STRUCTURAL MODEL OPTIMIZATION USING STATISTICAL EVALUATION

INTERIM REPORT

Prepared For

GEORGE C. MARSHALL SPACE FLIGHT CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MARSHALL SPACE FLIGHT CENTER, ALABAMA

Under Contract NAS8-27331

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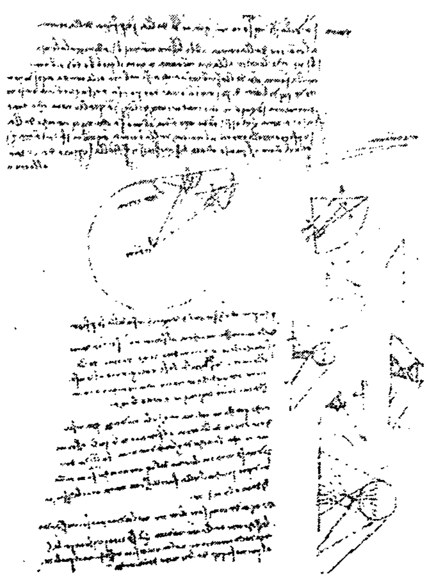

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- FOREWORD -

The work in this report was sponsored by the George C. Marshall Space Flight Center under NASA Contract NAS8-27331. The work was performed under the technical direction of Homer Pack, MSFC Code S&E Aero-DDS

The work on the contract has been the collective effort of four people, all of whom share authorship of the report: Jon D. Collins was project manager; Gary C. Hart and R. T. Gabler participated in the technical review, formulation and development; and Bruce Kennedy was responsible for the computer program development.

- ABSTRACT -

This report presents the results of research in applying statistical methods to the problem of structural dynamic system identification. The study is in three parts: a review of previous approaches by other researchers, a development of various linear estimators which might find application, and the design and development of a computer program which uses a Bayesian estimator.

The method is tried on two models and is successful where the predicted stiffness matrix is a proper model. e.g. a bending beam is represented by a bending model. Difficulties are encountered when the model concept varies. There is also evidence that nonlinearity must be handled properly to speed the convergence.

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- NOMENCLATURE -

Structure and Eigenproperty Definitions

x_i - i^{th} eigenvector

x_{ui} - u^{th} component of i^{th} eigenvector equivalent to x(i,u) in computer output

λ_i - i^{th} eigenvalue

K or [K] - stiffness matrix

k_{rs} - $(rs)^{th}$ component of the stiffness matrix

M or [M] - mass matrix

m_{rs} - $(rs)^{th}$ component of the mass matrix

Definitions Used in Estimation Theory

$x = \left\{ \frac{k_p}{m_p} \right\}$ - vector of physical properties

x^* - statistical best estimate of x based on acquired data and some estimation technique

$y = \left\{ \frac{d\lambda_i}{dx_{ji}} \right\}$ - vector of eigenproperties, in most cases only those quantities which have been measured in a test

[A] - matrix relating changes in physical properties of a structural system to changes in the mass and stiffness matrix elements

$$[A] = \left[\frac{\partial (k_{rs}, m_{rs})}{\partial (k_p, m_p)} \right]$$

[B] - matrix relating changes in eigenproperties to changes in mass and stiffness

$$[B] = \left[\frac{\partial (\lambda_i, x_{ji})}{\partial (k_{rs}, m_{uv})} \right]$$

$[T]$, T - matrix product, $[B][A]$

R - covariance matrix of the errors in measurement (ε). These correspond with the elements of the measured eigenproperties in the vector y.

S - covariance matrix of the uncertainty in the elements of the estimated vector x

W - weight matrix

$E\langle\ \rangle$ - expected value

$P(\ )$ - probability of an event

$p(\ )$ - probability density function of a random variable

ε - vector of measurement errors

σ - standard deviation

$(\ )'$ - transpose of the vector or matrix

$(\ )_{ls}$ - least squares

$(\ )_{mv}$ - minimum variance

$(\ )_{wls}$ - weighted least squares

$(\ )_{ml}$ - maximum likelihood

$(\ )_{c}$ - combined (used in the Bayes estimator development)

$(\ )_{b}$ - Bayes estimator approach

$(\ )_{x^*}$ - estimate of x

$(\ )_{x_p}$ - prior estimate of x

1.0 INTRODUCTION

Since December 1969, the Marshall Space Flight Center has been sponsoring research in the area of identification and modeling of uncertainty in structural dynamic models. In 1970 the J. H. Wiggins Company delivered to MSFC a computer program (VIDAP) which was able to trace identified uncertainty in mass and stiffness through to uncertainty in the frequencies and mode shapes. This program was particularly valuable when unusual elements had large stiffness uncertainty and this uncertainty could lead to problems in accurately predicting critical modes and frequencies.

The successful demonstration of the statistical method lead to other possibilities. For years there had been the continual problem of trying to make an analytical model match the data obtained from test. Many methods had been tried but none had operated successfully enough to receive general acceptance. Consequently the door was open to try a new approach which identified uncertainty in the model prior to trying to match the test data. This identified uncertainty would then aid in identifying where the largest changes would be applicable in the model.

This new study called "Structural Modes Accuracy Analysis," NAS8-27331 was initiated with the intent of using a multivariate conditional distribution to identify the revised masses and stiffnesses as a function of the measured modes and frequencies. Problems rose, however, in treating measurement uncertainty and a shift was made to the use of a Bayesian estimator which in format is very similar to a one-stage Kalman filter. The following sections describe this development and demonstrate the resulting computer program (MOUSE) on two problems with varying success.

From the work done to date we have developed confidence that the method selected is perhaps the most consistent and accurate system identification method found to date. However much work needs to be done in refinement as the problems of special applications, nonlinearity, non-orthogonality, proper models, etc. need more attention.

2. BACKGROUND REVIEW OF DYNAMIC MODEL OPTIMIZATION TECHNIQUES

2.1 INTRODUCTION

The formulation and verification of an analytical model which best represents the modal characteristics of a structure has been the goal of many engineers in recent years. The problem in this area which has received the most attention concerns itself with the use of experimentally established natural frequencies and mode shapes to estimate a structure's stiffness and/or mass matrices. In this chapter we summarize the various methods proposed by researchers for solution of this problem.

The methods surveyed fall into two basic categories. They either attempt to construct a simple low order structural model using only the measured modal characteristics of the structure or they seek to verify or modify a pre-established analytical model using available experiment data. The basic argument for the former approach is that it constructs a simple model which accurately represents the modal response over the range of the frequencies surveyed in the experimental test. In the latter approach, a considerable amount of pre-experiment effort is placed upon the formulation of an analytical structural model (usually a finite element model). Then, these analytically derived models of the structure's stiffness and mass matrices are altered by some rational scheme until their modal characteristics correspond to those obtained from the experimental test.

2.2 SURVEY OF METHODS

A simple direct procedure for the estimation of structural stiffness and mass matrices is possible when the number of structural degrees of freedom exactly corresponds to the number of modal properties measured. A structural model so formed from test data is said to be complete and in fact, it is also mathematically unique. Most of the references cited in this report describe this procedure. While the formulation of a unique complete structural model is most desirable it is seldom possible. The reason being that we seldom, if ever, have a one-to-one relationship between structural degrees of freedom and measured modal characteristics.

Considerable research has been done by Raney and Howlett (1)*, Ibanez et all (2), and Hillyer (3) with respect to the formulation of an incomplete mass and/or stiffness matrix which represents a structural model response characteristics within the frequency limits of the measured modal data. In particular, these efforts use the measured experimental data to construct an "equivalent" structural model whose number of degrees of freedom is equivalent to the number of measured modes. The shortcomings of the approaches are two-fold. First, they do not utilize recent advances in the finite element matrix modeling of structures and second, they do not allow for the direct use of the subjective judgment of the engineer.

Methods have been proposed by Gravitz (4), Rodden (5), Ross (6) and Berman (7) which formulate a structural stiffness matrix given the mass matrix and modal characteristics. Gravitz formulates the inverses of the stiffness matrix by averaging a generated stiffness matrix (which is not symmetric) and its transpose. This method seems to yield good results when the number of measured modal properties is significantly less than the order of the mass matrix and hence the number of modeled system degrees of freedom (Rodden (5)). However, Ross (6) notes that Gravitz's method seems to produce unacceptable results when the number of measured modes increases toward the number of model degrees of freedom. Ross suggests a method in which one calculates the characteristic shapes of the mass matrix and then adds them columnwise to the measured mode shapes to obtain a square positive definite modal matrix. Such a modal matrix may then be inverted and the formulation of a corresponding stiffness matrix follows directly. Alternately, Berman (7) defines an incomplete stiffness matrix by a matrix series summation. Each term in the summation contains the structure mass matrix and a single estimated natural frequency and mode shape.

If prior analytically established structural stiffness and mass matrices are available to the structural engineer then there exist methods where he may use this additional information as well as the measured modal characteristics to formulate the structural model. Ross (6) developed an algorithm which uses the measured modal characteristics plus as many characteristic shapes of the prior mass and/or stiffness matrices as necessary to yield a square, positive definite modal matrix. With this square modal matrix

*References used in Section 2 of this report are listed on pages 2-5 and 2-6.

the engineer then solves for a revised structural mass and stiffness matrix. A basic shortcoming of the procedure is that it does not categorize the changes in the stiffness and mass matrices with the uncertain elements of the structural model.

An alternate procedure which utilizes analytically derived mass and stiffness matrices has been proposed by Hall, Calkin and Sholar (8). The procedure uses a steepest decent gradient search algorithm to solve the problem in a minimization format. In particular, the authors form a special weighted least squares penalty function which they then minimize. The magnitude of the penalty function is a function of the difference between the measured and theoretical natural frequencies and mode shapes. In order to carefully classify the unknown parameters in the stiffness matrix, the procedure expresses the stiffness matrix as a matrix sum of a constant matrix and a series of unknown scalar variables times square position matrices. It is these scalar variables which are used to minimize the penalty function in the optimization routine. The basic disadvantage of this method is expense. In particular, one eigenvalue problem must be solved for each iteration and hence if the order of the matrices involved becomes large, so does the cost.

