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INVESTIGATION OF THE UTILITY OF MEAN SQUARE APPROXIMATION SYSTEMS AND IN SYSTEM RESPONSE PREDICTIONS

by J. L. Bogdanoff and A. J. Schiff

Prepared by
MIDWEST APPLIED SCIENCE CORPORATION
West Lafayette, Ind. 47906
for Goddard Space Flight Center

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16. Abstract

A method is presented for estimating the variability of a system's natural frequencies arising from the variability of the system's parameters. The only information required to obtain the estimates is the member variability, in the form of second order properties, and the natural frequencies and mode shapes of the mean system. Several examples are worked out in detail to illustrate how the method is applied.

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1. Resonant frequencies
2. Frequency response
3. Calculus of variations

TABLE OF CONTENTS

1. Introduction

The engineer who must rely on natural frequency determination in predicting response in linear structural vibration frequently needs a simple method of estimating how the accuracy of these determinations are related to the accuracy of his estimations of the masses and stiffnesses in the structure. He also may wish to determine which of the masses and/or stiffnesses have major influence on the location of certain natural frequencies. In this paper, we present a method directed towards meeting these needs.

The method provides an estimator for the standard deviations (S.D.) of a natural frequency in terms of only second order properties of the system parameters. A simplified form of this estimator is frequently an upper bound; when it is small, the S.D. of a natural frequency (n.f.) is generally small; when it is large, the S.D. of a natural frequency is not necessarily large.

Before describing our work, some comments on background are in order.

Rayleigh [1]* considered the following problem:
He could easily determine the n.f.'s and normal modes of systems having regular properties. He wanted to be able to use these results to compute n.f.'s and normal modes of systems which did not differ appreciably from one of these regular systems, but which were difficult to deal with by themselves. He proposed a perturbation method that he could use to estimate the n.f.'s and

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* Numbers in brackets refer to the references at the end of the report.

normal modes of the second system using the normal modes and n.f.'s of the first system and the amounts by which the parameters of the second system differed from the first. He proposed a first order formula for correction of the modes and a second order formula for correction of the n.f.'s. Clearly, Rayleigh's scheme could be employed to estimate n.f. change due to change in parameter values. However, if one puts Rayleigh's scheme in a suitable form for this purpose, it soon becomes apparent that the changes in the parameter values become involved in the computations in a fairly complex manner, making it difficult to develop an understanding of how they influence a given n.f. and corresponding normal mode. In all events, we cannot recall a recent use of Rayleigh's formulas.

Soong [2] in a series of papers in the early and mid 60's considered the changes on S.D. of natural frequencies of a linear chain due to changes in parameter values. By exploiting the form of the frequency equation for a N-degree of freedom chain having the same mean masses and springs and using a perturbation technique, he was able to determine the S.D.'s of the n.f.'s; these results became reasonably simple when the mass changes were regarded as independent random variables having the same distributions and when the spring changes were of similar type. His results indicated that the S.D.'s would increase with increasing mean n.f. When one thinks of hundreds of n.f.'s with raising frequencies, this is a disturbing result even if the S.D.'s of the parameter values are reasonably small. Because he made use of the special form of the frequency equation for a N-degree of freedom linear chain with identical masses and identical springs, it is difficult to extend Soong's approach to more general types of systems.

In his thesis [3] Collins proposed a perturbation scheme applicable to general N-degree of freedom linear systems by which it is possible to estimate the S.D.'s of the n.f.'s in terms of the covariances of system parameters and a set of $2N^2$ partial derivatives. He started from the matrix equation

$$\{K\} \underline{X}_i = \underline{\lambda}_i \{M\} \underline{X}_i$$

and developes the result

$$\delta\lambda_i = \frac{\underline{X}_i^T \{\delta K\} \underline{X}_i}{\underline{X}_i^T \{M\} \underline{X}_i} - \underline{\lambda}_i \frac{\underline{X}_i^T \{\delta M\} \underline{X}_i}{\underline{X}_i^T \{M\} \underline{X}_i}$$

From this he arrives at his formula for Variance λ_i or rather

$$E \{(\lambda_i - \underline{\lambda}_i)^2\} = \sum^{n^2} \sum^{n^2} \frac{\partial\lambda_i}{\partial k_u} \frac{\partial\lambda_i}{\partial k_u} \mathrm{Cov}(k_{u_1} k_v) + \text{--}$$

For a linear chain his results agree with those of Soong. For more complex systems, we have not had the time to examine his results. It is our tentative view that as in Soong's case it will be relatively difficult to obtain results by this method.

Wilkinson in his book "The Algebraic Eigenvalue Problem" [4] gives some general results which are as follows: let

A & B be $n \times n$ matrices;

α_s be the eigenvalues of A ,

α'_s be the eigenvalues of $A + B$;

if for all i & j $|b_{ij}| \leq \varepsilon$

then for all s

$$|\alpha_s - \alpha'_s| \leq n\varepsilon$$

If $n\varepsilon << 1$ and $|\alpha_s| >> 1$, this is a very general result; it is independent of s , however, and so is not of great interest to us.

The recent book [5] by Mehta on Random Matrices presents the physicists view of the eigenvalues of random matrices. As these interests largely center on the density of the eigenvalues rather than on the distribution of individual eigenvalues, they are not of direct interest to us.

2. On the Natural Frequencies of Two Coupled Linear (Lumped Parameter) Systems.

Let us consider two linear dynamical systems defined as follows:

(I) $$2T_I = a_{ij}\dot{q}_i\dot{q}_j, \quad 2V_I = c_{ij}q_iq_j \quad i,j = 1,\dots N_1$$

(II) $$2T_{II} = \alpha_{rs}\dot{p}_r\dot{p}_s, \quad 2V_{II} = \gamma_{rs}p_rp_s \quad r,s = 1,\dots N_2$$

where summation convention is used. These systems, initially unconnected, will be joined by springs $k_1, k_2 \dots k_x$ which will connect q_1 with p_1, q_2 with p_2, ... and q_x with p_x respectively.

Let the natural frequencies and normal modes of I be

(2.1) $$\omega_\ell;\ \xi_j^{(\ell)} \qquad \ell, j = 1, \dots N_1$$

$$a_{ij}\xi_i^{(\ell)}\xi_j^{(m)} = \delta_{\ell m}, \qquad c_{ij}\xi_i^{(\ell)}\xi_j^{(m)} = \omega^2\delta_{\ell m}$$

and let those of II be

(2.2) $$\lambda_u;\ \eta_r^{(u)} \qquad u,r = 1, \dots N_2$$

$$\alpha_{rs}\eta_r^{(u)}\eta_s^{(v)} = \delta_{uv}, \qquad \gamma_{rs}\eta_r^{(u)}\eta_s^{(v)} = \lambda_u^2\delta_{uv}$$

where δ_{ij} is the Kronecker delta, and repeated subscript summation convention is used.

The natural frequencies of the coupled systems are found as follows:

Assume forces $P_1\cos\sigma t, \ldots\ldots P_x\cos\sigma t$ act at $q_1, \ldots q_x$ and forces $-P_1\cos\sigma t, \ldots\ldots -P_x\cos\sigma t$ act at $p_1, \ldots p_x$. In terms of the original uncoupled systems coordinates $\phi_\ell(t)$ and $\theta_u(t)$ we then have

$$(2.3) \qquad q_i = \sum_{u=1}^{N_1} \xi_i^{(u)} \phi_u(t), \qquad 2T_I = \sum_{\ell=1}^{N_1} \dot{\phi}_\ell^2$$

$$2V_I = \sum_{\ell=1}^{N_1} \omega_\ell^2 \phi_\ell^2 , \qquad \delta W_I = \sum_{\ell=1}^{N_1} \sum_{h=1}^{X} P_h \xi_h^{(\ell)} \cos\sigma t \; \delta\phi_\ell$$

and

$$(2.4) \qquad p_r = \sum_{u=1}^{N_2} \eta_r^{(u)} \theta_u(t), \qquad 2T_{II} = \sum_{u=1}^{N_2} \dot{\theta}_u^2$$

$$2V_{II} = \sum_{u=1}^{N_2} \lambda_u^2 \theta_u^2, \qquad \delta W_{II} = - \sum_{u=1}^{N_2} \sum_{h=1}^{X} P_h \eta_h^{(u)} \cos\sigma t \; \delta\theta_u$$

for I and II respectively. The equations of motion are

$$(2.5) \qquad \ddot{\phi}_\ell + \omega_{(\ell)}^2 \phi_{(\ell)} = \sum_{h=1}^{X} P_h \xi_h^{(\ell)} \cos\sigma t$$

$$\ddot{\theta}_u + \lambda_{(u)}^2 \theta_{(u)} = \sum_{h=1}^{X} P_h \eta_h^{(u)} \cos\sigma t$$

where there is no summation on the same subscript when enclosed in the parenthesis, hence,

$$(2.6) \qquad \phi_\ell = \frac{\sum_{i=1}^{X} P_i \xi_i^{(\ell)}}{\omega_\ell^2 - \sigma^2} \cos\sigma t, \qquad q_i = \sum_{\ell=1}^{N_1} \frac{\sum_{h=1}^{X} P_h \xi_i^{(\ell)} \xi_h^{(\ell)}}{\omega_\ell^2 - \sigma^2} \cos\sigma t$$

$$\theta_u = \frac{\sum_{s=1}^{X} P_s \eta_s^{(u)}}{\lambda_u^2 - \sigma^2} \cos\sigma t, \qquad p_r = - \sum_{u=1}^{N_2} \frac{\sum_{h=1}^{X} P_h \eta_r^{(u)} \eta_h^{(u)}}{\lambda_u^2 - \sigma^2} \cos\sigma t$$

where

$$i = 1,\dots N_1 \quad \text{and} \quad r = 1,\dots N_2$$

In obtaining equations (2.6), we have solved (2.5) and made use of the first equation of (2.3) and (2.4) respectively. If we write

$$(2.7) \qquad q_i = A_i \cos\sigma t, \qquad p_r = B_r \cos\sigma t$$

then

$$(2.8) \qquad A_i = \sum_{\ell=1}^{N_1} \frac{\sum_{h=1}^{X} \xi_i^{(\ell)} P_h \xi_h^{(\ell)}}{\omega_\ell^2 - \sigma^2}, \quad i = 1,\dots N_1$$

$$B_r = - \sum_{u=1}^{N_2} \frac{\sum_{h=1}^{x} \eta_r^{(u)} P_h \eta_h^{(u)}}{\lambda_u^2 - \sigma^2}, \quad r = 1, \ldots N_2$$

The frequency equations are determined by substituting (2.8) into the equations

$$(2.9) \qquad P_y = - k_{(y)}[A_{(y)} - B_{(y)}], \quad y = 1, \ldots x$$

The case where $x = 1$, with only one connecting spring, is of special interest and also simplicity. The pertinent formulas from (2.8) and (2.9) are

$$(2.10) \qquad A_1 = \sum_{\ell=1}^{N_1} \frac{P_1 (\xi_1^{(\ell)})^2}{\omega_\ell^2 - \sigma^2}$$

$$B_1 = - \sum_{u=1}^{N_2} \frac{P_1 (\eta_1^{(u)})^2}{\lambda_u^2 - \sigma^2}$$

$$- P_1 = k_1 (A_1 - B_1)$$

or

$$(2.11) \qquad - \frac{1}{k_1} = \sum_{\ell=1}^{N_1} \frac{(\xi_1^{(\ell)})^2}{\omega_\ell^2 - \sigma^2} + \sum_{u=1}^{N_2} \frac{(\eta_1^{(u)})^2}{\lambda_u^2 - \sigma^2}$$

Equation (2.11) is the frequency equation for the coupled system.

The roots of this equation are the (angular) natural frequencies; we shall denote the roots by $\sigma_1, \ldots \sigma_{N_1+N_2}$.

The corresponding mode shapes are proportional to

$$(2.12) \qquad A_i^{(w)} \equiv \sum_{\ell=1}^{N_1} \frac{\xi_i^{(\ell)} P_1 \xi_1^{(\ell)}}{\omega_\ell^2 - \sigma_w^2} \qquad i = 1, \ldots N_1$$

$$w = 1, \ldots N_1 + N_2$$

$$B_r^{(w)} \equiv - \sum_{u=1}^{N_2} \frac{\eta_r^{(u)} P_1 \eta_1^{(u)}}{\lambda_u^2 - \sigma_w^2} \qquad r = 1, \ldots N_2$$

For later convenience we write

$$(2.13) \qquad g_I(\sigma) = \sum_{\ell=1}^{N_1} \frac{(\xi_1^{(\ell)})^2}{\omega_\ell^2 - \sigma^2} = A_1/P_1$$

$$g_{II}(\sigma) = \sum_{u=1}^{N_2} \frac{(\eta_1^{(u)})^2}{\lambda_u^2 - \sigma^2} = B_1/P_1$$

Let us now discuss the implications of the above formulas in the special case when $N_1 = 2$, $N_2 = 4$.

The case we have in mind initially is that in which the two degree of freedom system, System I, is to be mounted on another system for which $N_2 = 4$. It is easy to see that

System I

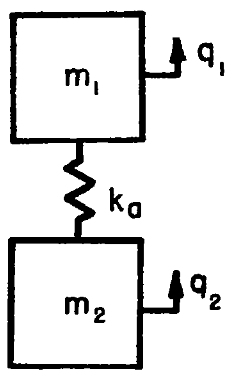

Figure 2.1

System II

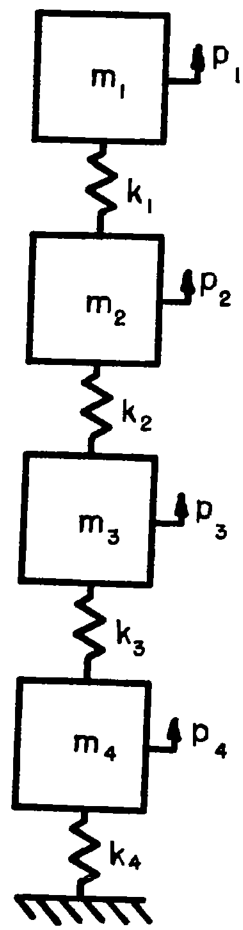

Figure 2.2

$$2T_I = m_1\dot{q}_1^2 + m_2\dot{q}_2^2$$

$$2V_I = k_a(q_1 - q_2)^2$$

$$\omega_1^2 = 0, \quad \omega_2^2 = \frac{k_a}{m_2}\left(1 + \frac{m_2}{m_1}\right)$$

$$\xi_1^{(1)} = \xi_2^{(1)} = \frac{1}{\sqrt{m_1 + m_2}}$$

$$\xi_1^{(2)} = \sqrt{\frac{m_2}{m_1^2 + m_1m_2}}, \quad \xi_2^{(2)} = -\frac{m_1}{m_2}\sqrt{\frac{m_2}{m_1^2 + m_1m_2}}$$

$$q_1 = \xi_1^{(1)}\phi_1 + \xi_1^{(2)}\phi_2, \quad q_2 = \xi_2^{(1)}\phi_1 + \xi_2^{(2)}\phi_2$$

$$2T_I = \dot{\phi}_1^2 + \dot{\phi}_2^2, \quad 2V_I = \omega_2^2\phi_2^2$$

$$\delta W_I = P_1\cos\sigma t\,\delta q_1 = P_1\cos\sigma t\,\xi_1^{(1)}\delta\phi_1 + P_1\cos\sigma t\,\xi_1^{(2)}\delta\phi_2$$

and hence, (2.7) becomes

$$q_1 = -\frac{[\xi_1^{(1)}]^2}{\sigma^2} P_1 \cos\sigma t + \frac{[\xi_1^{(2)}]^2}{\omega_2^2 - \sigma^2} P_1 \cos\sigma t$$

Thus, the ratio of the response to forcing function is

$$g_I = -\frac{[\xi_1^{(1)}]^2}{\sigma^2} + \frac{[\xi_1^{(2)}]^2}{\omega_2^2 - \sigma^2}$$

Figure 2.2 shows System II. Equation 2.13 gives g_{II} .

Figure 2.3 shows typical plots for g_I and g_{II} where we have set $N_2 = 4$ and placed ω_2 in the midst of λ_1, λ_2, λ_3, and λ_4 . When the coupling k_1 is light, $-1/k_1$ is large, negative, and plots as the solid horizontal line at the bottom of the figure. The roots of (2.11) for k_1 small are shown as solid dots on the σ-axis and denoted by $\sigma_1, \sigma_2, \ldots \sigma_6$. Since k_1 is small, providing only weak coupling between the two systems, $\sigma_1 \approx \omega_1 = 0$, $\sigma_2 \approx \lambda_1$, $\sigma_3 \approx \omega_2$, $\sigma_4 \approx \lambda_2$, $\sigma_5 \approx \lambda_3$, and $\sigma_6 \approx \lambda_4$. The mode shapes are determined from (2.12); when written out for this case they are

$$A_1^{(w)} = \frac{P_1(\xi_1^{(1)})^2}{-\sigma_w^2} + \frac{P_1(\xi_1^{(2)})^2}{\omega_2^2 - \sigma_w^2} \tag{2.14}$$

$$A_2^{(w)} = \frac{\xi_2^{(1)} P_1 \xi_1^{(1)}}{-\sigma_w^2} + \frac{\xi_2^{(1)} P_1 \xi_1^{(2)}}{\omega_2^2 - \sigma_w^2}$$

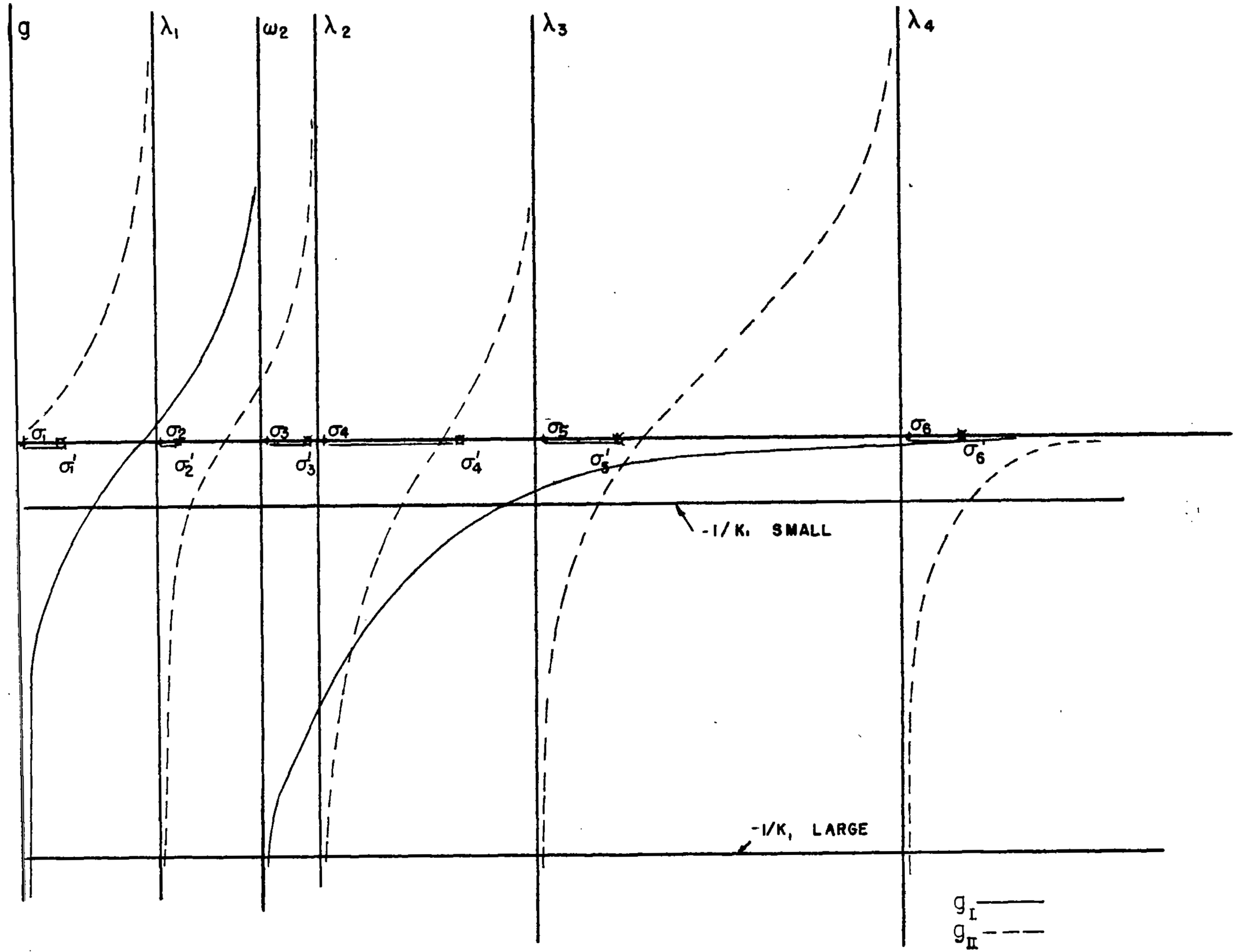
g
λ1
ω2
λ2
λ3
λ4
σ1
σ2
σ3
σ4
σ5
σ6
σ1'
σ2'
σ3'
σ4'
σ5'
σ6'
-1/K1 SMALL
-1/K1 LARGE
gI
gII

Figure 2.3

$$B_1^{(w)} = -\frac{P_1(\eta_1^{(1)})^2}{\lambda_1^2 - \sigma_w^2} - \frac{P_1(\eta_1^{(2)})^2}{\lambda_2^2 - \sigma_w^2} - \frac{P_1(\eta_1^{(3)})^2}{\lambda_3^2 - \sigma_w^2} - \frac{P_1(\eta_1^{(4)})^2}{\lambda_4^2 - \sigma_w^2}$$

$$B_2^{(w)} = -\frac{\eta_2^{(1)}P_1\eta_1^{(2)}}{\lambda_1^2 - \sigma_w^2} - \frac{\eta_2^{(2)}P_1\eta_1^{(2)}}{\lambda_2^2 - \sigma_w^2} - \frac{\eta_2^{(3)}P_1\eta_1^{(3)}}{\lambda_3^2 - \sigma_w^2} - \frac{\eta_2^{(4)}P_1\eta_1^{(4)}}{\lambda_4^2 - \sigma_w^2}$$

$$B_3^{(w)} = -\frac{\eta_3^{(1)}P_1\eta_1^{(1)}}{\lambda_1^2 - \sigma_w^2} - \frac{\eta_3^{(2)}P_1\eta_1^{(2)}}{\lambda_2^2 - \sigma_w^2} - \frac{\eta_3^{(3)}P_1\eta_1^{(3)}}{\lambda_3^2 - \sigma_w^2} - \frac{\eta_3^{(4)}P_1\eta_1^{(4)}}{\lambda_4^2 - \sigma_w^2}$$

$$B_4^{(w)} = -\frac{\eta_4^{(1)}P_1\eta_1^{(1)}}{\lambda_1^2 - \sigma_w^2} - \frac{\eta_4^{(2)}P_1\eta_1^{(2)}}{\lambda_2^2 - \sigma_w^2} - \frac{\eta_4^{(3)}P_1\eta_1^{(3)}}{\lambda_3^2 - \sigma_w^2} - \frac{\eta_4^{(4)}P_1\eta_1^{(4)}}{\lambda_4^2 - \sigma_w^2}$$

For $w = 1$, all terms in (2.14) are small except for the first term on the right of each of the first two equations; this is due to the fact that all denominators are large compared to σ_1 which is small. Thus, for this choice of w

$$A_1^{(1)} \sim \xi_1^{(1)},\ A_2^{(1)} \sim \xi_2^{(1)},\ B_1^{(1)} \sim \ldots \sim B_4^{(1)} \sim 0$$

For $w = 2$, all terms are small except for those which are first on the right of the last four of (2.14), since now $\lambda_1^2 - \sigma_2^2$ is small in all the denominators. Hence,

$$A_1^{(2)} \sim A_2^{(2)} \sim 0, \ B_1^{(2)} \sim \eta_1^{(1)}, \ B_2^{(2)} \sim \eta_2^{(1)}$$

$$B_3^{(2)} \sim \eta_3^{(1)}, \ B_4^{(2)} \sim \eta_4^{(1)}$$

Similar results are obtained when w equals 3, 4, 5, and 6. When the coupling is small, we conclude there is little interaction between the two systems according to our results; this is what one expects on physical grounds.

