CONVERGENCE OF THE DISCRETIZATION EQUATIONS WHICH ARISE IN STOCHASTIC CONTROL

by

H. J. Kushner+

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island

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ABSTRACT

for

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We consider a problem which arises in the numerical analysis of the degenerate partial differential equations which arise in stochastic control theory. The result will also shed some light on the relationship between diffusions and finite state Markov chains, and illustrates a role of probability theory in numerical analysis.

Consider the degenerate linear elliptic equations

$$V + k(x) = 0, \quad x \in G,$$

(1) $$V(x) = \varphi(x), \; x \in \partial G$$

$$\mathcal{L} \equiv \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i f_i(x) \frac{\partial}{\partial x_i},$$

where G is a bounded open set, k and φ are continuous and $a_{ij}(x) = \sum_k \sigma_{ik}(x)\sigma_{jk}(x)$, where σ_{ij} and f_i satisfy a Lipschitz condition in x. Equation (1) arises in stochastic control as a formal representation of the cost

(2) $$C(x) = E_x \int_0^\tau k(x_s)ds + E_x\varphi(x_\tau), \; x \in G,$$

for the diffusion

(3) $$dx = f(x)dt + \sigma(x)dz,$$

where z_t is a vector valued Wiener process and $\tau = \inf\{t: x_t \notin G\}$.

Similar representations can be developed for discounted, and time dependent costs, and our convergence results can be extended to these cases.

We consider the problem: Discretize (1) with finite difference interval h (or h_i if it depends on the direction). Solve the finite difference equations, to obtain a function $V_h(x)$ at the finite difference points. Then what happens to $V_h(x)$ as $h \to 0$? The question is important since (1) does not usually satisfy the classical conditions which classical numerical analysis is based upon, and if we are to have confidence in the finite difference solution, it is essential to know that $V_h(x) \to C(x)$ irrespective of whether (1) has strong derivatives or not; i.e., that (1) can legitimately serve as an intermediary in obtaining (2), or an approximation to (2).

It is shown that, whether or not (1) has a smooth solution, the solutions of finite difference equations converge to (2) under quite broad conditions, if the finite difference schemes are carefully chosen. The analysis is purely probabilistic, and uses only probabilistic properties of (2), (3), and the solutions of the finite difference equations.

Convergence of the Discretization Equations Which Arise in Stochastic Control

H. J. Kushner

Introduction. We will consider a problem which arises in the numerical analysis of the degenerate partial differential equations which arise in stochastic control theory. The result will also shed some light on the relationship between diffusions and finite state Markov chains, and illustrates a role of probability theory in numerical analysis. The discussion will be informal. See H. J. Kushner, "Probability Limit Theorems and the Convergence of Finite Difference Approximations to Partial Differential Equations", J. Math. Anal. and Appl., 1970, for more results.

The Problem. Consider the degenerate linear elliptic equations

$$(1) \qquad \mathcal{L}V + k(x) = 0, \quad x \in G,$$

$$V(x) = \varphi(x), \quad x \in \partial G$$

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where G is a bounded open set, k and φ are continuous and $a_{ij}(x) = \sum \sigma_{ik}(x)\sigma_{jk}(x)$, where σ_{ij} and f_i satisfy a Lipschitz condition in x. Equation (1) arises in stochastic control as a formal representation of the cost

$$C(x) = E_x \int_0^\tau k(x_s)ds + E_x\varphi(x_\tau), \quad x \in G, \tag{2}$$

for the diffusion

$$dx = f(x)dt + \sigma(x)dz,$$

where z_t is a vector valued Wiener process and $\tau = \inf\{t: x_t \notin G\}$. Similar representations can be developed for discounted, and time dependent costs, and our convergence results can be extended to these cases.

We consider the problem: Discretize (1) with finite difference interval h (or h_i if it depends on the direction). Solve the finite difference equations, to obtain a function $V_h(x)$ at the finite difference points. Then what happens to $V_h(x)$ as $h \to 0$? The question is important since (1) does not usually satisfy the classical conditions which classical numerical analysis is based upon, and if we are to have confidence in the finite difference solution, it is essential to know that $V_h(x) \to C(x)$ irrespective of whether (1) has strong derivatives or not; i.e., that (1) can legitimately serve as an intermediary in obtaining (2), or an approximation to (2).