The methods previously discussed only utilize estimates of the structural modal characteristics to formulate the structure mass and/or stiffness matrices. There also exists several procedures which utilize special force-response measurements to estimate the structures mass and stiffness matrices. Kozin and Kozin (9, 10, 11) have proposed a method which uses statistical temporal averages. In their procedure they formulate as many equations in terms of temporal statistical moments as mass, damping or stiffness coefficients. The method seems particularly interesting and one can see where the least squares part of the procedure could be improved to include prior uncertainty in the parameters. However, the method seems to be particularly expensive for large structural systems. Also, it is necessary to obtain an extremely large number of time dependent measurements in order to define the structural parameters.

Other special force-response methods which have been proposed are in the transfer function category. Such methods include Berman and Flannelly (12), and Raney and Howlett (1).

Also of notice is a paper presented by Berg (13) which uses a least squares estimation procedure to develop tables and guides for special classes of simple coupled systems.

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3. DEVELOPMENT OF A STATISTICAL MODEL FOR ANALYTICAL MODEL OPTIMIZATION

3.1 General

The statistical model used in this program development operates on a base that was laid by Gauss 176 years ago.[(1)*] It was at that time (1795) that the 18 year old Gauss formulated a least squares method to estimate the parameters necessary to characterize the motion of heavenly bodies from measurement data. His work, followed by Legendre, formed the basis for the generalized least squares. 120 years later Fisher conceived the maximum likelihood method of determining parameters, and more recently (1955 to the present) a series of authors have contributed to least-square estimation. The discussion in this section surveys each of the major estimation approaches and then presents the particular method which best fits the structural dynamic problem.

3.2 Linear Relationships

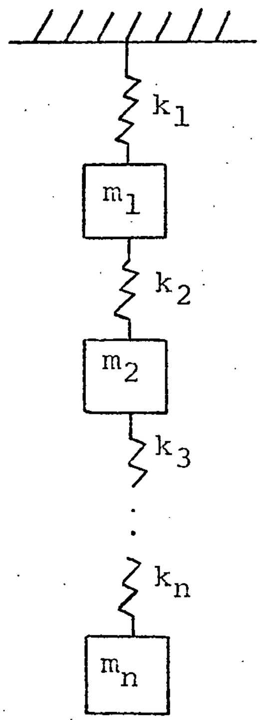
The objective of linear estimation theory is to make an estimate of the value of a set of numbers (defined by a vector) based on a set of observations. In the structural dynamic problem we want to revise the elements of the mass and stiffness matrix based on a set of observations of frequencies and mode shapes. In previous work[(2)] it has been established that small changes in the mass and stiffness elements of a structure can be related to changes in eigenvalues and eigenvectors by the equation:

$$\begin{Bmatrix} d\lambda_1 \\ d\lambda_2 \\ \cdot \\ dx_{11} \\ dx_{21} \\ \cdot \end{Bmatrix} = \begin{bmatrix} \frac{\partial\lambda_1}{\partial k_{11}} & \frac{\partial\lambda_1}{\partial k_{12}} & \frac{\partial\lambda_1}{\partial k_{13}} & \cdots & \frac{\partial\lambda_1}{\partial m_{nn}} \\ \frac{\partial\lambda_2}{\partial k_{11}} & \frac{\partial\lambda_2}{\partial k_{12}} & \frac{\partial\lambda_2}{\partial k_{13}} & \cdots & \frac{\partial\lambda_2}{\partial m_{nn}} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{\partial x_{11}}{\partial k_{11}} & \frac{\partial x_{11}}{\partial k_{12}} & \frac{\partial x_{11}}{\partial k_{13}} & \cdots & \frac{\partial x_{11}}{\partial m_{nn}} \\ \frac{\partial x_{21}}{\partial k_{11}} & \frac{\partial x_{21}}{\partial k_{12}} & \frac{\partial x_{21}}{\partial k_{13}} & \cdots & \frac{\partial x_{21}}{\partial m_{nn}} \\ \cdot & \cdot & \cdot & \cdots & \end{bmatrix} \begin{Bmatrix} dk_{11} \\ dk_{12} \\ \cdot \\ \cdot \\ \cdot \\ dm_{nn} \end{Bmatrix}$$

*References used in Section 3 of this report are listed on page 3-21 or

$$\left\{ \frac{d\lambda_i}{dx_{ji}} \right\} = [B] \left\{ \frac{dk_{rs}}{dm_{rs}} \right\} \tag{3-1}$$

The changes in the mass and stiffness matrix elements are linearly related to physical properties and to independent mass and stiffness elements. Consider the spring mass chain drawn below:



$$[M] = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & m_n \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdot & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdot & 0 \\ 0 & -k_3 & k_3+k_4 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & k_n \end{bmatrix}$$

Changes in the elements of $[K]$ and $[M]$ can be expressed as linear functions of k_1, k_2, ..., m_1, ..etc., as follows:

$$
\begin{Bmatrix} dk_{11} \\ dk_{12} \\ dk_{21} \\ dk_{22} \\ dk_{23} \\ dk_{32} \\ \cdot \\ \cdot \\ dm_{11} \\ \cdot \\ \cdot \end{Bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & \cdot & 0 & \cdot \\
0 & -1 & 0 & \cdot & 0 & \cdot \\
0 & -1 & 0 & \cdot & 0 & \cdot \\
0 & 1 & 1 & \cdot & 0 & \cdot \\
0 & 0 & -1 & \cdot & 0 & \cdot \\
0 & 0 & -1 & \cdot & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\begin{Bmatrix} dk_{1} \\ dk_{2} \\ dk_{3} \\ \cdot \\ dm_{1} \\ \cdot \end{Bmatrix}
$$

or

$$
\begin{Bmatrix} dk_{rs} \\ \text{-----} \\ dm_{rs} \end{Bmatrix} = [A] \begin{Bmatrix} dk_{p} \\ \text{---} \\ dm_{p} \end{Bmatrix} \tag{3-2}
$$

Combining (3-1) and (3-2) we obtain

$$
\begin{Bmatrix} d\lambda_{i} \\ \text{-----} \\ dx_{ji} \end{Bmatrix} = [B]\,[A] \begin{Bmatrix} dk_{p} \\ \text{---} \\ dm_{p} \end{Bmatrix} = [T] \begin{Bmatrix} dk_{p} \\ \text{---} \\ dm_{p} \end{Bmatrix} \tag{3-3}
$$

where $$[T] \equiv [B][A]$$

Therefore, we can express changes in a vector of modal properties as a linear function of changes in basic physical properties of the system. There is no dimensional requirement in (3-3), the matrix $[T]$ is rectangular with dimension $n_1 \times n_2$ where n_1 is the number of elements (rows) in $\left\{ \frac{d\lambda_i}{dx_{ji}} \right\}$ and n_2 is the number of elements (rows) in $\left\{ \frac{dk_p}{dm_p} \right\}$

No constrant is placed on the relative magnitude of n_1 and n_2.

When we optimize a structure analytical model using measured modal data, we are estimating $\left\{ \frac{dk_p}{dm_p} \right\}$ based on measured $\left\{ \frac{d\lambda_i}{dx_{ji}} \right\}$. Henceforth we denote the measured variables by the vector y, i.e. $y \equiv \left\{ \frac{d\lambda_i}{dx_{ji}} \right\}$ and let the variables to be estimated, be represented by the vector x, i.e. $x \equiv \left\{ \frac{dk_p}{dm_p} \right\}$.

Hence $$y = BAx$$

$$\text{or } y = Tx \tag{3-4}$$

3.3 Least Squares Estimation†

In the preceeding section, the linear relation $y = Tx$ was developed where y is a vector of observed data, T is a matrix of partial derivatives, and x is a vector to be estimated. Our objective is to develop the equation

$$x^* = Wy \tag{3-5}$$

where W is called the weight matrix and the superscript $(\ )^*$ represents an estimate of the quantity.

Considering the fact that the vector of observations, y, can have errors, we rewrite (3-4) to include this as

†Note material in this and subsequent sections is based on derivations found in Reference (3).

$$y = Tx + \varepsilon \tag{3-6}$$

where ε is the error vector of the observed data y. We assumed that the elements of ε are random variables with mean zero.

Substituting (3-6) into (3-5) we obtain

$$x^* = WTx + W\varepsilon \tag{3-7}$$

If an estimate x^* is unbiased, its expected value is equal to its true value, i.e. $E\langle x^*\rangle = x$. Therefore, this places a requirement on the development of W. Taking the expected value of (3-7) and setting equal to x so as to have an unbiased estimate it follows that

$$\begin{aligned} x = E\langle x^*\rangle &= E\langle WTx\rangle + E\langle W\varepsilon\rangle \\ &= WTE\langle x\rangle + WE\langle\varepsilon\rangle \\ &= WTx \end{aligned}$$

Therefore, if x^* is an unbiased estimate of x we must have a W which satisfies the criteria

$$WT = I$$

This constraint on W is referred to as the "exactness constraint".

We note that for the special case where T is a square positive definite matrix the solution for W is trivial. However, such a case implies that when we are estimating n unknowns we must have n data points. Such a case will seldom exist.

The least squares principle states that x^* should be chosen so that the sum of the squared components of the residual vector is minimized. If we define

$$\Delta^* \equiv y - Tx^* \tag{3-9}$$

as the residual vector (true observation minus simulated observations based on that estimate), then the objective is to minimize δ^* where

$$\begin{aligned} \delta^* &\equiv (\Delta^*)'(\Delta^*) \\ &= (Tx^* - y)'(Tx^* - y) \end{aligned} \tag{3-10}$$

We minimize δ^* by taking its derivative with respect to x* and setting it equal to zero†, i.e.

$$\frac{\partial \delta^*}{\partial x^*} = 0 = 2T'(Tx^* - y)$$

$$= T'Tx^* - T'y \tag{3-11}$$

$$T'Tx^* = T'y$$

$$\text{and } x^* = (T'T)^{-1}T'y = x^*_{\ell s} \tag{3-12}$$

Note that we denote the estimate using the least squares technique as $x^*_{\ell s}$ to distinguish it from the estimates obtained in the following sections.