The plot of $-1/k_1$ when k_1 is large is shown in Figure 2.3 by the dashed line just below the σ-axis. The roots of (2.11) are now shown by circled crosses and labeled $\sigma_1', \sigma_2', \ldots, \sigma_6'$; they differ substantially from the $\sigma_1, \sigma_2, \ldots \sigma_6$ respectively. The heavy segments below the σ-axis denote intervals that contain the σ_w as k_1 varies from a small value to a large one. Equations (2.14) still determine the mode shapes. The only general conclusion which can be drawn at this point is that the σ_w and corresponding mode shapes are close to $\omega_1, \omega_2, \lambda_1, \ldots, \lambda_4$ and their mode shapes, respectively, when k_1 is small, with differences becoming larger as k_1 increases.

It is now easy to see how the results to this point can be used to determine the variability in the natural frequencies of a system due to variability in one spring, provided that spring, when cut, separates the original system into two separate subsystems.

To summarize, let the two subsystems be denoted by System I and System II, with defining equations (I) and (II), respectively. Determine the natural frequencies and normal modes of each as in (2.1) and (2.2). Next graph the functions $g_I(\sigma)$ and $g_{II}(\sigma)$ which are defined in (2.13). Let (k_1', k_1'') denote the range of k_1. Then determine the natural frequencies $\sigma_1', \sigma_2', \ldots \sigma_{N_1+N_2}'$ corresponding to k_1', and $\sigma_1'', \sigma_2'', \ldots. \sigma_{N_1+N_2}''$ corresponding to k_1'', by the same procedure used to determine the natural frequencies in the above example, (See Fig. 2.3). The ranges in the natural frequencies due to the range (k_1', k_1'') in k_1 are

$$(\sigma_1', \sigma_1''), (\sigma_2', \sigma_2''), \ldots\ldots, (\sigma_{N_1+N_2}', \sigma_{N_1+N_2}'')$$

The spacings of the natural frequencies in System I and II, their corresponding mode shapes, and the slope of the function of on the right of (2.11) determines these ranges.

A second general conclusion can be drawn when one system, say System II, is very massive when compared with the other. For under this circumstance, the α_{rs} of (2.2) will be large in comparison with the a_{ij} of (2.1); this means that the $\eta_r^{(u)}$ will be small in comparison with the $\xi_i^{(\ell)}$ and the $g_{II}(\sigma)$ function will have a small slope everywhere except in the neighborhood of the λ_u, (See Figure 2.3). It is now easy to see that only the natural frequencies corresponding to those of System I will have appreciable variability due to variability in k_1.

While a digital computer utilizing the Monte Carlo approach can be used to determine the variability in natural frequencies due to parameter variability, the method just described does provide a useful adjunct to physical understanding when a subsystem I is to be attached somewhere on a large system II. For then, all information needed for evaluating g_I and g_{II} (at different possible attachment points) would be available, and it would be relatively simple to explore the influence of variability in the attachment spring constant on the natural frequencies on this graphical basis. Moreover, a graphical result is much simpler to comprehend than a mass of numbers, providing as it does a simple visual summary of a vast amount of numerical information.

3. Rayleigh's Corrections to Natural Frequency and Corresponding Mode Shape Due to Changes in Parameter Values.

Rayleigh [1] calculates the changes in natural frequency and corresponding mode due to changes in the stiffness and mass parameters of a system. Although his motivation for such a study was different* from ours, it is instructive to review his results and see what light they cast on our problem.

Let $q_1, \ldots q_N$ be the independent coordinates of the system with T and V given by

$$(3.1) \qquad 2T = a_{ij}\dot{q}_i\dot{q}_j \quad , \quad 2V = c_{ij}q_i q_j$$

Let $\theta_1, \ldots \theta_N$ be the corresponding normal coordinates with

$$(3.2) \qquad 2T = a_r\dot{\theta}_r^2 \quad , \quad 2V = c_r\theta_r^2$$

where

$$(3.3) \qquad q_i = \sum_r \eta_i^{(r)}\theta_r$$

$$a_r = a_{ij}\eta_i^{(r)}\eta_j^{(r)}$$

$$c_r = c_{ij}\eta_i^{(r)}\eta_j^{(r)}$$

---

* He uses approximations to simplify the analysis; changes in parameter values are necessary to change from simplified system to real system.

[1] See Rayleigh, "Theory of Sound", Section 90, Vol. 1.

Then the equations of motion in terms of the θ's are

(3.4) $$a_{(r)}\ddot{\theta}_{(r)} + c_{(r)}\theta_{(r)} = 0 \quad , \quad r = 1, \ldots\ldots N$$

The modal solutions of (3.4) are

(3.5) $$\theta_r \sim \cos(\omega_r t + \phi_r), \ \theta_s = 0 \text{ if } s \neq r; \ r,s = 1, \ldots, N$$

where

(3.6) $$\omega_r = \sqrt{c_{(r)}/a_{(r)}} \quad , \quad \phi_r = \text{const.}, \ r = 1, \ldots\ldots N$$

Thus, in any modal solution of (3.4) only one normal coordinate is non-zero at a time.

Now let us assume the parameters are changed. In terms of the θ's, this means that

(3.7) $$2T' \equiv 2(T + \delta T) = a_r\dot{\theta}_r^2 + \delta a_{rs}\dot{\theta}_r\dot{\theta}_s$$

$$2V' \equiv 2(V + \delta V) = c_r\theta_r^2 + \delta c_{rs}\theta_r\theta_s$$

where the δa_{rs} and δc_{rs} represent the changes, respectively, in the mass and stiffness parameters in terms of the θ-coordinates. The equations of motion of the changed system, that is, the new equation similar to (3.4), are

$$(3.8)\qquad a_{(s)}\ddot{\theta}_{(s)} + \delta a_{st}\ddot{\theta}_t + c_{(s)}\theta_{(s)} + \delta c_{st}\theta_t = 0$$

$$s = 1, \ldots . N$$

The solutions of the coupled equations (3.8) are

$$(3.9)\qquad \theta_s = u_s \cos(\lambda t + \phi), \quad s = 1, \ldots . N$$

where u_s , λ and ϕ are constants. The substitution of (3.9) into (3.8) yields

$$(3.10)\qquad (c_{(s)} - \lambda^2 a_{(s)})u_{(s)} + (\delta c_{(s)t} - \lambda^2 \delta a_{(s)t})u_t = 0, \quad s = 1, \ldots N$$

We assume at this point that the δa_{st} and δc_{st} are of first order in small quantities with respect to the a_s and c_s respectively. If the δa_{st} and δc_{st} were zero, all but one of the u_s , say u_r , would be zero. When the δa_{st} and δc_{st} are small, it is reasonable to assume initially that all u_s for $s \neq r$ are small with respect to u_r ; we make this assumption. Thus, to repeat, we are assuming that the δa_{st} , δc_{st} and u_s for $s \neq r$ are of first order in small quantities.

Let us now examine those of equations (3.10) for which $s \neq r$.

$$(3.11)\qquad a_{(s)}(\omega^2_{(s)} - \lambda^2)u_s + (\delta c_{s(r)} - \lambda^2 \delta a_{s(r)})u_{(r)}$$

$$+ \sum_t{}' (\delta c_{st} - \lambda^2 \delta a_{st})u_t = 0$$

where the summation $\sum'$ excludes $t = r$. The terms under the summation $\sum'$ are of second order in small quantities while the other terms are of first order.

Next consider the equation of (3.10) for which $s = r$.

$$(c_{(r)}-\lambda^2 a_{(r)})u_{(r)}$$

$$+ (\delta c_{(r)(r)}-\lambda^2\delta a_{(r)(r)})u_{(r)}$$

$$+ \sum_s{}' (\delta c_{rs}-\lambda^2\delta a_{rs})u_s = 0$$

The second line and third line are of first and second order in small quantities, respectively. If we rewrite this equation as

$$c_{(r)} + \delta c_{(r)(r)} - \lambda^2(a_r + \delta a_{(r)(r)})$$

$$= - \sum_s{}' (\delta c_{(r)s} - \lambda^2\delta a_{(r)s}) \frac{u_s}{u_{(r)}}$$

and use (3.11), we obtain

$$\lambda^2(a_r + \delta a_{(r)(r)}) = c_r + \delta c_{rr} - \sum_s{}' \frac{(\delta c_{rs}-\lambda^2\delta a_{rs})^2}{a_s(\omega_s^2 - \lambda^2)} \tag{3.12}$$

$$+ \text{ higher order terms}$$

With (3.11) and (3.12), we have a scheme for evaluating changes in natural frequency and its normal mode arising from changes in parameter values.

Rayleigh uses (3.11) and (3.12) as follows: First, he neglects the second order terms in (3.11), obtaining

$$\frac{u_s}{u_r} = -\frac{\delta c_{sr} - \lambda^2 \delta a_{sr}}{a_{(s)}(\omega^2_{(s)} - \lambda^2)}, \quad s = 1, \dots N; \; s \neq r$$

Then, since for the unchanged or initial system $\lambda^2 \approx \omega_r^2$,

$$(3.13) \qquad \frac{u_s}{u_r} \simeq \frac{\delta c_{sr} - \omega_r^2 \delta a_{sr}}{a_s(\omega_s^2 - \omega_r^2)} \qquad s = 1, \dots N; \; s \neq r$$

Equation (3.12), or the one before it, determines the corrected natural frequency corresponding to

$$(3.14) \qquad (\omega_r^2)_{\text{corrected}} = \frac{c_r + \delta c_{rr}}{a_r + \delta a_{rr}} - \sum_s{}' \frac{(\delta c_{rs} - \omega_r^2 \delta a_{rs})^2}{a_r a_s (\omega_s^2 - \omega_r^2)}$$

Equations (3.13) and (3.14) are due to Rayleigh.

We note that the change in mode shape (3.13) is of first order in small quantities provided ω_s is not near to ω_r for $s = 1, \dots N; \; s \neq r$. The first order correction term in (3.14) is the first term on the right; this term represents the correction to ω_r^2 arising from only the changes in a_r and c_r but not due to changes in mode shape. Changes in ω_r^2 due to changes in mode shape are of second order in small quantities.

If we write the equation before (3.12) and the equation after (3.12) in the forms

$$(3.15) \qquad \lambda^2 = \frac{\omega_r^2\left(1 + \frac{\delta c_{rr}}{c_r}\right) + \frac{1}{a_r}\sum_s{}' \; \delta c_{rs}\left(\frac{u_s}{u_r}\right)}{1 + \frac{\delta a_{rr}}{a_r} + \frac{1}{a_r}\sum_s{}' \; \delta a_{rs}\left(\frac{u_s}{u_r}\right)}$$

$$\left(\frac{u_s}{u_r}\right) = \frac{\delta c_{rs} - \lambda^2 \delta a_{rs}}{a_s(\omega_s^2 - \lambda^2)} \quad , \quad s = 1, \ldots . \; N; \; s \neq r$$

respectively, an approximation scheme can be set up in order to improve the accuracy of the corrections. We shall not pursue this point here.

Computationally, there is not much difficulty involved in evaluating any of the above expressions provided we have the natural frequencies ω_r and the corresponding mode shapes $\eta_j^{(r)}$ of the original system, and the δa_{rs} and δc_{rs} are evaluated. The ω_r and $\eta_j^{(r)}$ must be evaluated from the original system (I), and, if Δa_{ij} and Δc_{ij} are the changes in the physical parameters a_{ij} and c_{ij}, respectively, we have

$$(3.16) \qquad \delta a_{rs} = \Delta a_{ij}\eta_i^{(r)}\eta_j^{(s)}$$

$$\delta c_{rs} = \Delta c_{ij}\eta_i^{(r)}\eta_j^{(s)}$$

At this point, a difficulty should be mentioned.

The a_{ij} and c_{ij} are obtained from the physical system in terms of the original (or physical) coordinates. Thus, these quantities can be given a physical interpretation in terms of the structure. The same is true for the δa_{ij} and the δc_{ij}. However, the δa_{rs} and the δc_{rs} come from (3.16) and involve the mode shapes of the original system. Physical interpretation of changes in natural frequencies and mode shapes as given by (3.15) in terms of the δa_{ij} and the δc_{ij} now become difficult. Hence, one loses at this point physical contact with the original problem, except in special cases. (See Rayleigh [1], Section 185, for example.)

If the δa_{ij} and δc_{ij} are regarded as r.v.'s with zero means and small variances, then the δa_{rs} and δc_{rs} will be r.v.'s with the same properties. Then, using the first of (3.15), we obtain

$$E\{\lambda\} = \omega_r E\left\{\left|\frac{1 + \frac{\delta c_{(r)(r)}}{c_{(r)}} + \frac{1}{a_{(r)}\omega^2_{(r)}}\sum_s{}' \delta c_{(r)s}\left(\frac{u_s}{u_r}\right)}{1 + \frac{\delta a_{(r)(r)}}{a_{(r)}} + \frac{1}{a_{(r)}}\sum_s{}' \delta a_{(r)s}\left(\frac{u_s}{u_r}\right)}\right|^{\frac{1}{2}}\right\}$$

where u_s/u_r must be obtained from the second of (3.15). A similar formula can be obtained for $E\{\lambda^2\}$. Upon expanding out the expressions in the curly brackets and retaining second products and second degree terms in the r.v.'s, δa_{rs} and δc_{rs}, it is possible to obtain estimates for mean frequencies and their variances. We have not done this because physical interpretation of results is difficult.

Rayleigh's method does provide a computational scheme for estimating change in n.f.'s due to parameter change, but it does not lend itself to engineering interpretation. Thus, we shall not pursue it further.

4. A Brief Review of Mean Square Approximate Systems*

Let us consider a holonomic system with N-degrees of freedom and independent coordinates $q_1, \dots, q_N$. Let the kinetic and potential energies be

$$2T = A_{ij}\dot{q}_i\dot{q}_j \quad , \quad 2V = C_{ij}q_iq_j \tag{4.1}$$

respectively, where dots denote time derivatives and summation convention is used. The random variables A_{ij} and C_{ij} form σ sample systems with sample values $a_{ij,\sigma}$ and $c_{ij,\sigma}$, respectively, with mean values

$$E\{A_{ij}\} = a_{ij} \qquad E\{C_{ij}\} = c_{ij} \tag{4.2}$$

For all sample values of the r.v.'s A_{ij} and C_{ij}, $a_{ij,\sigma} = a_{ji,\sigma}$, $c_{ij,\sigma} = c_{ji,\sigma}$ and $a_{ij,}\ \dot{q}_i\dot{q}_j$ and $c_{ij,\sigma}q_iq_j$ are positive definite quadratic forms for all σ.

$$I(u,g) \equiv E\{\int_{\lambda_1}^{\lambda_2} g(\lambda) \sum_{i=1}^{N} \varepsilon_i^2 \, d\lambda\} \geq 0 \tag{4.3}$$

where

$$\sum_{i=1}^{N} \varepsilon_i^2 = [C_{ij}C_{ik} - \lambda^2(A_{ij}C_{ik}+A_{ik}C_{ij}) + \lambda^4 A_{ij}A_{ik}]u_ju_k$$

* A more detailed development of mean squared approximate systems can be found in [6].

$$g(\lambda) \geq 0 \quad \text{for} \quad \lambda_1 < \lambda < \lambda_2$$
$$= 0 \text{ otherwise}$$

$$\int_{\lambda_1}^{\lambda_2} g(\lambda)\, d\lambda = 1$$

and the u_j are the coordinate amplitudes. It should be noted that $g(\lambda)$ is an arbitrary weighting function satisfying the above condition and λ_1 and λ_2 are arbitrary bounds. The mean square oscillator associated with the interval (λ_1, λ_2) has a mode shape associated with this interval which is obtained by minimizing $I(u,g)$ subject to the condition

(4.4) $$a_{ij}u_iu_j = 1$$

Let ξ_j denote the set of u_j which minimize $I(u,g)$. It is clear that these ξ_j constitute the eigenvector of the smallest eigenvalue of $I(u,g)$; we define

(4.5) $$\mathcal{I}(\xi,g) \equiv I(\xi,g) = \min_u I(u,g)$$

If $g(\lambda)$ is a delta function $\delta(\lambda-\omega_k)$ centered at one of the natural frequencies, ω_k , of the mean system defined by the equation

(4.6) $$2\underline{I} = a_{ij}\dot{q}_i\dot{q}_j \qquad 2\underline{V} = c_{ij}q_iq_j$$

and if the ξ_j are proportional to the k^{th} mode shape $\eta_j^{(k)}$ of the mean system, then for the mean system

$$(4.7)\qquad I(\eta^{(k)},\ \delta(\lambda-\omega_k)) = 0$$

Only in this particular case is $I(\cdot)$ ever zero.

If, with $g(\lambda)$ still a δ-function centered on ω_k with (4.3) written in the form

$$(4.8)\qquad I(u,\delta(\lambda-\omega_k)) = I_{0,0}(u,\omega_k) + I_{1,0}(u,\omega_k)$$

where

$$(4.9)\qquad I_{0,0}(u,\omega_k) = [c_{ij}c_{i\ell}-\omega_k^2(a_{ij}c_{i\ell}+c_{ij}a_{i\ell}) + \omega_k^4 a_{ij}a_{i\ell}]\ u_j u_\ell$$

$$I_{1,0}(u,\omega_k) = [\text{Cov } C_{ij}C_{i\ell}-\omega_k^2(\text{Cov } A_{ij}C_{i\ell}+\text{Cov } C_{ij}A_{i\ell}) + \omega_k^4 \text{ Cov } A_{ij}A_{i\ell}]\ u_j u_\ell$$

then

$$(4.10)\qquad I(\xi,\delta(\lambda-\omega_k)) \equiv \min_u I(u,\delta(\lambda-\omega_k)) \equiv I_{0,0}(\xi,\omega_k) + I_{1,0}(\xi,\omega_k)\ .$$

With disorder small, the ξ_j which minimize $I(u,\delta(\lambda-\omega_k))$ will be close to the $\eta_j^{(k)}$. It now follows that

$$(4.11) \qquad I(\eta^{(k)}, \delta(\lambda-\omega_k)) = I_{0,0}(\eta^{(k)},\omega_k) + I_{1,0}(\eta^{(k)},\omega_k)$$

$$\geq I(\xi, \delta(\lambda-\omega_k))$$

Because of (4.7) and the fact that $I_{0,0}(u,\omega_k)$ is the $I(u,\delta(\lambda-\omega_k))$ of the mean system we note that

$$(4.12) \qquad I_{0,0}(\eta^{(k)}, \omega_k) = 0$$

Thus,

$$(4.13) \qquad I_{1,0}(\eta^{(k)}, \omega_k) \geq I(\xi, \delta(\lambda-\omega_k))$$

where $I_{1,0}(\eta^{(k)}, \omega_k)$ is easily computed from the information available on the mean system and on the second moments of the parameters.

