinates Defining the Motion of the Missile

Equations (2.11) and (2.12) necessitate the calculation of the accelerations $\ddot{\underset{\sim}{R}}_c$ and $\ddot{\underset{\sim}{R}}_i$. Actually, we shall work with Lagrange type equations of motion instead of Eqs. (2.11) and (2.12) and, to this end, we must calculate the system kinetic energy which is a function of the velocities $\dot{\underset{\sim}{R}}_c$ and $\dot{\underset{\sim}{R}}_i$ rather than the accelerations. In the following we shall derive expressions for $\dot{\underset{\sim}{R}}_c$ and $\dot{\underset{\sim}{R}}_i$.

The motion $\dot{\underset{\sim}{R}}_c$ of a missile element, as shown in Figure 2a, is composed of the rigid-body motion, as defined by the motion of the body axes x,y,z, and the motion of the element relative to the body axes. The motion of the system x,y,z is defined by the velocity vector $\dot{\underset{\sim}{R}}_0$ of the origin 0 and the angular velocity $\underset{\sim}{\omega}$ of these axes with respect to the inertial space X_0, Y_0, Z_0. Assuming planar motion, we can let the z-axis be parallel to Z_0 at all times, choose the origin 0 to coincide with the aft end of the missile, and denote by X(t) and Y(t) the two components of $\underset{\sim}{R}_0$ along the body axes x and y, respectively. The third rigid body coordinate $\Theta(t)$ represents the inclination of the body longitudinal axis x relative to the inertial axis X_0, as shown in Figure 3. The elastic motion

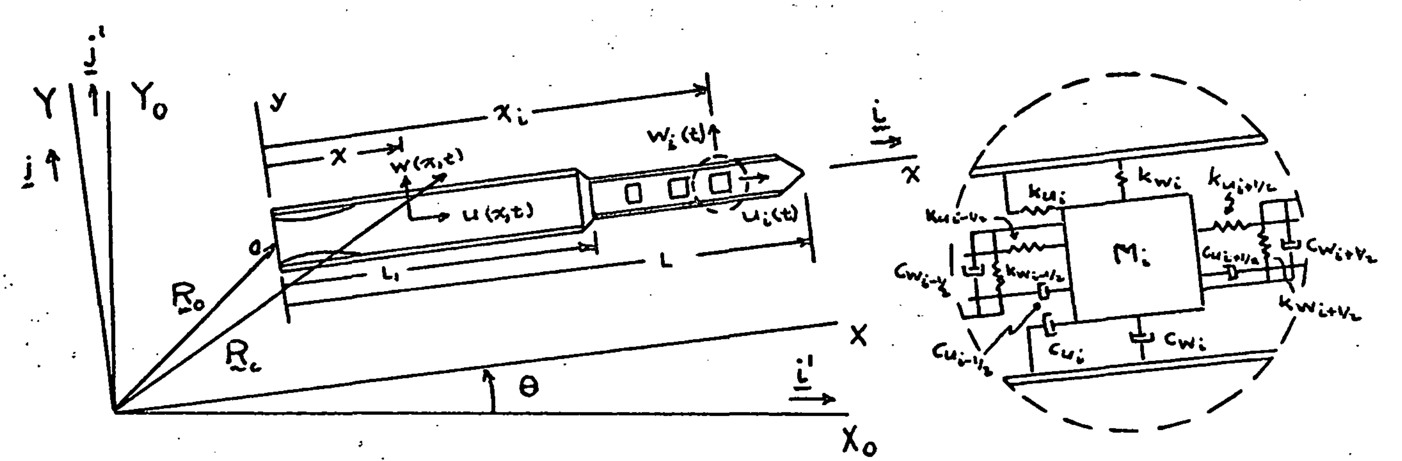


Figure 3

of a case element relative to axes x,y is assumed to take the form of the axial displacement $u(x,t)$ and the transverse displacement $w(x,t)$. The displacement components of the discrete masses M_i along the same directions are $u_i(t)$ and $w_i(t)$, respectively. The case element and the discrete masses, when at rest, have the positions x, $y = y(x)$ and x_i, $y_i = y(x_i)$, respectively, with respect to the body axes which implies that the center of the case element or the discrete masses may be offset relative to the x-axis. Hence the position of the case element relative to the inertial space can be written

$$\underline{R}_c = \underline{R}_o + \underline{r}_c = (X + x + u)\underline{i} + (Y + y + w)\underline{j} \tag{3.1}$$

whereas the positions of masses M_i have the form

$$\underline{R}_i = \underline{R}_o + \underline{r}_i = (X + x_i + u_i)\underline{i} + (Y + y_i + w_i)\underline{j} \qquad i = 1,2,---,n \tag{3.2}$$

where $\underline{i}$ and $\underline{j}$ are unit vectors rotating together with the body axes.

Because the motion is planar, there is only one component of rotation

$$\underline{\omega} = \dot{\theta}\,\underline{k} \tag{3.3}$$

which is also the angular velocity of the unit vectors $\underline{i}$ and $\underline{j}$. It follows that

$$\frac{d\underline{i}}{dt} = \underline{\omega} \times \underline{i} = \dot{\theta}\underline{j} \;, \quad \frac{d\underline{j}}{dt} = \underline{\omega} \times \underline{j} = -\dot{\theta}\,\underline{i} \;. \tag{3.4}$$

Since $\underline{R}_c$ and $\underline{R}_i$ are expressed in terms of components along the body axes, the velocity vector of the case element becomes

$$\begin{aligned} \dot{\underline{R}}_c &= \dot{\underline{R}}_{crel} + \underline{\omega} \times \underline{R}_c \\ &= [\dot{X} + \dot{u} - \dot{\theta}(Y + y + w)]\underline{i} + [\dot{Y} + \dot{w} + \dot{\theta}(X + x + u)]\underline{j} \end{aligned} \tag{3.5}$$

where $\dot{\underline{R}}_{crel}$ denotes the time derivative of $\underline{R}_c$ assuming that the unit vectors $\underline{i}$ and $\underline{j}$ are fixed. In a similar way, the velocity vectors for the masses M_i have the form

$$\dot{\underline{R}}_i = [\dot{X} + \dot{u}_i - \dot{\theta}(Y + y_i + w_i)]\underline{i} + [\dot{Y} + \dot{w}_i + \dot{\theta}(X + x_i + u_i)]\underline{j} \qquad i = 1,2,---,n \;. \tag{3.6}$$

3.2 The Variational Principle

Equation (2.11) represents the force equation of motion of a missile element and Eqs. (2.12) are the force equations for the discrete masses contained in the second stage. These equations can be used to derive the moment equations of motion about any desired point. Certain manipulations can lead to all the system differential equations. Instead of following this route, however, we shall derive all the system differential equations by means of a variational principle akin to Hamilton's principle[17]. To this end, let us write Eq. (2.11) in the form

$$\hat{\underset{\sim}{E}}(x,t) = \underset{\sim}{f}_S + \underset{\sim}{f}_B + (\underset{\sim}{f}_C + \underset{\sim}{f}_U + \underset{\sim}{f}_R)[h(x) - h(x - L_1)] - m\ddot{\underset{\sim}{R}}_c \tag{3.7}$$

which is merely an expression of dynamic equilibrium for the missile element. Some of the components of the surface force $\underset{\sim}{f}_S$ are present only over the segment $0 \le x \le L_1$, but this fact will be accounted for later. The equations of motion for the discrete masses can be given a similar treatment by writing Eqs. (2.12) as follows

$$\hat{\underset{\sim}{E}}(x_i,t) = (\underset{\sim}{F}_{Si} + \underset{\sim}{F}_{Bi} - M_i\ddot{\underset{\sim}{R}}_i)\,\delta(x - x_i) \tag{3.8}$$

where the spatial Dirac delta function $\delta(x - x_i)$ imparts to $\hat{\underset{\sim}{E}}(x_i,t)$ units of distributed force so as to make it compatible with $\hat{\underset{\sim}{E}}(x,t)$. We note that Eq. (3.8) contains no forces associated with the gas flow.

The virtual work density associated with the missile element has the expression

$$\delta\hat{W}_c(x,t) = \hat{\underset{\sim}{E}}(x,t)\cdot\delta\underset{\sim}{R}_c \tag{3.9}$$

whereas the one associated with the discrete masses can be written

$$\delta\hat{W}(x_i,t) = \hat{\underset{\sim}{E}}(x_i,t)\cdot\delta\underset{\sim}{R}_i \tag{3.10}$$

where $\delta\underset{\sim}{R}_c$ and $\delta\underset{\sim}{R}_i$ are arbitrary virtual displacements associated with $\underset{\sim}{R}_c$ and $\underset{\sim}{R}_i$, respectively, satisfying

$$\delta\underset{\sim}{R}_c = \delta\underset{\sim}{R}_i = 0 \qquad \text{at } t = t_0, t_1. \tag{3.11}$$

The variational principle then can be stated in the form

$$\int_{t_0}^{t_1}\int_0^L [\,\delta\hat{W}_c(x,t) + \delta\hat{W}(x_i,t)]\,dx\,dt = 0\,. \tag{3.12}$$

Equation (3.12) will be used to derive the Lagrange equations of motion of the system.

We note that the terms $\delta\hat{W}_c(x,t)$ and $\delta\hat{W}(x_i,t)$ do not imply the existence of functions $\hat{W}_c(x,t)$ and $\hat{W}(x_i,t)$ so that these terms should not be interpreted as representing the variations of $\hat{W}_c(x,t)$ and $\hat{W}(x_i,t)$, respectively, but mere infinitesimal expressions. In the case of the potential energy density function $\hat{PE}$ and the kinetic energy density function $\hat{KE}$ the symbols $\delta\hat{PE}$ and $\delta\hat{KE}$ do imply the variations of these functions.

3.3 The Lagrange - Type Equations of Motion

To derive the equations of motion of the system, it will prove convenient to consider the various terms entering into expressions (3.9) and (3.10) individually. This will also provide us with the opportunity to examine the nature of the surface and body forces.

The distributed surface force $\underset{\sim}{f}_S$ on the rocket element generally consists of forces external to the vehicle, such as the aerodynamic forces, and internal forces. The latter are due to stresses within the casing material throughout the entire length of the vehicle as well as gas pressure throughout the first stage, including the nozzle. The surface forces $\underset{\sim}{F}_{Si}$ on the mass M_i consist of restoring forces due to the spring action of the supports. Although forces resulting from viscous damping are also surface forces exerted on the discrete masses, we will find it convenient to treat them separately, namely by means of the Rayleigh's dissipation function. In view of this, we can write

$$\underset{\sim}{f}_S\cdot\delta\underset{\sim}{R}_c + \underset{\sim}{F}_{Si}\,\delta(x-x_i)\cdot\delta\underset{\sim}{R}_i = -\delta\hat{PE} + \delta\hat{W}_A + \delta\hat{W}_P\,[h(x)-h(x-L_1)] \tag{3.13}$$

where $\delta\hat{PE}$ is the variation of the potential energy density associated with internal stresses in the casing material and the spring, supporting the discrete masses, whereas $\delta\hat{W}_A$ and $\delta\hat{W}_P$ are virtual work densities associated with the aerodynamic forces and gas pressure, respectively.

The body forces are simply due to the Earth's gravitational field. Their effect can be written in the form

$$\underset{\sim}{f}_B \cdot \delta \underset{\sim}{R}_c + \underset{\sim}{F}_{Bi}\, \delta(x-x_i) \cdot \delta \underset{\sim}{R}_i = \delta \hat{W}_B \tag{3.14}$$

where $\delta\hat{W}_B$ is the corresponding virtual work density.

The forces $\underset{\sim}{f}_C$, $\underset{\sim}{f}_U$, and $\underset{\sim}{f}_R$ are all associated with the gas flow in the first stage. The expression

$$(\underset{\sim}{f}_C + \underset{\sim}{f}_U + \underset{\sim}{f}_R)[h(x)-h(x-L_1)] \cdot \delta \underset{\sim}{R}_c = (\delta\hat{W}_C + \delta\hat{W}_U + \delta\hat{W}_R)[h(x)-h(x-L_1)] \tag{3.15}$$

represents the virtual work densities due to the Coriolis force, the force due to the flow unsteadiness, and the reactive force, respectively.

Finally the terms involving the acceleration lead to

$$\begin{aligned} -m\ddot{\underset{\sim}{R}}_c \cdot \delta \underset{\sim}{R}_c - M_i \ddot{\underset{\sim}{R}}_i\, \delta(x-x_i) \cdot \delta \underset{\sim}{R}_i &= \tfrac{1}{2} m\, \delta(\dot{\underset{\sim}{R}}_c \cdot \dot{\underset{\sim}{R}}_c) + \tfrac{1}{2} M_i\, \delta(\dot{\underset{\sim}{R}}_i \cdot \dot{\underset{\sim}{R}}_i)\, \delta(x-x_i) \\ &+ \dot{m}\dot{\underset{\sim}{R}}_c [h(x)-h(x-L_1)] \cdot \delta \underset{\sim}{R}_c - \frac{d}{dt}(m\dot{\underset{\sim}{R}}_c \cdot \delta \underset{\sim}{R}_c) - \frac{d}{dt}[M_i \dot{\underset{\sim}{R}}_i\, \delta(x-x_i) \cdot \delta \underset{\sim}{R}_i] \\ &= \delta\hat{KE} + \dot{m}\dot{\underset{\sim}{R}}_c [h(x)-h(x-L_1)] \cdot \delta \underset{\sim}{R}_c - \frac{d}{dt}(m\dot{\underset{\sim}{R}}_c \cdot \delta \underset{\sim}{R}_c) - \frac{d}{dt}[M_i \dot{\underset{\sim}{R}}_i\, \delta(x-x_i) \cdot \delta \underset{\sim}{R}_i] \end{aligned} \tag{3.16}$$

where $\hat{KE}$ is the kinetic energy density.

Introducing the Lagrangian density

$$\hat{L} = \hat{KE} - \hat{PE} \tag{3.17}$$

substituting Eqs. (3.13) through (3.17) into Eq. (3.12), and using conditions (3.11), we obtain

$$\begin{aligned} \int_{t_0}^{t_1} \int_0^L \Big\{ \delta\hat{L} + \delta\hat{W}_A + (\delta\hat{W}_P + \delta\hat{W}_C + \delta\hat{W}_U + \delta\hat{W}_R)[h(x)-h(x-L_1)] \\ + \delta\hat{W}_B + \dot{m}\dot{\underset{\sim}{R}}_c [h(x)-h(x-L_1)] \cdot \delta \underset{\sim}{R}_c \Big\}\, dx\, dt = 0 \end{aligned} \tag{3.18}$$

which can be used to derive all the equations of motion.

To perform the operations specified by Eq. (3.18), we assume that the Lagrangian density has the functional form

$$\hat{L}=\hat{L}(X,Y,\theta,u,w,\dot{X},\dot{Y},\dot{\theta},\dot{u},\dot{w},u',w',w'',u_i,w_i,\dot{u}_i,\dot{w}_i) \tag{3.19}$$

where primes denote differentiations with respect to x and the subscript i takes on values 1,2,---, n. Moreover, the Lagrangian is simply

$$L=\int_0^L \hat{L}\,dx \tag{3.20}$$

The variation of the Lagrangian density can be written

$$\begin{aligned}\delta\hat{L} &= \frac{\partial\hat{L}}{\partial X}\delta X+\frac{\partial\hat{L}}{\partial Y}\delta Y+\frac{\partial\hat{L}}{\partial\theta}\delta\theta+\frac{\partial\hat{L}}{\partial u}\delta u+\frac{\partial\hat{L}}{\partial w}\delta w+\frac{\partial\hat{L}}{\partial\dot{X}}\delta\dot{X}\\ &+\frac{\partial\hat{L}}{\partial\dot{Y}}\delta\dot{Y}+\frac{\partial\hat{L}}{\partial\dot{\theta}}\delta\dot{\theta}+\frac{\partial\hat{L}}{\partial\dot{u}}\delta\dot{u}+\frac{\partial\hat{L}}{\partial\dot{w}}\delta\dot{w}+\frac{\partial\hat{L}}{\partial u'}\delta u'+\frac{\partial\hat{L}}{\partial w'}\delta w'\\ &+\frac{\partial\hat{L}}{\partial w''}\delta w''+\sum_{i=1}^{n}\left(\frac{\partial\hat{L}}{\partial u_i}\delta u_i+\frac{\partial\hat{L}}{\partial w_i}\delta w_i+\frac{\partial\hat{L}}{\partial\dot{u}_i}\delta\dot{u}_i+\frac{\partial\hat{L}}{\partial\dot{w}_i}\delta\dot{w}_i\right)\end{aligned} \tag{3.21}$$

and a similar expression can be written for the variation of L.