For Equation (1) let the difference interval be h (in any coordinate direction, for convenience only) and let e_i be the unit vector in the i-th coordinate direction. Suppose that G is strictly contained in a hypercube H with sides $[-A,A]$. Define the set of nodes R_h^r in R^r by $R_h^r = \{(n_1h,\ldots,n_rh),\ n_i$ ranging over $0, \pm 1, \pm 2,\ldots\}$. Define $G_h = G \cap R_h^r$.

In order to expose the method, and not get involved with the rather long finite difference equations arising when mixed second derivatives occur, we let $a_{ij} = 0$ for $i \neq j$. There is no trouble in extending the method to the more general case.

The following finite difference approximations will be used.

$$V_{x_i} \sim \frac{1}{h}\begin{Bmatrix} V(x + e_i h) - V(x) \\ V(x) - V(x - e_i h) \end{Bmatrix}, \tag{3a}$$

where the upper term of (3a) is used if $f_i(x) \geq 0$, and the lower otherwise. (This usage will be carried throughout)

$$V_{x_i x_i}(x) \sim \frac{V(x + e_i h) - 2V(x) + V(x - e_i h)}{h^2} \tag{3b}$$

The reason for the choice (3a) will appear shortly.

If $V_h(x)$ denotes the solution to the finite difference equations, then using (3) for $x \in G_h$, (1) yields

$$V_h(x) = \sum_i \frac{V_h(x+e_i h)}{Q_h(x)} \begin{Bmatrix} h|f_i| + a_{ii} \\ a_{ii} \end{Bmatrix} + \sum_i \frac{V_h(x-e_i h)}{Q_h(x)} \begin{Bmatrix} a_{ii} \\ h|f_i| + a_{ii} \end{Bmatrix} + k(x)\frac{h^2}{Q_h(x)} \tag{1'}$$

where

$$Q_h(x) = 2\sum_i a_{ii} + \sum_i h|f_i|.$$

Define $V_h(x) = \varphi(x)$ for $x \in R_h^r - G_h$. Rewrite (1') as (with the obvious identification of terms)

$$V_h(x) = \sum_i V_h(x + e_i h) p_h(x, x + e_i h)$$

$$(1'') \qquad + \sum_i V_h(x - e_i h) p_h(x, x - e_i h) + \rho_h(x) k(x)$$

$$\rho_h(x) = h^2/Q_h(x), \quad V(x) = \varphi(x), \text{ for } x \in R_h^r - G_h.$$

Now the reason for the choice (3) will become clear. Note that since the $p_h(x,y) \geq 0$ and sum to at most unity, and can be defined for all $x, y \in R_h^r$, they can be considered to be transition probabilities for a Markov chain on the grid R_h^r.

Denote the sequence of random variables of this Markov chain by $\{\xi_k^h\}$. Thus $P\{\xi_{k+1}^h = \xi_k^h + he_i\} = p_h(x, x + e_i h)$, etc. Define

$$N_h = \inf\{k: \xi_k^h \notin G_h\}.$$

Now we proceed to investigate the behavior as $h \to 0$. Suppose (not very restrictive - see example) $E_\xi N_h < K_h < \infty$. The solution to (1") can be written as

$$(1''') \qquad V_h(x) = E_x \sum_{k=0}^{N_h - 1} \rho_h(\xi_k^h) k(\xi_k^h) + E_x \varphi(\xi_{N_h}^h).$$

Equation (1"') looks something like a Reimann sum approximation to (2), and, indeed this is the idea to be pursued. We will show that the measures of a suitably interpolated $\{\xi_n^h\}$ converge to those of x_t in a certain sense, and that this implies that $V_h(x)$ converges to $C(x)$ for x on any G_h. This holds irrespective of whether strong derivatives of (1) exist or not.

A key fact is Theorem 1, drawn from Gikhman and Skorokhod, "Introduction to Random Processes", (in Russian), Izdatelctvo Nauka, Moscow, 1966.