Equation (3-12) is the classical least squares formula. The weight matrix W, Equation (3-5), is therefore

$$W = (T'T)^{-1}T' \equiv W_{\ell s} \tag{3-13}$$

where the notation $W_{\ell s}$ is used to distinguish it from the weight matrixes developed in the remaining sections of this text. Now, if we let R_ε be the diagonal covariance matrix of the independent measurement errors, i.e.,

$$R_\varepsilon = \begin{bmatrix} \sigma_{\varepsilon_1}^2 & & & & \\ & \sigma_{\varepsilon_2}^2 & & & \\ & & \sigma_{\varepsilon_3}^2 & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}$$

†Normally in least squares developments, the derivative is taken with respect to the parameters which in this case would be elements of T. However, the vector x can be treated as a parameter in the minimization process of δ* and the manipulation is easier. The proof of the equivalence of the two approaches and the vector differentiation are available in Reference (3).

then it follows that the covariance matrix for the estimate of $x^*_{\ell s}$ is:

$$S_{x^*_{\ell s}} = W_{\ell s} R_\varepsilon W'_{\ell s}$$

$$S_{x^*_{\ell s}} = (T'T)^{-1} T' R_\varepsilon T (T'T)^{-1} \qquad (3\text{-}14)$$

3.4 Minimum Variance Estimation

Although the weight matrix given by (3-13) for the least squares case satisfied the "exactness constraint" it is possible to find other weight matrices which also satisfy the constraint. In particular, it would be desirable to find a weight matrix which, besides satisfying the constraint, also minimized the covariance matrix of x^*, i.e.

$$(S_{x^*_{mv}})_{ii} \leq (S_{x^*})_{ii} \qquad (3\text{-}15)$$

where $(\ )_{ii}$ signifies that we are only considering the diagonal elements of the covariance matrix in the development.

In effect we are trying to find W_{mv} such that the diagonals of the matrix

$$S_{x^*_{mv}} = W_{mv} R_\varepsilon W'_{mv} \qquad (3\text{-}16)$$

are a minimum, where

$$x^*_{mv} = W_{mv} y$$

and

$$W_{mv} T = I$$

The solution for this weight matrix is obtained using Lagrange's method of undetermined multipliers. To demonstrate this, let us consider the special case where R_ε is a 3 x 3 and T is a 3 x 2. Let us denote

$$W = \begin{bmatrix} W_0 \\ -- \\ W_1 \end{bmatrix} = \begin{bmatrix} W_{00} & W_{01} & W_{02} \\ W_{10} & W_{11} & W_{12} \end{bmatrix} \tag{3-17}$$

and

$$T = \begin{bmatrix} T_0 \,\vdots\, T_1 \end{bmatrix} = \begin{bmatrix} t_{00} & t_{01} \\ t_{10} & t_{11} \\ t_{20} & t_{21} \end{bmatrix} \tag{3-18}$$

then the covariance matrix $S_X{}^*$ becomes

$$S_X{}^* = WR_{\varepsilon}W'$$

$$= \begin{bmatrix} W_0 \\ -- \\ W_1 \end{bmatrix} \begin{bmatrix} R_{\varepsilon} \end{bmatrix} \begin{bmatrix} W_0' \,\vdots\, W_1 \end{bmatrix} = \begin{bmatrix} W_0 R_{\varepsilon} W'_0 & W_0 R_{\varepsilon} W'_1 \\ W_1 R_{\varepsilon} W'_0 & W_1 R_{\varepsilon} W'_1 \end{bmatrix} \tag{3-19}$$

We are trying to minimize the diagonal elements of S_X*; i.e.: $W_0 R_{\varepsilon} W'_0$ and $W_1 R_{\varepsilon} W'_1$. Using the subject to the matrix constraint equation WT = I or the four scaler equations:

$$W_0 T_0 = 1$$

$$W_1 T_0 = 0$$

$$W_0 T_1 = 0$$

$$W_1 T_1 = 1$$

where to minimize $W_0 R_{\varepsilon} W'_0$ we use $W_0 T_0 = 1$ and $W_0 T_1 = 0$, and to minimize $W_1 R_{\varepsilon} W'_1$ we use $W_1 T_0 = 0$ and $W_1 T_1 = 1$.

Now, define a matrix of undetermined Lagrangian multipliers

$$\Lambda = \left[\Lambda_0 \,\vdots\, \Lambda_1\right] = \begin{bmatrix} \lambda_{00} & \lambda_{01} \\ \lambda_{10} & \lambda_{11} \end{bmatrix} \tag{3-20}$$

and form the Lagrangian function

$$L_0 \equiv W_0 R_\varepsilon W'_0 - (W_0 T_0 - 1) 2\lambda_{00} - (W_0 T_1 - 0) 2\lambda_{10} \tag{3-21}$$

Next differentiate with respect to each of the components of W and set equal to zero

$$\begin{aligned} 0 &= 2 \{1\ 0\ 0\}\ R_\varepsilon W'_0 - (\{1\ 0\ 0\}\ T_0 - 1)\ 2\lambda_{00} \\ &\qquad - \{1\ 0\ 0\}\ T_1\ 2\lambda_{10} \\ 0 &= 2 \{0\ 1\ 0\}\ R_\varepsilon W'_0 - (\{0\ 1\ 0\}\ T_0 - 1)\ 2\lambda_{00} \\ &\qquad - \{0\ 1\ 0\}\ T_1\ 2\lambda_{10} \\ 0 &= 2 \{0\ 0\ 1\}\ R_\varepsilon W'_0 - (\{0\ 0\ 1\}\ T_0 - 1)\ 2\lambda_{00} \\ &\qquad - \{0\ 0\ 1\}\ T_1\ 2\lambda_{10} \end{aligned} \tag{3-22}$$

Combining we have the vector equation

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = 2\ R_\varepsilon W'_0 - 2\ T_0 \lambda_{00} - 2\ T_1 \lambda_{10} \tag{3-23}$$

Then

$$R_\varepsilon W'_0 = \left[T_0 \,\vdots\, T_1\right] \begin{Bmatrix} \lambda_{00} \\ \lambda_{10} \end{Bmatrix} \tag{3-24}$$

Next repeat the procedure for $W_1 R_\varepsilon W_1'$

$$L_1 \equiv W_1 R_\varepsilon W_1' - (W_1 T_0 - 0) 2\lambda_{01} - (W_1 T_1 - 1) 2\lambda_{11} \tag{3-25}$$

which yields

$$R_\varepsilon W_1' = \left[T_0 \,\vdots\, T_1 \right] \begin{pmatrix} \lambda_{01} \\ \lambda_{11} \end{pmatrix} \tag{3-26}$$

These equations combine to give

$$R_\varepsilon W' = T\Lambda$$

or

$$WR_\varepsilon = \Lambda' T' \qquad \text{(R symmetric)}$$

It then follows that

$$W = \Lambda' T' R_\varepsilon^{-1} \tag{3-27}$$

and

$$WT = \Lambda' T' R_\varepsilon^{-1} T \tag{3-28}$$

Now reintroducing the exactness constraint

$$WT = I$$

we find that

$$I = \Lambda' T' R_\varepsilon^{-1} T$$

and

$$\Lambda' = (T' R_\varepsilon^{-1} T)^{-1} \tag{3-29}$$

Upon substituting Λ' into (3-27) we obtain W_{mv}.

$$W_{mv} = (T' R_\varepsilon^{-1} T)^{-1} T' R_\varepsilon^{-1} \tag{3-30}$$

and

$$S_{x^*_{mv}} = W_{mv} R_\varepsilon W'_{mv}$$

$$= (T' R_\varepsilon^{-1} T)^{-1} T' R_\varepsilon^{-1} R_\varepsilon \left[(T' R_\varepsilon^{-1} T)^{-1} T' R_\varepsilon^{-1} \right]'$$

$$= (T' R_\varepsilon^{-1} T)^{-1} \tag{3-31}$$

Our minimum variance estimate for x* then becomes

$$x^*_{mv} = W_{mv} Y \tag{3-32}$$

3.5 Weighted Least Squares

In Section 3.3 the least squares procedure was developed based on minimizing the equation

$$\delta^* = (Tx^* - y)'(Tx^* - y) \tag{3-33}$$

This equation can be modified by a symmetrical least squares weighting matrix P which is introduced as shown

$$\delta^* = (Tx^* - y)' P (Tx^* - y) \tag{3-34}$$

One possible form of this least squares weighting matrix is related to the measurement accuracies of y. Naturally more weight should be given to those observations in y where the measurement error is small and less weight where the error is large. The covariance matrix R_ε expresses the accuracies of the measurements in terms of the variances of the observations. Hence the weighting of the observations should be the inverse of the observation accuracies and thus it is logical to let $P = R_\varepsilon^{-1}$. Equation (3-34) now becomes

$$\delta^* = (Tx^* - y)' R_\varepsilon^{-1} (Tx^* - y) \tag{3-35}$$

To minimize (3-35) we take its derivative with respect to x* and set it equal to zero, i.e.

$$\frac{\partial \delta^*}{\partial x^*} = 0 = 2T'R_\varepsilon^{-1}(Tx^* - y)$$

$$0 = T'R_\varepsilon^{-1}Tx^* - T'R_\varepsilon^{-1}y$$

$$x^* = (T'R_\varepsilon^{-1}T)^{-1}T'R_\varepsilon^{-1}y \tag{3-36}$$

which means that

$$W_{w\ell s} = (T'R_\varepsilon^{-1}T)^{-1}T'R_\varepsilon^{-1} \tag{3-37}$$

because

$$x^*_{w\ell s} = W_{w\ell s}y \tag{3-38}$$

Therefore, if we compare (3-37) with (3-30) we see that this weighted least squares estimate is, in fact, the minimum variance estimate.

3.6 Principle of Maximum Likelihood

The maximum likelihood function can be expressed as the joint probability density function of all the measurement errors, ε. If we assume that the errors are normally distributed, than the likelihood function is given by

$$L_{m\ell} = f(\varepsilon)$$

$$= \frac{1}{(2\Pi)^{n/2}|R_\varepsilon|} \exp\left[-\tfrac{1}{2}\varepsilon' R_\varepsilon^{-1}\varepsilon\right] \tag{3-39}$$

where n is the total number of measurements.