The virtual work density due to aerodynamic forces has the form

$$\delta\hat{W}_A=\underset{\sim}{f}_A\cdot\delta\underset{\sim}{R}_c=-(f_{Ax}\underset{\sim}{i}+f_{Ay}\underset{\sim}{j})\cdot\delta\underset{\sim}{R}_c \tag{3.22}$$

where f_{Ax} and f_{Ay} are force components per wetted unit length of missile in the axial and transverse directions, respectively. The minus sign indicates that the forces oppose the motion.

The gas pressure leads to the virtual work density

$$\begin{aligned}\delta\hat{W}_p[h(x)-h(x-L_1)] &= -\left\{\left[\frac{\partial}{\partial x}(pA_f\underset{\sim}{i})\right][h(x)-h(x-L_1)]\right.\\ &\left.+\Delta(pA_f\underset{\sim}{i})[\delta(x)+\delta(x-L_1)]\right\}\cdot\delta\underset{\sim}{R}_c\end{aligned} \tag{3.23}$$

which accounts for the fact that across the two ends of the combustion chamber there are abrupt changes in the pressure force. In an analogous way, the virtual work due to the reactive forces has the form

$$\delta \hat{W}_R [h(x)-h(x-L_1)] = -\{[\frac{\partial}{\partial x}(m_f v \underset{\sim}{v})][h(x)-h(x-L_1)] + \Delta(m_f v \underset{\sim}{v})[\delta(x)+\delta(x-L_1)]\}\cdot \delta \underset{\sim}{R}_c \tag{3.24}$$

Equations (3.23) and (3.24) can be combined to yield the virtual work density due to the engine thrust

$$\delta \hat{W}_T [h(x)-h(x-L_1)] = (\delta \hat{W}_p + \delta \hat{W}_R)[h(x)-h(x-L_1)]$$

$$= -\{[\frac{\partial}{\partial x}(pA_f \underset{\sim}{i} + m_f v \underset{\sim}{v})][h(x)-h(x-L_1)] + \Delta(pA_f \underset{\sim}{i} + m_f v \underset{\sim}{v})[\delta(x)+\delta(x-L_1)]\}\cdot \delta \underset{\sim}{R}_c . \tag{3.25}$$

From the first of Eqs. (2.8), we obtain the virtual work density due to the Coriolis force

$$\delta \hat{W}_C [h(x)-h(x-L_1)] = -(2\underset{\sim}{\omega} \times \underset{\sim}{v}\, m_f)[h(x)-h(x-L_1)]\cdot \delta \underset{\sim}{R}_c \tag{3.26}$$

and from the second of Eqs. (2.8) follows the one due to the flow unsteadiness

$$\delta \hat{W}_U [h(x)-h(x-L_1)] = -[\frac{\partial}{\partial t}(m_f \underset{\sim}{v})][h(x)-h(x-L_1)]\cdot \delta \underset{\sim}{R}_c \tag{3.27}$$

Finally, the virtual work density due to the body forces is simply

$$\delta \hat{W}_B = m\underset{\sim}{g}\cdot \delta \underset{\sim}{R}_c + M_i \underset{\sim}{g}\, \delta(x-x_i)\cdot \delta \underset{\sim}{R}_i \tag{3.28}$$

where $\underset{\sim}{g}$ is the acceleration due to gravity.

The operations indicated by Eq. (3.18) necessitate expressions for $\delta \underset{\sim}{R}_c$ and $\delta \underset{\sim}{R}_i$ in terms of the system generalized coordinates. Noting that $\delta \underset{\sim}{i} = \delta\theta\, \underset{\sim}{j}$ and $\delta \underset{\sim}{j} = -\delta\theta\, \underset{\sim}{i}$, we obtain from Eqs. (3.1) and (3.2)

$$\delta \underset{\sim}{R}_c = [\delta X + \delta u - \delta\theta(Y+y+w)]\underset{\sim}{i} + [\delta Y + \delta w + \delta\theta(X+x+u)]\underset{\sim}{j} \tag{3.29}$$

and

$$\delta \underset{\sim}{R}_i = [\delta X + \delta u_i - \delta\theta(Y+y_i+w_i)]\underset{\sim}{i} + [\delta Y + \delta w_i + \delta\theta(X+x_i+u_i)]\underset{\sim}{j} \tag{3.30}$$

Equations (3.22) and (3.29) enable us to write

$$\int_0^L \delta \hat{W}_A\, dx = -\int_0^L (f_{Ax}\underset{\sim}{i} + f_{Ay}\underset{\sim}{j})\cdot \delta \underset{\sim}{R}_c\, dx$$

$$= F_{AX}\delta X + F_{AY}\delta Y + F_{A\theta}\delta\theta + \int_0^L f_{Au}\delta u\, dx + \int_0^L f_{Aw}\delta w\, dx \tag{3.31}$$

where

$$F_{AX} = -\int_0^L f_{Ax}\,dx \quad , \quad F_{AY} = -\int_0^L f_{Ay}\,dx$$

$$F_{A\theta} = \int_0^L [\, f_{Ax}(Y+y+w) - f_{Ay}(X+x+u)\,]\,dx \tag{3.32}$$

$$f_{Au} = -f_{Ax} \quad , \qquad f_{Aw} = -f_{Ay}$$

are generalized aerodynamic forces. Similarly, the virtual work due to the engine thrust has the form

$$\int_0^L \delta\hat{W}_T[h(x)-h(x-L_1)]\,dx = -\int_0^L \{[\tfrac{\partial}{\partial x}(pA_f\underline{i} + m_f v\underline{v})[h(x)-h(x-L_1)]$$

$$+ \Delta(pA_f\underline{i} + m_f v\underline{v})[\delta(x)+\delta(x-L_1)]\}\cdot\delta\underline{R}_c\,dx$$

$$= F_{TX}\,\delta X + F_{TY}\,\delta Y + F_{T\theta}\,\delta\theta + \int_0^L f_{Tu}\,\delta u\,dx + \int_0^L f_{Tw}\,\delta w\,dx \tag{3.33}$$

in which

$$F_{TX} = -[(pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{0+}^{L_1^-} + (pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{0-}^{0+} + (pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{L_1^-}^{L_1^+}$$

$$= (p_e - p_a)A_e + m_f(0)\,v_e^2\cos\gamma$$

$$F_{TY} = m_f(0)\,v_e^2\sin\gamma$$

$$F_{T\theta} = -F_{TX}Y + F_{TY}X + \int_0^{L_1}\{[\tfrac{\partial}{\partial x}(pA_f + m_f v\underline{v}\cdot\underline{i})](y+w)$$

$$-[\tfrac{\partial}{\partial x}(m_f v\underline{v}\cdot\underline{i})](x+u)\}\,dx - (pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{L_1^-}^{L_1^+}[y(L_1)+w(L_1,t)] \tag{3.34}$$

$$+(pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{0-}^{0+}[y(0)+w(0,t)] - (m_f v\underline{v}\cdot\underline{j})\Big|_{L_1^-}^{L_1^+}[L_1 + u(L_1,t)]$$

$$-(m_f v\underline{v}\cdot\underline{j})\Big|_{0-}^{0+}u(0,t)$$

$$f_{Tu} = -[\tfrac{\partial}{\partial x}(pA_f + m_f v\underline{v}\cdot\underline{i})][h(x)-h(x-L_1)] - (pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{L_1^-}^{L_1^+}\delta(x-L_1)$$

$$-(pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{0-}^{0+}\delta(x)$$

$$f_{Tw} = -[\tfrac{\partial}{\partial x}(m_f v\underline{v}\cdot\underline{j})][h(x)-h(x-L_1)] - (m_f v\underline{v}\cdot\underline{j})\Big|_{0-}^{0+}\delta(x)$$

where the subscript e denotes quantities evaluated at the nozzle exit, γ is the angle the engine makes with the positive direction of the x-axis, L_1+ indicates a position immediately inside the second stage and L_1- a position immediately inside the first stage both in the vicinity of $x = L_1$, etc. The virtual work due to the Coriolis force is simply

$$\int_0^L \delta\hat{W}_C\,dx = F_{CX}\delta X + F_{CY}\delta Y + F_{C\theta}\delta\theta + \int_0^L f_{Cu}\delta u\,dx + \int_0^L f_{Cw}\delta w\,dx \tag{3.35}$$

where

$$\begin{aligned} &F_{CX}=0, \quad F_{CY}=2\dot{\theta}\int_0^{L_1} m_f v\,dx, \quad F_{C\theta}=F_{CY}X+2\dot{\theta}\int_0^{L_1} m_f v(x+u)\,dx \\ &F_{Cu}=0, \quad F_{Cw}=2\dot{\theta}\,m_f v[h(x)-h(x-L_1)] \end{aligned} \tag{3.36}$$

In an anlogous way, we obtain the virtual work performed by the forces resulting from the flow unsteadiness

$$\int_0^L \delta\hat{W}_U\,dx = F_{UX}\delta X + F_{UY}\delta Y + F_{U\theta}\delta\theta + \int_0^L f_{Uu}\delta u\,dx + \int_0^L f_{Uw}\delta w\,dx \tag{3.37}$$

in which

$$\begin{aligned} &F_{UX}=-\int_0^L \frac{\partial}{\partial t}(m_f v)\,dx, \quad F_{UY}=0, \quad F_{U\theta}=-F_{UX}Y+\int_0^{L_1}\left[\frac{\partial}{\partial t}(m_f v)\right](y+w)\,dx \\ &F_{Uu}=-\left[\frac{\partial}{\partial t}(m_f v)\right][h(x)-h(x-L_1)], \qquad F_{Uw}=0 \end{aligned} \tag{3.38}$$

The virtual work due to body forces has the form

$$\begin{aligned} \int_0^L \delta\hat{W}_B\,dx &= F_{BX}\delta X + F_{BY}\delta Y + F_{B\theta}\delta\theta + \int_0^L f_{Bu}\delta u\,dx + \int_0^L f_{Bw}\delta w\,dx \\ &\quad + \sum_{i=1}^{n}(F_{Bi}\delta u_i + F_{Bw_i}\delta w_i) \end{aligned} \tag{3.39}$$

where

$$F_{BX}=(M_0+\sum_{i=1}^{n}M_i)\,\underset{\sim}{g}\cdot\underset{\sim}{i}\,,\quad F_{BY}=(M_0+\sum_{i=1}^{n}M_i)\,\underset{\sim}{g}\cdot\underset{\sim}{j}$$

$$F_{B\theta}=-F_{BX}Y+F_{BY}X-\int_0^L m[(y+w)\underset{\sim}{i}-(x+u)\underset{\sim}{j}]\cdot\underset{\sim}{g}\,dx$$
$$-\sum_{i=1}^{n}M_i[(y_i+w_i)\underset{\sim}{i}-(x_i+u_i)\underset{\sim}{j}]\cdot\underset{\sim}{g} \tag{3.40}$$

$$f_{Bu}=m\underset{\sim}{g}\cdot\underset{\sim}{i}\,,\quad f_{Bw}=m\underset{\sim}{g}\cdot\underset{\sim}{j}\,,\quad F_{Bu_i}=M_i\underset{\sim}{g}\cdot\underset{\sim}{i}\,,\ F_{Bw_i}=M_i\underset{\sim}{g}\cdot\underset{\sim}{j}\quad i=1,2,\text{--},n$$

in which M_0 is the total mass of the missile shell and

$$M=M_0+\sum_{i=1}^{n}M_i \tag{3.41}$$

is the total mass of the missile including the discrete masses. Finally, the last term in Eq.(3.18) leads to

$$\int_0^L \dot{m}\,\dot{\underset{\sim}{R}}_c[h(x)-h(x-L_1)]\cdot\delta\underset{\sim}{R}_c\,dx$$
$$=F_{MX}\delta X+F_{MY}\delta Y+F_{M\theta}\delta\theta+\int_0^L f_{Mu}\delta u\,dx+\int_0^L f_{Mw}\delta w\,dx \tag{3.42}$$

where the expressions

$$F_{MX}=\int_0^{L_1}\dot{m}\,[\dot{X}+\dot{u}-\dot{\theta}\,(Y+y+w)]\,dx$$

$$F_{MY}=\int_0^{L_1}\dot{m}\,[\dot{Y}+\dot{w}+\dot{\theta}\,(X+x+u)]\,dx$$

$$F_{M\theta}=-F_{MX}Y+F_{MY}X-\int_0^{L_1}\dot{m}\{[\dot{X}+\dot{u}-\dot{\theta}(Y+y+w)](y+w)$$
$$-[\dot{Y}+\dot{w}+\dot{\theta}(X+x+u)](x+u)\}\,dx \tag{3.43}$$

$$f_{Mu}=\dot{m}\,[\dot{X}+\dot{u}-\dot{\theta}\,(Y+y+w)][h(x)-h(x-L_1)]$$

$$f_{Mw}=\dot{m}\,[\dot{Y}+\dot{w}+\dot{\theta}\,(X+x+u)][h(x)-h(x-L_1)]$$

are equivalent forces due to the mass rate of change in the first stage.

At this point, we assume that the damping forces can be derived from the Rayleigh dissipation density function

$$\hat{\mathcal{F}} = \hat{\mathcal{F}}(\dot{u}, \dot{w}, \dot{u}_1, \dot{u}_2, ---, \dot{u}_n, \dot{w}_1, \dot{w}_2, ---, \dot{w}_n) \tag{3.44}$$

by writing

$$\hat{\mathcal{F}}_u = -\frac{\partial\hat{\mathcal{F}}}{\partial\dot{u}}, \qquad \hat{\mathcal{F}}_w = -\frac{\partial\hat{\mathcal{F}}}{\partial\dot{w}}$$
$$\mathcal{F}_{u_i} = -\frac{\partial\mathcal{F}}{\partial\dot{u}_i}, \qquad \mathcal{F}_{w_i} = -\frac{\partial\mathcal{F}}{\partial\dot{w}_i} \tag{3.45}$$

where

$$\mathcal{F} = \int_0^L \hat{\mathcal{F}}\, dx \tag{3.46}$$

is the Rayleigh dissipation function.

Introducing Eq. (3.21), as well as Eqs. (3.31) through (3.46), into Eq. (3.18), we obtain the Lagrange-type equations of motion

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{X}}\right) + \frac{\partial L}{\partial X} + F_{AX} + F_{TX} + F_{CX} + F_{UX} + F_{BX} + F_{MX} = 0$$
$$-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{Y}}\right) + \frac{\partial L}{\partial Y} + F_{AY} + F_{TY} + F_{CY} + F_{UY} + F_{BY} + F_{MY} = 0$$
$$-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) + \frac{\partial L}{\partial\theta} + F_{A\theta} + F_{T\theta} + F_{C\theta} + F_{U\theta} + F_{B\theta} + F_{M\theta} = 0$$
$$-\frac{\partial}{\partial t}\left(\frac{\partial\hat{L}}{\partial\dot{u}}\right) + \frac{\partial\hat{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial u'}\right) + f_{Au} + f_{Tu} + f_{Cu} + f_{Uu} + f_{Bu} + f_{Mu} + \hat{\mathcal{F}}_u = 0 \tag{3.47}$$
$$-\frac{\partial}{\partial t}\left(\frac{\partial\hat{L}}{\partial\dot{w}}\right) + \frac{\partial\hat{L}}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial w'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial\hat{L}}{\partial w''}\right) + f_{Aw} + f_{Tw} + f_{Cw} + f_{Uw} + f_{Bw} + f_{Mw} + \hat{\mathcal{F}}_w = 0$$
$$-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{u}_i}\right) + \frac{\partial L}{\partial u_i} + F_{Bu_i} + \mathcal{F}_{u_i} = 0$$
$$-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{w}_i}\right) + \frac{\partial L}{\partial w_i} + F_{Bw_i} + \mathcal{F}_{w_i} = 0 \qquad i = 1, 2, ---, n$$

where the equations for u and w are subject to the boundary conditions

$$\frac{\partial\hat{L}}{\partial u'} = 0 \quad \text{at } x = 0, L \tag{3.48}$$

and

$$\frac{\partial\hat{L}}{\partial w''} = 0, \quad \frac{\partial\hat{L}}{\partial w'} - \frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial w''}\right) = 0 \quad \text{at } x = 0, L. \tag{3.49}$$

3.4 The Equations of Motion in Explicit Form

The equations of motion, Eqs. (3.47), can be written in an explicit form in terms of the system generalized coordinates. This necessitates explicit expressions for the Lagrangian density $\hat{L}$ and Rayleigh's dissipation density function $\hat{\mathcal{F}}$, as well as the functions themselves, L and $\mathcal{F}$.