Theorem 1. Let $C[0,T] \equiv \Omega$ be the set of R^r valued continuous functions on the interval $[0,T]$. Let $y^n(t)$, $y(t)$, $t \in [0,T]$ be continuous processes with paths in the (topological) space Ω. Let μ_n and μ be the measures induced on Ω by the processes $y^n(\cdot)$ and $y(\cdot)$, resp. Let (for $0 \leq t' \leq t'' \leq T$)

$$(4) \qquad \lim_{\delta \to 0} \overline{\lim_{n}} P_x\{ \sup_{|t'-t''| \leq \delta} |y^n(t')-y^n(t'')| \geq \epsilon > 0\} = 0$$

for any $\epsilon > 0$. Let the finite dimensional distributions of $\{y^n(t)\}$ converge to those of $y(t)$. Let $F(\cdot)$ be a bounded and continuous (w.p.1.) Borel functional on the topological space Ω. Then

$$EF(y^n(\cdot)) \to EF(y(\cdot)).$$

In order to exploit Theorem A, the process $\{\xi_k^h\}$ must be related to a suitable continuous time process $\{\xi^h(t)\}$.

By a comparison of $(1''')$ and (2), we note that the "discrete time" cost rate is $\rho_h(x)$ times the continuous time cost. In an intuitive sense, one step of the discrete process ξ_k^h should take $\rho_h(\xi_k^h)$ units of real time. Thus the following definition is natural. Define the time sequence $\{t_k^h\}$ by (sometimes arguments of functions are deleted for simplicity, and $\rho_h(\xi_k^h)$, Δt_k^h are used interchangeably)

$$\Delta t_k^h \equiv \Delta t^h(\xi_k^h) \equiv \rho_h(\xi_k^h)$$

$$t_0 = t_0^h = 0, \quad t_k^h = \sum_{0 \le s < k} \Delta t_s^h.$$

Define a process $\xi^h(t)$ by

$$\xi^h(t_k^h) = \xi_k^h$$

at the times $\{t_k^h\}$, and for $t_k^h \le t < t_{k+1}^h$, by the linear interpolation

$$\xi^h(t) = \xi_{k+1}^h \frac{(t - t_k^h)}{\Delta t(\xi_k^h)} + \xi_k^h \frac{(t_{k+1}^h - t)}{\Delta t(\xi_k^h)} .$$

Thus the continuous process $\xi^h(t)$ is piecewise linear and changes slope at the random break points $\{t_k\}$ only.

The use of $\xi^h(t)$ is a natural way of relating $\{\xi_k^h\}$ and x.

The main results are:

Let measures μ_h and μ (on the topological space $\Omega = C[0,T]$) correspond to processes $\xi^h(t)$ and x_t, resp. $t \in [0,T]$. The conditions (C1)-(C4) used in the sequel are quite natural for a large class of problems, and are illustrated in the example.

Theorem 2. *Assume*

(C1) $f_i(\xi)$ *and* $\sigma_{ij}(\xi)$ *are uniformly bounded and satisfy a uniform Lipschitz condition.* (*Recall* $a = \sigma\sigma'$.)

(C2) *Let* δ_h *equal* h *or* h^2. *For real positive* K_i *let*

$$K_1\delta_h \geq \Delta t^h(\xi) \geq K_2\delta_h.$$

(C3) *Let* $a(\xi)$, *have the form*

$$a(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_0(\xi) \end{bmatrix}, \quad \sigma(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0(\xi) \end{bmatrix}, \quad \Sigma_0(\xi) = \sigma_0(\xi)\sigma_0'(\xi)$$

where $\Sigma_0^{-1}(\xi)$ *has uniformly bounded terms, and* $\sigma_0(\xi)$ *is an* $r_0 \times r_0$ *matrix.*

Then the finite dimensional distributions of the processes $\xi^h(t)$ *converge to those of the process* x_t *and, for* $0 \leq t' \leq t'' \leq T$, *and* $\epsilon > 0$

$$\lim_{\delta \to 0} \overline{\lim_{h \to 0}} P_x\{\sup_{|t'-t''| \leq \delta} |\xi^h(t') - \xi^h(t'')| \geq \epsilon\} = 0.$$