Details on applications and uniqueness of M. S. approximate systems are given in [6] and [7], respectively.

5. Derivation of Estimator for Upper Bound of the Standard Deviation of a Natural Frequency.

It is possible to derive our estimator in a number of ways under a variety of conditions. The derivations we shall give here are selected because of their simplicity and because they are the simplest to use in practice.

Let us first consider the case in which the masses are deterministic; we then select (or determine) coordinates so that the system is defined by the equations

$$(5.1) \qquad 2T = \sum_{i=1}^{N} \dot{q}_i^2 \qquad 2V = C_{ij} q_i q_j$$

where

$$(5.2) \qquad \text{Prob}\{C_{ij} = c_{ij,\sigma} = c_{ji,\sigma}\} = p_\sigma \geq 0$$

$$\sum_{\sigma=1}^{m} p_\sigma = 1 \quad , \quad E\{C_{ij}\} = c_{ij}$$

We thus contemplate a set of m sample systems each with N degrees of freedom defined by the equations

$$(5.3) \qquad 2T_\sigma = \sum_{i=1}^{N} \dot{q}_i^2 \quad , \quad 2V_\sigma = c_{ij,\sigma}\, q_i q_j$$

which have probability p_σ of occurrence. Let the natural frequencies of a sample system be

$$\omega_{1,\sigma} < \omega_{2,\sigma} < \dots < \omega_{N,\sigma}$$

with corresponding mode shapes $\eta^{(r)}_{j,\sigma}$; $r,j=1, \dots N$; $\sigma=1, \dots m$.

Then we have (σ is fixed)

$$(5.4) \qquad c_{ij,\sigma}\eta^{(r)}_{j,\sigma} = \omega^2_{(r),\sigma}\,\delta_{ij}\eta^{(r)}_{j,\sigma}$$

$$\delta_{ij}\eta^{(r)}_{i,\sigma}\eta^{(s)}_{j,\sigma} = \delta_{rs}$$

$$u_j = \sum_{r=1}^{N} \eta^{(r)}_{j,\sigma}\,\nu_{r,\sigma}$$

where the u_j are coordinate amplitudes as in Section 4, and the $\nu_{r,\sigma}$ are the corresponding normal coordinate amplitudes for the σ-sample system.

For the mean system defined by the equations

$$(5.5) \qquad 2\underline{T} = \delta_{ij}\dot{q}_i\dot{q}_j \quad , \qquad 2\underline{V} = c_{ij}q_iq_j$$

we have for natural frequencies and corresponding normal modes

$$c_{ij}\eta^{(r)}_{j} = \omega^2_{(r)}\delta_{ij}\eta^{(r)}_{j}$$

$$\delta_{ij}\eta^{(r)}_{i}\eta^{(s)}_{j} = \delta_{rs}$$

Using (4.3), we find that

$$(5.7) \qquad I(u,\delta(\lambda-\omega_k)) = \sum_{\sigma=1}^{m} p_\sigma \sum_{r=1}^{N} (\omega_{r,\sigma}^2-\omega_k^2)^2 \nu_{r,\sigma}^2$$

where we have employed the last two of (5.4) to transform to the normal coordinate amplitudes $\nu_{r,\sigma}$.

If there is no disorder,

$$(5.8) \qquad I(\xi,\omega_k) \equiv \min_u I(u,\delta(\lambda-\omega_k)) = I_{0,0}(\eta^{(k)},\omega_k) = 0$$

where

$$(5.9) \qquad \xi_j = \eta_j^{(k)} , \qquad \omega_{r,\sigma} = \omega_r$$

$$\nu_{k,\sigma} = 1 , \qquad \nu_{r,\sigma} = 0 \qquad \text{for} \quad r \neq k$$

$$\sum \nu_{r,\sigma}^2 = 1$$

If there is disorder,

$$(5.10) \qquad I(\xi',\omega_k) \equiv \min_u I(u,\delta(\lambda-\omega_k))$$

$$= \sum_{\sigma=1}^{m} p_\sigma \sum_{r=1}^{N} (\omega_{r,\sigma}^2-\omega_k^2)^2 \nu_{r,\sigma}'^2$$

$$\nu_{r,\sigma}'^2 = \eta_{j,\sigma}^{(r)} \xi'_j$$

We now have from (4.13),

$$(5.11) \qquad I_{1,0}(\eta^{(k)},\omega_k) \geq I(\xi',\omega_k)$$

$$= \sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma}^2-\omega_k^2)^2\nu_{k,\sigma}'^2$$

$$+ \sum_{\sigma=1}^{m} p_\sigma \sum_{r=1}^{N}{}' (\omega_{r,\sigma}^2-\omega_k^2)^2\nu_{r,\sigma}'^2$$

where $\sum_{r=1}^{N}{}'$ denotes the sum excluding $r = k$. For disorder sufficiently small, the $\nu_{r,\sigma}'$ will satisfy approximately the second of (5.9), i.e.

$$(5.12) \qquad \nu_{k,\sigma}'^2 \geq 1-\varepsilon_1 \approx 1$$

$$\nu_{r,\sigma}'^2 \approx 0 \qquad \text{for all } \sigma$$

Thus,

$$(5.13) \qquad I_{1,0}(\eta^{(k)},\omega_k) \geq I(\xi',\omega_k)$$

$$\geq (1-\varepsilon_1) \sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma}^2-\omega_k^2)^2$$

We now wish to show that there exists an ε_2 such that

$$(5.14) \qquad \int_{\alpha_1}^{\alpha_2} (x^2-a^2)^2 f(x)dx \geq (4-\varepsilon_2)a^2 \int_{\alpha_1}^{\alpha_2} (x-a)^2 f(x)dx$$

where a is in the interval (α_1, α_2) and

$$(5.15) \qquad f(x) \geq 0 \quad , \qquad \int_{\alpha_1}^{\alpha_2} f(x)dx = 1$$

If we rewrite (5.14) as

$$(5.16) \qquad \int_{\alpha_1}^{\alpha_2} (x-a)^2[x^2+2ax-3a^2+\varepsilon_2 a^2]\, f(x)dx \geq 0 \quad ,$$

we note that this inequality will be satisfied if the term in square brackets is non negative in the interval (α_1, α_2) . A graph of this term is shown in the adjacent figure. The roots are at

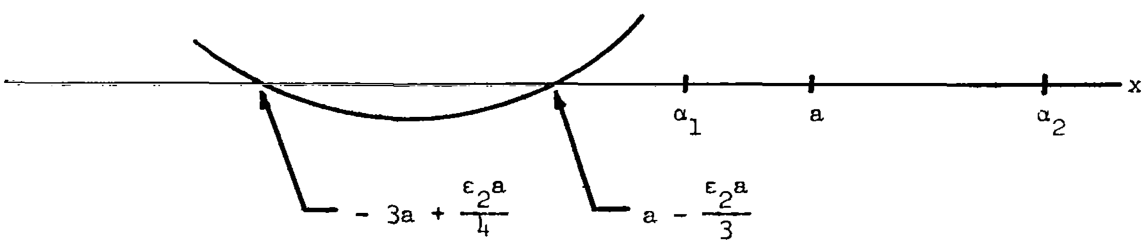

Figure 5.1

$-3a + \frac{\varepsilon_2 a}{4}$ and $a - \frac{\varepsilon_2 a}{4}$. If ε_2 is chosen sufficiently large so that the interval (α_1, α_2) is also as shown, the inequality (5.16) is obtained. Thus, for disorder sufficiently small

$$(5.17) \qquad \sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma}^2 - \omega_k^2)^2 \geq (4-\varepsilon_2)\omega_k^2 \sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma} - \omega_k)^2$$

In particular, we note that for $\varepsilon_2 = 1$ the above discussion implies that (5.17) will hold if the disorder is restricted so that $3\omega_k/4 < \omega_{k,\sigma}$ for all k and σ ; disorder will indeed be large for this to occur. We also know that the second moment

$$\sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma} - \omega_k)^2$$

about any point ω_k is not less than the variance, σ_k^2 , of ω_k ; thus,

$$\sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma} - \omega_k)^2 \geq \sigma_k^2$$

and, hence, we may write (5.13) as

$$(5.18) \qquad I_{1,0}(\eta^{(k)}, \omega_k) \geq (1-\varepsilon_1)(4-\varepsilon_2)\omega_k^2\sigma_k^2$$

or

$$(5.19) \qquad \sigma_k \leq \frac{1}{\omega_k} \sqrt{\frac{I_{1,0}(\eta^{(k)}, \omega_k)}{(1-\varepsilon_1)(4-\varepsilon_2)}}$$

The quantities ε_1 and ε_2 approach zero as the disorder approaches zero. An estimate for ε_1 can be obtained from the first of (5.12) and the second of (5.10); with a specified amount of disorder, calculate several $\eta^{(r)}_{j,\sigma}$; the $\eta^{(k)}_j$ are known for the mean system; then

$$(5.20) \qquad \nu_{k,r} = \eta^{(k)}_{j,\sigma}\eta^{(k)}_j$$

can be evaluated and from $1-\nu^2_{k,j}$ we can estimate ε_1 as $\min_\sigma(1-\nu^2_{k,\sigma})$. By reducing the disorder to a lower level and repeating the above calculations, we obtain another estimate of ε_1 . In this way, we find a satisfactory value for ε_1 for disorder sufficiently small.

As mentioned earlier, a value of $\varepsilon_2 = 1$ will be satisfactory for small disorder, a check being provided by the few $\omega_{r,\sigma}$ computed when evaluating the $\eta^{(r)}_{j,\sigma}$ needed in (5.20).

We note, however, that in two steps, going from (5.11) to (5.13) and going from (5.13) to (5.17), positive quantities have been removed from the left of (5.19). For these reasons, we shall use in most cases an uncorrected form of (5.19); namely

$$(5.21) \qquad \sigma_k \leq \frac{\sqrt{I_{1,0}(\eta^{(k)},\omega_k)}}{2\omega_k} \equiv \delta_k$$

The ease in evaluating this expression, in which only properties of the mean system and 2nd moments of parameters are required, also recommends its use.

We also note that our estimator (5.19) could be improved with additional effort by employing the quantity $I(\xi',\omega_k)$ instead of $I_{1,0}(\eta^{(k)},\omega_k)$.

Consider now the more general case with disorder in the masses and springs by replacing (5.1) with

$$(5.22) \qquad 2T = A_{ij}\dot{q}_i\dot{q}_j \quad , \quad 2V = C_{ij}q_iq_j$$

where

$$(5.23) \qquad \mathrm{Prob}\{A_{ij} = a_{ij,\sigma} = a_{ji,\sigma},\ C_{ij} = c_{ij,\sigma} = c_{ji,\sigma}\} = p_\sigma \geq 0$$

$$\sum_{\sigma=1}^{m} p_\sigma = 1, \quad E\{A_{ij}\} = a_{ij}, \quad E\{C_{ij}\} = c_{ij}$$

We first determine the non-singular linear transformation*

$$(5.24) \qquad q_i = f_{ij}y_j$$

which reduces the mean system to the form

$$(5.25) \qquad 2\underline{T} = \sum_{i=1}^{N} \dot{y}_i^2 \quad , \quad 2\underline{V} = c'_{ij}y_iy_j$$

where

$$(5.26) \qquad c'_{ij} = c_{rs}f_{ri}f_{sj}$$

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* In certain cases obtaining this transformation may be a formidable task.

Then we write

$$(5.27) \qquad a'_{ij,\sigma} = a_{rs,\sigma} f_{ri} f_{sj} \quad , \quad c'_{ij,\sigma} = c_{rs,\sigma} f_{ri} f_{sj}$$

$$E\{A'_{ij}\} = \delta_{ij} \quad , \quad E\{C'_{ij}\} = c'_{ij}$$

We note that the transformation (5.24) has been chosen so that the mean system mass matrix $\{a_{ij}\}$ reduces to $\{\delta_{ij}\}$; this means that the sample values $a_{ij,\sigma}$ will be approximately equal to δ_{ij} for all σ , the $a'_{ii,\sigma}$ being approximately equal to unity while the $a'_{ij,\sigma}$ for $i \neq j$ will be approximately equal to zero.

In terms of the y-coordinates, let the natural frequencies and corresponding normal modes of the mean system satisfy the equations

$$(5.28) \qquad c'_{ij} \eta_j^{(r)} = \omega_r^2 \delta_{ij} \eta_j^{(r)} \quad , \quad \delta_{ij} \eta_i^{(r)} \eta_j^{(s)} = \delta_{rs}$$

and let the natural frequencies and corresponding normal modes of the sample systems satisfy the equations

$$(5.29) \qquad c'_{ij,\sigma} \eta_{j,\sigma}^{(r)} = \omega_{(r)}^2 a'_{ij,\sigma} \eta_{j,\sigma}^{(r)} \quad , \quad a'_{ij,\sigma} \eta_{i,\sigma}^{(r)} \eta_{j,\sigma}^{(s)} = \delta_{rs}$$

Equation (5.11) must now be replaced by

(5.30) $$I_{1,0}(\eta^{(k)},\omega_k) \geq I(\xi',\omega_k)$$

$$= \sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma}^2-\omega_k^2)^2\nu_{k,\sigma}'^2 \sum_{i=1}^{N}(a_{ij,\sigma}'\eta_{j,\sigma}^{(k)})^2$$

$$+ \sum_{\sigma=1}^{m} p_\sigma \sum_{r,s=1}^{N} (\omega_{r,\sigma}^2-\omega_k^2)(\omega_{s,\sigma}^2-\omega_k^2)\nu_{r,\sigma}'\nu_{s,\sigma}' \sum_{i=1}^{N}(a_{ij,\sigma}'\eta_{j,\sigma}^{(r)})(a_{i\ell,\sigma}'\eta_{\ell,\sigma}^{(s)})$$

where

(5.31) $$\nu_{r,\sigma}' = \eta_{j,\sigma}^{(r)}\xi_j'$$

In virtue of the properties of $a_{ij,\sigma}'$ and for small system disorder, we have

(5.32) $$\sum_{i=1}^{N}(a_{ij,\sigma}'\eta_{j,\sigma}^{(k)})^2 \simeq 1$$

$$\nu_{k,\sigma}'^2 \simeq 1$$

$$\nu_{r,\sigma}'^2 \simeq 0 \quad \text{for} \quad r \neq k$$

and thus

(5.33) $$I_{1,0}(\eta^{(k)},\omega_k) \simeq \sum_{\sigma=1}^{m} p_\sigma(\omega_{k,\sigma}^2-\omega_k^2)^2 \simeq 4\omega_{(k)}^2\sigma_{(k)}^2$$

Equation (5.19) is therefore replaced by

$$(5.34)\qquad \sigma_k \simeq \frac{\sqrt{I_{1,0}(\eta^{(k)},\omega_k)}}{2\omega_k} \equiv \delta_k$$

which is an estimator for σ_k . The lack of inequality is due to the fact that terms discarded on the right of (5.30) are not individually positive. At this time we have not been able to establish that the second term on the right of (5.30) is always positive as was the corresponding term on the right of (5.11).

There is another possible way to proceed which might prove useful, although we have not as yet explored it. We recall that

$$(5.35)\qquad \varepsilon_{i,\sigma} = (c_{ij,\sigma} - \omega_k^2 a_{ij,\sigma})u_j$$

Now consider (non singular linear transformation)

$$(5.36)\qquad e_{r,\sigma} = \varepsilon_{i,\sigma}\eta_{i,\sigma}^{(r)} = (\omega_{(r),\sigma}^2 - \omega_k^2)a_{ij,\sigma}\eta_{i,\sigma}^{(r)}u_j$$

$$= \sum_{s=1}^{N} (\omega_{(r),\sigma}^2 - \omega_k^2)a_{ij,\sigma}\eta_{i,\sigma}^{(r)}\eta_{j,\sigma}^{(s)}\nu_{s,\sigma}$$

$$= (\omega_{(r),\sigma}^2 - \omega_k^2)\nu_{(r),\sigma}$$

We have

$$(5.37)\qquad E\{\sum_r e_r^2\} = \phi(u,\omega_k)$$

$$= \sum_{\sigma=1}^{m} p_\sigma \sum_{r=1}^{N} (\omega_{r,\sigma}^2 - \omega_k^2)\nu_{r,\sigma}^2$$

While the right hand side of this equation is of the same form as the right hand side of (5.10) and readily leads to the right of (5.18), it is difficult to evaluate $\phi(u, \omega_k)$ as sample mode shapes are involved. This will be pursued further in Section 7.

Appendix 1, Section 5.

To establish (5.12), we proceed as follows with an obvious change in notation:

We must show that an $\varepsilon_2 > 0$ exists such that

$$\text{(a)} \qquad \int (x^2-a^2)^2 f(x)dx = \int (x+a)^2(x-a)^2 f(x)dx$$

$$\geq (4-\varepsilon_2)a^2 \int (x-a)^2 f(x)dx$$

where $f(x)$ is a density function whose non-zero range is small and includes a . To establish (a) requires that

$$\text{(b)} \qquad \int (x-a)^2[(x+a)^2-a^2(4-\varepsilon_2)]f(x)dx \geq 0$$

or

$$\text{(c)} \qquad \int (x-a)^2[x^2+2ax-3a^2+\varepsilon_2 a^2]f(x)dx \geq 0$$

Let

$$x = y + a$$

then (c) becomes

$$\text{(d)} \qquad \int y^2(y^2+4ya+\varepsilon_2 a^2)f(y)dy > 0$$

For positive limits on y, the integrand is positive. For negative limits, the integrand is positive provided the limit is larger than

$$-2a + 2a\sqrt{1-\varepsilon_2/4}$$

Thus, given the lower limit for which $f(y) \geq 0$ we can find an $\varepsilon_2 > 0$ such that $(y^2+4ya+\varepsilon_2 a^2)$ is positive for

$$y \geq -2a + 2a\sqrt{1-\varepsilon_2/4}$$

6. Examples

a) One Degree of Freedom (See Figure 6.1)

We have

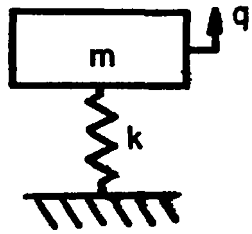


Figure 6.1

(6.1) $2T = m\dot{q}^2, \quad 2V = kq^2$

or with the substitution

$$p = q\sqrt{m}$$

(6.2) $2T = \dot{p}^2, \quad 2V = k/m \; p^2$

For this system, the natural frequency is

(6.3) $\omega_1 = \sqrt{k/m}$

Now let the spring constant be the random variable (r.v.) defined by

(6.4) $K = k(1 + B)$

where B is a dimensionless r.v. with the properties

(6.5) $E[B] = 0, \quad Var[B] = \sigma_B^2$

The natural frequency of the disordered system is

(6.6) $$\Omega_1 = \sqrt{\frac{k}{m}(1 + B)} \approx \omega_1(1 + \frac{B}{2} - \frac{B^2}{8})$$

Clearly,

(6.7) $$E[\Omega_1] = E[\omega_1(1 + \frac{B}{2} - \frac{B^2}{8})] = \omega_1(E[1] + \frac{E[B]}{2} - \frac{E[B^2]}{8})$$

$$= \omega_1(1 - \frac{\sigma_B^2}{8})$$

$$E[\Omega_1^2] = E[\omega_1^2(1 + B + ..)] = \omega_1^2(E[1] + E[B]) = \omega_1^2$$

$$Var[\Omega_1] = E[\Omega_1^2] - E^2[\Omega_1] \approx \frac{\omega_1^2\sigma_B^2}{4}$$

$$S.D.[\Omega_1] = \sqrt{Var[\Omega_1]} \approx \frac{\omega_1\sigma_B}{2}$$

The estimate of the $S.D.[\Omega_1]$, δ_1, will be evaluated using (5.34). This requires the evaluation of $I_{1,0}$ for which the second of (4.9) will be used. This is the easiest expression to use. Recall that in deriving this expression $\delta(\omega_1)$ has been used for $g(\lambda)$ and for the case at hand the only disordered element is the spring constant, thus all covariances involving the A's will be identically zero. We easily obtain

$$I_{1,0} \simeq \frac{k^2\sigma_B^2}{m^2}$$

(6.8) and

$$\delta_1 = \frac{\omega_1\sigma_B}{2}$$

Thus, in this case, our estimate agrees with the S.D. $[\Omega_1]$ when terms of B^2 are retained.

b) Two Degrees of Freedom System

Consider the two degree of freedom system shown in Figure 6.2. The springs are the random variables $K_i = k(1 + B_i)$

(6.9) $E[B_i] = 0 \qquad E[B_i^2] = \sigma_B^2 \qquad i = 1, 2$

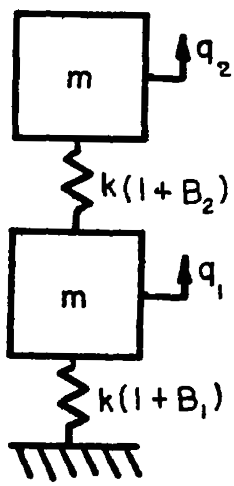


Figure 6.2

Then

$$\Omega_{1,2}^2 = \frac{K_1 + 2K_2 \pm \sqrt{K_1^2 + 4K_2^2}}{2m} \tag{6.10}$$

where the r.v. Ω_1 and Ω_2 are the natural frequencies of the system shown in Figure 6.2.