The Lagrangian density can be shown to have the expression

$$\hat{L} = \hat{KE} - \hat{PE} = \frac{1}{2} m \dot{\underset{\sim}{R}}_c \cdot \dot{\underset{\sim}{R}}_c + \frac{1}{2} M_i \delta(x-x_i) \dot{\underset{\sim}{R}}_i \cdot \dot{\underset{\sim}{R}}_i - \frac{1}{2} [EI(w'')^2 + EA_c(u')^2 + P(w')^2] - \frac{1}{2}\{k_{u_i}[u_i(t) - u(x,t)]^2 + k_{w_i}[w_i(t) - w(x,t)]^2\}\delta(x-x_i) - \frac{1}{2}[k_{u_{i+1/2}}(u_i - u_{i+1})^2 + k_{w_{i+1/2}}(w_i - w_{i+1})^2]\,\delta(x - x_{i+1/2}) \tag{3.50}$$

Where $\dot{\underset{\sim}{R}}_c$ and $\dot{\underset{\sim}{R}}_i$ are given by Eqs. (3.5) and (3.6), respectively,

$$P = EA_c \frac{\partial u}{\partial x} \tag{3.51}$$

is the axial force at any point x on the vehicle longitudinal axes, k_{u_i} and k_{w_i} are the springs connecting mass M_i to the vehicle shell, and $k_{u_{i+1/2}}$ and $k_{w_{i+1/2}}$ are the springs between masses M_i and M_{i+1}. We note that terms corresponding to subscripts larger than n must be ignored. Integrating Eq. (3.50), we obtain the Lagrangian

$$L = \frac{1}{2}\int_0^L \{m \dot{\underset{\sim}{R}}_c \cdot \dot{\underset{\sim}{R}}_c - [EI(w'')^2 + EA_c(u')^2 + P(w')^2]\}\,dx + \frac{1}{2}\sum_{i=1}^{n}[M_i \dot{\underset{\sim}{R}}_i \cdot \dot{\underset{\sim}{R}}_i - \{k_{u_i}[u_i(t) - u(x_i,t)]^2 + k_{w_i}[w_i(t) - w(x_i,t)]^2\}] - \frac{1}{2}\sum_{i=1}^{n-1}[k_{u_{i+1/2}}(u_i - u_{i+1})^2 + k_{w_{i+1/2}}(w_i - w_{i+1})^2] \tag{3.52}$$

Next introduce the operator $\mathcal{M}$ defined by

$$\mathcal{M}\{f\} = \int_0^L m f\,dx + \sum_{i=1}^{n} M_i f_i \tag{3.53}$$

where $f = f(x,t)$ and $f_i = f(x_i,t)$ denote any of the variables associated with the position of the case element and mass M_i, respectively. Introducing Eqs. (3.50) and (3.53) into Eqs. (3.47), and recalling the expressions for the generalized forces derived in the preceding section we obtain the equations corresponding to the rigid body motion in the explicit form

$$-M(\ddot{X}-\ddot{\theta}Y-2\dot{\theta}\dot{Y}-\dot{\theta}^2X)-m\{\ddot{u}\}+\ddot{\theta}m\{y+w\}+2\dot{\theta}m\{\dot{w}\}+\dot{\theta}^2m\{x+u\}$$
$$-\int_0^L f_{Ax}dx+(p_e-p_a)A_e+m_f(0)v_e^2\cos\gamma-\int_0^{L_1}\left[\frac{\partial}{\partial t}(m_fv)\right]dx-Mg\sin\theta=0$$
$$-M(\ddot{Y}+\ddot{\theta}X+2\dot{\theta}\dot{X}-\dot{\theta}^2Y)-m\{\ddot{w}\}-\ddot{\theta}m\{x+u\}-2\dot{\theta}m\{\dot{u}\}+\dot{\theta}^2m\{y+w\}$$
$$-\int_0^L f_{Ay}dx+m_f(0)v_e^2\sin\gamma+2\dot{\theta}\int_0^{L_1}m_fv\,dx-Mg\cos\theta=0$$
$$-(\ddot{\theta}Y+\dot{\theta}^2X-\ddot{X}+2\dot{\theta}\dot{Y})m\{y+w\}-(\ddot{\theta}X-\dot{\theta}^2Y+\ddot{Y}+2\dot{\theta}\dot{X})m\{x+u\} \tag{3.54}$$
$$+m\{\ddot{u}(y+w)-\ddot{w}(x+u)\}-\ddot{\theta}m\{(x+u)^2+(y+w)^2\}-2\dot{\theta}m\{\dot{u}(x+u)+\dot{w}(y+w)\}$$
$$+\int_0^L[f_{Ax}(y+w)-f_{Ay}(x+u)]dx+\int_0^{L_1}\left\{\left[\frac{\partial}{\partial x}(pA_f+m_fv\underset{\sim}{v}\cdot\underset{\sim}{i})+\frac{\partial}{\partial t}(m_fv)\right](y+w)\right.$$
$$\left.-\left[\frac{\partial}{\partial x}(m_fv\underset{\sim}{v}\cdot\underset{\sim}{j})-2\dot{\theta}m_fv\right](x+u)\right\}dx-(pA_f+m_fv\underset{\sim}{v}\cdot\underset{\sim}{i})\Big|_{L_1^-}^{L_1^+}[y(L_1)+w(L_1,t)]$$
$$+(pA_f+m_fv\underset{\sim}{v}\cdot\underset{\sim}{i})\Big|_{0-}^{0+}[y(0)+w(0,t)]-(m_fv\underset{\sim}{v}\cdot\underset{\sim}{j})\Big|_{L_1^-}^{L_1^+}[L_1+u(L_1,t)]$$
$$-(m_fv\underset{\sim}{v}\cdot\underset{\sim}{j})\Big|_{0-}^{0+}u(0,t)+g\,m\{(y+w)\sin\theta-(x+u)\cos\theta\}=0$$

The remaining equations of motion necessitate the definition of the Rayleigh dissipation density function, which is assumed to have the form

$$\hat{\mathcal{F}}=\frac{1}{2}\Big[c_u\dot{u}^2+c_w\dot{w}^2+\{c_{u_i}[\dot{u}_i(t)-\dot{u}(x,t)]^2+c_{w_i}[\dot{w}_i(t)$$
$$-\dot{w}(x,t)]^2\}\delta(x-x_i)+[c_{u_{i+1/2}}(\dot{u}_i-\dot{u}_{i+1})^2+c_{w_{i+1/2}}(\dot{w}_i-\dot{w}_{i+1})^2]\delta(x-x_{i+1/2})\Big] \tag{3.55}$$

with the function itself having the expression

$$\mathcal{F}=\frac{1}{2}\int_0^L(c_u\dot{u}^2+c_w\dot{w}^2)dx+\frac{1}{2}\sum_{i=1}^{n}\{c_{u_i}[\dot{u}_i(t)-\dot{u}(x_i,t)]^2+c_{w_i}[\dot{w}_i(t)$$
$$-\dot{w}(x_i,t)]^2\}+\frac{1}{2}\sum_{i=1}^{n-1}[c_{u_{i+1/2}}(\dot{u}_i-\dot{u}_{i+1})^2+c_{w_{i+1/2}}(\dot{w}_i-\dot{w}_{i+1})^2] \tag{3.56}$$

This enables us to derive the equations for the discrete masses

$$-M_i[\ddot{X}+\ddot{u}_i-\ddot{\theta}(Y+y_i+w_i)-2\dot{\theta}(\dot{Y}+\dot{w}_i)-\dot{\theta}^2(X+x_i+u_i)]-M_i g\sin\theta$$

$$-k_{u_i}[u_i(t)-u(x_i,t)]-k_{u_{i+1/2}}(u_i-u_{i+1})-k_{u_{i-1/2}}(u_i-u_{i-1})$$

$$-c_{u_i}[\dot{u}_i(t)-\dot{u}(x_i,t)]-c_{u_{i+1/2}}(\dot{u}_i-\dot{u}_{i+1})-c_{u_{i-1/2}}(\dot{u}_i-\dot{u}_{i-1})=0 \quad i=1,2,---,n$$

$$-M_i[\ddot{Y}+\ddot{w}_i+\ddot{\theta}(X+x_i+u_i)+2\dot{\theta}(\dot{X}+\dot{u}_i)-\dot{\theta}^2(Y+y_i+w_i)]-M_i g\cos\theta \qquad (3.57)$$

$$-k_{w_i}[w_i(t)-w(x_i,t)]-k_{w_{i+1/2}}(w_i-w_{i+1})-k_{w_{i-1/2}}(w_i-w_{i-1})$$

$$-c_{w_i}[\dot{w}_i(t)-\dot{w}(x_i,t)]-c_{w_{i+1/2}}(\dot{w}_i-\dot{w}_{i+1})-c_{w_{i-1/2}}(\dot{w}_i-\dot{w}_{i-1})=0 \quad i=1,2,---,n$$

where terms with subscripts smaller than one and larger than n must be omitted.

Finally, the equations for the elastic motion can be shown to have the form

$$\frac{\partial}{\partial x}(EA_c u')-m[\ddot{X}+\ddot{u}-\ddot{\theta}(Y+y+w)-2\dot{\theta}(\dot{Y}+\dot{w})-\dot{\theta}^2(X+x+u)]$$

$$-f_{Ax}-\left[\frac{\partial}{\partial x}(pA_f+m_f v\underset{\sim}{v}\cdot\underset{\sim}{i})+\frac{\partial}{\partial t}(m_f v)\right][h(x)-h(x-L_1)]$$

$$+(pA_f+m_f v\underset{\sim}{v}\cdot\underset{\sim}{i})\Big|_{L_1^-}^{L_1^+}\delta(x-L_1)-(pA_f+m_f v\underset{\sim}{v}\cdot\underset{\sim}{i})\Big|_{0-}^{0+}\delta(x)-mg\sin\theta$$

$$+k_{u_i}(u_i-u)\,\delta(x-x_i)+c_{u_i}(\dot{u}_i-\dot{u})\,\delta(x-x_i)=0 \qquad (3.58)$$

$$-\frac{\partial^2}{\partial x^2}(EIw'')+\frac{\partial}{\partial x}(Pw')-m[\ddot{Y}+\ddot{w}+\ddot{\theta}(X+x+u)+2\dot{\theta}(\dot{X}+\dot{u})$$

$$-\dot{\theta}^2(Y+y+w)]-f_{Ay}-\left[\frac{\partial}{\partial x}(m_f v\underset{\sim}{v}\cdot\underset{\sim}{j})-2\dot{\theta}m_f v\right][h(x)-h(x-L_1)]$$

$$-(m_f v\underset{\sim}{v}\cdot\underset{\sim}{j})\Big|_{0-}^{0+}\delta(x)-mg\cos\theta+k_{w_i}(w_i-w)\,\delta(x-x_i)$$

$$+c_{w_i}(\dot{w}_i-\dot{w})\,\delta(x-x_i)=0$$

which are subject to the boundary conditions

$$EA_c u'=0 \quad \text{at } x=0,L \qquad (3.59)$$

and

$$EI w'' = \frac{\partial}{\partial x}(EI w'') = 0 \quad \text{at } x = 0,L \tag{3.60}$$

where in the second of boundary conditions (3.60) we took into account that $P = 0$ at $x = 0,L$.

4. Calculation of the Engine Thrust and Internal Gas Flow

The purpose of a nozzle is twofold. First it restricts the rate of escape of the gas from the combustion chamber to a rate suitable for the reaction of the propellant, and second it changes the energy of the propellant gas from internal energy to kinetic energy. A convergent-divergent nozzle is used to convert all of the available enthalpy into kinetic energy. In the portion before the throat the flow is subsonic whereas after the throat the flow is supersonic, with sonic conditions occurring at the throat. Although losses may exist in the nozzle we assume that they are negligible and regard the flow as one-dimensional isentropic steady flow.

Consider the flow of Figure 4 and assume that the stagnation

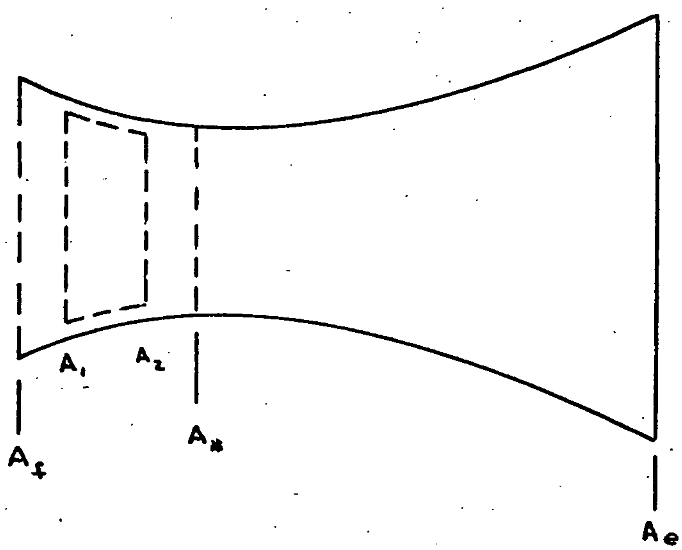


Figure 4

conditions denoted by the subscript 0, are known. We may then write the governing physical equations as follows.

First the flow must satisfy the first law of thermodynamics. Considering the control volume of Figure 4 and denoting the enthalpy per unit mass by h, this law may be stated as

$$h_0 = h_1 + \frac{1}{2} V_1^2 = h_2 + \frac{1}{2} V_2^2 \tag{4.1}$$

Assuming no friction or heat transfer, the second law of thermodynamics reduces to

$$S_0 = S_1 = S_2 = \text{constant} \tag{4.2}$$

where S is the entropy.

Furthermore, since no mass is added within the nozzle the flow must satisfy the continuity equation which may be written as

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 = \text{constant} \tag{4.3}$$

where ρ_1 (ρ_2) is the density at station 1(2).

If we denote the force exerted by the nozzle wall on the gas F_T, the momentum equation can be written as

$$F_T + p_1 A_1 - p_2 A_2 = \rho_2 A_2 v_2^2 - \rho_1 A_1 v_1^2 \tag{4.4}$$

Equations (4.1) through (4.4) must be supplemented by the equation of state which for a perfect gas has the form

$$p = \rho R T \tag{4.5}$$

in which R is the universal gas constant and T the temperature.

On introducing the definition of the speed of sound for a perfect gas

$$c = [kRT]^{1/2} \tag{4.6}$$

where

$$k = c_p / c_v \tag{4.7}$$

in which c_p and c_v are the specific heats, we may, after some manipulations of the above equations, arrive at the following expressions

$$T_0 / T = 1 + \frac{k-1}{2} M^2 \tag{4.8}$$

$$p_0 / p = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}} \tag{4.9}$$

$$\rho_o/\rho = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}} \tag{4.10}$$

where M = v/c is the Mach number. Finally, the cross-sectional area A at any point is related to the cross-sectional area A_* at the throat by

$$A/A_* = G_*/G = \frac{1}{M}\left[\frac{2}{k+1}\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k+1}{2(k-1)}} \tag{4.11}$$

where

$$G = \rho v \tag{4.12}$$

and

$$G_* = \left[\frac{k p_o^2}{RT_o}\right]^{1/2} \left[\frac{2}{k+1}\right]^{\frac{k+1}{2(k-1)}} \tag{4.13}$$

are the mass flow rates per unit area at an arbitrary point and the throat, respectively.