Remark. (C2) means the following. Either we allow $\Sigma_i a_{ii}(\xi) \geq \epsilon_0 > 0$ for some real ϵ_0, in which case $\delta_h = h^2$, or we allow $\Sigma_i a_{ii}(\xi) \equiv 0$ and $\Sigma_i |f_i| \geq \epsilon_0 > 0$, in which case $\delta_h = h$. Thus 2 cases are considered - one case in which there is always some diffusion somewhere, and one case in which there is no diffusion - but where the velocity of x_t is never 0. In the intermediate case the ratio

$$\frac{\max_\xi \Delta t^h(\xi)}{\min_\xi \Delta t^h(\xi)}$$

may be infinite, invalidating our proofs. The first case is one of great importance.

Corollary 1. Assume (C1)-(C5). Let $F(\cdot)$ be a bounded continuous function on $C[0,T]$ w.p.1. (relative to μ). Then (with $\xi_0^h = x = x_0$)

$$E_x F(\xi^h(\cdot)) \to E_x F(x(\cdot)).$$

Theorem 2 uses condition (C6): There is an $h_0 > 0$ so that for $h < h_0$, G satisfies: Let $\ell = (a,b)$ be a line connecting two adjacent (along coordinate directions) points (a,b) of the grid R_h^r. If a and b are both in G, then so is the line ℓ connecting them.

(C6) can be weakened in many ways - but there seems little point in complicating the condition here. It is certainly satisfied (for any h_0) for convex G. (C6) is used to assure that the first passage times from G, of both $\xi^h(t)$ and ξ_k^h are approximately the same time. I.e., if $\xi^h(t)$ leaves the G between the n-th and n+1-st steps of ξ_k^h, then $N_h = n$. It is used to avoid the possibility illustrated in the Figure (for all small h), where if the discrete process ξ_k^h jumps from a to b at time n, it has not actually left G, but the interpolated process $\xi^h(t)$ leaves right after time n. Let $N_h^T = \max\{n: t_n^h < T\}$ and $M_h = \min(N_h, N_h^T)$.

Theorem 3. Assume (C1)-(C3) and (C6). Let $k(\cdot)$ and $\varphi(\cdot)$ be uniformly continuous and bounded on some open set containing $\bar{G} = G + \partial G$. Let τ denote the first (random) time that the process x_t leaves G

($\tau = \inf\{t: x_t \notin G\}$), and suppose that $T \cap \tau \equiv \min(T,\tau)$ is continuous w.p.1. (The w.p.1. statement is relative to μ) on $C[0,T]$. Denote $\tau_h = \inf\{t: \xi^h(t) \notin G\}$. Then

$$E_x \int_0^{T\cap\tau} k(x_s)ds = \lim_{h\to 0} E_x \int_0^{T\cap\tau_h} k(\xi^h(s))ds$$

$$= \lim_{h\to 0} E_x \sum_{s=0}^{M_h-1} k(\xi^h_s)\rho_h(\xi^h_s)$$

$$E_x\varphi(x_{T\cap\tau}) = \lim_{h\to 0} E_x\varphi(\xi^h(T\cap\tau_h)) = \lim_{h\to 0} E_x\varphi(\xi^h_{M_h}).$$

Theorem 4. Assume the conditions of Theorem 2, and let, for some $t_0 < \infty$,

(5) $$P_x\{\xi^h_k \text{ leaves } G_h \text{ at least once by time } t_0/K_2\delta_h\} \geq M_0 > 0$$

where M_0 is independent of $x \in G_h$ and $h > 0$, for small h. Then the $V_h(x)$ given by (1"') converge to (2) as $h \to 0$, uniformly in x in G; i.e., the solutions of the finite difference equations (1', 1") converge to the weak solution (2) of Equation (1), as $h \to 0$.