The errors, ε, can be written as a function of y and x*, i.e.

$$\varepsilon = y - Tx^*$$

and using this it follows from (3-39) that

$$L_{m\ell} = \frac{1}{(2\Pi)^{n/2}|R_\varepsilon|} \exp\left[-\tfrac{1}{2}(y-Tx^*)'R_\varepsilon^{-1}(y-Tx^*)\right] \tag{3-40}$$

When the peak value of this distribution is maximized, its variance is minimized. Maximization can be made on the basis of x^* since we are seeking an x^* in our solution which minimizes ε.

To obtain a maximum on x^* we first take the logarithm of both sides of (3-40) and then differentiate with respect to the optimizing parameter x^*.

So doing, it follows that

$$\ln(L_{m\ell}) = -\ln\left[(2\Pi)^{n/2}|R_\varepsilon|\right] - \tfrac{1}{2}(y-Tx^*)'R_\varepsilon^{-1}(y-Tx^*) \quad (3\text{-}41)$$

and

$$\frac{\partial\left[\ln(L_{m\ell})\right]}{\partial x^*} = 0 = 0 + T'R_\varepsilon^{-1}(Tx^* - y)$$

It then follows that

$$x^* = (T'R_\varepsilon^{-1}T)^{-1}T'R_\varepsilon^{-1}y \quad (3\text{-}42)$$

and finally that

$$W_{m\ell} = (T'R_\varepsilon^{-1}T)^{-1}T'R_\varepsilon^{-1} \quad (3\text{-}43)$$

because

$$x^* = W_{m\ell}y \quad (3\text{-}44)$$

(3-43) is identical to (3-30) and (3-37). Hence the maximum likelihood estimate is also the minimum variance estimate and equivalent to the weighted least squares estimate in the situation where the errors ε are assumed to be normally distributed.

3.7 Incorporation of Prior Information

In the methods described in the preceeding sections, no consideration was given to the fact that the analyst may have useful prior information which could be used in obtaining an optimum x^* (in our case optimum revised stiffness and mass matrix elements). Consequently, all data were weighted either equally or by the inverse of the measurement error and no attempt was made to acknowledge the fact that some estimated stiffnesses are more certain than others and should be less subject to change. It is desirable to give consideration to prior predictability of the stiffness and mass characteristics of the structure.

To use prior information, we can make use of the principle of Bayes rule which can be expressed as follows:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

where $P(B|A)$ = probability of the occurrence of B given that A has occurred

$P(A|B)$ = probability of the occurrence of A given that B has occurred

$P(A)$ = probability that A will occur.

$P(B)$ = probability that B will occur.

Translated into the language of our problem, this becomes:

$$P(k \;\&\; m|\text{test eigenvalues and eigenvectors})$$

$$= \frac{P\;(\text{test eigenvalues and eigenvectors}|k\&m)\;P(k\&m)}{P\;(\text{test eigenvalues and eigenvectors})} \quad (3\text{-}45)$$

P (test eigenvalues and eigenvectors|k&m) is the probability distribution for λ and x developed in the computer program called VIDAP; P (k & m) is the probability distribution describing expected means and variances for k & m; P (k & m | test eigenvalues and eigenvectors) is the probability distribution which contains the revised mean values and covariances of the stiffnesses and masses in light of the test information.

If, (1) the observation errors are assumed to be normally distributed, (2) the prior distribution is assumed to be normally distributed, and (3) the prior distribution and the measurement errors are independent, the conditional probability density function can be written as

$$p(x|y) = p(y|x)p(y) \quad (3\text{-}46)$$

and at this point substitutions can be made for y and x, multivariate density functions can be constructed, and the means and variance can be obtained for the estimate x^*. (Note $p(x)$ is a probability density function whereas $P(A)$ is a probability of an event).

A rather direct approach which produces the same results and is computationally easier can be used in place of the Bayes formulation. This second formulation utilizes the minimum variance principle to find a W which minimizes the variance of x^* around both the observation errors and the prior estimate.

$$\text{Define } x_p = x + \varepsilon_p \tag{3-47}$$

where x_p is a vector of random variables describing the estimated mean values and covariances of x (stiffness and mass elements), x is the true value of the stiffness and mass elements (unknown) and ε_p is the vector representing errors in the estimate of x.

Now if we let $y = Tx + \varepsilon$ as first described in Equation (3-6) then equations (3-6) and (3-47) can be combined into a single matrix equation

$$\begin{Bmatrix} x_p \\ y \end{Bmatrix} = \begin{bmatrix} I \\ \hline T \end{bmatrix} x + \begin{Bmatrix} \varepsilon_p \\ \varepsilon \end{Bmatrix}$$

$$y_c = T_c x + \varepsilon_c \tag{3-48}$$

Define $y_c \equiv \begin{Bmatrix} x_p \\ y \end{Bmatrix}$

$$T_c \equiv \begin{bmatrix} I \\ \hline T \end{bmatrix}$$

$$\varepsilon_c \equiv \begin{Bmatrix} \varepsilon_p \\ \varepsilon \end{Bmatrix}$$

Denote

$$R_c = \left[\begin{array}{c|c} S_{x_p} & 0 \\ \hline 0 & R_\varepsilon \end{array}\right] \tag{3-49}$$

where S_{x_p} is the covariance matrix of the prior distribution of x. From a user's standpoint S_{x_p} is the covariance matrix that must be developed to define uncertainty in estimated mass and stiffness properties.

It then follows from the minimum variance formulation, Equation (3-30),

$$x_{b^*} = (T_c' R_c^{-1} T_c)^{-1} T_c' R_c^{-1} y_c \tag{3-50}$$

and

$$S_{x_{b^*}} = (T_c' R_c^{-1} T_c)^{-1} \tag{3-51}$$

where

$$R_c^{-1} = \left[\begin{array}{c|c} S_{x_p}^{-1} & 0 \\ \hline 0 & R_\varepsilon^{-1} \end{array}\right] \tag{3-52}$$

Noting that

$$T_c' T_c^{-1} = \left[I \mid T' \right] \left[\begin{array}{c|c} S_{x_p}^{-1} & 0 \\ 0 & R_\varepsilon^{-1} \end{array}\right]$$

$$= \left[S_{x_p}^{-1} \mid T' R_\varepsilon^{-1} \right] \tag{3-53}$$

and

$$T_c' R_\varepsilon^{-1} T = \left[S_{x_p}^{-1} \mid T' R_\varepsilon^{-1} \right] \left[\begin{array}{c} I \\ \hline T \end{array}\right]$$

$$= S_{x_p}^{-1} + T' R_\varepsilon^{-1} T \tag{3-54}$$

we obtain

$$S_{x_{b^*}} = \left[S_{x_p}^{-1} + T' R_\varepsilon^{-1} T \right]^{-1} \tag{3-55}$$

Similarly, noting that

$$T_c' R_c^{-1} y = \left[S_{x_p}^{-1} \mid T' R_\varepsilon^{-1} \right] \left\{\begin{array}{c} x_p \\ \hline y \end{array}\right\}$$

$$= S_{x_p}^{-1} x_p + T' R_\varepsilon^{-1} y \tag{3-56}$$

it follows that

$$x_b^* = \left[S_{x_p}{}^{-1} + T'R_\varepsilon{}^{-1}T\right]^{-1}\left[S_{x_p}{}^{-1}x_p + T'R_\varepsilon{}^{-1}y\right]$$

$$= \left[S_{x_p}{}^{-1} + T'R_\varepsilon{}^{-1}T\right]^{-1}\left[S_{x_p}{}^{-1}x_p + T'R_\varepsilon{}^{-1}y + T'R_\varepsilon{}^{-1}Tx_p - T'R_\varepsilon{}^{-1}Tx_p\right]$$

$$= \left[S_{x_p}{}^{-1} + T'R_\varepsilon{}^{-1}T\right]^{-1}\left\{\left[S_{x_p}{}^{-1} + T'R_\varepsilon{}^{-1}T\right]x_p + T'R_\varepsilon{}^{-1}\left[y - Tx_p\right]\right\}$$

$$= x_p + \left[S_{x_p}{}^{-1} + T'R_\varepsilon{}^{-1}T\right]^{-1}T'R_\varepsilon{}^{-1}\{y - Tx_p\}$$

$$= x_p + S_{x_b}{}^* \ T'R_\varepsilon{}^{-1}\{y - Tx_p\} \qquad (3\text{-}57)$$

Note that the primary shortcoming in (3-57) is the computation of $S_{x_{b*}}$ which involves the inversion of a large matrix (see (3-55)). However, a convenient identity, sometimes attributed to Householder[5], eliminates most of the inversion problem. The identity is

$$[S_{x_p}{}^{-1} + T'R_\varepsilon{}^{-1}T]^{-1} = S_{x_p} - S_{x_p}T'[R_\varepsilon + TS_{x_p}T']^{-1}TS \qquad (3\text{-}58)$$

which substituted into (3-55) and 3-57) yields

$$x_{b*} = x_p + [S_{x_p} - S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1}TS_{x_p}]T'R_\varepsilon{}^{-1}\{y - Tx_p\}$$

$$x_{b*} = x_p + [S_{x_p}T'R_\varepsilon{}^{-1} - S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1}TS_{x_p}T'R_\varepsilon{}^{-1}] \cdot\{y - T_{x_p}\}$$

$$= x_p + [S_{x_p}T'R_\varepsilon{}^{-1} - S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1} \cdot(R_\varepsilon + TS_{x_p}T' - R_\varepsilon)R_\varepsilon{}^{-1}]\{y - T_{x_p}\}$$

$$= x_p + [S_{x_p}T'R_\varepsilon^{-1} - S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1}(R_\varepsilon + TS_{x_p}T')R_\varepsilon^{-1}$$

$$+ S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1}R_\varepsilon R_\varepsilon^{-1}]\cdot\{y - Tx_p\}$$

$$= x_p + [S_{x_p}T'R_\varepsilon^{-1} - S_{x_p}T'R_\varepsilon^{-1} + S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1}]$$

$$\cdot\{y - T_{x_p}\}$$

$$= x_p + S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1}\{y - T'x_p\} \qquad (3\text{-}59)$$

Finally we obtain that

$$x_{b*} = x_p + W_b\{y - Tx_p\} \qquad (3\text{-}60)$$

where $\quad W_b = S_{x_p}T'(R_\varepsilon + TS_{x_p}T')^{-1}$

the posterior covariance matrix of the estimates is then

$$S_{x_b}{}^* = S_{x_p} - S_{x_p}T'\left[R_\varepsilon + TS_{x_p}T'\right]^{-1}TS_{x_p} \qquad (3\text{-}61)$$

Equations (3-59), (3-60), and (3-61) form the basis for the MOUSE computer program developed under this contract. The matrix T is the product BA in Equation (3-3), S_{x_p} is the covariance matrix describing the uncertainty in estimates of the vector $\left\{\frac{dk_p}{dm_p}\right\}$, x_p is the initial estimate of the physical properties $\left\{\frac{k_p}{m_p}\right\}$, y is the vector of observed eigenvalues and eigenvectors, $\left\{\frac{\lambda i}{x_{ji}}\right\}$, and R_ε is the diagonal matrix of variances of the observations.