The above expressions suffice to describe the flow in terms of the parameters of the nozzle. Since the interest lies in the conditions at the nozzle exit and because in the analysis presented here the nozzle exit area is usually given, we may find the Mach number at the exit from Eq. (4.11), where k is assumed known. Using Eqs. (4.8) through (4.13), the other parameters at the exit section may then be found. This enables us to determine the rocket thrust as

$$F_T = p_e A_e + \rho_e A_e v_e^2 = p_e A_e (1 + k M_e^2) \tag{4.14}$$

Eq. (4.14) only considers thrust due to the internal gas flow and if conditions other than in vacuum are considered, then the thrust due to atmospheric pressure acting on A_e must be subtracted from the above equation.

So far it has been assumed that the stagnation conditions are available for evaluating the flow parameters at the exit section. This assumption must next be considered. The stagnation conditions are determined by events occurring upstream of the nozzle. In our case there is a propellant charge ablating, thus producing heat and gas. This gas enters the nozzle only after proceeding along the propellant grain and the nozzle plenum chamber.

The processes occurring upstream, therefore, are not isentropic and the stagnation conditions are not constant, but decrease as the nozzle is approached. The stagnation conditions, however, may be obtained in terms of conditions at the forward end of the propellant charge by an analysis similar to that done above for the nozzle. In Reference[11] we can find a plot of the stagnation pressure ratio as a function of a reduced mass flux ratio $\dot{m}/\dot{m}_*$ which indicates that for a Mach number of less than 0.4 the drop in stagnation pressure may not be significant. Therefore, we assume that the stagnation pressure as well as the other stagnation conditions, occurring at the forward end of the propulsion stage, are constant and apply also to the nozzle. These assumptions may have to be reconsidered if a more precise model is desired.

5. Numerical Solutions

5.1 General Remarks

In seeking numerical solutions of differential equations, it is frequently more advantageous to work with first-order rather than second-order equations. With this in mind, we introduce a set of 5 + 2n new variables defined by

$$\dot{X} = R, \quad \dot{Y} = S, \quad \dot{\theta} = T, \quad \dot{u} = p, \quad \dot{w} = q$$
$$\dot{u}_i = p_i, \quad \dot{w}_i = q_i, \quad i = 1, 2, ---, n \tag{5.1}$$

which can be regarded as a set of 5 + 2n first-order differential equations. Another 5 + 2n first-order differential equations in time can be obtained by introducing the variables defined by Eqs. (5.1) into Eqs. (3.54), (3.57), and (3.58) with the result

$$-M(\dot{R} - \dot{T}Y - 2TS - T^2X) - \mathfrak{m}\{\dot{p}\} + \dot{T}\mathfrak{m}\{y+w\} + 2T\mathfrak{m}\{q\} + T^2\mathfrak{m}\{x+u\}$$
$$-\int_0^L f_{Ax}\,dx + (p_e - p_a)A_e + m_f(0)v_e^2\cos\gamma - \int_0^L \left[\frac{\partial}{\partial t}(m_f v)\right]dx - Mg\sin\theta = 0 \tag{5.2}$$

$$-M(\dot{S} + \dot{T}X + 2TR - T^2Y) - \mathfrak{m}\{\dot{q}\} - \dot{T}\mathfrak{m}\{x+u\} - 2T\mathfrak{m}\{p\} + T^2\mathfrak{m}\{y+w\}$$
$$-\int_0^L f_{Ay}\,dx + m_f(0)v_e^2\sin\gamma + 2T\int_0^{L_1} m_f v\,dx - Mg\cos\theta = 0 \tag{5.3}$$

$$-(\dot{T}Y + T^2X - \dot{R} + 2TS)\mathfrak{m}\{y+w\} - (\dot{T}X - T^2Y + \dot{S} + 2TR)\mathfrak{m}\{x+u\}$$
$$+\mathfrak{m}\{\dot{p}(y+w) - \dot{q}(x+u)\} - \dot{T}\mathfrak{m}\{(x+u)^2 + (y+w)^2\} - 2T\mathfrak{m}\{p(x+u) + q(y+w)\}$$
$$+\int_0^L [f_{Ax}(y+w) - f_{Ay}(x+u)]\,dx + \int_0^L \Big\{\Big[\frac{\partial}{\partial x}(pA_f + m_f v\,\underset{\sim}{v}\cdot\underset{\sim}{i}) + \frac{\partial}{\partial t}(m_f v)\Big](y+w)$$
$$-\Big[\frac{\partial}{\partial x}(m_f v\,\underset{\sim}{v}\cdot\underset{\sim}{j}) - 2Tm_f v\Big](x+u)\Big\}dx - (pA_f + m_f v\,\underset{\sim}{v}\cdot\underset{\sim}{i})\Big|_{L_1^-}^{L_1^+}[y(L_1) + w(L_1,t)]$$
$$+(pA_f + m_f v\,\underset{\sim}{v}\cdot\underset{\sim}{i})\Big|_{0-}^{0+}[y(0) + w(0,t)] - (m_f v\,\underset{\sim}{v}\cdot\underset{\sim}{j})\Big|_{L_1^-}^{L_1^+}[L_1 + u(L_1,t)]$$
$$-m_f v\,\underset{\sim}{v}\cdot\underset{\sim}{j}\,\Big|_{0-}^{0+} u(0,t) + g\mathfrak{m}\{(y+w)\sin\theta - (x+u)\cos\theta\} = 0 \tag{5.4}$$

$$\frac{\partial}{\partial x}(EA_c u') - m[\dot{R} + \dot{p} - \dot{T}(Y+y+w) - 2T(\dot{S}+\dot{q}) - T^2(X+x+u)]$$

$$-f_{Ax} - [\frac{\partial}{\partial x}(pA_f + m_f v \underline{v}\cdot\underline{i}) + \frac{\partial}{\partial t}(m_f v)][h(x) - h(x-L_1)]$$

$$-(pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{L_1-}^{L_1+}\delta(x-L_1) - (pA_f + m_f v\underline{v}\cdot\underline{i})\Big|_{0-}^{0+}\delta(x) - mg\sin\theta$$

$$+k_{u_i}(u_i - u)\,\delta(x-x_i) + c_{u_i}(p_i - p)\,\delta(x-x_i) = 0 \qquad (5.5)$$

$$-\frac{\partial^2}{\partial x^2}(EI\,w'') + \frac{\partial}{\partial x}(Pw') - m[\dot{S} + \dot{q} + \dot{T}(X+x+u) + 2T(R+p)$$

$$-T^2(Y+y+w)] - f_{Ay} - [\frac{\partial}{\partial x}(m_f v\underline{v}\cdot\underline{j}) - 2Tm_f v][h(x) - h(x-L_1)]$$

$$-m_f v\underline{v}\cdot\underline{j}\Big|_{0-}^{0+}\delta(x) - mg\cos\theta + k_{w_i}(w_i - w)\delta(x-x_i) + c_{w_i}(q_i - q)\delta(x-x_i) = 0$$

$$(5.6)$$

$$-M_i[\dot{R} + \dot{p}_i - \dot{T}(Y+y_i+w_i) - 2T(S+q_i) - T^2(X+x_i+u_i)] - M_i g\sin\theta$$

$$-k_{u_i}(u_i - u) - k_{u_{i+1/2}}(u_i - u_{i+1}) - k_{u_{i-1/2}}(u_i - u_{i-1}) - c_{u_i}(p_i - p)$$

$$-c_{u_{i+1/2}}(p_i - p_{i+1}) - c_{u_{i-1/2}}(p_i - p_{i-1}) = 0 \qquad i = 1,2,---,n \qquad (5.7)$$

$$-M_i[\dot{S} + \dot{q}_i + \dot{T}(X+x_i+u_i) + 2T(R+p_i) - T^2(Y+y_i+w_i)] - M_i g\cos\theta$$

$$-k_{w_i}(w_i - w) - k_{w_{i+1/2}}(w_i - w_{i+1}) - k_{w_{i-1/2}}(w_i - w_{i-1}) - c_{w_i}(q_i - q)$$

$$-c_{w_{i+1/2}}(q_i - q_{i+1}) - c_{w_{i-1/2}}(q_i - q_{i-1}) = 0 \qquad i = 1,2,---,n \qquad (5.8)$$

The boundary conditions, Eqs. (3.59) and (3.60), retain their form.

In light of the complexity of the differential equations of motion, no closed-form solution can be expected. Hence, we shall attempt a numerical solution based on the finite-difference approach. It will prove advantageous to consider unequal time increments but equal spatial increments. With this in mind, we introduce the idea of unequal segments for the purpose of evaluating derivatives of any arbitrary function f = f(x,t). From Figure 5 and the definition of the central differences, we obtain the derivative of f at the

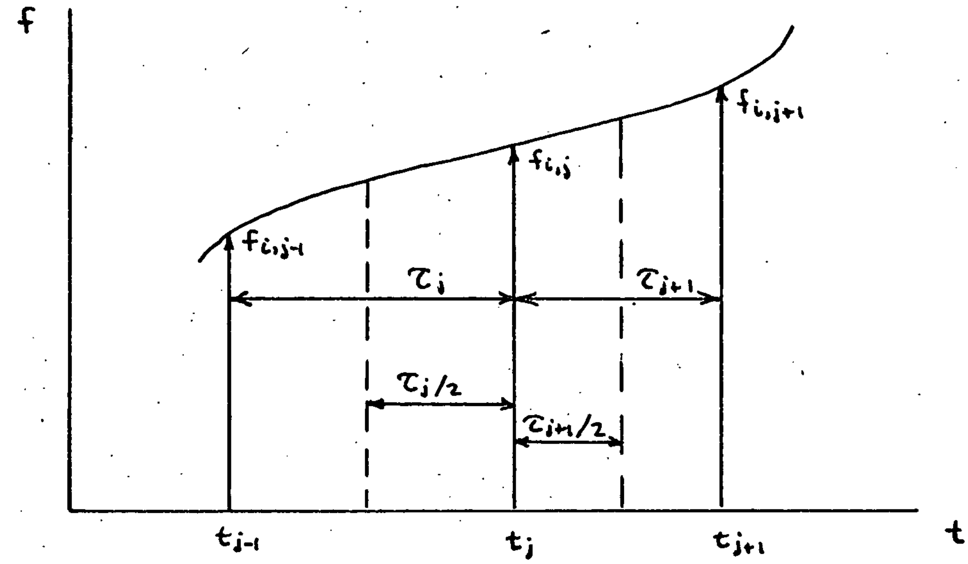


Figure 5

jth time increment for the ith station in the form

$$\left.\frac{\partial f}{\partial t}\right|_{\substack{x=x_i\\ t=t_j}} = \frac{f_{i,j+1/2} - f_{i,j-1/2}}{\frac{1}{2}\tau_j + \frac{1}{2}\tau_{j+1}} \tag{5.9}$$

However, since the interest lies in integer values of the subscript, we average the function with noninteger subscripts as follows

$$\begin{aligned} f_{i,j+1/2} &= \frac{1}{2}\left(f_{i,j+1} + f_{i,j}\right) \\ f_{i,j-1/2} &= \frac{1}{2}\left(f_{i,j} + f_{i,j-1}\right) \end{aligned} \tag{5.10}$$

so that expression (5.9) becomes

$$\left.\frac{\partial f}{\partial t}\right|_{\substack{x=x_i\\ t=t_j}} = \frac{f_{i,j+1} - f_{i,j-1}}{\tau_j + \tau_{j+1}} \tag{5.11}$$

Assuming equal increments for the spatial variables, in analogy with expression (5.11) we conclude that

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_i\\ t=t_j}} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} \tag{5.12}$$

where h is the spatial increment.

Since Eqs. (5.2) through (5.4) contain integrals, we approximate them by the simple trapazoidal rule, which for an arbitrary function $f = f(x,t)$ has the form

$$\int_0^L f(x,t)\,dx = \sum_{i=1}^{r} \frac{h}{2}(f_{i,j} + f_{i+1,j}) \tag{5.13}$$

where r is the number of segments of the missile (one less than the number of stations, $rh = L$). Using expression (5.13) we conclude that the operator $\mathfrak{M}$ of Eq. (3.53) has the form

$$\mathfrak{M}\{f_{i,j}\} = \sum_{i=1}^{r} \frac{h}{2}(m_{i,j}f_{i,j} + m_{i+1,j}\,f_{i+1,j}) + \sum_{i=1}^{n} M_i F_{i,j} \tag{5.14}$$

where $F_{i,j}$ represents any of the variables describing the ith mass at time $t = \sum_{k=1}^{j} \tau_k$.

Next we consider the real and equivalent forces appearing in Eqs. (5.2) through (5.6). Assuming that the flight of the missile takes place in vacuum, the generalized aerodynamic forces vanish

$$F_{AX} = F_{AY} = F_{A\theta} = f_{Au} = f_{Aw} = 0. \tag{5.15}$$

The generalized forces due to the engine thrust are calculated under the assumption of complete expansion of the nozzle to vacuum, so that $p_e = p_a = 0$. Under these conditions Eqs. (3.34) become

$$\begin{aligned} &F_{TX} = m_f(0)v_e^2 \cos\gamma\ , \qquad F_{TY} = m_f(0)v_e^2 \sin\gamma \\ &F_{T\theta} = m_f(0)v_e^2\{[y(0)+w(0,t)]\cos\gamma - u(0,t)\sin\gamma\} \\ &\qquad - pA_f(L_1)[y(L_1)+w(L_1,t)] \\ &F_{Tu} = pA_f\,\delta(x-L_1) + [m_f(0)v_e^2\cos\gamma - pA_f(L_1)]\,\delta(x) \\ &F_{Tw} = m_f(0)v_e^2 \sin\gamma\ \delta(x) \end{aligned} \tag{5.16}$$

The Coriolis forces are given by Eqs. (3.36). Moreover, since the internal flow can safely be assumed to be steady, Eqs. (3.38) become

$$F_{UX} = F_{UY} = F_{U\theta} = F_{Uu} = F_{Uw} = 0. \tag{5.17}$$

It is still necessary to find expressions for $m_f v$ and v_e. The first is found by considering the continuity equation, which, for uniform burning over the entire length of the combustion chamber, takes the form

$$m_f v = -\int_x^{L_1} \dot{m}\, dx = m_f(0) v_e (1 - x/L_1) \; . \qquad (5.18)$$

The expression for the exhaust velocity, v_e may be obtained by making use of Eq. (4.1) as well as certain definitions introduced in Section 4, with the result

$$v_e = \left\{ \frac{2gk}{k-1} RT\left[1 - \left(\frac{p_e}{p}\right)^{\frac{k-1}{k}}\right]\right\}^{1/2} . \qquad (5.19)$$

However, since we assumed that $p_e = 0$ Eq. (5.19) reduces to

$$v_e = \left\{ \frac{2gk}{k-1} RT \right\}^{1/2} . \qquad (5.20)$$

5.2 Finite-Difference Equations

To avoid any possible confusion, we introduce the following notation: Subscripts i generally refer to position and subscripts j to time. For example, $U_{i,j}$ refers to the ith mass at time $t = \sum_{k=1}^{j} \tau_k$, while X_j refers to the rigid body translation at time $t = \sum_{k=1}^{j} \tau_k$. Similarily $u_{i,j}$ refers to the element in the position $x = x_i$ at time $t = \sum_{k=1}^{j} \tau_k$. With this in mind and recalling Eq. (5.11) we write Eqs. (5.1) in the form

$$\begin{aligned}
X_{j+1} &= X_{j-1} + (\tau_j + \tau_{j+1}) R_j \\
Y_{j+1} &= Y_{j-1} + (\tau_j + \tau_{j+1}) S_j \\
\Theta_{j+1} &= \Theta_{j-1} + (\tau_j + \tau_{j+1}) T_j \\
u_{i,j+1} &= u_{i,j-1} + (\tau_j + \tau_{j+1}) p_{i,j} \qquad i = 1,2,---,r+1 \\
w_{i,j+1} &= w_{i,j-1} + (\tau_j + \tau_{j+1}) q_{i,j} \\
U_{i,j+1} &= U_{i,j-1} + (\tau_j + \tau_{j+1}) P_{i,j} \qquad i = 1,2,---,n \\
W_{i,j} &= W_{i,j-1} + (\tau_j + \tau_{j+1}) Q_{i,j}
\end{aligned} \qquad (5.21)$$

Next we consider the equation for axial motion of the case element, Eq. (5.5), and use the thrust terms defined in Section 5.1 to write the equation in the form

$$\frac{\partial}{\partial x}(EA_c u') - m(\dot{R} + \dot{p} - \dot{T}(Y+y+w) - 2T(S+q) - T^2(X+x+u)]$$
$$+ pA_f(L_1)\delta(x-L_1) - (pA_f(L_1) - m_f(0)v_e^2\cos\gamma)\delta(x) - mg\sin\theta$$
$$+ k_{u_i}(u_i - u)\,\delta(x-x_i) + c_{u_i}(p_i - p)\,\delta(x-x_i) = 0 \tag{5.22}$$