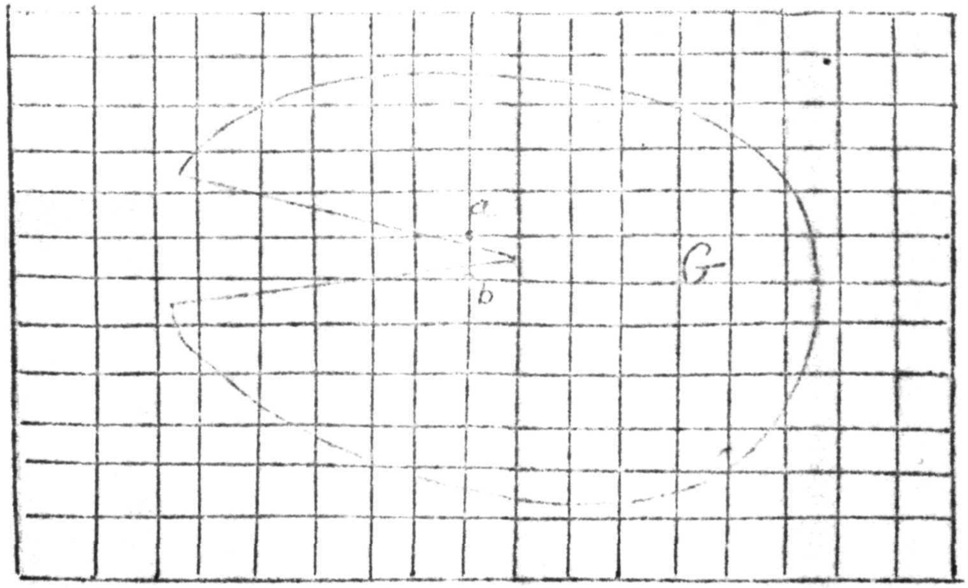

Figure 1

The conditions imposed in Theorems 2-4 are rather natural for a large class of problems. In order to illustrate this, a brief check of their validity will be made for a 2-dimensional problem. It should be clear that the example is typical of a large class. Although the basic problem arose in numerical analysis, the approach taken here, as well as the conditions, are probabilistic. Hence, the checking of the conditions involves probabilistic calculations on the underlying processes. Let

$$dx_1 = f_1(x_2)dt \qquad 2a = \begin{bmatrix} 0 & 0 \\ 0 & v^2 \end{bmatrix} = \sigma\sigma'$$

$$dx_2 = f_2(x)dt + v\,dz$$

where v is a constant and the f_i satisfy (C1). Let $f_1(x_2) = x_2$ in $\bar{G}$. We seek to solve

$$\mathcal{L}V(x) + k(x) = 0 \quad \text{in} \quad G$$

$$V(x) = \varphi(x) \quad \text{on} \quad \partial G$$

where $k(\cdot)$ and $\varphi(\cdot)$ are continuous and bounded,

$$\mathcal{L} = \frac{v^2}{2} \frac{\partial^2}{\partial x_2^2} + f_1(x) \frac{\partial}{\partial x_1} + f_2(x) \frac{\partial}{\partial x_2},$$

and G is the box

$$G = \{x: |x_i| \leq A\}.$$

Thus (C6) holds for all $h_0 > 0$.

Note that $\mathcal{L}$ is degenerate and G has corners; hence, classical theory cannot be used to solve the convergence problem as $h \to 0$.

We need only show that $T \cap \tau$ is continuous w.p.1. (relative to μ) on $C[0,T]$, and that (5) holds. Let ω be a generic point of $C[0,T] = \Omega$. $T \cap \tau$ is continuous at any path which is not tangent to the boundary at $T \cap \tau$.

Returning to the problem, refer to Fig. 2. It is clear that if tangencies at the boundary occur only w.p. zero, then, by virtue of the continuity of $x_t(\omega)$ w.p.1., $(T \cap \tau)(\omega)$ will be continuous w.p.1. This will now be shown to be the case.

We observe that

(a) for $x_{2t} > 0$ x_{1t} must increase (as time increases) since $dx_1 = x_2 dt$ in G. Hence, w.p.1., points on the boundary section L_4 (Fig. 2) are not accessible. Similarly for L_3.