Again make a change of variables, as in the last example, to transform the kinetic energy to a unit matrix.

For the mean system, i.e., with $B_i = 0$ we easily find that

$$\omega_1 = .616\ \omega_o \quad , \quad \eta_1^{(1)} = .526 \quad , \quad \eta_2^{(1)} = .851 \tag{6.11}$$

$$\omega_2 = 1.616\ \omega_o \quad , \quad \eta_1^{(2)} = .851 \quad , \quad \eta_2^{(2)} = -\ .526$$

where $\omega_o = \sqrt{k/m}$ and $(\eta_1^{(1)}, \eta_2^{(1)})$; $(\eta_1^{(2)}, \eta_2^{(2)})$ are the mode shapes for the first and second mode, respectively.

Two cases can be considered, where the B_i are independent and when they are equal.

A. Consider the case where the B_i are independent, i.e.,

$$E[B_i B_j] = \sigma_B^2 \delta_{ij} : \quad i,j = 1,2$$

Using equation (6.10) expressions for the first and second moments can be obtained.

(6.12) $$E[\Omega_1] = E[(\frac{K_1 + 2K_2 + \sqrt{K_1^2 + 4K_2^2}}{2m})^{\frac{1}{2}}]$$

$$E[\Omega_1^2] = E[\frac{K_1 + 2K_2 + \sqrt{K_1^2 + 4K_2^2}}{2m}]$$

Expanding the terms in the brackets and retaining moments of second order, and using equation (6.9) and the third and fourth of (6.7) the S.D. of Ω_1 can be obtained.

$$E[\Omega_1] = (.6180 - .1910\ \sigma_B^2)\omega_o$$

$$S.D.\ [\Omega_1] = (Var\ [\Omega_1])^{\frac{1}{2}} = \{E[\Omega_1^2] - E^2[\Omega_1]\}^{\frac{1}{2}} = .239\ \sigma_B\omega_o$$

and in a similar fashion for Ω_2

(6.13) $$E[\Omega_2] = E[(\frac{K_1 + 2K_2 - \sqrt{K_1^2 + 4K_2^2}}{2m})^{\frac{1}{2}}]$$

$$E[\Omega_2^2] = E[\frac{K_1 + 2K_2 - \sqrt{K_1^2 + 4K_2^2}}{2m}]$$

$$E[\Omega_1] = (1.618 - .066\ \sigma_B^2)\ \omega_o$$

$$S.D.\ [\Omega_2] = (Var\ [\Omega_2])^{\frac{1}{2}} = \{E[\Omega_2^2] - E^2[\Omega_2]\}^{\frac{1}{2}} = .6265\ \sigma_B\omega_o$$

These calculations indicate that there is more variation in Ω_2 than in Ω_1 .

Recall that the variance of a random variable is the second moment about the mean of the random variable, that is

$$\text{Var}\ [\Omega] = E[(\Omega - E[\Omega])^2]$$

It is interesting to note that for this example the second moment about the natural frequency of the <u>mean system</u> is equal to the second moment about the <u>system mean</u> when terms of second order are retained.

Thus for this example

$$E[(\Omega_1 - E[\Omega_1])^2] = E[(\Omega_1 - \omega_1)^2]$$

and

$$E[(\Omega_2 - E[\Omega_2])^2] = E[(\Omega_2 - \omega_2)^2]$$

The estimates for S.D. $[\Omega_1]$ and S.D. $[\Omega_2]$ δ_1 and δ_2 respectively, can be obtained by using equation (5.34) for δ and equation (4.9) to evaluate $I_{1,0}$. Again it is noted that equation (4.9) provides the easiest way to get $I_{1,0}$ and that the only disordered elements in the system are the springs. Thus in using equation (4.9) to evaluate $I_{1,0}$ all terms containing the covariances of the masses, that is, the covariance of the A's will vanish. The resulting δ's are

(6.14) $$\delta_1 = .5647\ \omega_o\sigma$$

$$\delta_2 = .6567\ \omega_o\sigma$$

Comparing S.D. $[\Omega_1]$ and S.D. $[\Omega_2]$ with δ_1 and δ_2, respectively, it can be seen that δ_2 is slightly larger than S.D. $[\Omega_2]$ but δ_1 is substantially larger than S.D. $[\Omega_1]$. To investigate the source of this discrepancy the derivation of the estimator contained in Section 5 was examined. The source of the error can best be seen from equation (5.11). In the derivation of the estimator the terms contained in the summation on the last line were dropped since from the second of equations (5.12) $\nu'^2_{r,\sigma} \approx 0$. While these terms are small the terms which they are multiplied by in (5.11) may be large. Indeed it is just these terms that account for the largest portions of $I_{1,0}$ as determined by (4.9). These calculations also reveal that the ξ_j which minimize $I(u, \delta)$ differ slightly from the $\eta_j^{(\cdot)}$; this small discrepancy also contributes to the magnitude of $I_{1,0}$. If these terms are eliminated from $I_{1,0}$, we find that the value of δ_1 so predicted is close to S.D. $[\Omega_1]$. In a latter section of this report a method of correcting these errors will be described.

B. Consider now the case where $B_1 = B_2$, i.e., the disorder in the springs is perfectly correlated.

$$E[B_i B_j] = \sigma_B^2 \quad i,j = 1,2$$

Following the same procedure as in A above the S.D. $[\Omega_1]$ and S.D. $[\Omega_2]$ can be obtained. The present case only differs from the last when terms of the form $E[B_1 B_2]$ are evaluated. Previously these terms were zero since the B's were independent where as now they yield a σ_B^2. Thus

$$(6.15) \qquad E[\Omega_1] = E[(\frac{K_1 + 2K_2 + \sqrt{K_1^2 + 4K_2^2}}{2m})^{\frac{1}{2}}]$$

$$= (.6180 - .0772\ \sigma_B^2)\ \omega_o$$

$$\text{S.D.}\ [\Omega_1] = (\text{Var}[\Omega_1])^{\frac{1}{2}} = \{E[\Omega_1^2] - E^2[\Omega_1]\}^{\frac{1}{2}} = .309\ \sigma_B\omega_o$$

and similarly

$$E[\Omega_2] = (1.6180 - .2022\ \sigma_B^2)\ \omega_o$$

$$\text{S.D.}\ [\Omega_2] = .809\ \sigma_B\omega_o$$

Following exactly the same procedures as in A with the exception of the evaluation of $\text{Cov}[B_1B_2]$ as noted above, the δ's can be determined

$$(6.16) \qquad \delta_1 = .309\ \sigma_B\omega_o$$

$$\delta_2 = .8096\ \sigma_B\omega_o$$

Thus when the springs are independent the S.D. $[\Omega]$ is less than that obtained when they are perfectly correlated. The increase in the S.D. $[\Omega]$ in the second case is reflected by an increase in the estimated δ_i . In this case the S.D. $[\Omega_i]$ is very close to the corresponding δ_i .

For purposes of comparison, a Monte Carlo simulation was done on a digital computer. Both cases were investigated, that is, the first, with the springs being independent, and the second, with the springs being perfectly correlated. In each case Monte Carlo estimates for the standard deviation of the first and second natural frequencies were obtained. In each case, B_1 and B_2 are taken as normally distributed with mean zero and standard deviation .1 .

For independent springs $\sigma_B = .1$ $\text{Cov}[B_1 B_2] = 0$

$$\text{S.D.}\ [\Omega_1] = .0239\ \omega_o \qquad \text{S.D.}\ [\Omega_2] = .0625\ \omega_o$$

$$\delta_1 = .05647\ \omega_o \qquad \delta_2 = .0657\ \omega_o$$

$$\widehat{\text{S.D.}}\ [\Omega_1] = .0246\ \omega_o \qquad \widehat{\text{S.D.}}\ [\Omega_2] = .0565\ \omega_o$$

For perfectly correlated springs, i.e., $B_1 = B_2$, $\sigma_B = .1$; $\text{Cov}[B_1 B_2] = .1$

$$\text{S.D.}\ [\Omega_1] = .0309\ \omega_o \qquad \text{S.D.}\ [\Omega_2] = .08090\ \omega_o$$

$$\delta_1 = .0309\ \omega_o \qquad \delta_2 = .08096\ \omega_o$$

$$\widehat{\text{S.D.}}\ [\Omega_1] = .0315\ \omega_o \qquad \widehat{\text{S.D.}}\ [\Omega_2] = .0825\ \omega_o$$

where $\widehat{\text{S.D.}}\ [\cdot]$ are the estimates obtained from the Monte Carlo experiments using 500 trials.

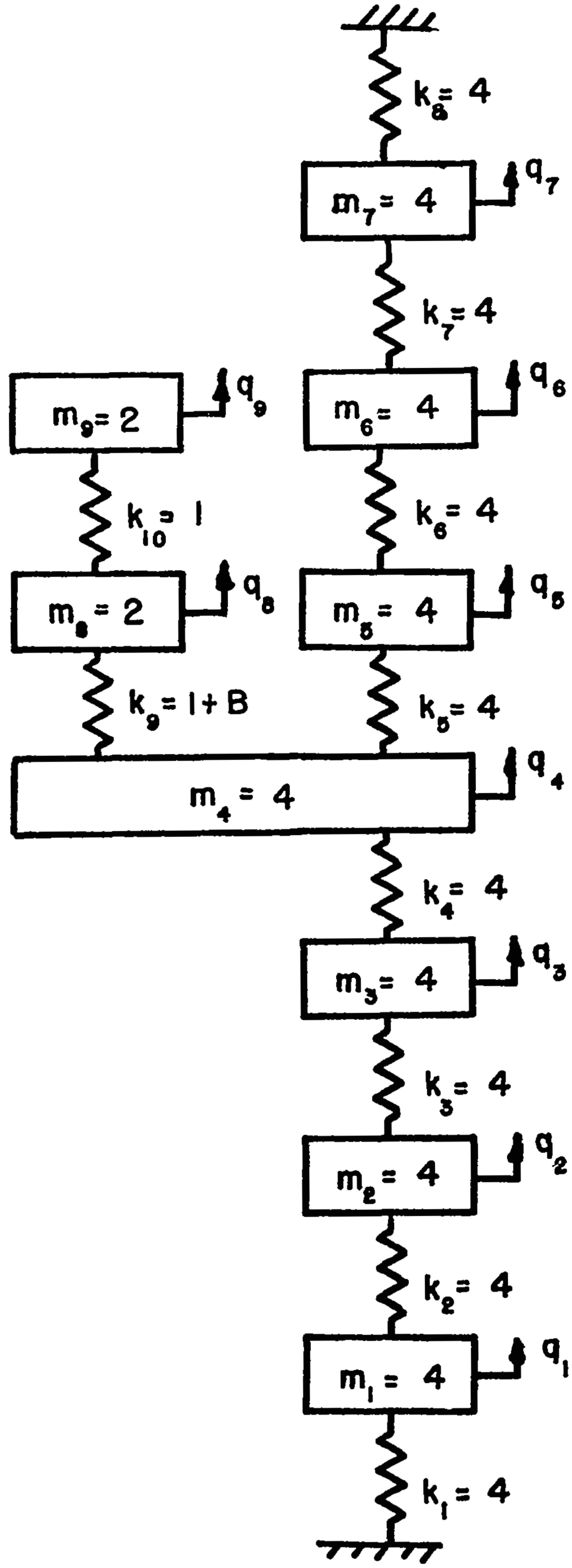
In both cases the standard deviation as calculated, when terms above second order in σ are dropped, compare well with those obtained by Monte Carlo estimates. When the springs are independent the δ's are larger than the calculated and the Monte Carlo estimates of the S.D. [Ω] . The overly conservating estimate of δ_1 was commented on earlier. When the springs are perfectly correlated the Monte Carlo estimates of the S.D. [Ω] exceed those obtained by the other methods by about 2%. Thus the mean square estimates, δ , are not strict upper bounds to the S.D. [Ω] . It should be noted that the δ's are very close to the Monte Carlo estimates.

c) Nine-Degree of Freedom System

The next system we consider is shown in Figure 6.3. Only one spring is regarded as a r.v.; it is k_9 and connects the 7 and 2-degree of freedom sub-systems. For the mean system, $B = 0$, the natural frequencies are given in the next Table.

Table 6.1

mode no.	n.f.	mode shapes	
k	ω_k	$\eta_4^{(k)}$	$\eta_8^{(k)}$
1	.3139	.364	.462
2	.5204	.2900	-.283
3	.7654	.0	.0
4	1.0757	.3540	.560
5	1.2031	.3190	-.615
6	1.4142	.0	.0
7	1.6870	-.495	.101
8	1.8478	.0	.0
9	1.9825	.558	-.069



$k_8 = 4$
$m_7 = 4$
q_7
$k_7 = 4$
$m_9 = 2$
q_9
$m_6 = 4$
q_6
$k_{10} = 1$
$k_6 = 4$
$m_8 = 2$
q_8
$m_5 = 4$
q_5
$k_9 = 1 + B$
$k_5 = 4$
$m_4 = 4$
q_4
$k_4 = 4$
$m_3 = 4$
q_3
$k_3 = 4$
$m_2 = 4$
q_2
$k_2 = 4$
$m_1 = 4$
q_1
$k_1 = 4$

Figure 6.3

Let us first compute δ_k for each natural frequency by again using equation (5.34). This requires the evaluation of $I_{1,0}$ for which equation (4.9) will again be used. Since the only disorder is associated with the springs, i.e., k_9, equation (4.9) reduces to

$$(6.17) \qquad I_{1,0} = [\text{Cov } C_{ij} \; C_{i\ell}] \; u_j u_\ell$$

the covariances of the A's vanishing. Using equation (6.9) equation (6.17) reduces to

$$(6.18) \qquad I_{1,0} = [.1875(\eta_4^{(k)})^2 - .5302\eta_4^{(k)}\eta_8^{(k)} + .3750 \; (\eta_8^{(k)})^2] \; \sigma_B^2$$

Utilizing the numerical results given in Table 6.1, (5.34) and letting $\sigma_B = .1$ the results given in Table 6.2 are obtained.

Appendix 6.1 contains a more detailed development for this problem.

Let us next estimate the S.D. $[\Omega_k]$ using a Monte Carlo technique. Fifty sample values of $1 + B$ were obtained; once with B Gaussian (mean zero and S.D. [0.10]) and twice with B uniformly distributed (mean zero and S.D. [0.10]) . For each set of sample value of B , the 9 natural frequencies were computed and their mean frequencies and standard deviations calculated. These S.D.'s are given in Table 6.2, and the mean natural frequencies are presented in Table 6.3.

Table 6.2

Standard Deviation of Ω_k

Monte Carlo Method

k	δ_k	Uniform Dist. S.D. $[\Omega_k]$	Uniform Dist. S.D. $[\Omega_k]$	Gaussian S.D. $[\Omega_k]$
1	.0199	.0036	.0039	.0037
2	.0287	.0117	.0128	.0116
3	.0	.0	.0	.0
4	.00882	.0024	.0026	.0021
5	.0214	.0146	.0161	.0143
6	.0	.0	.0	.0
7	.00817	.0030	.0033	.0029
8	.0	.0	.0	.0
9	.00716	.0027	.0030	.0026

Table 6.3

Natural Frequencies

k	Exact for Mean System ω_k	Monte Carlo Techniques: Uniform Dist. Mean Ω_k	Uniform Dist. Mean Ω_k	Gaussian Mean Ω_k
1	.3139	.3124	.3126	.3125
2	.5204	.5160	.5169	.5166
3	.7654	.7654	.7654	.7654
4	1.0757	1.0747	1.0748	1.0748
5	1.2031	1.1984	1.1996	1.1990
6	1.4142	1.4142	1.4142	1.4142
7	1.6869	1.6860	1.6863	1.6861
8	1.8478	1.8478	1.8478	1.8478
9	1.9825	1.9817	1.9819	1.9818

The results given in Table 6.2 are interesting. We first note that the S.D. $[\Omega_k]$ as computed by the Monte Carlo method do not deviate significantly among each other for the same k even though two distinct distributions were employed. Second, we note that the results for the uniform distribution also do not deviate appreciably between each other for the same k, thus indicating small sample variability with 50 sample values. It is clear that all non-zero δ_k are larger than the non-zero S.D. $[\Omega_k]$; this illustrates a point made in Section 2. Put another way, when the δ_k are small, so are the corresponding S.D. $[\Omega_k]$ when the δ_k are large, the corresponding S.D. $[\Omega_k]$ are not necessarily large. In particular, we finally note that δ_1 is considerably larger than S.D. $[\Omega_1]$; the same point was noted in the 2-degree of freedom example and the explanation for the difference is the same. Except for $k = 1$, the δ_k predict the correct relative magnitude of the S.D. $[\Omega_k]$.

It should also be noted that it is only necessary to calculate the covariances given in (4.9) once for all of the δ_k . Thus having evaluated the covariances equation (6.18) can be used to evaluate all of the δ_k , the only difference is evaluating $I_{1,0}$ arising from the use of appropriate modal displacements.

d) Twenty-Degree of Freedom System

As the next example, we shall consider the system shown in Figure 6.4

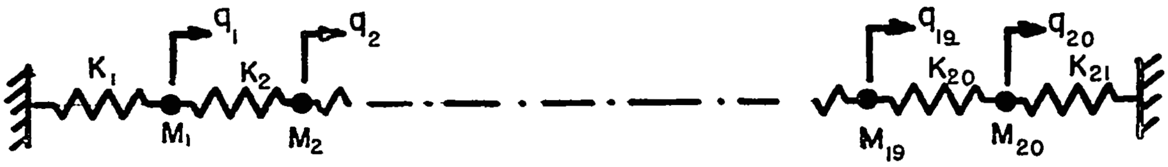


Figure 6.4

We shall assume that the r.v.'s M_j and K_ℓ are independent and independent of each other, and satisfy the relations

$$(6.19) \qquad E\{M_j\} = m \quad , \quad E\{K_\ell\} = k \qquad j = 1, 2, \dots 20 \quad \ell = 1, 2, \dots 21$$

$$\mathrm{Var}[M_j] = m^2\sigma_m^2 \quad , \quad \mathrm{Var}[K_\ell] = k^2\sigma_\ell^2$$

Disorder is thus the same in the springs and in the masses of the system.

The δ_k's will be evaluated using the same procedures as in the previous examples. Equation (5.34) will be used to evaluate δ_k and (4.9) will be used to evaluate $I_{1,0}(\eta^{(k)}, \omega_k)$. Evaluation of (4.9) requires knowledge of the system matrix in tranformed coordinates.

$$2T = \sum_{i=1}^{20} \sum_{j=1}^{20} A_{ij}\dot{q}_i\dot{q}_j = \sum_{i=1}^{20} M_i\dot{q}_i^2$$

$$2V = \sum_{i=1}^{20} \sum_{j=1}^{20} C_{ij}q_iq_j =$$

$$= K_1q_1^2 + \sum_{i=1}^{19} K_{i+1} (q_{i+1} - q_i)^2 + K_{21}q_{20}^2$$

Again making a change of variables to transform the mass matrix into a unit matrix requires

$$p_i = \sqrt{m_i}\, q_i = \sqrt{m}\, q_i$$

It should be noted that this transformation involves the mean masses m_i and not the random variables M_i . Thus typical stiffness and mass terms after the transformation will be

$$C_{ii} = \frac{K_i + K_{i+1}}{m_i} = \frac{K_i + K_{i+1}}{m}$$

$$C_{ij} = \frac{- K_{i+1}}{\sqrt{m_i m_i +1}} = - \frac{K_{i+1}}{m}$$

$$A_{ii} = \frac{M_i}{m_i} = \frac{M_i}{m}$$

Equation (4.9) is more formidable in this example since the covariance of the A's no longer vanishes. It should be noted that terms of the form $Cov[A_{ij} C_{i\ell}]$ still vanish since the masses and stiffnesses are independent. The banding in the stiffness matrix further simplifies the evaluation of (4.9). Using expressions similar to (6.9)

$$K_i = k(1 + B_i) \quad ; \quad E[B_i] = 0 \quad ; \quad E[B_2^2] = \sigma_{B_i}^2 = \sigma_k^2$$

$$M_i = m(1 + D_i) \quad ; \quad E[D_i] = 0 \quad ; \quad E[D_2^2] = \sigma_{D_i}^2 = \sigma_m^2$$

Equation (4.9) reduces to

$$(6.20) \quad I_{1,0}(3\ \sigma_k^2\ \omega_o^4 + \sigma_m^2\ \omega_k^4)(\eta_1^{(k)2} + \eta_{20}^{(k)2})$$

$$+ \sum_{i=2}^{19} (4\ \sigma_k^2\ \omega_o^4 + \sigma_m^2\ \omega_k^4)\ \eta_i^{(k)2}$$

$$- \sum_{i=1}^{19}\ 2\ \sigma_k^2\ \omega_o^4\ \eta_i^{(k)}\ \eta_{i+1}^{k}$$

where $\omega_o^2 = \frac{k}{m}$

Equation (6.20) can be evaluated under three sets of conditions:

a) $\sigma_m^2 = .01$; $\sigma_k^2 = 0$

b) $\sigma_m^2 = 0$; $\sigma_k^2 = .01$

c) $\sigma_m^2 = .01$; $\sigma_k^2 = .01$

These results are shown in Figure 6.5, where the ordinate values have been connected by smooth lines for ease in reading. The main point to observe is that with uniform disorder throughout the system δ_j does not decrease with increasing j , the disorder in the masses actually producing a marked increase in δ_j . Again the disorder associated with the springs shows a marked increase for longer modes. The δ_j for $j = 1, 2, 3$ for $\sigma_k = 0.1$ are .475, .240 and .163 and for $\sigma_k = \sigma_m = 0.1$ are .4745, .2404 and .165 respectively, which have been deleted from the figure for convenience of scaling.