3.8 Summary of Estimation Techniques

Table 3-1 summarizes the estimation formulas developed in the preceeding sections. The methods can be divided easily into three groups:

(1) estimation without statistical consideration,

(2) estimation with consideration of observation error,

(3) estimation with consideration of observation error and prior information.

Table 3-1 Summary of Estimation Techniques

Name	Use observation errors?	Use prior estimate?	Estimation formula	Estimator, W	Covariance of Estimate S_x^*
Least Squares	No	No	$x^*=Wy$	$W=(T'T)^{-1}T$	$S_x^*=(T'T)^{-1}T'R_\varepsilon T\ (T'T)^{-1}$
Minimum Variance Weighted Least Squares Maximum Likliehood	Yes	No	$x^*=Wy$	$W=(T'R_\varepsilon^{-1}T)^{-1} \cdot T'R_\varepsilon^{-1}$	$S_x^*=(T'\ R_\varepsilon^{-1}T)^{-1}$
Bayes	Yes	Yes	$x^*=x_p +W(y-Tx_p)$	$W=S_{x_p}T'(R_\varepsilon+ TS_{x_p}T')^{-1}$	$S_x^*=S_{x_p} -S_{x_p}T'\left[R_\varepsilon+TS_{x_p}T'\right]^{-1}TS_{x_p}$

x^* - estimate of vector x

y - vector of observations

T - matrix relating x and y

x_p - prior estimate of x

R_ε - covariance matrix of observation errors

S_{x_p} - covariance matrix describing uncertainty of prior estimate of x

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4. DEVELOPMENT OF THE MOUSE PROGRAM

4.1 Approach

The Bayes estimator described in the previous section was selected for use in MOUSE. This estimator permits the user to make a judgment on the quality of various parts of his model. Although he may be wrong, the method provides the opportunity for preferential treatment so that stiffnesses that are easily estimated are weighted over stiffnesses that are not.

The basic equations used in MOUSE are all derived in Section 3. The equations for the Bayes estimator are shown in Table 3-1, page 3-20. The operations in MOUSE are described by the flow diagram in Figure 4-1.

4.2 Inputs

Other than options and certain special instructions the inputs to MOUSE are described by those boxes in Figure 4-1 having identification starting with I-__. A discussion of these inputs follows.

I-1

The first inputs are the stiffness and mass matrices. The mass matrix is restricted in this case to a diagonal matrix. There are no special restrictions, the system can have rigid body motion.

I-2

Matrix [A] must be entered to identify the relationships between the elements of the mass and stiffness matrices with the independent mass and stiffness properties. The development of the [A] matrix for a spring-mass chain is shown in Section 3-2. The [A] matrix is equivalent to

$$\left[{}^{(i)} \frac{\partial (k)}{\partial (p)} \right] \text{ in VIDAP.}$$

I-3

$[S_{km_p}]$ is the prior estimate of the covariance of the independent mass and stiffness properties. In the case of a spring-mass chain this matrix would contain along

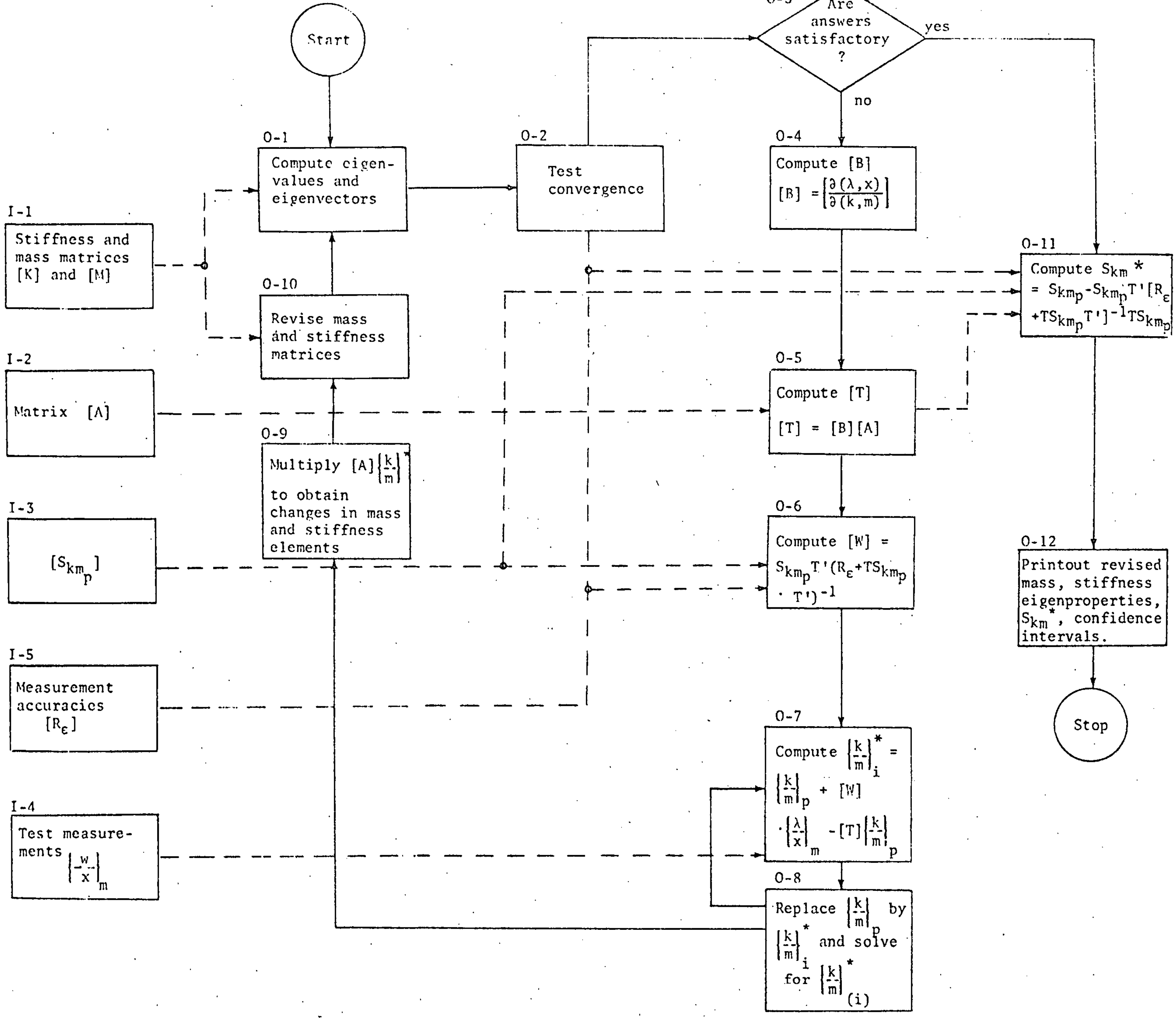


Figure 4-1 MOUSE Operations

the diagonal the variances (σ^2) of each of the masses and stiffnesses of the chain. If mass and stiffness elements are statistically correlated $[S_{km_p}]$ will have off-diagonal elements representing the covariances, otherwise $[S_{km_p}]$ is diagonal. The development of $[S_{km_p}]$ is identical to $[{}^{(i)}\Sigma_p]$ in VIDAP[1].

I-4

All of the test measurements of interest form a vector

$$\left\{\frac{\omega}{x}\right\}_m = \left\{\begin{array}{c} \omega_1 \\ \omega_2 \\ \cdot \\ \cdot \\ \cdot \\ \hline x_{11} \\ x_{21} \\ \cdot \\ \cdot \\ x_{ij} \\ \cdot \\ \cdot \end{array}\right\}$$

The frequencies are immediately transformed to eigenvalues and are used in that form throughout. Only eigenvectors of interest need to be input and then only parts of the eigenvectors if the test measurements are restricted. There is no minimum to the number of measurements, a single frequency or eigenvector component can be entered and the program will optimize on the data.

I-5

The accuracies of the measurements must be entered to establish the quality of the data. Since the accuracies will generally be uncorrelated, variances are estimated for each measurement and the variances form the elements of the diagonal matrix $[R_\varepsilon]$. The sequential order of the variances must match the sequential order of the measurements in I-4.

4.3 Operations

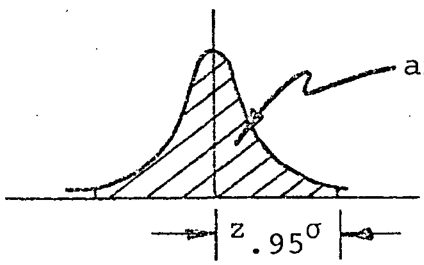
Operations in Figure 4-1 are those boxes having identifications starting with O-_. Solid lines in the flow diagram represent movement to one activity or another and dashed lines represent flow of information.

O-1 Eigenvalue/vector Computation

The mass and stiffness matrices are entered into a Givens-Householder eigenvalue routine and eigenvalues and eigenvectors are computed which correspond with those measured in the test.