Using expressions (5.11) and (5.12), Eq. (5.22) takes the finite difference form

$$\frac{EA_{c_i}}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{4h^2}(EA_{c_{i+1}} - EA_{c_{i-1}})(u_{i+1,j} - u_{i-1,j}) - \frac{m_{i,j}}{\tau_j + \tau_{j+1}}[R_{j+1} - R_{j-1}$$
$$+ p_{i,j+1} - p_{i,j-1} - (T_{j+1} - T_{j-1})(Y_j + y_i + w_{i,j})] + m_{i,j}[2T_j(S_j + q_{i,j}) + T_j^2(X_j + x_i + u_{i,j})]$$
$$+ \frac{pA_f(L_1)}{h}\delta_{im} - \frac{1}{h}(pA_f(L_1) - m_f(0)v_e^2\cos\gamma)\delta_{io} - m_{i,j}\,g\sin\theta_j$$
$$+ \frac{k_{u_{i_k}}}{h}(U_{i_k,j} - u_{i,j})\delta_{i_k i} + \frac{c_{u_{i_k}}}{h}(P_{i_k,j} - p_{i,j})\delta_{i_k i} = 0 \tag{5.23}$$

where the subscript m refers to the station corresponding to $x = L_1$ and δ_{ij} is the Kronecker delta. The subscript i_k refers to the kth discrete mass. Upon rearranging, Eq. (5.23) becomes

$$p_{i,j+1} = p_{i,j-1} - (R_{j+1} - R_{j-1}) + (T_{j+1} - T_{j-1})(Y_j + y_i + w_{i,j}) + (\tau_j + \tau_{j+1})[2T_j(S_j$$
$$+ q_{i,j}) + T_j^2(X_j + x_i + u_{i,j}) + \frac{1}{4m_{i,j}h^2}(EA_{c_{i+1}} - EA_{c_{i-1}})(u_{i+1,j} - u_{i-1,j})$$
$$+ \frac{EA_{c_i}}{h^2 m_{i,j}}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{pA_f(L_1)}{m_{i,j}h}\delta_{im} - \frac{1}{m_{i,j}h}(pA_f(L_1) - m_f(0)v_e^2\cos\gamma)\delta_{io}$$
$$- g\sin\theta_j + \frac{k_{u_{i_k}}}{m_{i,j}h}(U_{i_k,j} - u_{i,j})\delta_{i_k,i} + \frac{c_{u_{i_k}}}{m_{i,j}h}(P_{i_k,j} - p_{i,j})\delta_{i_k i} \tag{5.24}$$

In a similar manner, using Eq. (5.6), the finite-difference expression for the transverse motion becomes

$$
\begin{aligned}
q_{i,j+1} = {} & q_{i,j-1} - (S_{j+1} - S_{j-1}) - (T_{j+1} - T_{j-1})(X_j + x_i + u_{i,j}) + (\tau_j + \tau_{j+1})\Big[T_i^2(Y_i + y_i \\
& + w_{i,j}) - 2T_j(R_j + p_{i,j}) - \frac{EI_i}{m_{i,j}h^4}(w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}) \\
& - \frac{1}{2m_{i,j}h^4}(EI_{i+1} - EI_{i-1})(w_{i+2,j} - 2w_{i+1,j} + 2w_{i-1,j} - w_{i-2,j}) - \frac{1}{m_{i,j}h^4}(EI_{i+1} - 2EI_i + EI_{i-1})(w_{i+1,j} \\
& - 2w_{i,j} + w_{i-1,j}) + \frac{1}{8m_{i,j}h^3}(EA_{c_{i+1}} - EA_{c_{i-1}})(u_{i+1,j} - u_{i-1,j})(w_{i+1,j} - w_{i-1,j}) \\
& + \frac{EA_{c_i}}{2m_{i,j}h^3}\left\{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})(w_{i+1,j} - w_{i-1,j}) + (u_{i+1,j} - u_{i-1,j})(w_{i+1,j} - 2w_{i,j} + w_{i-1,j})\right\} \\
& + \frac{2T_j(m_f v)_i}{m_{i,j}} - \frac{m_f V_e^2}{m_{i,j}}\sin\gamma\,\delta_{io} - g\cos\theta_j + \frac{k_{wk}}{m_{i,j}h}(W_{i_k,j} - w_{i,j})\delta_{i_k i} \\
& + \frac{c_{wk}}{m_{i,j}h}(Q_{i_k,j} - q_{i,j})\delta_{i_k i}\Big]
\end{aligned}
\tag{5.25}
$$

Using (5.14), the rigid body equations can be written in finite-difference form

$$
\begin{aligned}
R_{j+1} - \Big[Y_j + \frac{1}{M_j}\mathfrak{m}\{y_i + w_{i,j}\}\Big]T_{j+1} = {} & R_{j-1} - \Big[Y_j + \frac{1}{M_j}\mathfrak{m}\{y_i + w_{i,j}\}\Big]T_{j-1} \\
& - \frac{1}{M_j}\mathfrak{m}\{p_{i,j+1} - p_{i,j-1}\} + (\tau_j + \tau_{j+1})\Big[2T_j S_j + T_j^2 X_j + \frac{2T_j}{M_j}\mathfrak{m}\{q_{i,j}\} \\
& + \frac{T_j^2}{M_j}\mathfrak{m}\{x_i + u_{i,j}\} + \frac{m_f(0)V_e^2}{M_j}\cos\gamma - g\sin\theta_j\Big]
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
S_{j+1} - \Big[X_j + \frac{1}{M_j}\mathfrak{m}\{x_i + u_{i,j}\}\Big]T_{j+1} = {} & S_{j-1} - \Big[X_j + \frac{1}{M_j}\mathfrak{m}\{x_i + u_{i,j}\}\Big]T_{j-1} \\
& - \frac{1}{M_j}\mathfrak{m}\{q_{i,j+1} - q_{i,j-1}\} + (\tau_j + \tau_{j+1})\Big[T_j^2 Y_j - 2T_j R_j - \frac{2T_j}{M_j}\mathfrak{m}\{p_{i,j}\} \\
& + \frac{T_j^2}{M_j}\mathfrak{m}\{y_i + w_{i,j}\} + \frac{m_f(0)V_e^2}{M_j}\sin\gamma + \frac{T_j h}{M_j}\sum_{i=1}^{r-m}[(m_f v)_i + (m_f v)_{i+1}] - g\cos\theta_j\Big]
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
& -R_{j+1}\mathfrak{m}\{y_i + w_{i,j}\} + S_{j+1}\mathfrak{m}\{x_i + u_{i,j}\} + T_{j+1}\Big[Y_j\mathfrak{m}\{y_i + w_{i,j}\} + X_j\mathfrak{m}\{x_i + u_{i,j}\} \\
& + \mathfrak{m}\{(x_i + u_{i,j})^2 + (y_i + w_{i,j})^2\}\Big] = -R_{j-1}\mathfrak{m}\{y_i + w_{i,j}\} + S_{j-1}\mathfrak{m}\{x_i + u_{i,j}\} \\
& + T_{j-1}\Big[Y_j\mathfrak{m}\{y_i + w_{i,j}\} + X_j\mathfrak{m}\{x_i + u_{i,j}\} + \mathfrak{m}\{(x_i + u_{i,j})^2 + (y_i + w_{i,j})^2\}\Big] \\
& + \mathfrak{m}\{(p_{i,j+1} - p_{i,j-1})(y_i + w_{i,j}) - (q_{i,j+1} - q_{i,j-1})(x_i + u_{i,j})\}
\end{aligned}
$$

$$+(\tau_j+\tau_{j+1})\left[(T_j^2 Y_j - 2T_j R_j)\mathcal{M}\{x_i+u_{i,j}\} - (T_j^2 X_j + 2T_j S_j)\mathcal{M}\{y_i+w_{i,j}\}\right.$$

$$-2T_j\mathcal{M}\{p_{i,j}(x_i+u_{i,j})+q_{i,j}(y_i+w_{i,j})\} - pA_f(L_1)(y_m+w_{m,j})$$

$$+(pA_f(L_1)-m_f(0)v_e^2\cos\gamma)(y_1+w_{1,j})-u_{1,j}m_f(0)v_e^2\sin\gamma+g\left[\mathcal{M}\{y_i\right.$$

$$\left.\left.+w_{i,j}\}\sin\theta_j - \mathcal{M}\{x_i+u_{i,j}\}\cos\theta_j\right]\right] \tag{5.28}$$

Finally the equations for the discrete masses lead to the finite-difference relations

$$P_{i,j+1}=P_{i,j-1}-(R_{j+1}-R_{j-1})+(T_{j+1}-T_{j-1})(Y_j+Y_i+W_{i,j})+(\tau_j+\tau_{j+1})\left[2T_j(S_j+Q_{i,j})\right.$$

$$+T_j^2(X_j+X_i+U_{i,j})-g\sin\theta_j-\frac{1}{M_i}\{k_{u_i}(U_{i,j}-u_{i_k,j})+k_{u_{i+1/2}}(U_{i,j}-U_{i+1,j})$$

$$\left.+k_{u_{i-1/2}}(U_{i,j}-U_{i-1,j})+c_{u_i}(P_{i,j}-p_{i_k,j})+c_{u_{i+1/2}}(P_{i,j}-P_{i+1,j})+c_{u_{i-1/2}}(P_{i,j}-P_{i-1,j})\}\right] \tag{5.29}$$

$$Q_{i,j+1}=Q_{i,j-1}-(S_{j+1}-S_{j-1})-(T_{j+1}-T_{j-1})(X_j+X_i+U_{i,j})+(\tau_j+\tau_{j+1})\left[T_j^2(Y_j+Y_i+W_{i,j})\right.$$

$$-2T_j(R_j+P_{i,j})-g\cos\theta_j-\frac{1}{M_i}\{k_{w_i}(W_{i,j}-w_{i_k,j})+k_{w_{i+1/2}}(W_{i,j}-W_{i+1,j})$$

$$\left.+k_{w_{i-1/2}}(W_{i,j}-W_{i-1,j})+c_{w_i}(Q_{i,j}-q_{i_k,j})+c_{w_{i+1/2}}(Q_{i,j}-Q_{i+1,j})+c_{w_{i-1/2}}(Q_{i,j}-Q_{i-1,j})\}\right] \tag{5.30}$$

The boundary conditions, Eqs. (3.59) and (3.60), become

$$\begin{aligned} u_{0,j} &= u_{2,j} \\ u_{r+2,j} &= u_{r,j} \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} w_{0,j} &= 2w_{1,j}-w_{2,j} \\ w_{r+2,j} &= 2w_{r+1,j}-w_{r,j} \\ w_{-1,j} &= w_{3,j}-2w_{2,j}+2w_{0,j} \\ w_{r+3,j} &= 2w_{r+2,j}-2w_{r,j}+w_{r-1,j} \end{aligned} \tag{5.32}$$

Eqs. (5.24) through (5.30) may be written in compact matrix notation as follows

$$[A]_j\{X\}_{j+1} = [A]_j\{X\}_{j-1} + \{C\}_j \tag{5.33}$$

where $[A]_j$ is a square matrix of order 10 + 4r + 4n whose elements are real, $\{X\}_{j+1}$ and $\{X\}_{j-1}$ are (10+4r+4n)-dimensional state vectors, and $\{C\}_j$ is a (10+4r+4n)-dimensional vector corresponding to forcing functions of various kinds. The subscripts indicate the time at which the components are evaluated. A recursive formula could be devised by simply premultiplying both sides of Eq. (5.33) by $[A]_j^{-1}$. It turns out, however, that the matrix $[A]_j$ is singular so that its inverse does not exist. The reason lies in the fact that the equations of motion are not independent. In fact Eqs. (5.26) through (5.28) represent certain integrals of the remaining equations. Hence a procedure is advised whereby the problem of calculating the inverse of $[A]_j$ is circumvented.

Let us write Eq. (5.33) in terms of partitioned matrices as follows

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]_j \left\{\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right\}_{j+1} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]_j \left\{\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right\}_{j-1} + \left\{\begin{array}{c} C_1 \\ \hline C_2 \end{array}\right\}_j \tag{5.34}$$

where the 6-dimensional vectors $\{X_1\}_{j+1}$ and $\{X_1\}_{j-1}$ correspond to the rigid-body motion and the (4+4r+4n)-dimensional vectors $\{X_2\}_{j+1}$ and $\{X_2\}_{j-1}$ to the elastic motion. Equation (5.34) can be separated into one for the rigid-body and one for the elastic motion in the form

$$[A_{11}]_j\{X_1\}_{j+1} + [A_{12}]_j\{X_2\}_{j+1} = [A_{11}]_j\{X_1\}_{j-1} + [A_{12}]_j\{X_2\}_{j-1} + \{C_1\}_j \tag{5.35}$$

and

$$[A_{21}]_j\{X_1\}_{j+1} + [A_{22}]\{X_2\}_{j+1} = [A_{21}]_j\{X_1\}_{j-1} + [A_{22}]_j\{X_2\}_{j-1} + \{C_2\}_j \tag{5.36}$$

The assumption that the elastic motion does not affect the rigid-body motion is equivalent to

$$[A_{12}]_j\{X_2\}_{j+1} = [A_{12}]\{X_2\}_{j-1} = \{0\} \tag{5.37}$$

which can be interpreted as the statement that the weighted average of the elastic motion is zero, where the weighting matrix is $[A_{12}]_j$. This enables us to solve Eq. (5.35) for the rigid-body motion

$$\{X_1\}_{j+1} = \{X_1\}_{j-1} + [A_{11}]_j^{-1}\{C_1\}_j \tag{5.38}$$

On the other hand, the rigid body does affect the elastic motion. Indeed, introducing Eq. (5.38) into (5.36), we obtain

$$\{X_2\}_{j+1} = \{X_2\}_{j-1} - [A_{22}]_j^{-1}[A_{21}]_j[A_{11}]_j^{-1}\{C_1\}_j + [A_{22}]_j^{-1}\{C_2\}_j \tag{5.39}$$

It turns out that $[A_{22}]_j$ is the unit matrix, denoted by $[1]$, so that Eq. (5.39) reduces to

$$\{X_2\}_{j+1} = \{X_2\}_{j-1} - [A_{21}]_j[A_{11}]_j^{-1}\{C_1\}_j + \{C_2\}_j \tag{5.40}$$

Equations (5.38) and (5.40) can be solved recursively for the rigid-body and elastic motions.

A question of particular importance is the stability of the finite-difference solution. Inherent in the finite-difference approximation to differential equations is the introduction of extraneous solutions which may be unstable. The problem of finding the conditions under which the finite-difference approximation gives the true solution and not a divergent one is in itself a difficult problem and will not be pursued here. However, in order to insure some semblance of stability, we invoke the analogy with a uniform, constant mass beam. It is indicated in Reference 13 that to ensure the stability of the resulting incremental expressions for the longitudinal vibration of a uniform beam, the following conditions must be met

$$\sigma^2 \leq \frac{h^2 m}{EA_c} \tag{5.41}$$

while for transverse vibration

$$\sigma^2 \leq \frac{h^4 m}{4EI} \tag{5.42}$$

where σ is the time increment and h is the spatial increment, m is the mass per unit length and EA_c and EI are the longitudinal and transverse stiffnesses of the beam respectively. But in the system under consideration we do not have a uniform beam. We do, however have uniform segments so that we may write Eqs. (5.41) and (5.42) for each segment as

$$\sigma_{i,j+1}^2 \leq \frac{h^2 m_{i,j+1}}{EA_{c_i}} \tag{5.43}$$

and

$$\sigma_{i,j+1}^2 \leq \frac{h^4 m_{i,j+1}}{4EI_i} \tag{5.44}$$

where the first subscript refers to the spatial increment and the second to the time increment. Equations (5.43) and (5.44) give two values of $\sigma_{i,j+1}$ for each spatial increment. If these are compared to those corresponding to all the spatial increments and the smallest among them retained, then the inequalities (5.43) and (5.44) are satisfied. This value, denoted by τ_{j+1}, is then used as the time increment. Mathematically we seek

$$\tau_{j+1} = \min_i \sigma_{i,j+1} \tag{5.45}$$

and note that this value may change from one increment of time to the next because the mass is time-dependent, m = m(t). Expression (5.45) does not ensure stability but it does give a reasonable estimate for the time increment, corresponding to known values of mass and stiffness distribution and assumed values for the spatial increments.