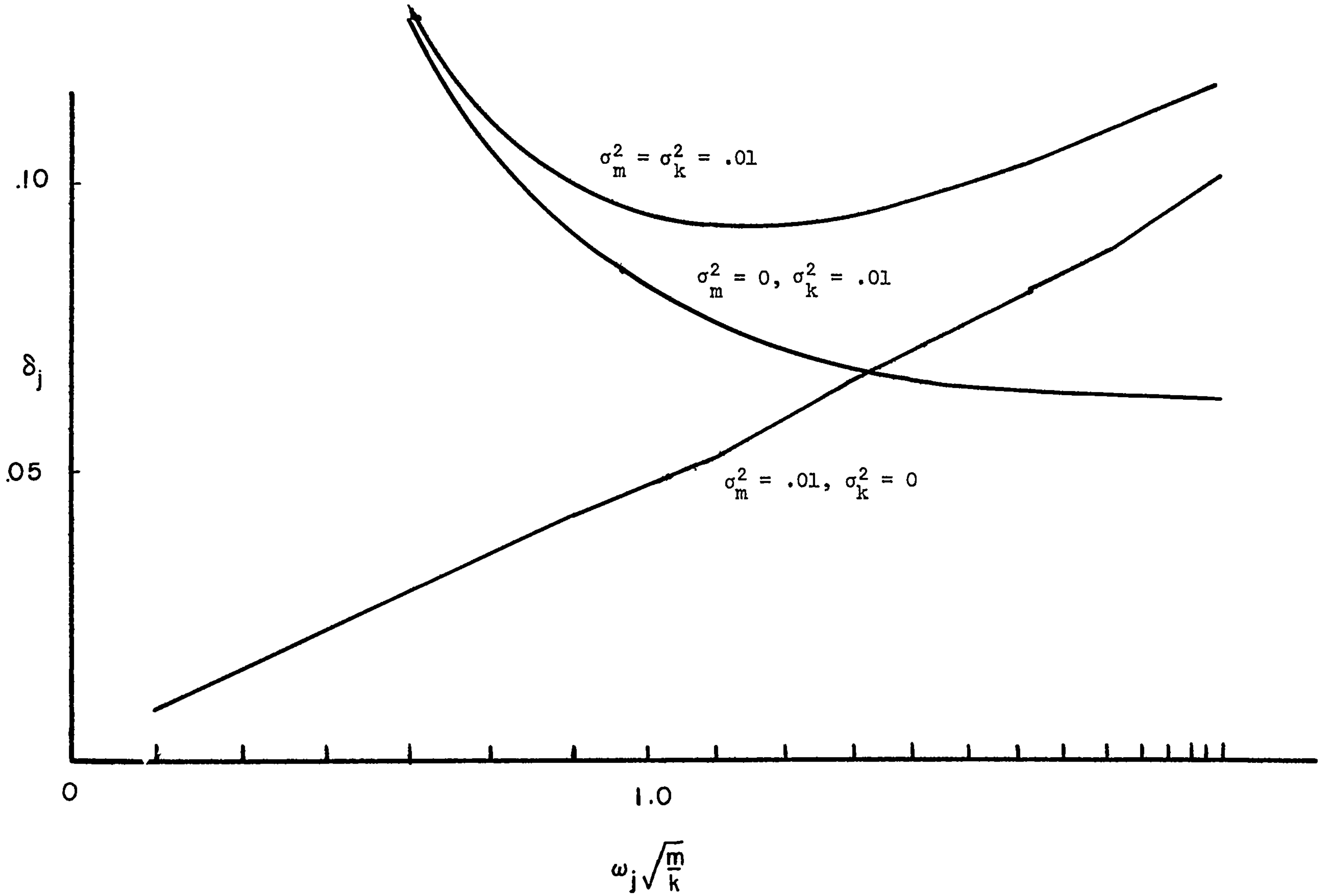


Figure 6.5

e) A Simple Truss with Seven-Degrees of Freedom.

The system is shown in Figure 6.6, with joints designated by numbers and rods by letters. The rod areas are considered as r.v.'s with half each bar mass concentrated at the rod ends. Each joint carries an added mass which is also considered as a r.v. The kinetic and potential energies are

$$(6.21) \qquad 2T = M_1(\dot{x}_1^2+\dot{y}_1^2) + M_2(\dot{x}_2^2+\dot{y}_2^2) + M_3(\dot{x}_3^2+\dot{y}_3^2) + M_4\dot{x}_4^2$$

$$(6.22) \qquad 2V = \frac{K_A}{4}(x_1^2 + 2\sqrt{3}\, x_1y_1 + 3y_1^2) + K_Bx_2^2$$

$$+ \frac{K_C}{4}[(x_2 - x_1)^2 - 2\sqrt{3}(x_2 - x_1)(y_2 - y_1) + 3(y_2 - y_1)^2]$$

$$+ K_D(x_3 - x_1)^2$$

$$+ \frac{K_E}{4}[(x_3 - x_2)^2 + 2\sqrt{3}(x_3 - x_2)(y_3 - y_2) + 3(y_3 - y_2)^2]$$

$$+ K_F(x_4 - x_2)^2$$

$$+ \frac{K_G}{4}[(x_4 - x_3)^2 + 2\sqrt{3}(x_4 - x_3)y_3 + 3y_3^2]$$

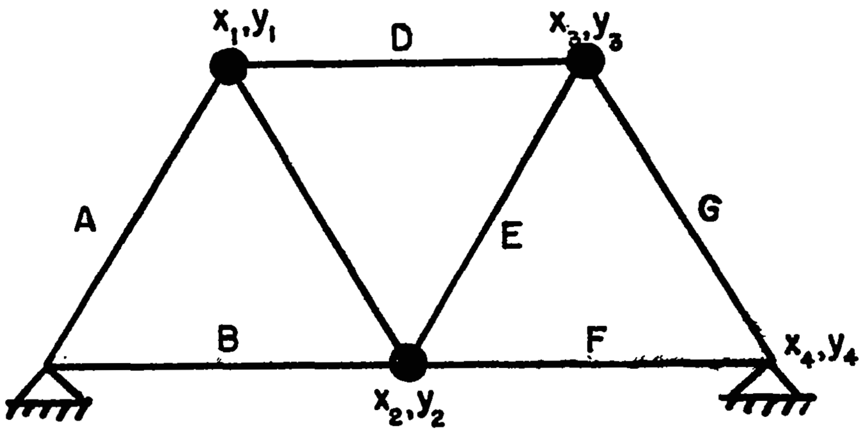


Figure 6.6

where

$$K_\alpha = k(1 + X_\alpha) \quad , \quad \alpha = A', B, \ldots G$$

$$k = \frac{EA'}{\ell} \quad , \quad E = \text{Young's modulus}$$

$$A' = \text{mean rod area} \quad , \quad \ell = \text{bar length}$$

$$M_\alpha = m(1 + X_\alpha) \quad , \quad m = \rho A' \ell$$

$$\rho = \text{mass per unit volume}$$

$$M_1 = (M_A + M_C + M_D)/2 + M_{01}$$

$$M_2 = (M_B + M_C + M_E + M_F)/2 + M_{o2}$$

$$M_3 = (M_D + M_E + M_G)/2 + M_{o3}$$

$$M_4 = (M_F + M_G)/2 + M_{o4}$$

$M_{oj} = m_{oj}(1 + Y_j)$ = mass carried by point j

$j = 1, 2, 3, 4$

$$E\{X_\alpha\} = 0 \quad , \quad E\{Y_j\} = 0 \quad , \quad \alpha = A, B, \dots G$$

$$\mathrm{Var}\ X_\alpha = \sigma_\alpha^2 \quad , \quad \mathrm{Var}\ Y_j = \sigma_j^2$$

We introduce the transformation

(6.23) $x_1\sqrt{m_1} = \underline{x}_1$ $\qquad y_1\sqrt{m_1} = \underline{y}_1$

$x_2\sqrt{m_2} = \underline{x}_2$ $\qquad y_2\sqrt{m_2} = \underline{y}_2$

$x_3\sqrt{m_3} = \underline{x}_3$ $\qquad y_3\sqrt{m_3} = \underline{y}_3$

$x_4\sqrt{m_4} = \underline{x}_4$

where

$$m_1 = \frac{3m}{2} + m_{o1} \qquad\qquad m_2 = 2m + m_{o2}$$

$$m_3 = \frac{3m}{2} + m_{o3} \qquad\qquad m_4 = m + m_{o4}$$

Equations (6.21) and (6.22) now become

$$(6.24) \qquad 2T = \frac{M_1}{m_1} (\dot{\underline{x}}_1^2 + \dot{\underline{y}}_1^2) + \frac{M_2}{m_1} (\dot{\underline{x}}_2^2 + \dot{\underline{y}}_2^2)$$

$$+ \frac{M_3}{m_3} (\dot{\underline{x}}_3^2 + \dot{\underline{y}}_3^2) + \frac{M_4}{m_4} \dot{\underline{x}}_4^2$$

$$(6.25) \qquad 2V = \frac{K_A}{4\, m_1} (\underline{x}_1^2 + 2\sqrt{3}\, \underline{x}_1 \underline{y}_1 + 3\underline{y}_1^2) + \frac{K_B}{m_2} \underline{x}_2^2$$

$$+ \frac{K_c}{4} \left[\left(\frac{\underline{x}_2}{\sqrt{m_2}} - \frac{\underline{x}_1}{\sqrt{m_1}}\right)^2 - 2\sqrt{3} \left(\frac{\underline{x}_2}{\sqrt{m_2}} - \frac{\underline{x}_1}{\sqrt{m_1}}\right)\left(\frac{\underline{y}_2}{\sqrt{m_2}} - \frac{\underline{y}_1}{\sqrt{m_1}}\right) + 3 \left(\frac{\underline{y}_2}{\sqrt{m_2}} - \frac{\underline{y}_1}{\sqrt{m_1}}\right)^2 \right]$$

$$+ K_D \left(\frac{\underline{x}_3}{\sqrt{m_3}} - \frac{\underline{x}_1}{\sqrt{m_1}}\right)^2$$

$$+ \frac{K_E}{4} \left[\left(\frac{\underline{x}_3}{\sqrt{m_3}} - \frac{\underline{x}_2}{\sqrt{m_2}}\right)^2 + 2\sqrt{3} \left(\frac{\underline{x}_3}{\sqrt{m_3}} - \frac{\underline{x}_2}{\sqrt{m_2}}\right)\left(\frac{\underline{y}_3}{\sqrt{m_3}} - \frac{\underline{y}_2}{\sqrt{m_2}}\right) + 3\left(\frac{\underline{y}_3}{\sqrt{m_3}} - \frac{\underline{y}_2}{\sqrt{m_2}}\right)^2 \right]$$

$$+ K_F \left(\frac{\underline{x}_4}{\sqrt{m_4}} - \frac{\underline{x}_2}{\sqrt{m_2}}\right)^2$$

$$+ \frac{K_G}{4} \left[\left(\frac{\underline{x}_4}{\sqrt{m_4}} - \frac{\underline{x}_3}{\sqrt{m_3}}\right)^2 + 2\sqrt{3} \left(\frac{\underline{x}_4}{\sqrt{m_4}} - \frac{\underline{x}_3}{\sqrt{m_3}}\right) \frac{\underline{y}_3}{\sqrt{m_3}} + \frac{3}{m_3} \underline{y}_3^2 \right] ,$$

In what follows, coordinate $\underline{x}_1$, $\underline{y}_1$, ..., $\underline{x}_4$ will always be employed.

The δ_k were calculated using formulas (4.9) and (5.34) for the following cases.

Case 1: $m_{10} = m_{20} = m_{30} = m_{40} = 10m$; $\sigma_j \equiv 0$ for $j = 1, 2, 3, 4$;

$$\sigma_\alpha \equiv 0 \quad \text{for} \quad \alpha = B, E, F, G,$$

and

$$\sigma_A = \sigma \quad , \quad \sigma_C = \sigma_D = 0 \quad ,$$

then with

$$\sigma_C = \sigma \; , \quad \sigma_A = \sigma_D = 0 \quad ,$$

then with

$$\sigma_D = \sigma \; , \quad \sigma_A = \sigma_C = 0 \quad ,$$

and finally with

$$\sigma_A = \sigma_C = \sigma_D = \sigma$$

The natural frequencies of the mean system for $\ell = 24''$, $A' = 1/8$ sq. in., $E = 30 \times 10^6$ psi, $\rho g = .286$ lbs./cu. in. are

$\omega_1 = 1,026$ rad/sec.

$\omega_2 = 1,421$ rad/sec.

$\omega_3 = 2,276$ rad/sec.

$\omega_4 = 2,616$ rad/sec.

$\omega_5 = 3,469$ rad/sec.

$\omega_6 = 4,057$ rad/sec.

$\omega_7 = 4,294$ rad/sec.

The results are presented in Figure 6.7. For each of the four subcases indicated above, we have plotted δ_k as ordinate vs ω_k as abscissa; for ease in identification, we have connected each point with its neighbor by means of a straight line segment; and we have labeled each polygonal line with the letter α corresponding to the subscript or subscripts for which $\sigma_\alpha \neq 0$. The most important conclusion to be drawn is that the δ_k do not necessarily increase with increasing k .

Case 2: $m_{10} = m_{20} = m_{30} = m_{40} = 10m$;

$$\sigma_\alpha = 0 \quad \text{for} \quad \alpha = A, B, C, D, E, F, G,;$$

and

$$\sigma_1 = \sigma \quad , \quad \sigma_2 = \sigma_3 = \sigma_4 = 0 \quad ,$$

then with

$$\sigma_2 = \sigma \quad , \quad \sigma_1 = \sigma_3 = \sigma_4 = 0$$

RANDOM BAR AREA

(DETERMINISTIC MASSES EXCEPT FOR BAR MASSES)

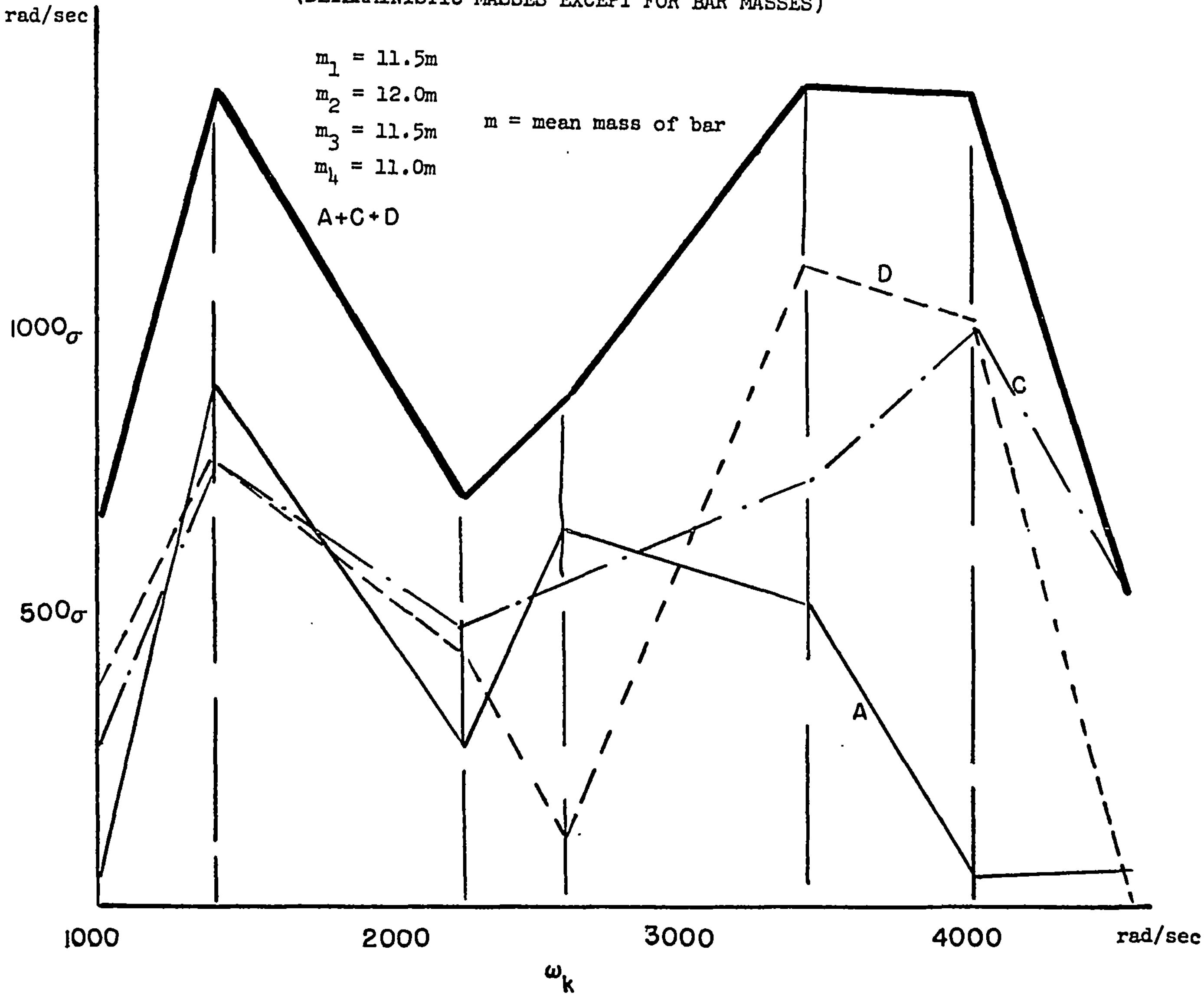


Figure 6.7

then with

$$\sigma_3 = \sigma \quad , \quad \sigma_1 = \sigma_2 = \sigma_4 = \text{\`{}}$$

then with

$$\sigma_4 = \sigma \quad , \quad \sigma_1 = \sigma_2 = \sigma_3 = 0$$

and finally with

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma \quad .$$

The natural frequenices of the mean system are the same as given in Case 1. The results are presented in Figure 6.8. Again, we have plotted for each subcase δ_k as ordinate vs ω_k as abscissa, have connected each point with its neighbor; and have labeled each polygonal line with the number of corresponding to the subscript or subcripts for which $\sigma \neq 0$. While there is no general trend with increasing ω_k when individual added pin masses have variability it is clear that when all the masses are independent and have the same variance, δ_k increases nearly linearly with ω_k ; this is similar to the situation encountered in Example (d).

f) A 21-Degree of Freedom Space Frame

The structure that we shall consider is shown in Figure 6.9. It is three-dimensional and has 12 joints and 32 rods. The structure is connected to rigid ground at four joints; there are four joints, A, B, C, and D at the next level; at the third level there are joints E, F, G, and H with EGH-JKL a rigid mass. The rods have elasticity and mass; we shall assume that half the mass of each bar is concentrated at its end

ADDED MASSES RANDOM BAR AREA DETERMINISTIC

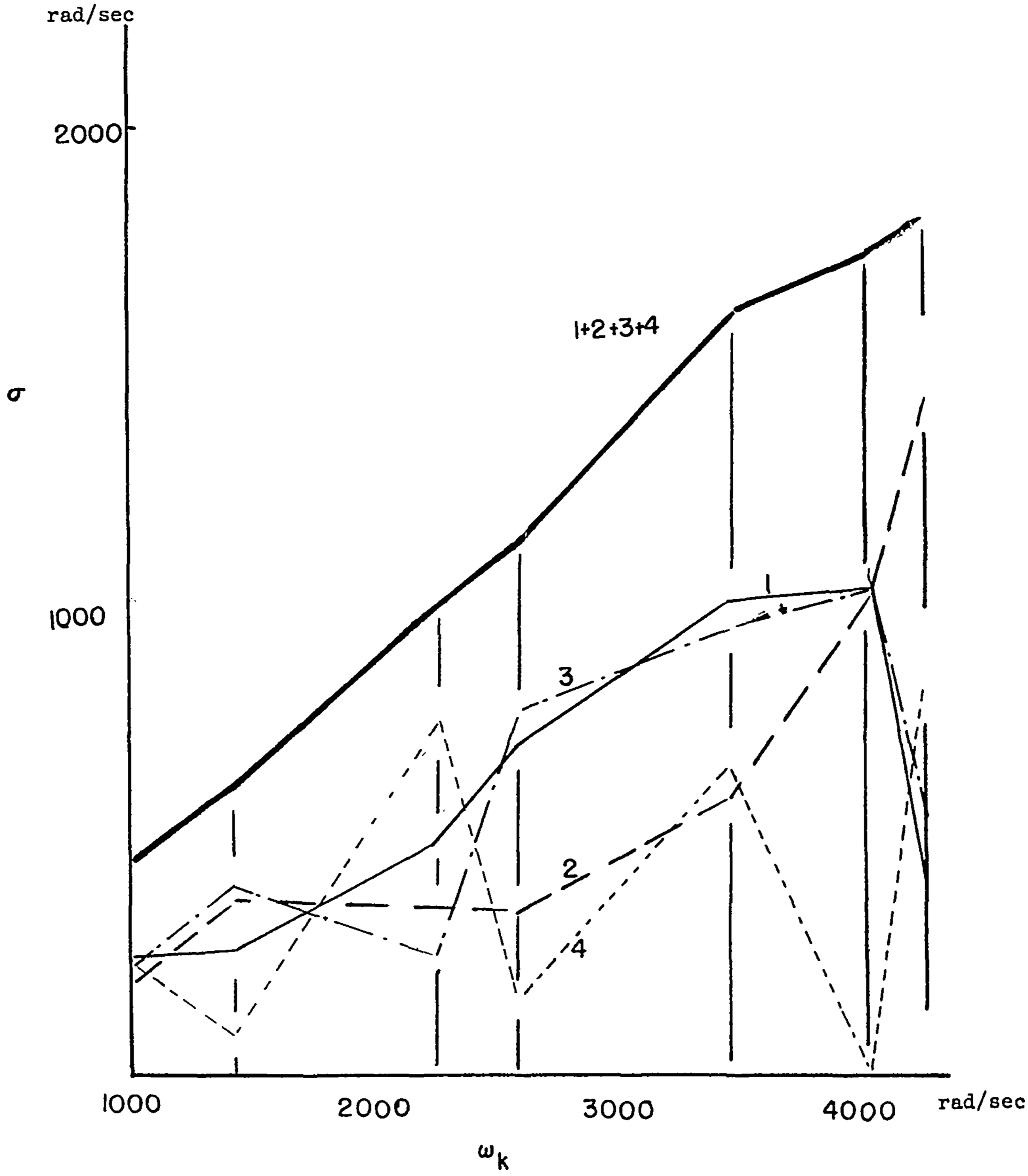


Figure 6.8

points. Each joint, except E, G, and H, may carry additional mass independent of rod masses. The joint masses are treated as particles. The structure is thus to be modeled as a lumped parameter system.