O-2 Convergence Test

Confidence intervals are developed from the measurement accuracies to test convergence. The standard deviation of the measurement accuracy is multiplied by the factor corresponding to the 90, 95, or 99 percentage levels of the normal distribution



Next the difference is computed between the measured and predicted eigenvalues and eigenvectors.

$$\begin{Bmatrix} \Delta\lambda_1 \\ \Delta\lambda_2 \\ \cdot \\ \cdot \\ \Delta x_{11} \\ \Delta x_{21} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ x_{11} \\ x_{21} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}_{\text{meas.}} - \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ x_{11} \\ x_{21} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}_{\text{predicted}}$$

The column of delta values are then compared with the $z_{.95}\sigma$'s

$$\begin{Bmatrix} \Delta\lambda_1 \\ \Delta\lambda_2 \\ \cdot \\ \cdot \\ \Delta x_{11} \\ \Delta x_{21} \end{Bmatrix} \text{vs} \begin{Bmatrix} z_{.95}\sigma_{\lambda_1} \\ z_{.95}\sigma_{\lambda_2} \\ \cdot \\ \cdot \\ z_{.95}\sigma_{x_{11}} \\ z_{.95}\sigma_{x_{21}} \end{Bmatrix}$$

If 95% confidence limits have been chosen, we can expect one in twenty $\Delta\lambda$ or Δx to exceed its corresponding $z_{.95}\sigma$. However, if more than one in twenty exceed $z_{.95}\sigma$ we can conclude then the answers have not properly converged and the program must pass through another convergence. The convergence test is therefore a ratioing of exceedences to non-exceedences to determine if too many lie outside the confidence interval.

O-3 Convergence Test

This block directs the program to recycle or to perform final computations depending upon the results from O-2.

O-4 Compute [B]

The matrix $[B] = \left[\frac{\partial(\lambda,x)}{\partial(k,m)}\right]$ is computed using the techniques developed for VIDAP. These partial derivatives correspond to the measured values of the eigenvectors and eigenvalues and the particular estimated values of mass and stiffness properties. The number of partial derivatives computed is not contingent upon the number of degrees of freedom in the problem but rather on the number of measurements and the number of elements allowed to vary in the structure.

O-5 Compute [T]

The linear transformation matrix, [T], relating eigenproperties to stiffness and mass is formed from the product [B][A].

O-6 Compute [W]

The Bayes estimator [W] is computed using the formula

$$W = S_{km_p} T' (R_\varepsilon + T S_{km_p} T')^{-1}$$

O-7 Compute $\left\{\frac{k}{m}\right\}^*$

The original mean values of the independent mass and stiffness properties having estimated uncertainty are now revised using the formula

$$\left\{\frac{k}{m}\right\}^*_i = \left\{\frac{k}{m}\right\}_p + [W]\left(\left\{\frac{\lambda}{x}\right\}_m - [T]\left\{\frac{k}{m}\right\}_p\right)$$

O-8 Resolution of $\left\{\frac{k}{m}\right\}^*$ using $\left\{\frac{k}{m}\right\}^*_i$

The $\left\{\frac{k}{m}\right\}^*_i$ computed in O-7 replaces $\left\{\frac{k}{m}\right\}_p$ to permit a cycling through O-7 several times.

O-9 Compute Changes in Mass and Stiffness Matrix Elements

After O-7 and O-8, the new $\left\{\frac{k}{m}\right\}^*$ is used to revise the mass and stiffness matrices. The revisions are computed using the [A] matrix

$$
\begin{Bmatrix} k_{11} \\ k_{12} \\ \cdot \\ \cdot \\ --- \\ m_{11} \\ m_{22} \\ \cdot \\ \cdot \end{Bmatrix} = [A] \left\{\frac{k}{m}\right\}^*
$$

O-10 Revise the Mass and Stiffness Matrices

The revisions computed in the prior step are incorporated into the mass and stiffness matrices prior to recomputation of the eigenproperties.

O-11 Compute $S_{km}{}^*$

Once the program has converged the covariance matrix of the values of the independent mass and stiffness properties can be recomputed based on the revised values.

$$
S_{km}{}^* = S_{km_p} - S_{km_p} T' [R_\varepsilon + T S_{km_p} T']^{-1} T S_{km_p}
$$

O-12 Printout

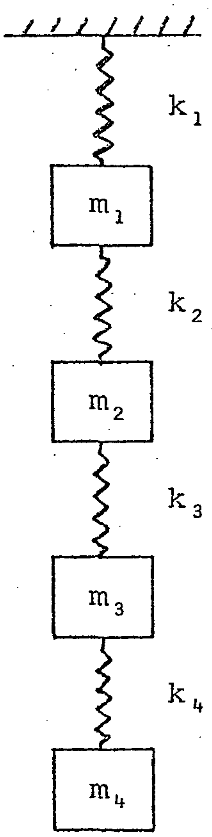
The final printout contains the revised mass and stiffness matrices, eigenproperties, $S_{km}{}^*$ and the confidence intervals of the eigenproperties.

The operations in the program which are the most time consuming are O-1 and O-4. Both are equivalent in length. Thus if the program requires three iterations in the outer loop the time required is equivalent to approximately six eigenvalue/vector computations.

5. DEMONSTRATION OF MOUSE

5.1 Spring-Mass Problems

The first problem selected for demonstration and checkout of MOUSE was a four degree-of-freedom spring-mass system. The basic reasons for the selection were the small number of degrees of freedom and the ease in which a set of hypothetical test data with errors could be derived. The system with associated mass and stiffness matrices is shown below.



$$[K] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

$$[M] = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & m_3 & \\ 0 & & & m_4 \end{bmatrix}$$

Note that the stiffnesses in the chain are distributed in the stiffness matrix, i.e. $k_{11} = k_1+k_2$, $k_{12} = -k_2$, etc. To maintain these relationships, an [A] matrix is developed.

$$\{dk_{ij}\} = \begin{bmatrix} dk_{11} \\ dk_{12} \\ dk_{22} \\ dk_{23} \\ dk_{33} \\ dk_{34} \\ dk_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dk_1 \\ dk_2 \\ dk_3 \\ dk_4 \end{bmatrix} = [A_{ij,r}]\{dk_r\}$$

To maintain physical significance, the changes must be make to k_1, k_2, k_3 and k_4. Consequently the matrix $[A_{ij,r}]$ plays a very important role in the stiffness matrix optimization.

Initially it may be difficult to predict certain stiffnesses because of elemental complexity (e.g. bending of a conical shell). Since the user of the program will have to estimate how well he can predict each of the stiffnesses, k_1, k_2, . . , he can make his estimate part relative and part absolute. For example the stiffness of a hollow circular tube will be more predictable than the stiffness of the conical shell. The analyst may feel that his uncertainty in the second case is five times the uncertainty in the first. He may then assign a 5% uncertainty (from experience) to the tube stiffness and a 25% uncertainty to the shell stiffness. These numbers are then input into the program in a covariance matrix for k and m

$$[S_{k,m}] = \begin{bmatrix} \sigma_{k_1}^2 & & & \\ & \sigma_{k_2}^2 & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix}$$

Note that the covariance matrix and mean values for k and m define the "prior distribution" of the mass and stiffness matrix. It is the objective of the program to update this prior distribution using test data.

Finally, the test measurements are input along with the measurement accuracies. Reasonable prediction of the accuracies is important because a bad test point with an unreasonably high estimate of accuracy would drive the program away from an optimum solution.

Sample results for a specific four degree-of-freedom chain are presented in Tables 5-1 through 5-3. Note that the examples shown use all of the eigenvector and eigenvalue data obtainable. This was convenient, but the program is not restricted with regard to the number of test measurements. A single frequency measurement could be used and it would have an influence on the stiffness and mass matrices.

From Table 5-2 it is obvious that perfect modal data will produce an almost perfect correction to the model. From 5-3 we find that imperfect data also corrects the model but the convergence is not exact. One additional point might be added: the mode shapes in Table 5-3 are perturbed from the true spring-mass model. If the data represented a slightly different model in concept (e.g. more coupling between masses) the results would not have been so positive. Therefore, at this point we can conclude that MOUSE does operate effectively on a small system where the analytical model is an adequate representation of the true system

TABLE 5-1 True System Characteristics

(1) Stiffness Matrix

$$\begin{bmatrix} 2.25 \times 10^5 & -1.40 \times 10^5 & 0 & 0 \\ -1.40 \times 10^5 & 2.55 \times 10^5 & -1.15 \times 10^5 & 0 \\ 0 & -1.15 \times 10^5 & 3.15 \times 10^5 & -2.0 \times 10^5 \\ 0 & 0 & -2.0 \times 10^5 & 2.0 \times 10^5 \end{bmatrix}$$

(2) Mass Matrix

$$\begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

(3)

	Eigenvalues	Frequencies
1	1.28872+04	1.80676+01
2	1.17816+05	5.46289+01
3	3.51608+05	9.43734+01
4	5.12689+05	1.13959+02

(4) Modal Matrix

Vector(1)	Vector(2)	Vector(3)	Vector(4)
2.93227-01	-6.70209-01	6.46597-01	-2.16220-01
4.44265-01	-5.13114-01	-5.84744-01	4.44316-01
5.78353-01	2.03809-01	-2.95936-01	-7.32388-01
6.18186-01	4.95980-01	3.90396-01	4.68445-01

TABLE 5-2 Deliberate Perturbation of k_1 and k_2 in the Spring-Mass Chain, Optimization with Perfect Test Data

(1) Stiffness Matrix Before Optimization

$$\begin{bmatrix} 2.50 \times 10^5 & -1.50 \times 10^5 & 0 & 0 \\ -1.50 \times 10^5 & 2.65 \times 10^5 & -1.15 \times 10^5 & 0 \\ 0 & -1.15 \times 10^5 & 3.15 \times 10^5 & -2.00 \times 10^5 \\ 0 & 0 & -2.00 \times 10^5 & 2.00 \times 10^5 \end{bmatrix}$$

(2) Mass Matrix - Same as Table 1 (Unperturbed)

(3) Eigenvalues/vectors - Same as Table 1

(4) Stiffness Covariance Matrix

$$[\Sigma_k] = \begin{bmatrix} \sigma_{k_1}^{2} & \sigma_{k_1 k_2} \\ \sigma_{k_1 k_2} & \sigma_{k_1}^{2} \end{bmatrix} = \begin{bmatrix} 7.2 \times 10^7 & 0 \\ 0 & 1.0 \times 10^8 \end{bmatrix}$$

(5) $[A]$ Matrix

$$\begin{Bmatrix} dk_{11} \\ dk_{12} \\ dk_{22} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} dk_1 \\ dk_2 \end{Bmatrix}$$