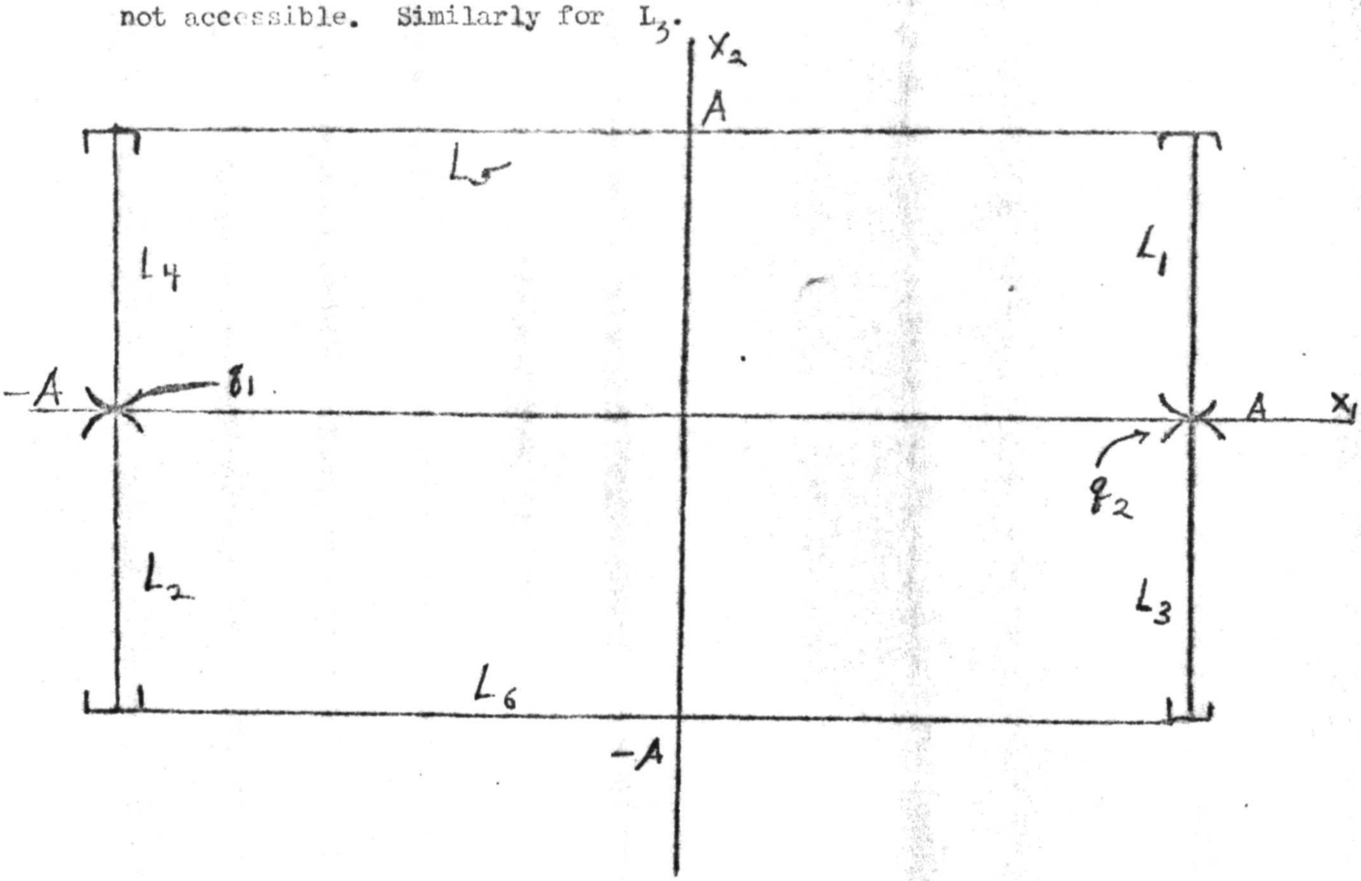

Figure 2

(b) Also, since $x_{2t} > 0$ on L_1, the path cannot be tangent on L_1, and similarly for L_2.

(c) Owing to the dominant effects of the diffusion on movement in the vertical direction, the points on L_5 and L_6 are regular in the sense of Dynkin; i.e.,

$$\overline{\lim_{\delta \to 0}} P_x\{x_\tau \in \partial G,\ x_{\tau+\epsilon} \in \overline{G} \text{ for all } \delta \geq \epsilon > 0\} = 0$$

(d) $P_x\{\tau = T\} = 0$

(e) $P_x\{x_\tau = q_1 \text{ or } q_2\} = 0$.