An OXYZ coordinate system is shown in Figure 6.9. The displacement components with respect to this coordinate system of each joint will be laveled u_A, v_A, w_A for example. The only joints that we must consider explicitly are A, B, C, D, and F . The rigid mass EGHJKL at the top has center of mass N ; this mass has displacements u_N, v_N, w_N, respectively, in the coordinate direction OX, OY, OZ, and angular rotations ϕ, θ, ψ about lines through N parallel to the coordinate directions. The displacements of E, G, and H will be expressed in terms of u_N, v_N, w_N and ϕ, θ, ψ . The 21 independent coordinates of this system are thus

$$
\begin{array}{l}
u_A,\ v_A,\ w_A \\
u_B,\ v_B,\ w_B \\
u_C,\ v_C,\ w_C \\
u_D,\ v_D,\ w_D \\
u_F,\ v_F,\ w_F \\
u_N,\ v_N,\ w_N \\
\phi,\ \theta,\ \psi
\end{array}
$$

and we assume that these are small. The system thus has 21 degrees of freedom.

The kinetic energy, T , of the structure is given by the formula

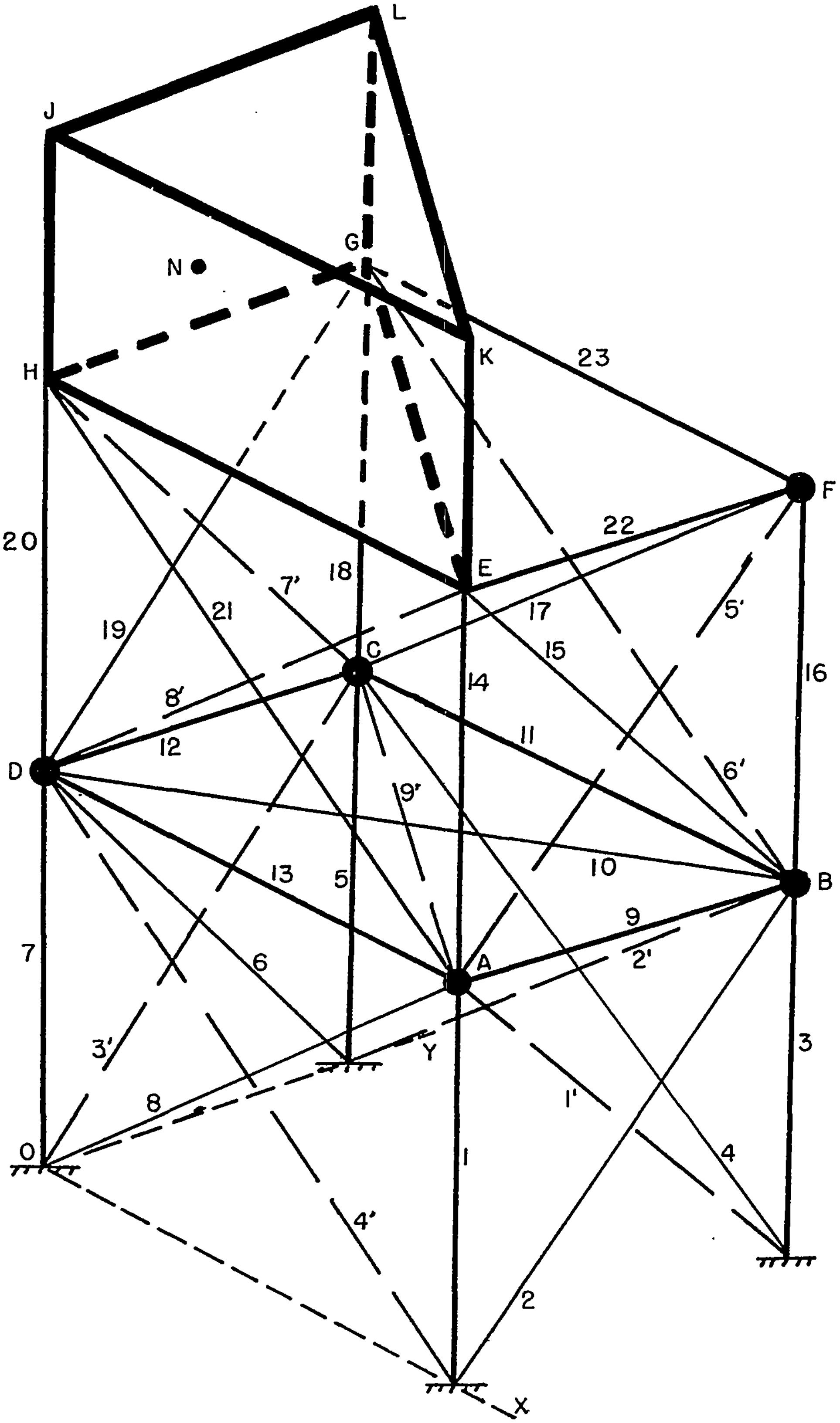

L
J
G
N
H
K
23
F
22
20
18
E
7'
17
5'
19
21
C
15
16
14
8'
11
12
D
6'
9'
10
13
5
B
9
A
2'
7
6
3
3'
Y
8
1'
4
O
1
4'
2
X

Figure 6.9

$$(6.26) \qquad 2T = m_A(\dot{u}_A^2 + \dot{v}_A^2 + \dot{w}_A^2) + m_B(\dot{u}_B^2 + \dot{v}_B^2 + \dot{w}_B^2)$$

$$+ m_C(\dot{u}_C^2 + \dot{v}_C^2 + \dot{w}_C^2) + m_D(\dot{u}_D^2 + \dot{v}_D^2 + \dot{w}_D^2)$$

$$+ m_F(\dot{u}_F^2 + \dot{v}_F^2 + \dot{w}_F^2) + m_N(\dot{u}_N^2 + \dot{v}_N^2 + \dot{w}_N^2)$$

$$+ I_1\dot{\phi}^2 - 2I_{12}\dot{\phi}\dot{\theta} - 2I_{13}\dot{\phi}\dot{\psi}$$

$$+ I_2\dot{\theta}^2 - 2I_{23}\dot{\theta}\dot{\psi} + I_3\dot{\psi}^2$$

where $m_{(\cdot)}$ is the mass of the joint or body labeled by the subscript $(\cdot)$. While we have shown the rigid mass EGHJKL as a triangular right cylinder, this has only been for convenience in illustration; thus, we assume it has moment and products of inertia I_1, I_2, I_3 , I_{12} , I_{13}, I_{23} with respect to rectangular axes with origin at N and parallel to OXYZ . The products of inertia I_{12}, I_{13}, I_{23} may not vanish, depending on the explicit nature of the rigid mass EGHJKL .

The potential energy of the $(\cdot)$ rod is

$$(6.27) \qquad 2V_{(\cdot)} = \left(\frac{AE}{\ell^3}\right)_{(\cdot)} [\Delta x^2_{(\cdot)}\Delta u^2_{(\cdot)} + 2\Delta x_{(\cdot)}\Delta y_{(\cdot)}\Delta u_{(\cdot)}\Delta v_{(\cdot)}$$

$$+ 2\Delta x_{(\cdot)}\Delta z_{(\cdot)}\Delta u_{(\cdot)}\Delta w_{(\cdot)} + \Delta y^2_{(\cdot)}\Delta v^2_{(\cdot)}$$

$$+ 2\Delta y_{(\cdot)}\Delta z_{(\cdot)}\Delta v_{(\cdot)}\Delta w_{(\cdot)} + \Delta z^2_{(\cdot)}\Delta w^2_{(\cdot)}]$$

(6.28) $$\ell^2\left(\frac{AE}{L^3}\right)_{(\cdot)} = k_{(\cdot)}$$

where now $(\cdot)$ denotes a number in this set; for rods numbered 1' to 9', we shall write

(6.29) $$\ell^2\left(\frac{AE}{L^3}\right)_{(\cdot\cdot)} = \kappa_{(\cdot)}$$

where $(\cdot\cdot)$ is one of the numbers 1', 2', ..., 9' and $(\cdot)$ is the corresponding number in 1, 2, ..., 9.

There is one more step needed before we can write down the expression for the potential energy. The displacements at E, G, and H must be expressed in terms of those of the rigid mass. Let the coordinates of E, G, and H with respect to NXYZ be, respectively,

$$x'_E, \quad y'_E, \quad z'_E$$

$$x'_G, \quad y'_G, \quad z'_G$$

$$x'_H, \quad y'_H, \quad z'_H$$

The potential energy is now seen to be

$$
\begin{aligned}
(6.30) \qquad 2V = {} & 9k_1 w_A^2 + k_2(16v_B^2 + 24v_B w_B + 9w_B^2) \\
& + 9k_3 w_B^2 + k_4(16u_C^2 - 24u_C w_C + 9w_C^2) \\
& + 9k_5 w_C^2 + k_6(16v_D^2 - 24v_D w_D + 9w_D^2) \\
& + 9k_7 w_D^2 + k_8(16u_A^2 + 24u_A w_A + 9w_A^2) \\
& + 16k_9(v_B - v_A)^2 + k_{10}[16(u_D - u_B)^2 \\
& - 32(u_D - u_B)(v_D - v_B) + 16(v_D - v_B)^2] \\
& + 16k_{11}(u_C - u_B)^2 + 16k_{12}(v_D - v_C)^2 \\
& + 16k_{13}(u_A - u_D)^2 \\
& + 9k_{14}(w_N + y_E'\phi - x_E'\theta - w_A)^2 \\
& + k_{15}[16(v_N + x_E'\psi - z_E'\phi - v_B)^2 \\
& - 24(v_N + x_E'\psi - z_E'\phi - v_B)(w_N + y_E'\phi - x_E'\theta - w_B) \\
& + 9(w_N + y_E'\phi - x_E'\theta - w_B)^2] \\
& + 9k_{16}(w_F - w_B)^2 + k_{17}[16(u_F - u_C)^2
\end{aligned}
$$

$$- 24(u_F - u_C)(w_F - w_C) + 9(w_F - w_C)^2]$$

$$+ 9k_{18}(w_N + y'_G\phi - x'_G\theta - w_C)^2$$

$$+ k_{19}[16(v_N + x'_G\psi - z'_G\phi - v_D)^2 - 24(v_N + x'_G\psi - z'_G\phi$$

$$- v_D)(w_N + y'_G\phi - x'_G\theta - w_D) + 9(w_N + y'_G\phi - x'_G\theta - w_D)^2]$$

$$+ 9k_{20}(w_N + y'_H\phi - x'_H\theta - w_D)^2$$

$$+ k_{21}[16(u_N + z'_H\theta - y'_H\psi - u_A)^2 - 24(u_N + z'_H\theta - y'_H\psi$$

$$- u_A)(w_N + y'_H\phi - x'_H\theta - w_A) + 9(w_N + y'_H\phi - x'_H\theta - w_A)^2]$$

$$+ 16k_{22}(v_F - v_N - x'_E\psi + z'_E\phi)^2$$

$$+ 16k_{23}(u_N + z'_G\theta - y'_G\psi - u_F)^2$$

$$+ \kappa_1(16v_A^2 - 24v_Aw_A + 9w_A^2) + \kappa_2(16u_B^2 + 24u_Bw_B + 9w_B^2)$$

$$+ \kappa_3(16v_C^2 + 24v_Cw_C + 9w_C^2) + \kappa_4(16u_D^2 - 24u_Dw_D + 9w_D^2)$$

$$+ \kappa_5[16(v_F - v_A)^2 + 24(v_F - v_A)(w_F - w_A) + 9(w_F - w_A)^2]$$

$$+ \kappa_6[16(u_N + z'_G\theta - y'_G\psi - v_B)^2 - 24(u_N + z'_G\theta - y'_G\psi$$

$$- v_B)(w_N + y_G'\phi - x_G'\theta - w_B) + 9(w_N + y_G'\phi - x_G'\theta - w_B)^2]$$

$$+ \kappa_7[16(v_N + x_H'\psi - z_H'\phi - v_C)^2 - 24(v_N + x_H'\psi - z_H'\phi$$

$$- v_C)(w_N + y_H'\phi - x_H'\theta - w_C) + 9(w_N + y_H'\phi - x_H'\theta - w_C)^2]$$

$$+ \kappa_8[16(u_N + z_E'\theta - y_E'\psi - u_D)^2 + 24(u_N + z_E'\theta - y_E'\psi$$

$$- u_D)(w_N + y_E'\phi - x_E'\theta - w_D) + 9(w_N + y_E'\phi - x_E'\theta - w_D)^2]$$

The equations of free motion are obtained using Lagrange's equations in the general form

$$(6.31) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial(\cdot)}\right) + \frac{\partial V}{\partial(\cdot)} = 0$$

where $(\cdot)$ represents one of the 21 generalized coordinates. If we assume in the equations of motion that each coordinate varies simple harmonically with angular frequency λ and use the coordinate symbol as coordinate amplitude (to save introducing new notation), the equations for the natural frequencies and the corresponding normal modes are

$$(6.32) \qquad [16(k_8 + k_{13} + k_{21} + \kappa_9) - m_A\lambda^2]u_A - 16\kappa_9 v_A$$

$$+ 12(k_8 - k_{21})w_A - 16\kappa_9 u_C + 16\kappa_9 v_C$$

$$- 16k_{13}u_D - 16k_{21}u_N + 12k_{21}w_N + 12k_{21}y'_H\phi$$

$$- 4k_{21}(4z'_H + 3x'_H)\theta + 16k_{21}y'_H\psi = 0$$

(6.33) $$- 16\kappa_9 u_A + [16(k_9 + \kappa_1 + \kappa_5 + \kappa_9) - m_A\lambda^2]v_A$$

$$+ 12(\kappa_5 - \kappa_1)w_A - 16k_9 v_B + 16\kappa_9 u_C - 16\kappa_9 v_C$$

$$- 16\kappa_5 v_F - 12\kappa_5 w_F = 0$$

(6.34) $$12(k_8 - k_{21})u_A + 12(\kappa_5 - \kappa_1)v_A$$

$$+ [9(k_1 + k_8 + k_{14} + k_{21} + \kappa_1 + \kappa_5) - m_A\lambda^2]w_A$$

$$- 12\kappa_5 v_F - 9\kappa_5 w_F + 12k_{21}u_N - 9(k_{14} + k_{21})w_N$$

$$- 9(k_{14}y'_E + k_{21}y'_H)\phi + (9k_{14}x'_E + 9k_{21}x'_H + 12k_{21}z'_H)\theta$$

$$- 12k_{21}y'_H\psi = 0$$

(6.35) $$[16(k_{10} + k_{11} + \kappa_2 + \kappa_6) - m_B\lambda^2]u_B + 16k_{10}v_B$$

$$+ 12(\kappa_2 - \kappa_6)w_B - 16k_{11}u_C - 16k_{10}u_D - 16k_{10}v_D$$

$$- 16\kappa_6 u_N + 12\kappa_6 w_N + 12\kappa_6 y_G' \phi - 4\kappa_6(4z_G' + 3x_G')\theta$$

$$+ 16\kappa_6 y_G' \psi = 0$$

(6.36) $$- 16k_9 v_A + 16k_{10} u_B + [16(k_2 + k_9 + k_{10} + k_{15}) - m_B \lambda^2] v_B$$

$$+ 12(k_2 - k_{15}) w_B - 16k_{10} u_D - 16k_{10} v_D - 16k_{15} v_N$$

$$+ 12k_{15} w_N + 4k_{15}(3y_E' + 4z_E')\phi - 12k_{15} x_E' \theta$$

$$- 16k_{15} x_E' \psi = 0$$

(6.37) $$12(\kappa_2 - \kappa_6) u_B + 12(k_2 - k_{15}) v_B + [9(k_2 + k_3 + k_{15}$$

$$+ k_{16} + \kappa_2 + \kappa_6) - m_B \lambda^2] w_B - 9k_{16} w_F + 12\kappa_6 u_N$$

$$+ 12k_{15} v_N - 9(k_{15} + \kappa_6) w_N - (12k_{15} z_E' + 9k_{15} y_E'$$

$$+ 9\kappa_6 y_G')\phi + (9k_{15} x_E' + 12\kappa_6 z_G' + 9\kappa_6 x_G')\theta$$

$$+ (12k_{15} x_E' - 12\kappa_6 y_G')\psi = 0$$

(6.38) $$- 16\kappa_9 u_A + 16\kappa_9 v_A - 16k_{11} u_B$$

$$+ [16(k_4 + k_{11} + k_{17} + \kappa_9) - m_C \lambda^2] u_C - 16\kappa_9 v_C$$

$$- 12(k_4 - k_{17}) w_C - 16k_{17} u_F - 12k_{17} w_F = 0$$

(6.39) $$16\kappa_9 u_A - 16\kappa_9 v_A - 16\kappa_9 u_C$$

$$+ [16(k_{12} + \kappa_3 + \kappa_7 + \kappa_9) - m_C\lambda^2]v_C$$

$$+ (12\kappa_3 - 12\kappa_7)w_C - 16k_{12}v_D - 16\kappa_7 v_N + 12\kappa_7 w_N$$

$$+ (16\kappa_7 z_H' + 12\kappa_7 y_H')\phi - 12\kappa_7 x_H'\theta - 16\kappa_7 x_H'\psi = 0$$