It is of some interest to determine the error introduced by approximating the differential equations of motion by the finite-difference expressions. Since the change in mass is small for the time increments used, we assume for simplicity that the time increments are constant $\tau_j = \tau_{i+1} = \tau$. With this in mind, we consider each term of Eq. (5.24) as a continuous function of x and t. For example, consider $u_{i,j+1}$ as denoting $u(x,t+\tau)$. Expanding each term of Eq. (5.24) in a Taylor series about the point (x,t) we obtain

$$\begin{aligned}
&\dot{R} + \frac{d^3R}{dt^3}\frac{\tau^2}{6} + --- + \dot{p} + \frac{\partial^3 p}{\partial t^3}\frac{\tau^2}{6} + --- - \dot{T}(Y+y+w) - \frac{d^3T}{dt^3}\frac{\tau^2}{6}(Y+y+w) + --- \\
&-2T(S+q) - T^2(X+x+u) - \frac{1}{m}\frac{\partial EA_c}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{m}\frac{\partial EA_c}{\partial x}\frac{\partial^3 u}{\partial x^3}\frac{h^3}{6} - \frac{1}{m}\frac{\partial^3 EA_c}{\partial x^3}\frac{\partial u}{\partial x}\frac{h^2}{6} \\
&----- - \frac{EA_c}{m}\frac{\partial^2 u}{\partial x^2} - \frac{EA_c}{m}\frac{\partial^4 u}{\partial x^4}\frac{h^2}{12} ----- \frac{pA_f(L)}{m}\delta(x-L_1) \\
&+\frac{1}{m}(pA_f(L_1) - m_f(0)V_e^2\cos\gamma)\delta(x) + g\sin\theta - \frac{k_{u_i}}{m}(u_i - u)\delta(x-x_i) \\
&-\frac{C_{u_i}}{m}(p_i - p)\,\delta(x-x_i) = 0
\end{aligned} \tag{5.46}$$

Comparing Eq. (5.46) and (5.5) we find that the truncation error, e_T, is

$$\begin{aligned}
e_T = &\left[\frac{d^3R}{dt^3} + \frac{\partial^3 p}{\partial t^3} - \frac{d^3T}{dt^3}(Y+y+w) + ---\right]\frac{\tau^2}{6} \\
&-\frac{1}{m}\left[\frac{\partial EA_c}{\partial x}\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 EA_c}{\partial x^3}\frac{\partial u}{\partial x} + \frac{EA_c}{2}\frac{\partial^4 u}{\partial x^4} + ---\right]\frac{h^2}{6} + ---
\end{aligned} \tag{5.47}$$

or

$$e_T = O(\tau^2) + O(h^2) \tag{5.48}$$

under the assumption that the terms in the brackets are <1. The notation $O(\tau^2)$ indicates that a term is of the order of τ^2. Using expression (5.41) we finally conclude that

$$e_T = O(\tau^2) \tag{5.49}$$

Similar conclusions may be reached by considering the other finite-difference expressions.

5.3 Results

A computer program to solve Eqs. (5.24) through (5.30) in the form of Eqs. (5.38) and (5.40) was written in the FORTRAN IV language and evaluated numerically on an IBM 360/50 computer.

Due to the exceedingly small time increments necessary for a stable solution (τ of the order of 10^{-6}), a large number of steps must be taken to obtain even a small burn time. As a result of the large number of steps, the problem of accumulation of round-off error is prevalent. For relatively small burn times, however, the round-off errors should be small and an indication of the system behavior should be obtainable.

The numerical values used in the computation were

$E_1 = E_2 = 30 \times 10^6$ psi, $L_1 = 100$ in., $L = 150$ in., $h = 2$ in.,
$A_f(0) = 36.4$ in.^{2}, $p = 2000$ psi, $\dot{m}_c g = 1.57$ lbs/in./sec.,
$A_{c1} = 7.53$ in.^{2}, $A_{c2} = 5.0$ in.^{2}, $I_1 = 93$ in.^{4}, $I_2 = 50$ in.^{4},
$m_1 g = 4.25$ lbs/in., $m_2 g = 3.0$ lbs/in.

where the subscripts 1 and 2 refer to the first and second stage characteristics respectively. The numerical values used to describe the three discrete masses were

$X_1 = 125$ in., $X_2 = 130$ in., $X_3 = 135$ in., $M_1 g = M_2 g = M_3 g = 6$ lbs.,
$k_{u_1} = k_{u_2} = k_{u_3} = k_{w_1} = k_{w_2} = k_{w_3} = 10^6$ lbs/ft.,
$k_{u_{1/2}} = k_{u_{3/2}} = k_{w_{1/2}} = k_{w_{3/2}} = 10^5$ lbs/ft.,
$c_{u_1} = c_{u_2} = c_{u_3} = c_{w_1} = c_{w_2} = c_{w_3} = c_{u_{1/2}} = c_{u_{3/2}} = 0$,
$c_{w_{1/2}} = c_{w_{3/2}} = 0$, Y_1, Y_2, Y_3, variable.

The initial conditions were

$$w(x,0) = 10^{-5}(\sin \pi x/L-2/\pi)+0.5x10^{-5}(3/\pi-6x/\pi^2 L-\sin 2\pi x/L) \text{ ft.}$$

$$X(0) = 0,\ R(0) = 0,\ Y(0) = 0,\ S(0) = 0,\ T(0) = 0,\ u(x,0) = 0$$

$$U_i(0) = W_i(0) = P_i(0) = Q_i(0) = 0 \quad i = 1,2,3$$

while the launch angle, θ_0, was varied by inputs to the program.

The axial and transverse translation of the missile as a function of time with θ_0 as a parameter is shown in Figure 6 and 7. The axial and transverse elastic displacements for several time increments are shown in Figure 8 for the following conditions: (I) the system parameters have the numerical values listed above, and (II) all the parameters are as in the first case with the exception of k_{u_2} and k_{w_2} which are zero, $k_{u_2} = k_{w_2} = 0$. Figure 9 shows the axial and transverse displacements of the discrete masses as a function of time for Case I while Figure 10 shows the motion for Case II.

The more rapid change occurring in the axial elastic displacements shown in Figure 8 is caused by the sudden increase in pressure following ignition. The subsequent motion in the axial direction is more rapid because disturbances in this direction are propagated more rapidly than in the transverse direction.

When the discrete masses are attached to the case as well as to one another, the axial displacements consist of synchronous motions lagging behind the motion of the rocket shell, as shown by the dashed line in Figure 9. On the other hand, the transverse displacements do not exhibit such lag and the discrete masses move so that their mean motion coincides with the motion of the case. The effect of setting $k_{u_2} = k_{w_2} = 0$ becomes apparent by comparing Figures 9 and 10. Since in this case the second discrete mass is not attached to the rocket case, the forces producing its motion are transmitted only through the springs connecting it to the adjacent masses. Whereas the cyclic time is seen to increase substantially from Figure 8 we conclude that the motion differs only by a small amount from the one in which all masses are attached to the case.