(6) Stiffness Matrix after Optimization

$$\begin{bmatrix} 2.25103 \times 10^5 & -1.40084 \times 10^5 & 0 & 0 \\ -1.40084 \times 10^5 & 2.55084 \times 10^5 & -1.15 \times 10^5 & 0 \\ 0 & -1.15 \times 10^5 & 3.15 \times 10^5 & -2.0 \times 10^5 \\ 0 & 0 & -2.0 \times 10^5 & 2.0 \times 10^5 \end{bmatrix}$$

TABLE 5-3 Deliberate Perturbation of k_1 and k_2 in the Spring-Mass Chain, Optimization with Imperfect Test Data

(1) Stiffness Matrix Before Optimization - Same as Table 2

(2) Mass Matrix - Same as Table 1 (Unperturbed)

(3) Eigenvalues/vectors

	Test Eigenvalues	Frequencies
1	1.27080+04	1.79415+01
2	1.17797+05	5.46245+01
3	3.60191+05	9.55183+01
4	5.24376+05	1.15250+02

Test Modal Matrix

Vector(1)	Vector(2)	Vector(3)	Vector(4)
3.20877-01	-5.97509-01	6.55247-01	-2.98320-01
4.83715-01	-5.60864-01	-5.70894-01	4.66466-01
5.58403-01	1.92409-01	-2.98186-01	-7.56338-01
6.52936-01	5.00780-01	3.83446-01	3.55445-01

(4) Stiffness Covariance Matrix - Same as Table 2

(5) [A] Matrix - Same as Table 2

(6) Measurement Errors

$\sigma_{\lambda_1}^2 = 6.6564 \times 10^4$

$\sigma_{\lambda_2}^2 = 5.5507 \times 10^6$

$\sigma_{\lambda_3}^2 = 4.9421 \times 10^7$

$\sigma_{\lambda_4}^2 = 1.0510 \times 10^8$

$\sigma_{x_{ij}}^2 = 2.5 \times 10^{-3}$

(all eigenvector elements have the same variance)

(7) Stiffness Matrix after Optimization

$$\begin{bmatrix} 2.30524 \times 10^5 & -1.48385 \times 10^5 & 0 & 0 \\ -1.48385 \times 10^5 & 2.63385 \times 10^5 & -1.13 \times 10^5 & 0 \\ 0 & -1.15 \times 10^5 & 3.15 \times 10^5 & -2.0 \times 10^5 \\ 0 & 0 & -2.0 \times 10^5 & 2.0 \times 10^5 \end{bmatrix}$$

5.2 Saturn Lateral Vibrations

A second test of MOUSE involved a 58 degree-of freedom model of Saturn. This model contains bending stiffness only. The primary uncertainty in the structure was believed to be in the upper half of the structure and it was this half that was permitted to fluctuate in the analysis. The vehicle is pictured in Figure 5-1.

The model that was provided contained bending stiffness (slender beam) only and no allowance was made for shear. Consequently the frequencies measured in test diverge considerably from the model starting with the third elastic mode. 100 elements of test data were originally input to MOUSE representing nineteen stations of the first five mode shapes and the first five frequencies. The program did not converge properly because the corrections were so large that stiffnesses became negative. This same problem occurred as long as data from the third, fourth and fifth modes was included. The program worked only with data from the first two modes.

After considerable experimentation with cutting of step size and modifying the iteration scheme, it was finally concluded that the shear behavior of the vehicle could not be matched by a strictly bending model. It has been recommended that shear stiffness be added to the model before trying the MOUSE application again.

The excellent agreement in the first two modes indicates that these modes probably involve relatively little shear. The results of the MOUSE application are shown in Tables 5-4 and 5-5.

Note that low numbered positions represent the uppermost part of the vehicle. The position numbering and the corresponding vehicle stations are given in Table 5-6.

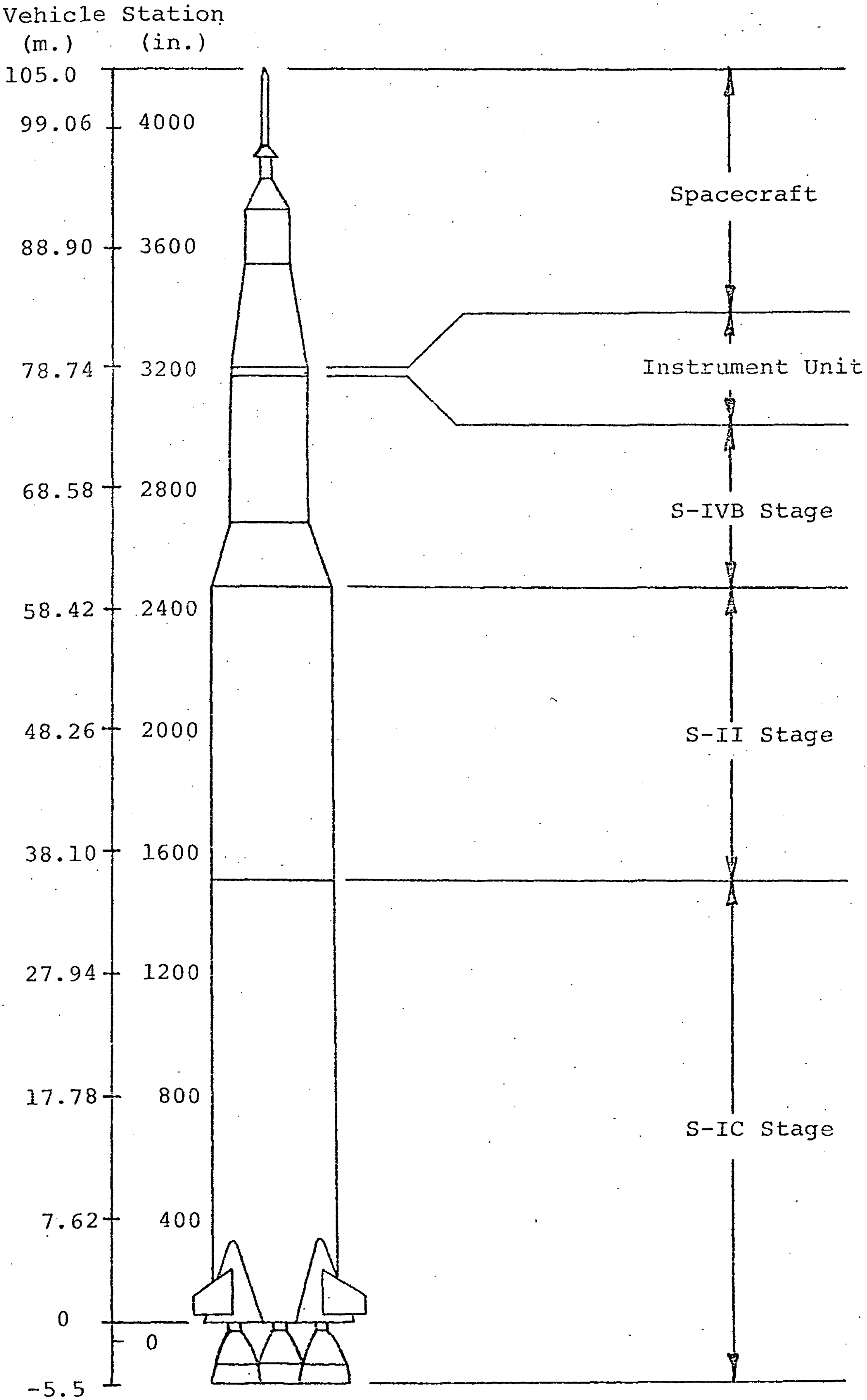


Figure 5-1 Vehicle Configuration

TABLE 5-4 Comparison of Frequencies

Frequencies measured in test	Frequencies predicted in the original model	Revised frequencies using data from first two modes and MOUSE
1.106 Hz	1.121 Hz	1.108 Hz
1.821 Hz	1.840 Hz	1.823 Hz
2.547 Hz	3.051 Hz	
3.443 Hz	4.747 Hz	
5.167 Hz	8.064 Hz	

TABLE 5-5 Comparison of Eigenvalues and Eigenvectors

MODE	REVISED EIGENVALUES	TEST EIGENVALUES
3	4.84492+01	4.82914+01
4	1.31180+02	1.30912+02

POSITION	MODE	EIGENVECTORS	EIGENVECTORS
1	3	1.22973-02	8.67537-03
5	3	9.91213-03	7.11380-03
9	3	5.28236-03	4.16418-03
13	3	4.22650-03	3.38340-03
17	3	2.39935-03	1.99534-03
21	3	1.67648-03	1.38806-03
25	3	1.11963-03	8.67537-04
27	3	6.18492-04	3.47015-04
29	3	2.63795-04	4.33769-05
33	3	-9.97579-05	-2.60261-04
37	3	-3.52664-04	-5.20522-04
39	3	-4.58497-04	-6.94030-04
41	3	-4.82939-04	-6.07276-04
43	3	-4.59299-04	-5.20522-04
45	3	-3.02682-04	-3.47015-04
49	3	-6.89009-05	-8.67537-05
51	3	2.26645-04	2.60261-04
53	3	4.88565-04	5.20522-04
55	3	7.56235-04	7.80783-04
1	4	1.92353-02	1.64340-02
5	4	1.28121-02	8.21699-03
9	4	1.44193-03	1.97208-03
13	4	-4.31323-04	6.57359-04
17	4	-1.85582-03	-8.21699-04
21	4	-1.93500-03	-1.31472-03
25	4	-1.79272-03	-1.47906-03
27	4	-1.53289-03	-1.64340-03
29	4	-1.24898-03	-1.47906-03
33	4	-7.81985-04	-1.31472-03
37	4	-3.00863-04	-6.57359-04
39	4	-5.81048-06	-8.21699-05
41	4	1.28824-04	1.64340-04
43	4	2.94450-04	3.28679-04
45	4	3.23037-04	4.10849-04
49	4	1.75429-04	3.28679-04
51	4	-9.91559-05	-8.21699-05
53	4	-3.74566-04	-4.93019-04
55	4	-6.63393-04	-6.57359-04

TABLE 5-6 Positions of Eigenvector Measurements

Vehicle Stations	Position No.
105.0 meters	1
102.0	5
95.0	9
92.5	13
85.0	17
80.5	21
76.0	25
71.0	27
66.5	29
59.5	33
51.5	37
45.5	39
42.0	41
35.5	43
27.0	45
19.5	49
12.0	51
6.0	53
0.0	55

6.0 CONCLUSIONS AND RECOMMENDATIONS

The following are conclusions resulting from the developmental and evaluation effort on MOUSE.