(6.40) $$- 12(k_4 - k_{17})u_C + 12(\kappa_3 - \kappa_7)v_C + [9(k_4 + k_5 + k_{17}$$

$$+ k_{18} + \kappa_3 + \kappa_7) - m_C\lambda^2]w_C - 12k_{17}u_F - 9k_{17}w_F$$

$$+ 12\kappa_7 v_N - 9(k_{18} + \kappa_7)w_N + (- 9k_{18}y_G'$$

$$- 12\kappa_7 z_H' - 9\kappa_7 y_H')\phi + 9(k_{18}x_G' + \kappa_7 x_H')\theta$$

$$+ 12\kappa_7 x_H'\psi = 0$$

(6.41) $$- 16k_{13}u_A - 16k_{10}u_B - 16k_{10}v_B$$

$$+ [16(k_{10} + k_{13} + \kappa_8 + \kappa_4) - m_D\lambda^2]u_D$$

$$+ 16k_{10}v_D + 12(\kappa_8 - \kappa_4)w_D - 16\kappa_8 u_N$$

$$- 12\kappa_8 w_N - 12\kappa_8 y_E'\phi + 4\kappa_8(3x_E' - 4z_E')\theta$$

$$+ 16\kappa_8 y_E'\psi = 0$$

$$(6.42) \quad - 16k_{10}u_B - 16k_{10}v_B - 16k_{12}v_C + 16k_{10}u_D$$

$$+ [16(k_6 + k_{10} + k_{12} + k_{19}) - m_D\lambda^2]v_D$$

$$- 12(k_6 - k_{19})w_D - 16k_{19}v_N - 12k_{19}w_N$$

$$+ 4k_{19}(- 3y'_G + 4z'_G)\phi + 12k_{19}x'_G\theta$$

$$- 16k_{19}x'_G\psi = 0$$

$$(6.43) \quad 4(- 3\kappa_4 + 3\kappa_8)u_D - 12(k_6 - k_{19}(v_D$$

$$+ [9(k_6 + k_7 + k_{19} + k_{20} + \kappa_4 + \kappa_8) - m_D\lambda^2]w_D$$

$$- 12\kappa_8u_N - 12k_{19}v_N - 9(k_{19} + k_{20} + \kappa_8)w_N$$

$$+ (+ 12k_{19}z'_G - 9k_{19}y'_G - 9k_{20}y'_H - 9\ {}_8y'_E)\phi$$

$$+ (9k_{19}x'_G + 9k_{20}x'_H - 12\kappa_8z'_E + 9\kappa_8x'_E)\theta$$

$$+ 12(- k_{19}x'_G + \kappa_8y'_E)\psi = 0$$

$$(6.44) \quad - 16k_{17}u_C - 12k_{17}w_C + [16(k_{17} + k_{23})$$

$$- m_F\lambda^2]u_F + 12k_{17}w_F - 16k_{23}u_N$$

$$- 16k_{23}z'_G\theta + 16k_{23}y'_G\psi = 0$$

$$(6.45) \quad -16\kappa_5 v_A - 12\kappa_5 w_A + [16(k_{22} + \kappa_5)$$

$$- m_F\lambda^2]v_F + 12\kappa_5 w_F - 16k_{22}v_N + 16k_{22}z_E'\phi$$

$$- 16k_{22}x_E'\psi = 0$$

$$(6.46) \quad -12\kappa_5 v_A - 9\kappa_5 w_A - 9k_{16}w_B - 12k_{17}u_C$$

$$- 9k_{17}w_C + 12k_{17}u_F + 12\kappa_5 v_F$$

$$+ [9(k_{16} + k_{17} + \kappa_5) - m_F\lambda^2]w_F = 0$$

$$(6.47) \quad -16k_{21}u_A + 12k_{21}w_A - 16\kappa_6 u_B + 12\kappa_6 w_B$$

$$- 16\kappa_8 u_D - 12\kappa_8 w_D - 16k_{23}u_F$$

$$+ [16(k_{21} + \kappa_8 + k_{23} + \kappa_6) - m\lambda^2]u_N$$

$$+ 12(-k_{21} + \kappa_8 - \kappa_6)w_N + (-12k_{21}y_H' + 12\kappa_8 y_E' - 12\kappa_6 y_G')\phi$$

$$+ (16k_{21}z_H' + 12k_{21}x_H' + 16\kappa_8 z_E' + 16k_{23}z_G' - 12\kappa_8 x_E'$$

$$+ 16\kappa_6 z_G' + 12\kappa_6 x_G')\theta + (-16k_{21}y_H' - 16\kappa_8 y_E'$$

$$- 16k_{23}y_G' - 16\kappa_6 y_G')\psi = 0$$

$$(6.48) \quad -16k_{15}v_B + 12k_{15}w_B - 16\kappa_7 v_C$$

$$+ 12\kappa_7 w_C - 16k_{19}v_D - 12k_{19}w_D - 16k_{22}v_F$$

$$+ [16(k_{15} + k_{19} + k_{22} + \kappa_7) - m\lambda^2]v_N$$

$$- 12(k_{15} - k_{19} + \kappa_7)w_N - (16k_{15}z'_E + 12k_{15}y'_E$$

$$+ 16k_{19}z'_G - 12k_{19}y'_G + 16k_{22}z'_E - 16\kappa_7 z'_H - 12\kappa_7 y'_H)\phi$$

$$+ 12(k_{15}x'_E - k_{19}x'_G + \kappa_7 x'_H)\theta$$

$$+ 16(k_{15}x'_E + k_{19}x'_G + k_{22}x'_E + \kappa_7 x'_H)\psi = 0$$

$$(6.49) \quad 12k_{21}u_A - 9(k_{14} + k_{21})w_A + 12\kappa_6 u_B + 12k_{15}v_B$$

$$- 9(k_{15} + \kappa_6)w_B + 12\kappa_7 v_C - 9(k_{17} + \kappa_7)w_C$$

$$- 12\kappa_8 u_D - 12k_{19}v_D - 9(k_{19} + k_{20} + \kappa_8)w_D$$

$$+ 12(- k_{21} + \kappa_8 - \kappa_6)u_N - 12(k_{15} - k_{19} + \kappa_7)v_N$$

$$+ [9(k_{14} + k_{15} + k_{18} + k_{19} + k_{20} + k_{21} + \kappa_6 + \kappa_7 + \kappa_8)$$

$$- m\lambda^2]w_N + (9k_{14}y'_E + 12k_{15}z'_E + 9k_{15}y'_E + 9k_{18}y'_G$$

$$- 12k_{19}z'_G + 9k_{19}y'_G + 9k_{20}y'_H + 9k_{21}y'_H + 9\kappa_6 y'_G$$

$$+ 12\kappa_7 z'_H + 9\kappa_7 y'_H + 9\kappa_8 y'_E)\phi + (- 9k_{14}x'_E - 9k_{15}x'_E$$

$$- 9k_{18}x'_G - 9k_{19}x'_G - 9k_{20}x'_H - 12k_{21}z'_H - 9k_{21}x'_H - 12\kappa_6 z'_G$$

$$- 9\kappa_6 x'_G - 9\kappa_7 x'_H + 12\kappa_8 z'_E - 9\kappa_8 x'_E)\theta + (- 12k_{15}x'_E + 12k_{19}x'_G$$

$$+ 12k_{21}y'_H + 12\kappa_6 y'_G - 12\kappa_7 x'_H - 12\kappa_8 y'_E)\psi = 0$$

(6.50) $$12k_{21}y'_H u_A - 9(k_{14}y'_E + k_{21}y'_H)w_A + 12\kappa_6 y'_G u_B + (16k_{15}z'_E + 12k_{15}y'_E)v_B$$

$$-(12k_{15}z'_E + 9k_{15}y'_E + 9\kappa_6 y'_G)w_B + (16\kappa_7 z'_H + 12\kappa_7 y'_H)v_C - (9k_{18}y'_G + 12\kappa_7 z'_H$$

$$+9\kappa_7 y'_H)w_C - 12\kappa_8 y'_E u_D + (16k_{19}z'_G - 12k_{19}y'_G)v_D - (-12k_{19}z'_G + 9k_{19}y'_G$$

$$+9k_{20}y'_H + 9\kappa_8 y'_E)w_D + 16k_{22}z'_E v_F + (-12k_{21}y'_H + 12\kappa_8 y'_E - 12\kappa_6 y'_G)u_N$$

$$-(16k_{15}z'_E + 12k_{15}y'_E + 16k_{19}z'_G - 12k_{19}y'_G + 16k_{22}z'_E - 16\kappa_7 z'_H - 12\kappa_7 y'_H)v_N$$

$$+[9k_{14}y'_E + 12k_{15}z'_E + 9k_{15}y'_E + 9k_{18}y'_G - 12k_{19}z'_G + 9k_{19}y'_G + 9k_{20}y'_H$$

$$+9k_{21}y'_H + 9\kappa_6 y'_G + 12\kappa_7 z'_H + 9\kappa_7 y'_H + 9\kappa_8 y'_E]w_N$$

$$+(9k_{14}y'^2_E + 16k_{15}z'^2_E + 24k_{15}y'_E z'_E + 9k_{15}y'^2_E + 9k_{18}y'^2_G + 16k_{19}z'^2_G$$

$$-24k_{19}y_G'z_G'+9k_{19}y_G'^2+9k_{20}y_H'^2+9k_{21}y_H'^2+16k_{22}z_E'^2$$

$$+9\kappa_6 y_G'^2+16\kappa_7 z_H'^2+24\kappa_7 y_H'z_H'+9\kappa_7 y_H'^2+9\kappa_8 y_E'^2-I_1\lambda^2)\phi$$

$$+(-9k_{14}x_E'y_E'-12k_{15}x_E'z_E'-9k_{15}x_E'y_E'-9k_{18}x_G'y_G'+12k_{19}x_G'z_G'-9k_{19}x_G'y_G'$$

$$-9k_{20}x_H'y_H'-12k_{21}y_H'z_H'-9k_{21}x_H'y_H'-12\kappa_6 y_G'z_G'-9\kappa_6 x_G'y_G'$$

$$-12\kappa_7 x_H'z_H'-9\kappa_7 x_H'y_H'+12\kappa_8 y_E'z_E'-9\kappa_8 x_E'y_E'+I_{12}\lambda^2)\theta$$

$$+(-16k_{15}x_E'z_E'-12k_{15}x_E'y_E'-16k_{19}x_G'z_G'+12k_{19}x_G'y_G'$$

$$+12k_{21}y_H'^2-16k_{22}x_E'z_E'+12\kappa_6 y_G'^2-16\kappa_7 x_H'z_H'$$

$$-12\kappa_7 x_H'y_H'-12\kappa_8 y_E'^2+I_{13}\lambda^2)\psi = 0$$

$$(6.51)\qquad -(16k_{21}z_H' + 12k_{21}x_H')u_A + (9k_{14}x_E' + 12k_{21}z_H' + 9k_{21}x_H')w_A$$

$$-(16\kappa_6 z_G' + 12\kappa_6 x_G')u_B - 12k_{15}x_E'v_B + (9k_{15}x_E' + 12\kappa_6 z_G' + 9\kappa_6 x_G')w_B$$

$$-12\kappa_7 x_H'v_C + (9k_{18}x_G' + 9\kappa_7 x_H')w_C + (-16\kappa_8 z_E' + 12\kappa_8 x_E')u_D$$

$$+12k_{19}x_G'v_D + (9k_{19}x_G' + 9k_{20}x_H' - 12\kappa_8 z_E' + 9\kappa_8 x_E')w_D$$

$$-16k_{23}z_G'u_F + (16k_{21}z_H' + 12k_{21}x_H' + 16k_{23}z_G' + 16\kappa_6 z_G'$$

$$+ 12\kappa_6 x_G' + 16\kappa_8 z_E' - 12\kappa_8 x_E')u_N + (12k_{15} x_E' - 12k_{19} x_G' + 12\kappa_7 x_H')v_N$$

$$+ (-9k_{14} x_E' - 9k_{15} x_E' - 9k_{18} x_G' - 9k_{19} x_G' - 9k_{20} x_H' - 12k_{21} z_H'$$

$$- 9k_{21} x_H' - 12\kappa_6 z_G' - 9\kappa_6 x_G' - 9\kappa_7 x_H' + 12\kappa_8 z_E' - 9\kappa_8 x_E')w_N$$

$$+ (-9k_{14} x_E' y_E' - 12k_{15} x_E' z_E' - 9k_{15} x_E' y_E' - 9k_{18} x_G' y_G'$$

$$+ 12k_{19} x_G' z_G' - 9k_{19} x_G' y_G' - 9k_{20} x_H' y_H' - 12k_{21} y_H' z_H'$$

$$- 9k_{21} x_H' y_H' - 12\kappa_6 y_G' z_G' - 9\kappa_6 x_G' y_G' - 12\kappa_7 x_H' z_H' - 9\kappa_7 x_H' y_H'$$

$$+ 12\kappa_8 y_E' z_E' - 9\kappa_8 x_E' y_E' + I_{12}\lambda^2)\phi$$

$$+ (9k_{14} x_E'^2 + 9k_{15} x_E'^2 + 9k_{18} x_G'^2 + 9k_{19} x_G'^2 + 9k_{20} x_H'^2 + 16k_{21} z_H'^2$$

$$+ 24k_{21} x_H' z_H' + 9k_{21} x_H'^2 + 16k_{23} z_G'^2 + 16\kappa_6 z_G'^2 + 12\kappa_6 x_G' z_G'$$

$$+ 12\kappa_6 x_G' z_G' + 9\kappa_6 x_G'^2 + 9\kappa_7 x_H'^2 - 24\kappa_8 x_E' z_E' + 16\kappa_8 z_H'^2 + 9\kappa_8 x_E'^2$$

$$- I_2\lambda^2)\theta + (12k_{15} x_E'^2 - 12k_{19} x_G'^2 - 16k_{21} y_H' z_H'$$

$$- 16k_{23} y_G' z_G' - 12k_{21} x_H' y_H' - 16\kappa_6 y_G' z_G' - 12\kappa_6 x_G' y_G' + 12\kappa_7 x_H'^2$$

$$-16\kappa_8 y_E' z_E' + 12\kappa_8 x_E' y_E' + I_{23}\lambda^2)\psi = 0$$

$$
\begin{aligned}
(6.52)\quad & 16k_{21}y'_H u_A - 12k_{21}y'_H w_A + 16\kappa_6 y'_G u_B - 16k_{15}x'_E v_B \\
& +(12k_{15}x'_E - 12\kappa_6 y'_G)w_B - 16\kappa_7 x'_H v_C + 12\kappa_7 x'_H w_C \\
& +16\kappa_8 y'_E u_D - 16k_{19}x'_G v_D + 12(-k_{19}x'_G + \kappa_8 y'_E)w_D + 16k_{23}y'_G u_F - 16k_{22}x'_E v_F \\
& +(-16k_{21}y'_H - 16k_{23}y'_G - 16\kappa_6 y'_G - 16\kappa_8 y'_E)u_N + (16k_{15}x'_E + 16k_{19}x'_G + 16k_{22}x'_E \\
& +16\kappa_7 x'_H)v_N + (-12k_{15}x'_E + 12k_{19}x'_G + 12k_{21}y'_H + 12\kappa_6 y'_G - 12\kappa_7 x'_H \\
& -12\kappa_8 y'_E)w_N + (-16k_{15}x'_E z'_E - 12k_{15}x'_E y'_E - 16k_{19}x'_G z'_G + 12k_{19}x'_G y'_G \\
& +12k_{21}y'^2_H - 16k_{22}x'_E z'_E + 12\kappa_6 y'^2_G - 12\kappa_7 x'_H y'_H - 16\kappa_7 x'_H z'_H - 12\kappa_8 y'^2_E + I_{13}\lambda^2)\phi \\
& +(12k_{15}x'^2_E - 12k_{19}x'^2_G - 16k_{21}y'_H z'_H - 12k_{21}x'_H y'_H - 16k_{23}y'_G z'_G \\
& -16\kappa_6 y'_G z'_G - 12\kappa_6 x'_G y'_G + 12\kappa_7 x'^2_H - 16\kappa_8 y'_E z'_E + 12\kappa_8 x'_E y'_E + I_{23}\lambda^2)\theta \\
& +(16k_{15}x'^2_E + 16k_{19}x'^2_G + 16k_{21}y'^2_H + 16k_{22}x'^2_E + 16k_{23}y'^2_G + 16\kappa_6 y'^2_G \\
& +16\kappa_7 x'^2_H + 16\kappa_8 y'^2_E - I_3\lambda^2)\psi = 0
\end{aligned}
$$

The effective spring constants for the primary and secondary members are given in Tables 6.4 and 6.5 respectively.

Table 6.4

EFFECTIVE SPRING CONSTANTS OF PRIMARY MEMBERS

k_1 = 46296.	k_9 = 19531.	k_{17} = 10000.
k_2 = 10000.	k_{10} = 6908.	k_{18} = 46296.
k_3 = 46296.	k_{11} = 19531.	k_{19} = 10000.
k_4 = 10000.	k_{12} = 19531.	k_{20} = 46296.
k_5 = 46296.	k_{13} = 19531.	k_{21} = 10000.
k_6 = 10000.	k_{14} = 46296.	k_{22} = 19531.
k_7 = 46296.	k_{15} = 10000.	k_{23} = 19531.
k_8 = 10000.	k_{16} = 46296.	

Table 6.5

EFFECTIVE SPRING CONSTANTS OF SECONDARY MEMBERS

κ_1 = 10000.	κ_4 = 10000.	κ_7 = 10000.
κ_2 = 10000.	κ_5 = 10000.	κ_8 = 10000.
κ_3 = 10000.	κ_6 = 10000.	κ_9 = 6908.

The lumped weight which is added to each joint in addition to the distributed weight associated with the structural member is 100 pounds. The weight of the upper structure is 3.75×10^4 pounds. The moments and products of inertia are given in Table 6.6.

Table 6.6

MOMENTS AND PRODUCTS OF INERTIA

	ϕ	θ	ψ
ϕ	54398.	24867.	0.
θ	24867.	54398.	0.
ψ	0.	0.	49735.

The natural frequencies of the system without and with the secondary members are given in Tables 6.7 and 6.8 respectively. The corresponding mode shapes are given in Tables 6.9 and 6.10.

Table 6.7

NATURAL FREQUENCIES WITHOUT SECONDARY MEMBERS

f_1 = 27.37	f_8 = 512.05	f_{15} = 1305.84
f_2 = 28.36	f_9 = 621.96	f_{16} = 1356.87
f_3 = 83.37	f_{10} = 712.95	f_{17} = 1397.75
f_4 = 193.56	f_{11} = 762.68	f_{18} = 1469.60
f_5 = 204.51	f_{12} = 914.27	f_{19} = 1489.47
f_6 = 456.16	f_{13} = 1052.25	f_{20} = 1501.89
f_7 = 489.57	f_{14} = 1287.82	f_{21} = 1673.00

Table 6.8

NATURAL FREQUENCIES WITH SECONDARY MEMBERS

$f_1 = 34.76$	$f_8 = 722.89$	$f_{15} = 1274.37$
$f_2 = 47.21$	$f_9 = 767.56$	$f_{16} = 1291.05$
$f_3 = 97.33$	$f_{10} = 774.92$	$f_{17} = 1400.65$
$f_4 = 239.17$	$f_{11} = 998.94$	$f_{18} = 1422.29$
$f_5 = 250.22$	$f_{12} = 1026.15$	$f_{19} = 1428.64$
$f_6 = 564.96$	$f_{13} = 1176.17$	$f_{20} = 1519.16$
$f_7 = 682.54$	$f_{14} = 1244.66$	$f_{21} = 1657.38$

Consider now estimating the standard deviations of the natural frequency. Equations (5.34) and the second of (4.9) will be used as estimates of the standard deviation of the natural frequencies.

(6.53) $$\delta_k = \frac{\sqrt{I_{1,0}}}{2\omega_k}$$

(6.54) $$I_{1,0} = [\mathrm{Cov}\, C_{ij}C_{i\ell} - \omega_k^2(\mathrm{Cov}\, A_{ij}C_{i\ell} + \mathrm{Cov}\, C_{ij}A_{i\ell}) + \omega_k^4\, \mathrm{Cov}\, A_{ij}A_{i\ell}]u_j u_\ell$$

The evaluation of $I_{1,0}$ requires knowledge of the covariances of the system members and the modal matrix of the mean system. The covariances of the system members must be evaluated from the members quality control history, or estimated from past experience with similar systems, or engineering judgment. The difficulty in evaluating $I_{1,0}$ is that the C_{ij} and A_{ij}, that is, the elements of the stiffness and mass matrices are complex functions of the system members as indicated by equations (6.32) through (6.52). For the proposed estimates to be of practical value equation (6.54) must be capable of being evaluated by means of the computer. This can be achieved by evaluating the mass and stiffness matrices when parameter values of all but one member are assumed to be zero.

The δ's for several choices of system disorder will now be evaluated and compared with corresponding Monte Carlo estimates of the S.D. $[\Omega_i]$. While the system shown in Figure 6.9 is a lumped mass system with pinned joints, members which share a joint will be assigned disorder. This is done to simulate, within the limitations of the model, a faulty joint and account for correlation between members. The coordinates used to evaluate the δ's will be those which transform the mass matrix to a unity matrix. For ease of notation, $\widehat{S.D.}\ [\Omega_i]$, will designate the Monte Carlo estimate of the standard deviation of the ith natural frequency.

The first case will consider members 1 and 8 having random spring constants of the form

$$K_1 = k_1(1 + B_1)$$

$$K_8 = k_8(1 + B_2)$$

in which

$$E[B_1] = 0 = E[B_2]$$

$$\text{Var}\ [B_1] = .01,\ \text{Var}\ [B_2] = .01 \qquad (6.55)$$

$$\text{Cov}\ [B_1 B_2] = .0025$$

With members 1 and 8 disordered the system will be considered with and without the secondary members. Table 6.9 gives the δ's with the spring constants correlated as in (6.55) and when they are independent. The first column gives the mode number. The remaining columns give the various ratios of δ to the corresponding mean system natural frequency times 100. That is, it is a percent variation. Since the presence of the secondary members significantly raises the lower natural frequencies the ratios will be used for comparison.

The following observations can be made from the δ's for members 1 and 8 disordered.

1) The effect of correlation with and without secondary members is small. That is, the differences between the corresponding ratios in Table 6.9 are small.
2) The presence of member correlation can raise or lower the δ's.
3) The δ's get larger for the lower frequencies although the increase is not monotonic. The δ's for the lowest frequencies are very large.
4) The general form of the δ's with and without the secondary members are similar although several terms show a marked difference. For example, the δ's for the highest and third highest frequency show significant variations.

Table 6.9

MEAN SQUARE ESTIMATES WITH MEMBERS 1 AND 8 DISORDERED.

	With κ		Without κ	
Mode	Correlated	Uncorrelated	Correlated	Uncorrelated
1	43.33	45.96	67.96	73.41
2	19.87	18.82	51.02	53.04
3	13.62	13.24	21.92	22.09
4	4.44	4.14	8.18	7.75
5	3.42	3.24	6.56	6.26
6	.77	.73	1.07	1.08
7	1.21	1.16	3.13	3.14
8	1.15	1.18	.18	.18
9	.82	.81	.16	.16
10	.30	.31	1.18	1.18
11	.53	.56	.86	.86
12	.54	.52	.09	.08
13	.67	.63	.15	.15
14	.53	.53	.36	.35
15	.37	.40	.92	.92
16	.38	.36	.16	.15
17	.03	.03	.01	.01
18	1.20	1.14	2.25	2.15
19	1.23	1.18	.24	.25
20	.22	.22	.20	.19
21	.21	.21	.004	.004

Table 6.10

MONTE CARLO ESTIMATES WITH MEMBERS 1 AND 8 DISORDERED AND CORRELATED

Mode	With κ	Without κ
1	.632	.662
2	.188	.506
3	.468	.773
4	.217	.415
5	.226	.464
6	.06001	.0882
7	.249	.764
8	.291	.00202
9	.156	.00229
10	.0244	.233
11	.0798	.137
12	.0703	.00142
13	.1606	.00874
14	.1814	.111
15	.06401	.443
16	.0551	.0144
17	.11409	.0002209
18	.5576	1.437
19	.945	.308
20	.0481	.619
21	.0376	.0000218

Table 6.10 gives similar results for Monte Carlo estimates, $\hat{S.D.}$ The ratio is the percent change with respect with mean system natural frequency. Twenty samples were used to obtain $\hat{S.D.}\ [\Omega_i]$. The member disorder is described by equations (6.55) with the underlying distribution being uniform. The Monte Carlo results are for correlated members.