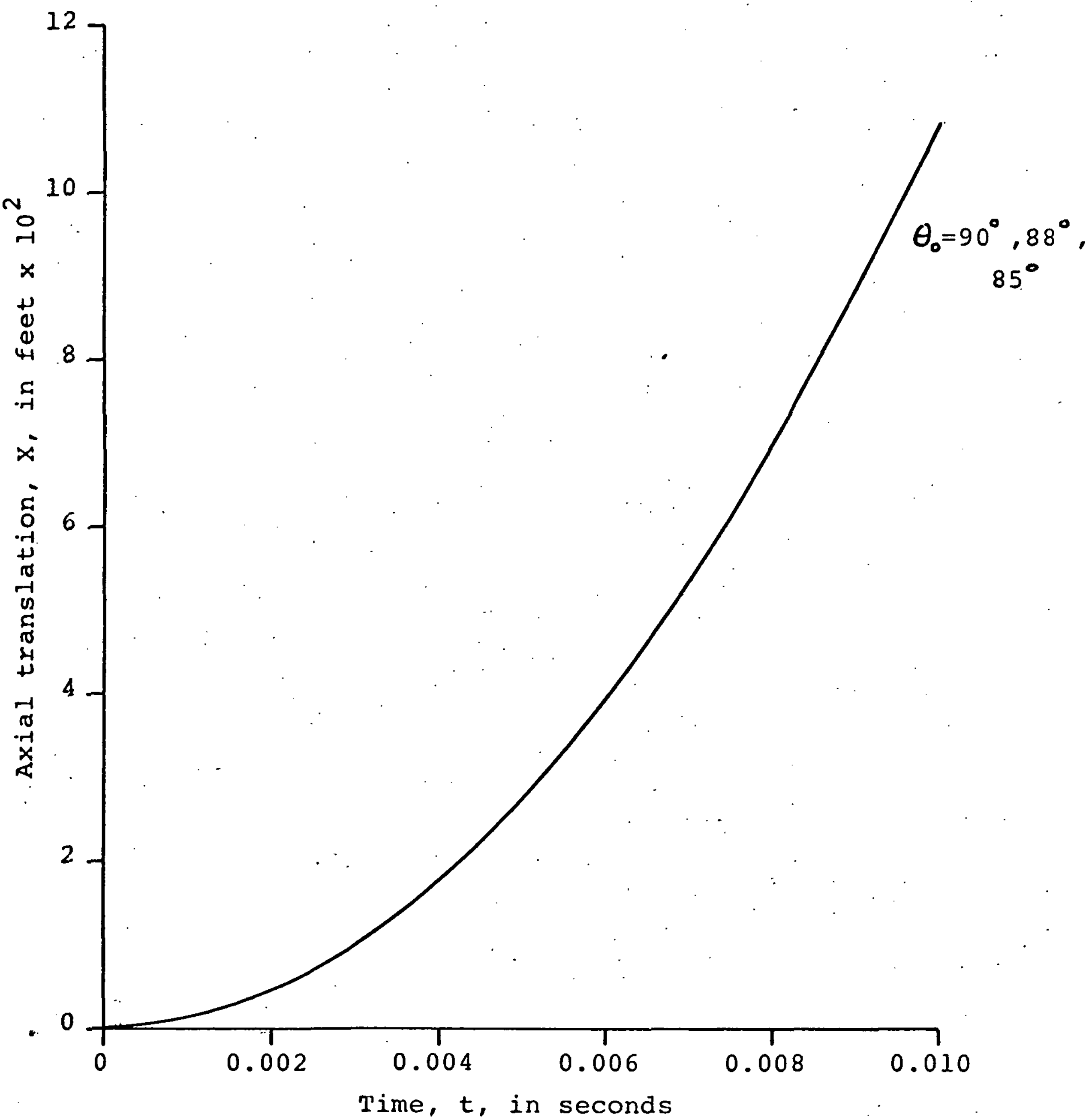


Figure 6 - Axial translation vs. time with θ_0 as parameter

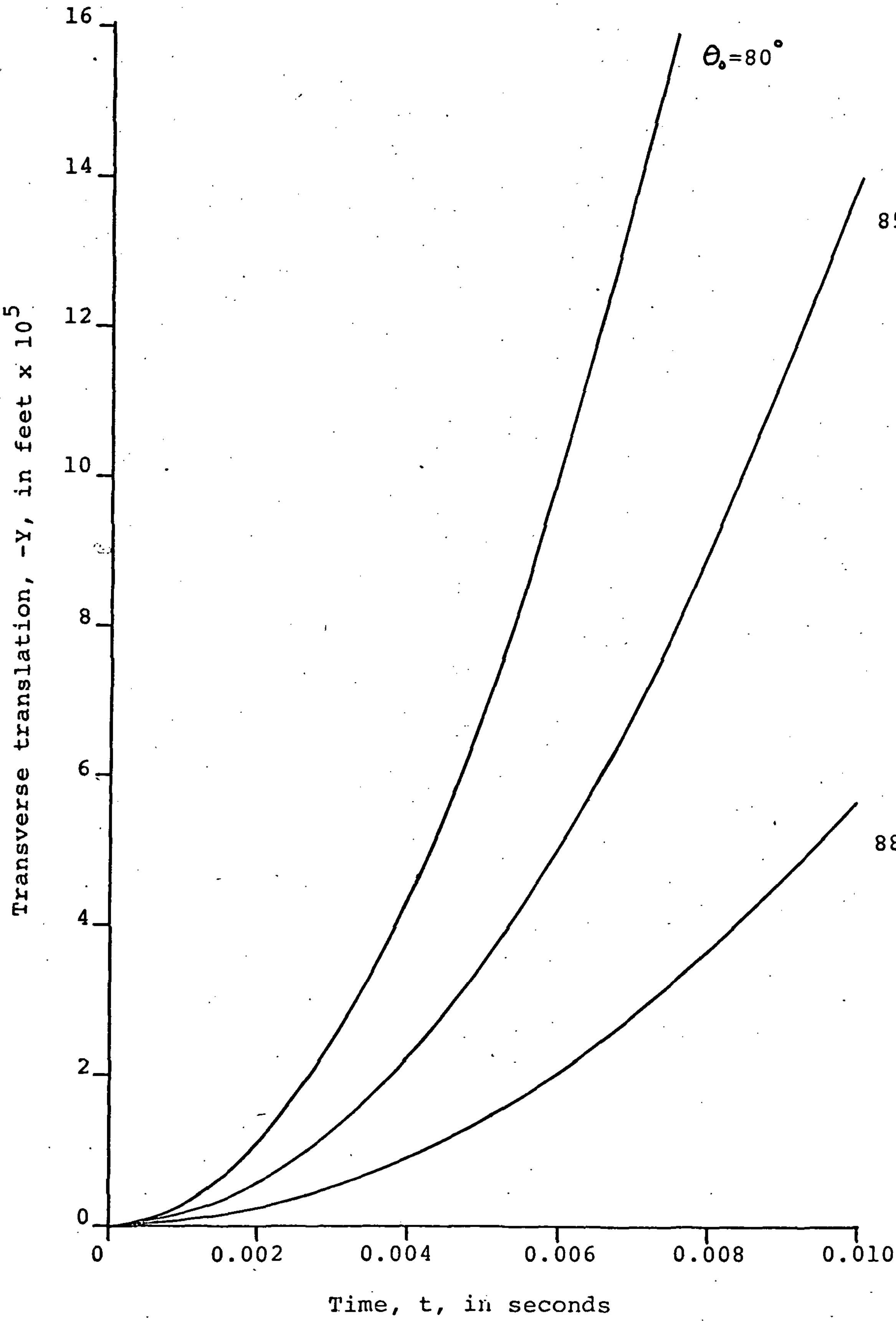


Figure 7 - Transverse translation vs time with θ_0 as parameter

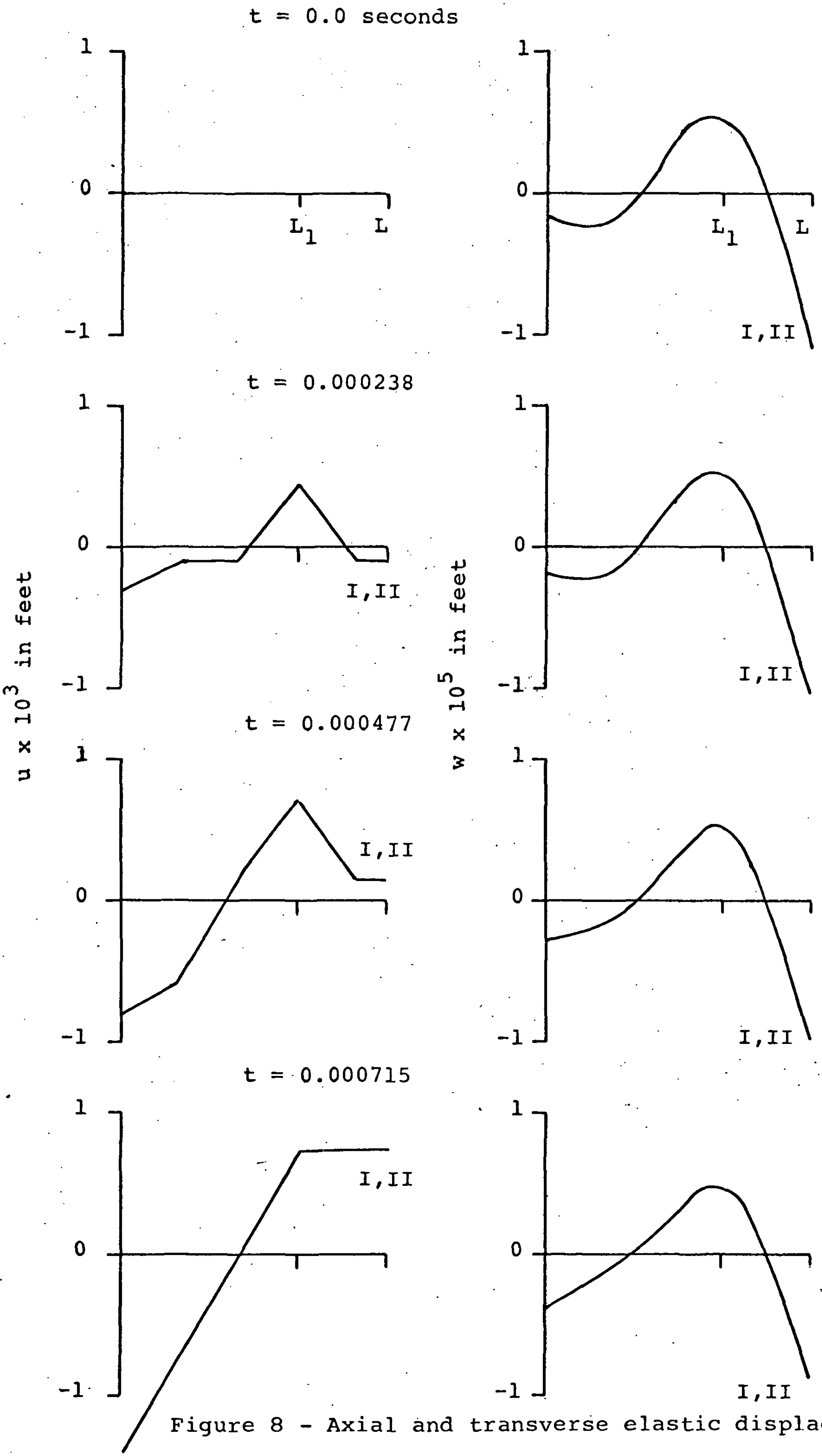


Figure 8 - Axial and transverse elastic displacements

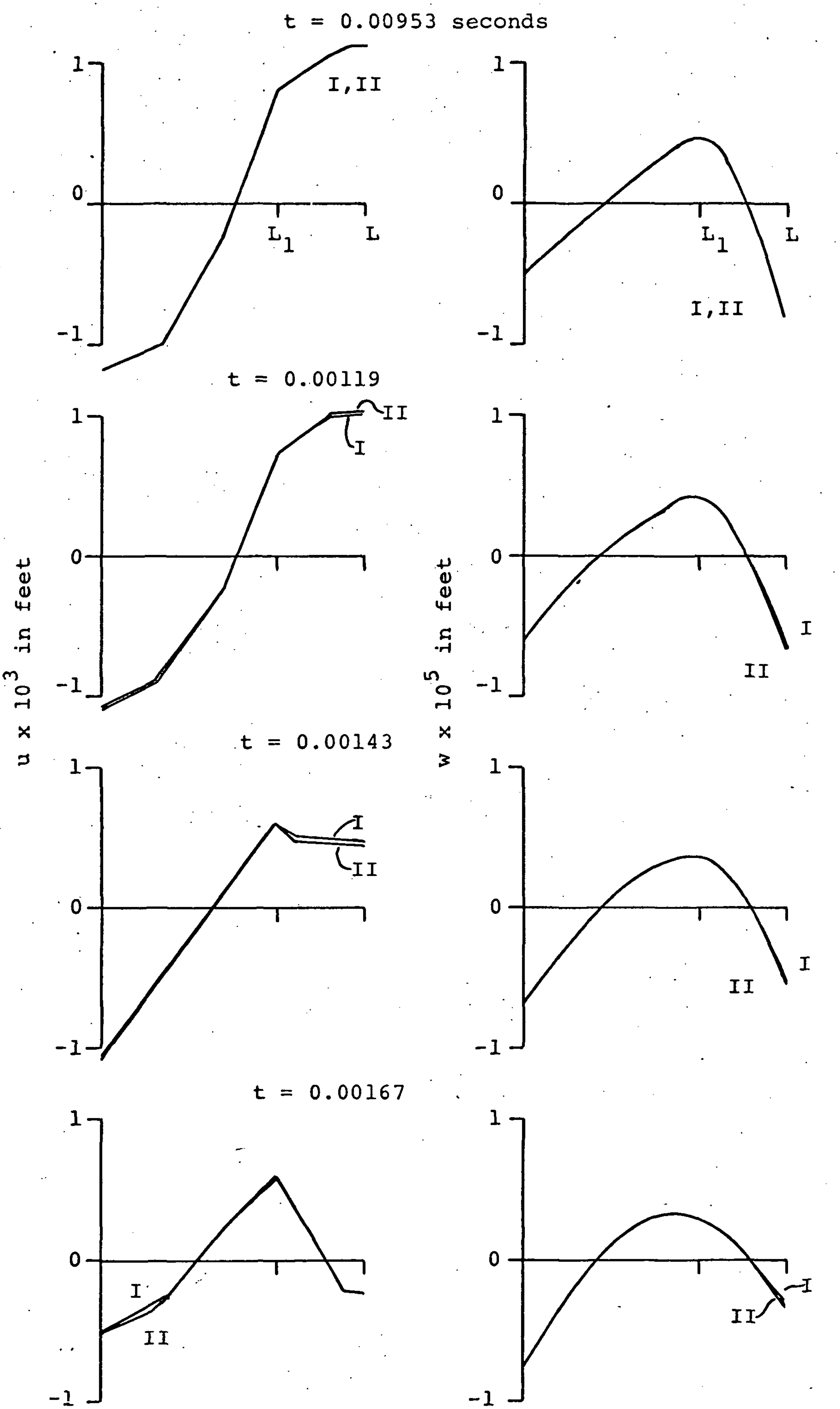


Figure 8 - Axial and transverse elastic displacements (cont.)

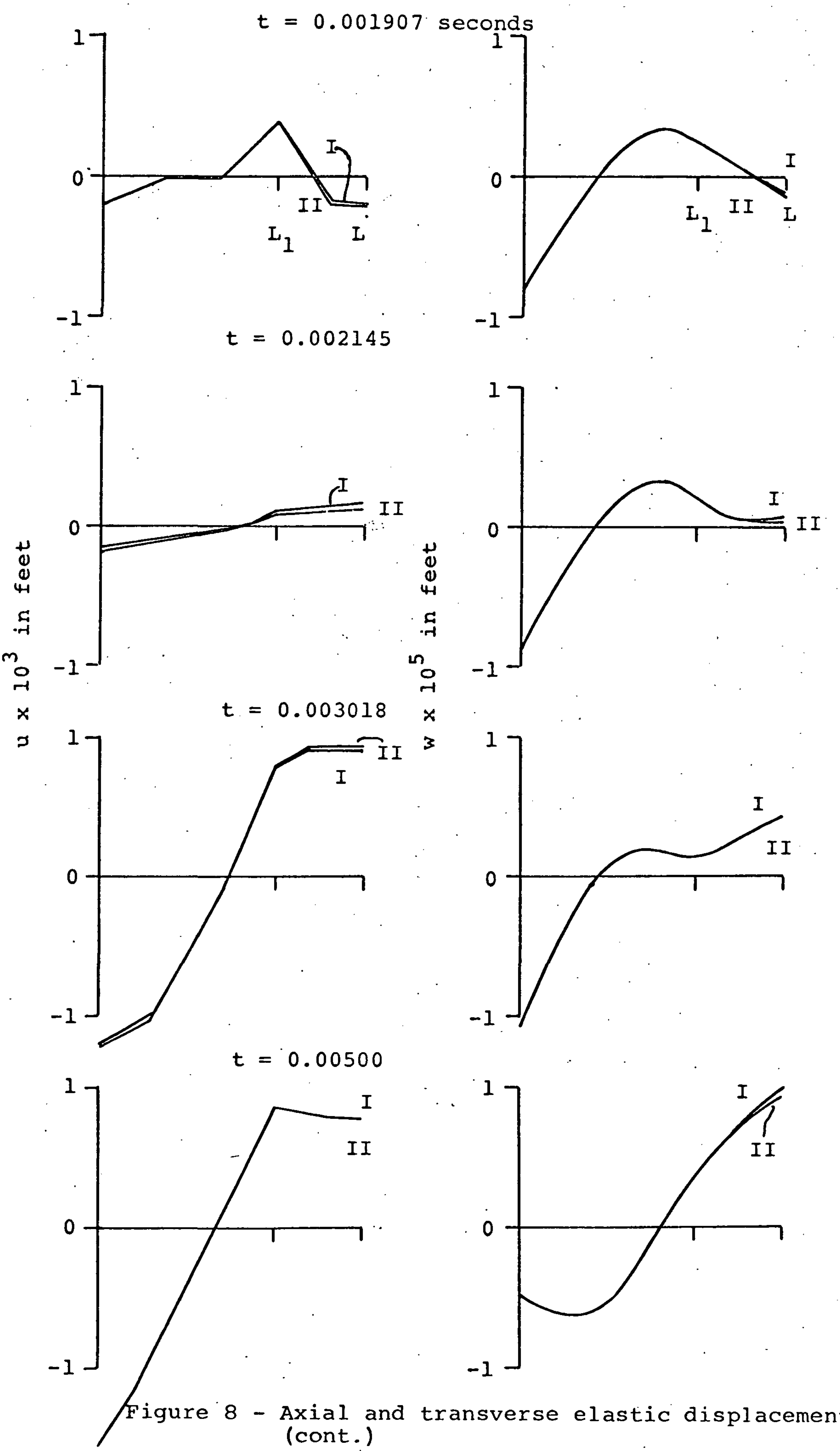


Figure 8 - Axial and transverse elastic displacements (cont.)

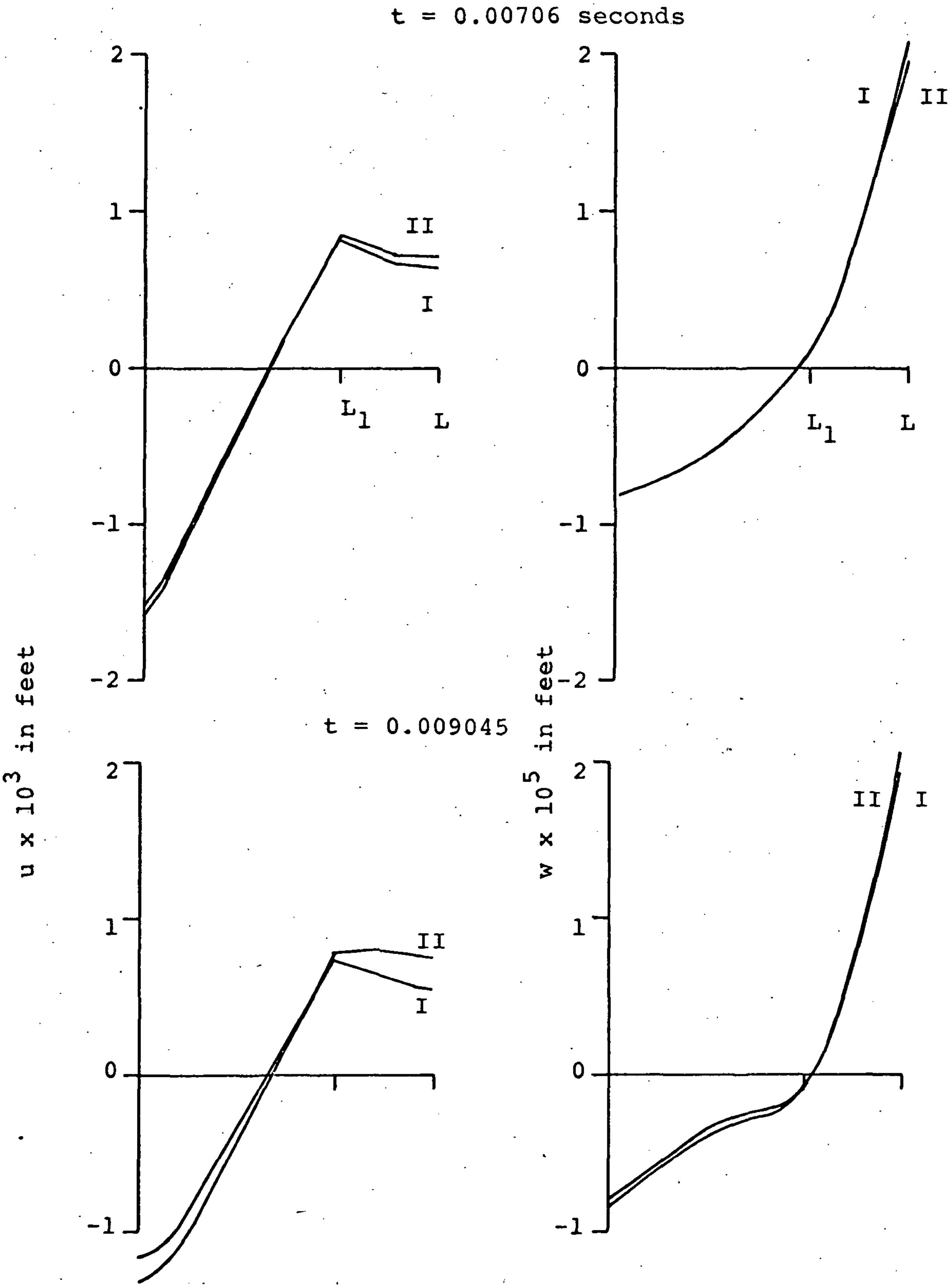


Figure 8 - Axial and transverse elastic displacements (cont.)

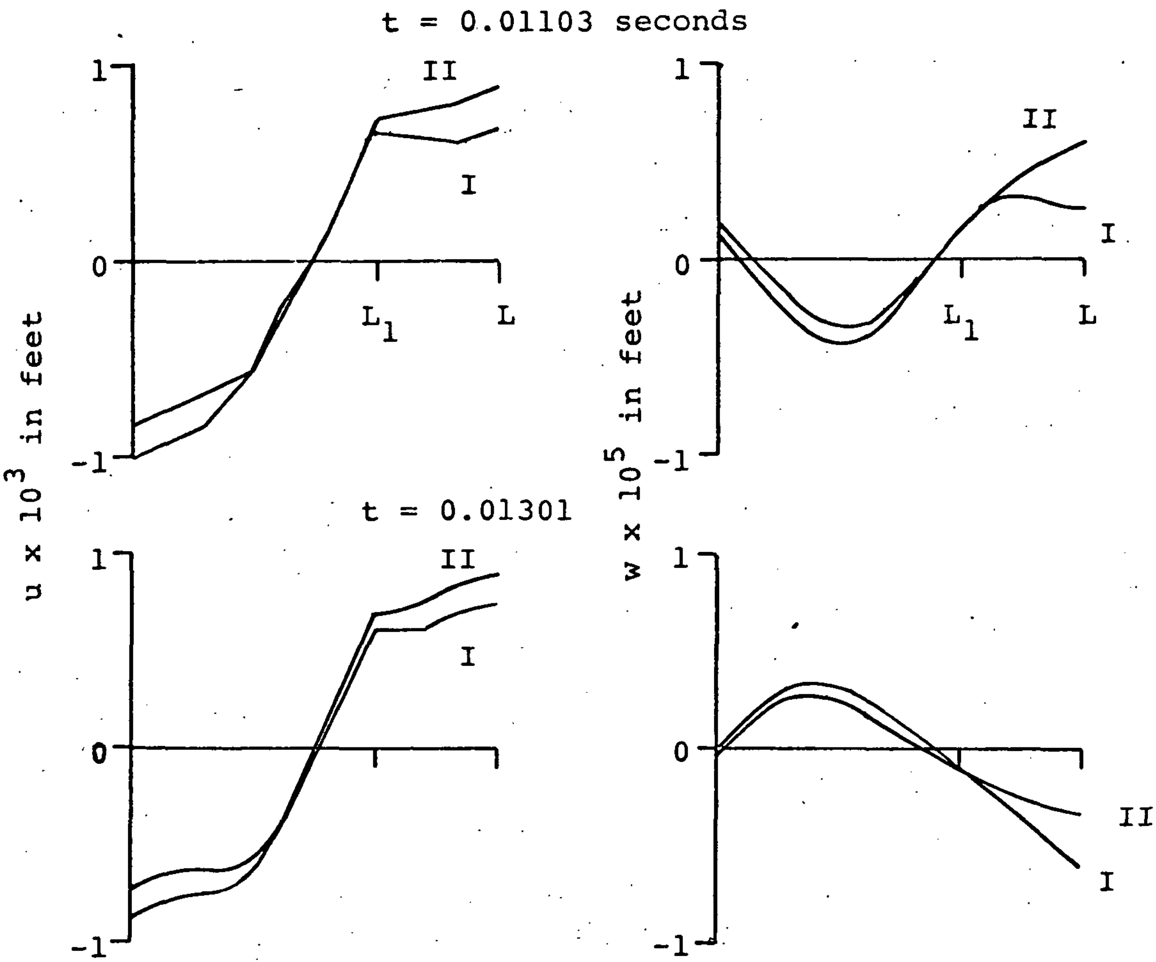


Figure 8 - Axial and transverse elastic displacements (cont.)