- the MOUSE Bayesian estimator approach is valid for true models in the linear range of perturbation.
- difficulties arise if the perturbations exceed the linear range, there is a need for handling quickly and efficiently the non-linearity in the convergence process.
- the measured modes strongly influence the convergence and non-orthogonality or un-accounted for inaccuracy can influence satisfactory convergence.

Areas which need further investigation are:

- the use of higher order derivatives
- the influence of improper models
- restraint of corrections within specified limits
- maintenance of total mass fixity while letting component mass vary.

- APPENDIX A -

USER'S MANUAL FOR MOUSE

A.1 INTRODUCTION

Modal Optimization Using Statistical Evaluators (MOUSE) was written in FORTRAN V for UNIVAC 1108.

- Mass, Stiffness, Model Covariance and A Matrices - These matrices shall not exceed 140 rows or columns.
- Control Flags - The control flags give the options of:
  - (1) printing or not printing the partial derivative matrix;
  - (2) selecting 90%, 95% or 99% confidence limits;
  - (3) maximum number of iterations before partials are recomputed; and
  - (4) maximum number of times partials may be computed.
- Input-Output Tapes (Disk) - Logical 5 is the input (card reader) and Logical 6 is the output (printer). Logicals 3, 4, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, and 19 are binary scratch tapes (disk) used during computation.
- Core Requirement - The program requires 56K storage.
- Random Elements - A maximum of 140 random elements are acceptable.

A.2 CARD PREPARATION

1. Case Heading (20A4)

   This card may contain any alphanumeric character in columns 1-80 for problem identification.

2. N, NM, NVEC, JFLAG, KONF, IROW, ITERMX, INTR, NRS, NPARD (10I8)

   Column 1-8 - Integer must be right justified

   N = Mass and stiffness matrix dimension

Column 9-16 - Integer must be right justified

NM = Number of modes (test data)

Column 17-24 - Integer must be right justified

NVEC = Number of highest mode in test data including rigid-body modes. A maximum of three rigid-body modes allowable.

Example: The system has two rigid-body modes and the test data includes the first five flexure modes. NM = 5, NVEC = 7 since the fifth flexure mode is the seventh system mode.

Column 25-32 - Integer must be right justified

JFLAG = 0 Do not print partial derivatives
JFLAG = 1 Print partial derivative matrix.

Column 33-40 - Integer must be right justified

KONF = 1 99% confidence limits
KONF = 2 95% confidence limits
KONF = 3 90% confidence limits

Column 41-48 - Integer must be right justified

IROW = number of measured elements per eigenvector

Column 49-56 - Integer must be right justified

ITERMX = Maximum number of iterations before recomputing partial derivatives.

Column 57-64 - Integer must be right justified

INTR = Number of variable properties, i.e. stiffness or mass in model.

Column 65-72 - Integer must be right justified

NRS = number of variable elements in mass and/or stiffness matrix.

Column 73-80 - Integer must be right justified

NPARD = Maximum number of partial derivative computations

3. MODE (I) 10I8

Column 1-8 - Integer must be right justified

MODE (I) = Ith mode

Column 73-80 - Integer must be right justified

MODE (I) = Ith mode

The mode numbers must be entered in ascending order of the test data and include the rigid-body modes. If the system has two rigid-body modes and the test data is of the first, third and fourth flexible modes, the mode numbers entered would be 3, 5, and 6. Up to 10 modes may be entered.

4. RS (I,J) 20(I4)

Column 1-4 - Integer must be right justified

I = Row number of random element

Column 5-8 - Integer must be right justified

J = Column number of random element

.

.

.

The associated I and J locate the position of the random element in the stiffness and mass matrices. The element locations in the stiffness matrix must be entered first. An I and J equal to zero terminates the stiffness location and starts the mass location read. If there are no stiffness random elements the first I and J must be zero.

There must be enough cards for NRS + 1 random elements. Example: if all the random elements are in the stiffness matrix and there are 30 such elements, 4 cards would be required. Three cards containing the 30 I's and J's and the last card with I and J equal to zero to indicate the end of stiffness elements. If there is a mixture of stiffness and mass elements the zero separator will take care of the NRS + 1 requirement.

5. INDIC (J) 20I4

Column 1-4 - Integer must be right justified

INDIC (J) = location of data in test eigenvector

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.

Each card contains 20 location values. If the last value on the card is zero no more cards will be read. A maximum of 7 cards will be read.

6. [COV (K,M)] INPUT ROUTINE

The standard deviations of stiffness and mass are entered with INPUT. INPUT reads cards in the following format (4(2I3,E14.8)).

Column 1-3 - Integer must be right justified,

Row location of element

Column 4-6 - Integer must be right justified

Column location of element

Column 7-20 - E14.8

Element value

Column 21-23 - Integer must be right justified

Row location of element

.

.

.

A blank card terminates INPUT read.

This matrix is the estimated error of the model. It currently is for uncorrelated properties but will contain correlation in the next edition. The values are squared by the program to generate a diagonal covariance matrix, when each diagonal element represents the randomness of a particular property of the system.

7. [A] INPUT ROUTINE

Same format as number 6.

Since this matrix is non-symmetrical, all non-zero values must be entered.

8. LAMTST(I) 4E20.5

Column 1-20 - E20.5 FORMAT

LAMTST(I) = Modal frequency from test

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.

.

The modal test frequencies must be entered in ascending and in units of H_Z.

9. [X(I,J)] INPUT ROUTINE

Same format as number 6

Since this matrix is non-symmetrical, all non-zero values must be entered. The test eigenvectors may be normalized any way desired since the program will use the theoretical values and mass matrix to renormalize the values suitable for operation in the program.

10. VAR(λ,X) 5E16.8

<u>Column 1-16</u> - E16.8

VAR(I) = measurement accuracy (standard deviation)

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The frequency measurement accuracy (standard deviation) is entered first, with one for each modal frequency. The eigenvector accuracy is entered next (one for each eigenvector element). If the frequency of the mode is measured to .01 H_z, .01 would be entered for each frequency entered. If the eigenvalues were normalized to 1 and were measured to 2% of full scale, .02 would be entered for each eigenvector element.

11. [M(I,J)] INPUT ROUTINE

Same format as number 6

Since the mass matrix is symmetric, only the diagonal and upper triangular non-zero elements need to be entered.

12. [K(I,J)] INPUT ROUTINE

Same format as number 6

Since the stiffness matrix is symmetric, only the diagonal and upper triangular non-zero elements need to be entered.

MOUSE

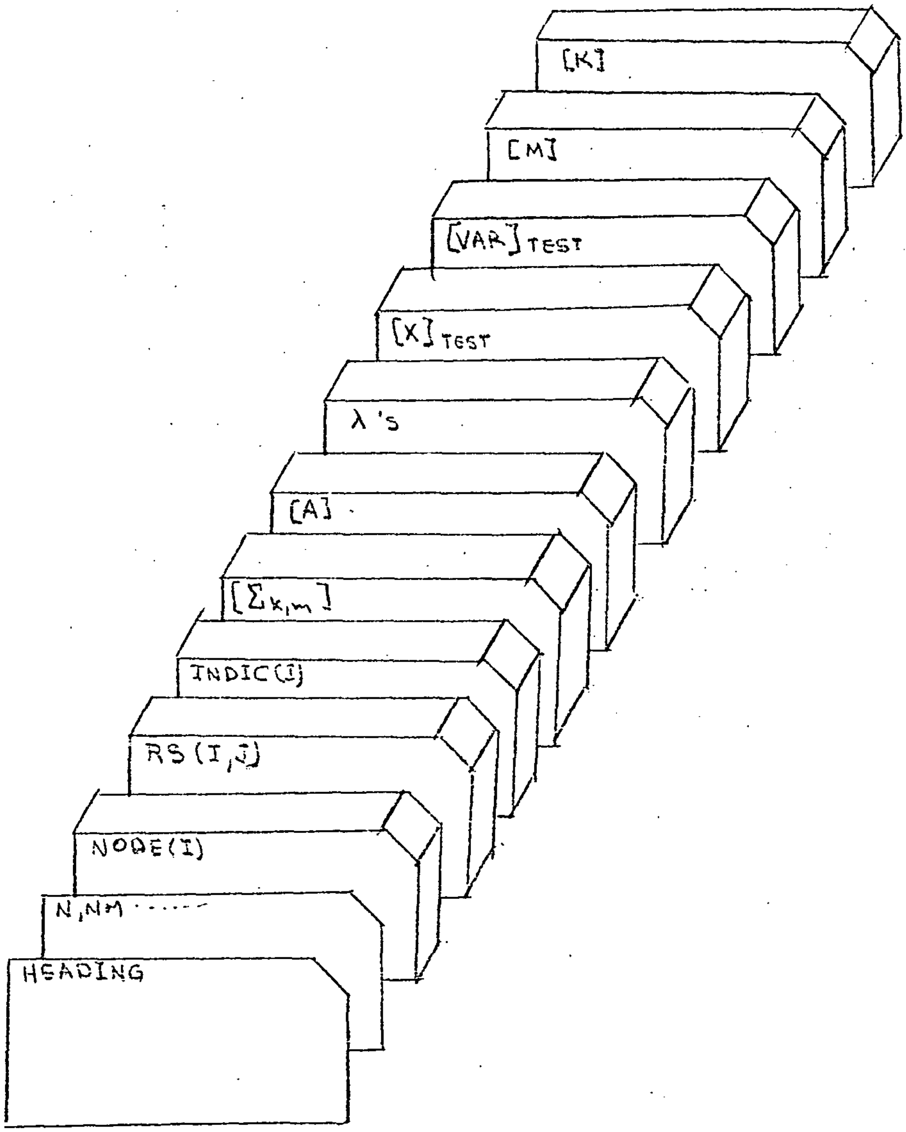

Figure 1A