(f) Since x_t is continuous, w.p.1., there are $\epsilon_3(\omega) > 0$, w.p.1., $\epsilon_4(\omega) > 0$ w.p.1. so that

$$\text{distance}(x_{t-\epsilon_3(\omega)}, \text{ exterior of } \overline{G}) \geq \epsilon_4(\omega),$$

for all $t \leq \tau(\omega)$ if $\tau(\omega) \leq T$.

(a)-(f) imply continuity w.p.1. of $T \cap \tau$.

Only (5) remains to be proved. Let $N = t_0/K_1h^2$, for any $t_0 > 0$, and define

$$M_+(\xi) = \text{number of positive steps of } \xi^h_{k,2}, \quad k \leq N$$

$$M_-(\xi) = \text{number of negative steps of } \xi^h_{k,2}, \quad k \leq N.$$

A sufficient condition for (5) is

$$q_h(x) \equiv P_x\{M_+(\xi) - M_-(\xi) \geq \frac{2A}{h}\} \geq M_0 > 0.$$

For some real K, we have the bounds,

$$\frac{1}{2} - Kh \leq P_x\{\xi^h_{k+1,2} - \xi^h_{k,2} = h\} \leq \frac{1}{2} + Kh.$$

Let $\{u^h_k\}$ be a Markov process on $\{0, \pm 1, \pm 2, \ldots\}$ with transition probability

$$P\{u^k_{k+1} = u^k_k + 1\} = \frac{1}{2} - Kh = 1 - P\{u^h_{k+1} = u^h_k - 1\}.$$

Define $M_{\pm}(u)$ analogously to $M_{\pm}(\xi)$. Then

$$q_h(x) \geq q_{h,u}(x) \equiv P\{M_+(u) - M_-(u) \geq \frac{2A}{h}\}.$$

The mean value of $u^h_{k+1} - u^h_k$ is $-2Kh$, and its variance is $1 - (2kh)^2$. Now

$$q_{h,u}(x) = P\left\{\frac{M_+(u) - M_-(u) - N(-2kh)}{\sqrt{N}\sqrt{1-(2Kh)^2}} \geq \frac{2A/h + 2Nkh}{\sqrt{N}\sqrt{1-(2Kh)^2}}\right\}.$$

The left term in brackets converges in distribution to the normal zero mean and unity variance random variable, and the right hand term in the brackets is strictly less than some $K_3 < \infty$ for small h. Thus, for all small h

$$q_h(x) \geq q_{h,u}(x) \geq \frac{1}{\sqrt{2\pi}} \int_{K_3}^{\infty} \exp - \frac{1}{2}y^2 dy,$$

which proves (5).

The proof of the convergence of the finite dimensional distributions of $\xi^h(t)$ to those of the diffusion x_t proceeds in a series of steps. First the representation (6) is proved, where $\{\tilde{u}^h_k\}$ are orthogonal for each fixed h, each covariance proportional to Δt_k^h and the $\tilde{\beta}_k^h$ are "negligible" for small h.

$$(6) \qquad \xi_{k+1}^h = \xi_k^h + \Delta t_k^h f(\xi_k^h) + \begin{Bmatrix} \hat{\beta}_k^h \\ \Sigma_{0,h}^{1/2} \tilde{\omega}_k^h \end{Bmatrix}$$

Then the interpolation of (6) is approximated by the interpolation of $\eta^h(t)$ of

$$(7) \qquad \eta_{k+1}^h = \eta_k^h + \Delta t_k^h f(\eta_k^h) + \begin{Bmatrix} 0 \\ \Sigma_0^{1/2} \tilde{\omega}_k^h \end{Bmatrix} .$$

The finite dimensional distributions of the interpolation of (7) are approximated by those of the interpolation of (as $h \to 0$, small Δ),

$$(8) \qquad \tilde{y}_{k+1}^h = \tilde{y}_k^h + f(\tilde{y}_k^h) \sum_s \Delta t_s^h + \begin{Bmatrix} 0 \\ \sigma_0(\tilde{y}_k^h) \sum_s \omega_s^h \end{Bmatrix}$$

where the sums of Δt_s^h are approximately Δ, and finally the interpolation of (8) by a diffusion.