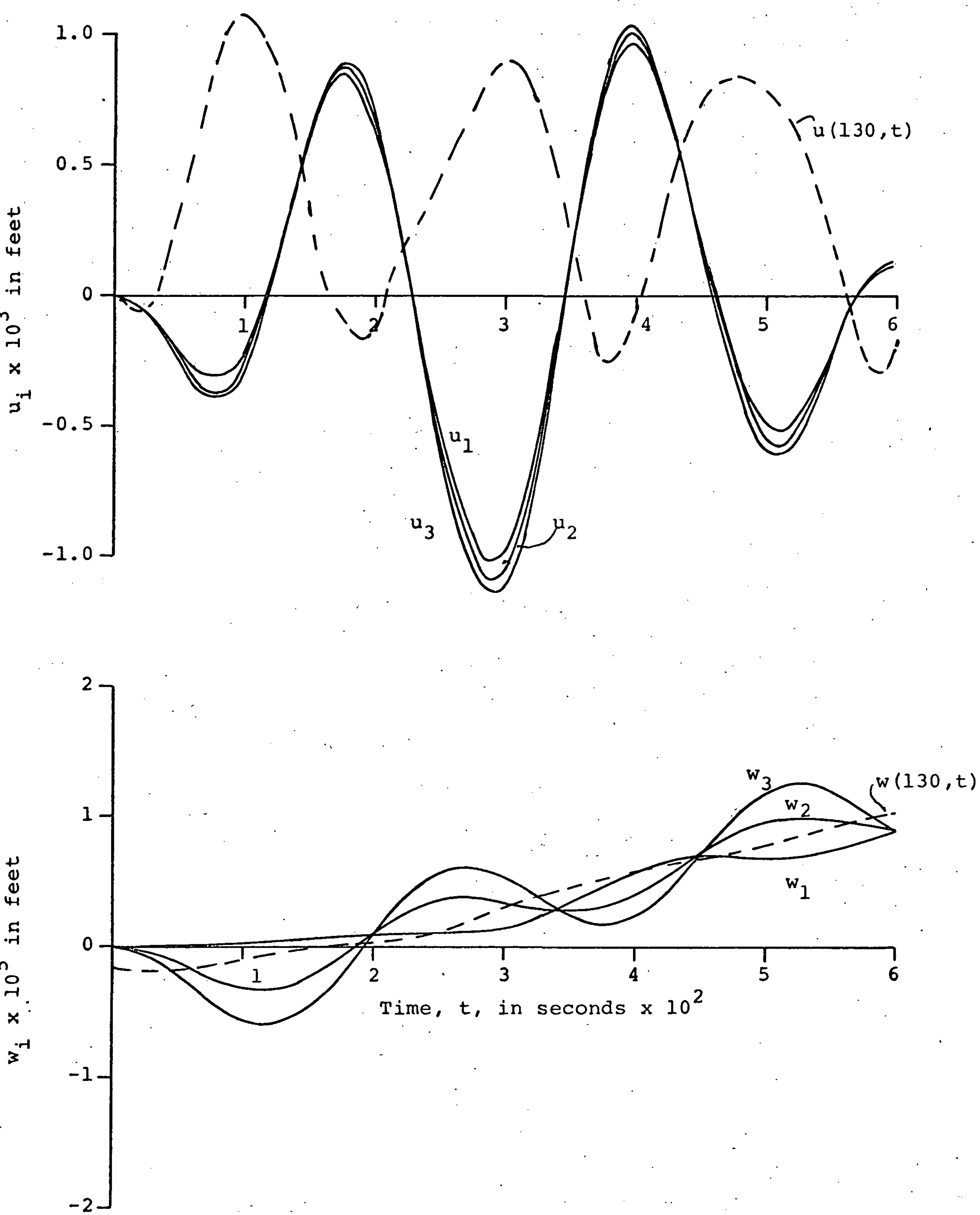
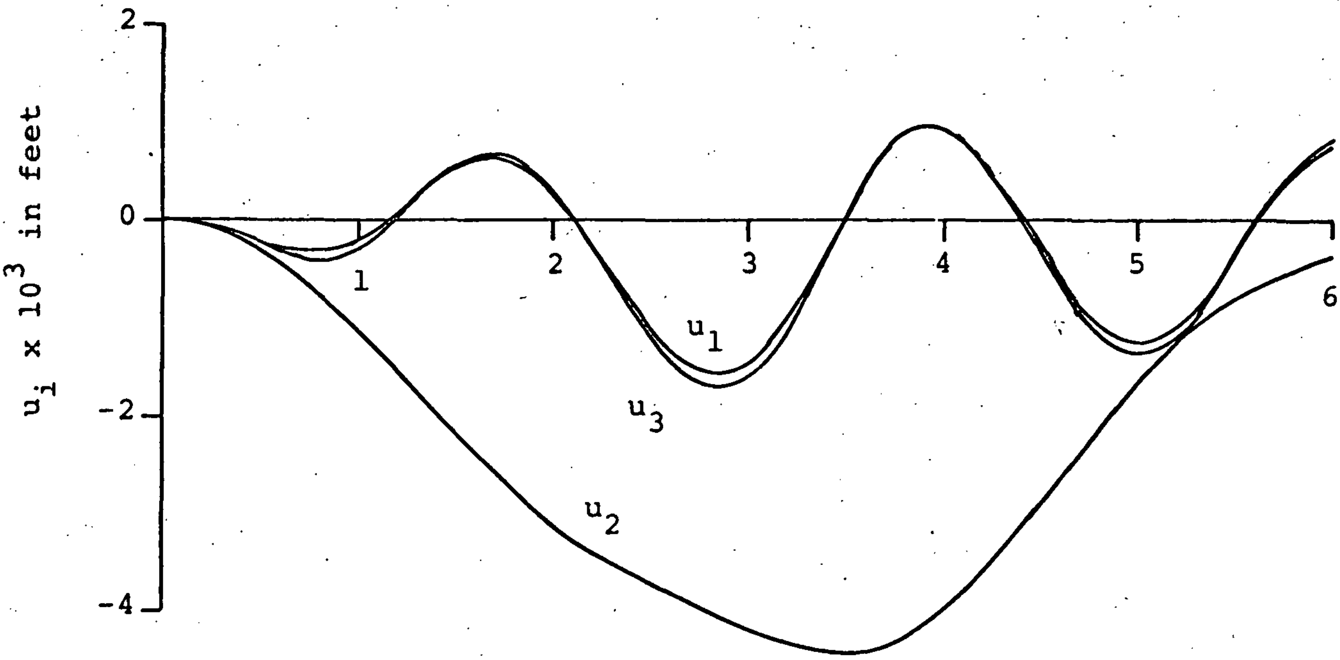


Figure 9 - Discrete mass axial and transverse displacements vs time (case I)



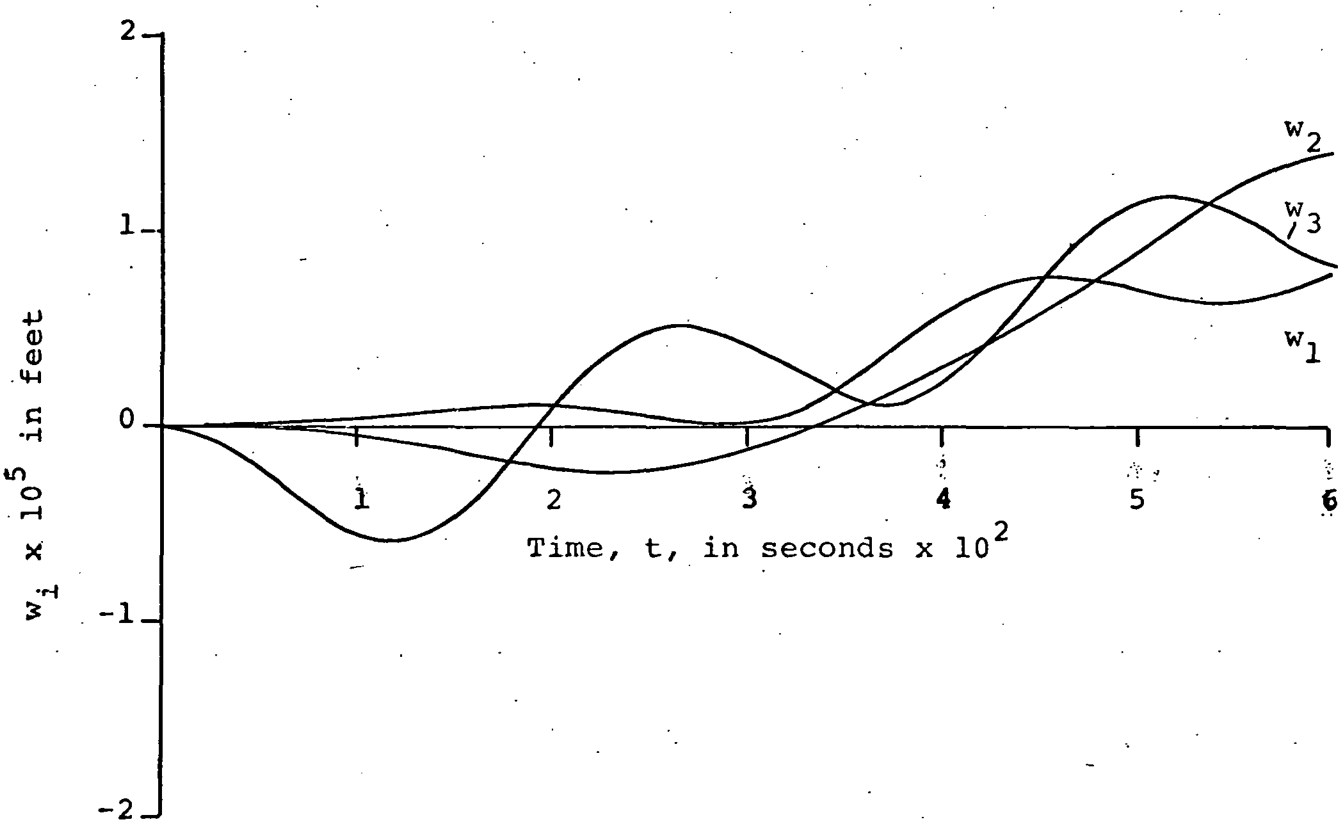


Figure 10 - Discrete mass axial and transverse displacements vs time (case II)

Within the limited cases considered and the relatively short burning time allowed, the launch angle θ_0 and the offsets of the discrete masses produced no noticeable effects. However, this is not to be interpreted as an implication that no significant effects of these parameters on the elastic motion of the case and the discrete masses should be expected when less limited cases are explored.

6. Summary and Conclusions

This investigation is concerned with the dynamic characteristics of a slender elastic body of variable mass. The analysis is applicable to a two-stage solid-fuel missile envisioned as a slender cylindrical body capable of both rigid-body motion as well as axial and transverse elastic deformations. The first stage contains the solid-fuel motor and posseses variable mass, whereas the second stage contains packaged instruments simulated by discrete masses.

The equations of motion were first written in vector form and then transformed by means of a variational principle into simultaneous equations of motion in terms of generalized coordinates.

Because no closed form solution of the equations is possible, a numerical solution based on finite difference relations had been pursued. A computer solution of a limited number of cases has been obtained. For the relatively small burning times used, the effect of the elastic motion on the rigid body motion was not noticeable. The motion of the discrete masses was different for different cases, depending on the manner in which they were attached to the case. If all masses were attached to the case as well as to each other by springs, the motion was synchronous. If a mass is disconnected from the case but remains attached to other masses through springs, the cyclic time increases.

Due to the nature of the finite difference method of solving the equations of motion, a very small time increment is necessary to obtain a stable solution and the resulting large computing time necessary to obtain any desired results makes this form of solution undesirable.

Several alternative methods to solve the equations of motion deserve investigation. One such method is to express the elastic displacements of the rocket case in terms of sets of eigenfunctions corresponding to the associated constant-mass system thereby converting the partial differential equations into ordinary differential equations. Methods of solving sets of ordinary differential equations are more adequate and the question of stability of the solution may prove less bothersome. This method should be attempted with the expectation that longer burn times will be possible.

References

1. Grubin, C. "Mechanics of Variable Mass Systems," Jour. Franklin Inst., Vol. 276, 1963, pp. 305-312.

2. Dryer, M., "Equations of Motion for an Arbitrary Body of Variable Mass," Martin-Denver Res. Dept. R-61-18, 1961.

3. Leitmann, G., "On the Equation of Rocket Motion," Jour. British Interplanetary Soc., Vol. 16, 1957, pp. 141-147.

4. Meriam, J.L. "Variable Mass Dynamics," Jour. Engineering Education, Vol. 51, 1960, pp. 241-243.

5. Davis, L. Jr., Follin, J.W. Jr., and Blitzer, L., Exterior Ballistics of Rockets, Van Nostrand Co., Princeton, N.J., 1958.

6. Seide, P., "Effect of Constant Longitudinal Acceleration on the Transverse Vibration of Uniform Beams," Aerospace Corp., TDR-169 (3560-30) TN-6, 1963.

7. Beal, T.R., "Dynamic Stability of a Flexible Missile Under Constant and Pulsating Thrusts," AIAA Jour., Vol. 3, 1965, pp. 486-494.

8. Miles, J.W., Young, D., Fowler, J.R., et al., "Generalized Missile Dynamics Series," Space Technology Labs., Vols. I - VII, GM - TR - 0165, 1958.

9. Birnbaum, J., "Bending Vibrations of a Perforated Grain Solid Propellant During Powered Flight., IAS Paper No. 61-30, 1961.

10. Edelen, D.G.B., "On the Dynamic Effects of Fuel Flow on the Motion of Boost Vehicles," Rand Corp., RM-3268 - NASA, 1962.

11. Price, E.W., "One-Dimensional, Steady Flow With Mass Addition and the Effects of Combustion Chamber Flow on Rocket Thrust," ARS Jour., Vol. 5, No. 2, 1955, pp. 61-68, 78.

12. Meirovitch, L., and Wesley, D.A., "On the Dynamic Characteristics of a Variable-Mass Slender Elastic Body Under High Acceleration," AIAA Jour., Vol. 5, No. 8, 1967, pp. 1439-1497.

13. Meirovitch, L., and Wesley, D.A., "On the Dynamic Characteristics of a Variable-Mass Slender Elastic Body Under High Acceleration," NASA CR-713, February, 1967.

14. Meirovitch, L., "Dynamic Characteristics of a Variable Mass Flexible Missile," Semi-Annual Technical Progress Report, NASA Research Grant NGR 36-004-030, February 1968.

15. Meirovitch, L., and Bankovskis, J., "Dynamic Characteristics of a Variable-Mass Flexible Missile," Semi-Annual Technical Progress Report, NASA Research Grant NGR 36-004-030, August, 1968.

16. Shames, I., Mechanics of Fluids, McGraw-Hill Book Company, Inc., N.Y., 1962.

17. Meirovitch, L., Analytical Methods in Vibrations, The Macmillan Company, N.Y., 1967.