The following observations can be made from the Monte Carlo estimates for members 1 and 8 disordered.

1) The variations show no obvious trend with mode number. This is at variance with the δ's .

2) For certain natural frequencies, for example the highest and 5th highest, the effects of the secondary members can be very large. It should be noted that for each sequence of Monte Carlo trials (i.e., with and without secondary members) the same random numbers were used. The statistical uncertainties in Monte Carlo estimates in evaluating the δ's will be considered later.

3) The variability without secondary members is larger as would be expected, i.e., the redundancy of the secondary members would be expected to reduce the overall variability in the system when the disorder in the primary system remains the same.

The following observations can be drawn when comparing the δ's and $\hat{S.D.}\ [\Omega]$ for members 1 and 8 disordered.

1) With secondary members present

$\delta_5 < \widehat{S.D.}\ [\Omega_5]$

and without the secondary members

$\delta_2 < \widehat{S.D.}\ [\Omega_2]$

$\delta_3 < \widehat{S.D.}\ [\Omega_3]$

Thus Monte Carlo estimates of some of the S.D. [Ω's] of the natural frequencies can be greater than the corresponding δ's . We also note that these Monte Carlo estimates can be larger than the true S.D.'s .

2) The δ's for low frequencies are very conservative.

3) When comparing the system with and without secondary members large variations for δ_1 , δ_4 and δ_5 were noted. The same variation is observed for the Monte Carlo estimates.

Consider a second case with members 2, 9, and 15 random. The covariance used are

$$\text{(6.56)} \qquad \text{Cov}\ [B_1 B_2] = \begin{vmatrix} .01 & .0025 & .0025 \\ .0025 & .010 & .0025 \\ .0025 & .0025 & .01 \end{vmatrix}$$

Many of the general observations for members 1 and 8, above, also apply here. From Table 6.11, which gives the δ's with and without the secondary members the following additional observations can be made.

1) The low frequency δ's are larger than for the previous results for random members 1 and 8.

2) The high frequency δ's are not significantly higher than those in 1 and 8. Additional runs made with 5 correlated random members show similar results.

3) The effects of covariance between members remains small. This is also true with as many as 5 correlated members.

Columns 2 and 3 of Table 6.12 show $\widehat{S.D.}\ [\Omega_i]$ with secondary members for 20 and 40 Monte Carlo trials respectively. Column 4 shows $\widehat{S.D.}\ [\Omega_i]$ without secondary members. The comparison of columns 2 and 3 gives an indication of the statistical variation associated with the Monte Carlo estimates. While many terms differ only by 20 percent several differ by a factor of two and for $\widehat{S.D.}\ [\Omega_8]$ the difference is several orders of magnitude. Note that for the 20 trial runs $\widehat{S.D.}\ [\Omega_8]$ is extremely small. This indicates the extent of the underlying variability and the inadequacy of Monte Carlo methods using a small number of trials for even the relatively small system under consideration. It also raises the question of the use of the $\widehat{S.D.}\ [\Omega]$ as a standard for the evaluation of the δ's .

Finally comparing the $\widehat{S.D.}\ [\Omega_i]$ with the corresponding δ's it can be seen that

$$\delta_i > \widehat{S.D.}\ [\Omega_i] \quad i = 1, \ldots, 21$$

Table 6.11

MEAN SQUARE ESTIMATES WITH MEMBERS 2, 9 AND 15 DISORDERED.

	With κ		Without κ	
Mode	Correlated	Uncorrelated	Correlated	Uncorrelated
1	23.30	24.11	61.16	63.36
2	17.14	18.68	22.05	22.84
3	7.00	7.43	6.92	7.17
4	4.04	4.21	5.07	5.27
5	1.79	1.90	1.19	1.16
6	1.51	1.55	3.01	3.25
7	1.79	1.86	3.64	3.98
8	1.31	1.35	5.00	5.46
9	1.36	1.39	2.32	2.52
10	1.19	1.21	2.62	2.71
11	1.02	1.01	2.04	2.08
12	1.54	1.40	.13	.13
13	1.23	1.12	.86	.84
14	.01	.01	2.12	1.98
15	1.41	1.35	2.12	1.98
16	2.32	2.18	1.31	1.22
17	.01	.01	.02	.02
18	.33	.34	.09	.09
19	.30	.29	1.41	1.32
20	1.27	1.20	1.42	1.32
21	.61	.61	.56	.58

Table 6.12

MONTE CARLO ESTIMATES WITH MEMBERS 2, 9 AND 15 DISORDERED.

Mode	With κ	With κ	Without κ
1	.08336	.106	.198
2	.298	.534	.108
3	.145	.219	.1279
4	.213	.390	.344
5	.0512	.0822	.0225
6	.229	.325	.741
7	.498	.662	.937
8	.207	.395	1.531
9	.226	.494	.561
10	.343	.394	.684
11	.179	.247	.462
12	.357	.471	.00504
13	.349	.571	.156
14	.0000837	.227	1.535
15	.686	.877	.773
16	1.455	1.261	.446
17	.0000882	.000131	.000134
18	.0578	.0973	.05007
19	.0363	.0339	.552
20	.699	.681	.719
21	.200	.312	.324

In evaluating the δ_k the terms in the summation in forming $I_{1,0}(u, \omega_k)$ of equation (4.9) must be collected. For each natural frequency the contributions to $I_{1,0}(u, \omega_k)$ associated with each component of the covariance of the disordered members in the system can be grouped to form an element of the so called sensitivity matrix. This matrix has one column and one row for each disordered member. Thus, each element in the matrix is associated with the corresponding covariance between members. Table 6.13 shows normalized sensitivity matrices for δ_1 and δ_4 for members 3, 10, 11, 15, and 16 disordered. Each matrix has been normalized so that its largest value is 1.0. For example, the first element of the first row, S_{11} , indicates the contribution of the variability of the third member, Cov $[K_3^2]$ to the total disorder in the first natural frequency. In this example it has the largest influence on the total variability of the first natural frequency. Element S_{14} corresponds to the contribution associated with the covariance between the third and fifteenth members, Cov$[K_3 \; K_{15}]$ These matrices give very detailed information which relates the variability of each member and between members to the variability of each natural frequency.

Several additional cases have been considered but with inconclusive results. These were concerned with observing the effects of small changes in the system. Monte Carlo estimates were obtained with the members uncorrelated. A second case investigated the effects of using a diagonal mass matrix; that is, equating the products of inertia of the system to zero. In both cases the statistical variation associated

Table 6.13

TYPICAL SENSITIVITY MATRICES

MEMBERS 3, 10, 11, 15, AND 16 DISORDERED.

First Natural Frequency

1.0	0.0	1.55×10^{-7}	-.231	-.165
	.056	-.00631	.0371	-1.49×10^{-7}
		.0214	0.0	0.0
			.637	.0768
				.251

Fourth Natural Frequency

.0129	0.0	0.0	.0181	-.00749
	1.0	.0713	-.107	0.0
		.1533	0.0	0.0
			.3006	-.0209
				.0398

with the Monte Carlo estimates masked small changes and the results were thus inconclusive.

The δ's were also obtained for several single members to see if they all exhibit the same character. Of particular interest are the δ's for the lower frequencies. Members 1, 3, 8, 9, 10, 11, 15, 16 have been evaluated and all except member 9 show the same general form of the δ's , increasing markedly for low frequencies. The δ's for member 9 were small for both high and low frequencies. Member 9 was the only horizontal member investigated.

Finally a study of the effects of coordinate transformations was initiated. Since the δ's are estimates of the standard deviations, i.e., the δ's differ from the S.D. in that some terms, hopefully small, have been deleted in evaluating them. If a different coordinate system is used, one would expect different results. As was mentioned earlier the δ's have been evaluated using coordinates which transform the mass matrix to a unity matrix. The question which remains is do results obtained from other coordinates differ significantly and do they still tend to yield an upper bound to the S.D. $[\Omega]$. Initial results using the original coordinates indicated that the δ's are close to those originally obtained but $\delta_i <$ S.D. $[\Omega_i]$ for more terms than in the earlier analysis.

Appendix 1, Section 6.

This appendix shall work out in more detail the steps performed in the nine-degree of freedom system example.

The kinetic and potential energies can be written as

$$2T = \sum_{i=1}^{9} \sum_{j=1}^{9} A_{ij}\dot{q}_i\dot{q}_j = \sum_{i=1}^{9} m_i\dot{q}_i^2$$

$$2V = \sum_{i=1}^{9} \sum_{j=1}^{9} C_{ij}q_iq_j$$

$$= \sum_{i=1}^{7} (k_i + k_{i+1})q_i^2$$

$$+ 2 \sum_{i=1}^{6} (- k_{i+1})q_iq_{i+1} + \underline{K_9q_4^2}$$

$$+ \underline{2 (- K_9)q_4q_8} + \underline{(K_9 + k_{10})q_8^2}$$

$$+ 2 (- k_{10})q_8q_9 + k_{10}q_9^2$$

The only disordered element in the system is

$$K_9 = k_9(1 + B)$$

where

$$E[B] = 0 \quad Var[B] = \sigma_B^2$$

The only terms in the system equations which contain K_9 have been underlined. Note that these terms only involve coordinates q_4 and q_8 . The coordinates are now transformed so that the mass matrix A_{ij} will be a unit matrix in terms of the transformed coordinates. For the case at hand the required transformation is given by

$$p_j = q_j \sqrt{m_j}$$

Table A 6.1 shows the stiffness matrix in transformed coordinates. This matrix forms the system matrix in transformed coordinates. Its eigenvalues will be the same as those for the original system and the corresponding eigen vectors will be used to find the δ_k , the estimates of S.D. $[\Omega_k]$. Table 6.1 in the text contains the eigen values and the fourth and eighth eigen vector component for each eigen frequency.

Equation (5.34) will again be used to evaluate δ_k . This in turn requires $I_{1,0}(\eta^{(k)}, \omega_k)$ to be evaluated for which the last of (4.9) is used. Note that the A_{ij} do not contain any disordered elements, thus (4.9) is reduced to $I_{1,0}(\eta^{(k)}, \omega_k) =$ Cov $C_{ij}C_{i\ell}\eta_j^{(k)}\eta_\ell^{(k)}$.

As was noted earlier, the only matrix elements which contain K_9 are C_{44}, C_{48}, C_{84}, and C_{88} . Thus covariance of all terms not containing one of these terms will vanish.

Thus

Table A 6.1

STIFFNESS MATRIX IN TRANSFORMED COORDINATES

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
p_1	$\frac{k_1+k_2}{m_1}$	$\frac{-k_2}{\sqrt{m_1 m_2}}$							
p_2	$\frac{-k_2}{\sqrt{m_1 m_2}}$	$\frac{k_2+k_3}{m_2}$	$\frac{-k_3}{\sqrt{m_2 m_3}}$						
p_3		$\frac{-k_3}{\sqrt{m_2 m_3}}$	$\frac{k_3+k_4}{m_3}$	$\frac{-k_4}{\sqrt{m_3 m_4}}$					
p_4			$\frac{-k_4}{\sqrt{m_3 m_4}}$	$\frac{k_4+k_5+k_9}{m_4}$	$\frac{-k_5}{\sqrt{m_4 m_5}}$			$\frac{-k_9}{\sqrt{m_4 m_8}}$	
p_5				$\frac{-k_5}{\sqrt{m_4 m_5}}$	$\frac{k_5+k_6}{m_5}$	$\frac{-k_6}{\sqrt{m_5 m_6}}$			
p_6					$\frac{-k_6}{\sqrt{m_5 m_6}}$	$\frac{k_6+k_7}{m_6}$	$\frac{-k_7}{\sqrt{m_6 m_7}}$		
p_7						$\frac{-k_7}{\sqrt{m_6 m_7}}$	$\frac{k_7+k_8}{m_7}$		
p_8				$\frac{-k_9}{\sqrt{m_4 m_8}}$				$\frac{k_9+k_{10}}{m_8}$	$\frac{-k_{10}}{\sqrt{m_8 m_9}}$
p_9								$\frac{-k_{10}}{\sqrt{m_8 m_9}}$	$\frac{k_{10}}{m_9}$

$$I_{1,0}(\eta^{(k)}, \omega_k) = [\mathrm{Cov}\ C_{44}C_{44} + \mathrm{Cov}\ C_{84}C_{84}]\eta_4^{(k)}\eta_4^{(k)}$$

$$+ [\mathrm{Cov}\ C_{44}C_{48} + \mathrm{Cov}\ C_{84}C_{88}]\eta_4^{(k)}\eta_8^{(k)}$$

$$+ [\mathrm{Cov}\ C_{48}C_{44} + \mathrm{Cov}\ C_{88}C_{84}]\eta_8^{(k)}\eta_4^{(k)}$$

$$+ [\mathrm{Cov}\ C_{48}C_{48} + \mathrm{Cov}\ C_{88}C_{88}]\eta_8^{(k)}\eta_8^{(k)}$$

Each of the above covariances must be evaluated. Noting that $C_{ij} = C_{ji}$ provides some simplification. To illustrate the procedure the first two terms will be evaluated.

$$\mathrm{Cov}\ C_{44}C_{44} = E[(C_{44} - \overline{C}_{44})(C_{44} - \overline{C}_{44})]$$

$$C_{44} = \frac{k_4 + k_5 + K_9}{m_4} = \frac{4 + 4 + 1 + B}{4} = \frac{9 + B}{4}$$

$$\overline{C}_{44} = E[C_{44}] = E[\tfrac{9+B}{4}] = \tfrac{9}{4} + \tfrac{1}{4}\,E[B] = \tfrac{9}{4}$$

Thus $C_{44} - \overline{C}_{44} = \frac{9}{4} + \frac{B}{4} - \frac{9}{4} = \frac{B}{4}$

$$\mathrm{Cov}\ [C_{44}C_{44}] = E[\tfrac{B}{4} \cdot \tfrac{B}{4}] = \tfrac{1}{16}\,E[B^2] = \frac{\sigma_B^2}{16}$$

$$\mathrm{Cov}\,[C_{84}C_{84}] = E[(C_{84} - \overline{C}_{84})(C_{84} - \overline{C}_{84})]$$

$$C_{84} = -\frac{K_9}{(m_4 m_8)^{1/2}} = -\frac{1 + B}{(4\cdot 2)^{1/2}} = -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}\,B}{4}$$

$$\overline{C}_{84} = E[-\frac{1+B}{2\sqrt{2}}] = -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} E[B] = -\frac{\sqrt{2}}{4}$$

$$C_{84} - \overline{C}_{84} = -\frac{\sqrt{2}}{4} \quad \frac{\sqrt{2}B}{4} \quad \frac{\sqrt{2}}{4} = -\frac{B\sqrt{2}}{8}$$

$$\text{Cov } C_{84}C_{84} = E[-\frac{\sqrt{2}B}{4} \cdot \frac{\sqrt{2}B}{4}] = \frac{1}{8} E[B^2] = \frac{\sigma_B^2}{8}$$

Thus the first term of $I_{1,0}(\eta^{(k)}, \omega_k)$ is

$$[\text{Cov } C_{44}C_{44} + \text{Cov } C_{84}C_{84}]\eta_4^{(k)}\eta_4^{(k)} =$$

$$[\frac{\sigma_B^2}{16} + \frac{\sigma_B^2}{8}][\eta_4^{(k)}]^2 = \frac{3}{16} \sigma_B^2(\eta_4^{(k)})^2$$

$$= .1875 \; (\eta_4^{(k)})^2 \; \sigma_B^2$$

Since there are no other terms with $(\eta_4^{(k)})^2$ this coefficient corresponds to the first of equation (6.18). In a similar fashion the remaining terms can be evaluated to yield (6.18).

As an illustration δ_4 will be evaluated

$$I_{1,0}(\eta^{(4)}, \omega_4) = [.1875(\eta_4^{(4)})^2 - .5302 \; \eta_4^{(4)}\eta_8^{(4)}$$

$$+ .375 \; (\eta_8^{(4)})^2] \; \sigma_B^2$$

On using $\sigma_B = .1$ and the values in Table 6.1 i.e. $\eta_4^{(4)} = .354$, $\eta_8^{(4)} = .560$

$$\omega_4 = 1.0757$$

$$I_{1,0}(\eta^{(4)}, \omega_4) = .0003599$$

and using equation (5.34)

$$\delta_4 = \frac{\sqrt{I_{1,0}(\eta^{(4)}, \omega_4)}}{2\omega_4} = \frac{\sqrt{3.599\text{x}10^{-4}}}{2\text{x}1.0757} = .00882$$

This is the value shown in Table 6.2. Similar calculation can be made for the other modes. Note that the covariances need only be calculated once since the only difference between $I_{1,0}(\eta^{(i)}, \omega_i)$ and $I_{1,0}(\eta^{(j)}, \omega_j)$ $(i \neq j)$ is the mode shape in equation (6.18)

7. Improvement in Results and Discussion.

The estimator δ_k has been consistantly high for low k in each of the examples presented. This section will discuss two ways of improving these estimates. The use of equation (5.34) to evaluate δ_k requires that $I_{1,0}(\cdot)$ be evaluated. The first methods are concerned with improving the evaluation of $I_{1,0}(\xi', \omega_k)$. Equation (4.9) has been used to approximate $I_{1,0}(\xi', \omega_k)$ by using the mode shapes, $\eta^{(k)}$, for the mean system for the ξ' . Two alternatives are possible. Evaluation of the covariances in (4.9) and substituting in a particular ω_k from the mean system establishes a coefficient matrix to $u_i u_j$ for each eigen frequency. The lowest eigenvalue of this matrix which we shall call I(matrix) is a minimum of $I_{1,0}$ for the given k . The eigen vector which corresponds to ξ' can be substituted into (4.9) to yield $I_{1,0}(\xi', \omega_k)$. The improvements obtained by these corrections have been evaluated using the nine-degree of freedom example and are shown in Table 7.1. While $I_{1,0}(\xi', \omega_k)$ is consistantly lower than either $I_{1,0}(\eta^{(k)}, \omega_k)$ or I(matrix) , this difference is relative insignificant.

Table 7.1
Comparison of estimates of $I(\xi', \omega_k)$

k	$I_{1,0}(\eta^{(k)}, \omega_k)$	$I_{1,0}(\xi', \omega_k)$	I(matrix)
1	.00157	.000147	.000152
2	.000892	.000878	.000885
3	0	0	0
4	.000360	.000337	.000349
5	.00264	.00262	.00263
6	0	0	0
7	.000760	.000757	.000759
8	0	0	0
9	.000807	.000805	.000806

The second method of improving the estimator δ_k makes a correction by accounting for terms which were dropped in the derivation. Returning to equation (5.11)

$$(5.11') \qquad I_{1,0}(\eta^{(k)}, \omega_k) \geq I(\xi', \omega_k)$$

$$= \sum_{\sigma=1}^{m} p_\sigma (\omega_{k,\sigma}^2 - \omega_k^2)^2 \nu'_{k,\sigma}$$

$$+ \sum_{\sigma=1}^{m} p_\sigma \sum_{r=1}^{N}{}' (\omega_{r,\sigma}^2 - \omega_k^2)^2 \nu'^2_{r,\sigma}$$

In forming the estimator, we used $I_{1,0}(\eta^{(k)}, \omega_k)$ since this is easy to compute from information on the mean system and covariances of the various terms. We discarded the second sum on the right of $I(\cdot)$ in (5.11'). A better estimator can be formed at the expense of additional effort by using in place of $I_{1,0}(\cdot)$

$$(7.1) \qquad I_{1,0}(\eta^{(k)}, \omega_k) - \sum_{\sigma=1}^{m} p_\sigma \sum_{r=1}^{N}{}' (\omega_{r,\sigma}^2 - \omega_k^2)^2 \nu'^2_{r,\sigma}$$

While any of the three methods for evaluating $I(\xi', \omega_k)$ can be used, $I_{1,0}(\eta^{(k)}, \omega_k)$ will be used since it is the easiest to evaluate. The terms within the double summation are obtained by contriving 2 ($\sigma = 2$) or more sample systems such that the variance of each element in the ensemble of systems is equal to that in the original calculation. Using the last of (5.10) the $\nu'_{r,\sigma}$ can be evaluated to form the correction. This has been done for the two-degree of freedom and the nine-degree of freedom systems considered in Section 6.

In the two-degree of freedom example, we considered Case A in which the B_i are independent. Two sample systems were formed such that VAR $B_i = .01$, $i = 1, 2$. The values $I_{1,0}(\eta^{(k)}, \omega_k)$, $\omega_{k,\sigma}$, $\eta^{(k)}_{j,\sigma}$, and $\nu'_{k,\sigma}$, $k = 1, 2$ were computed; and expression (7.1) was evaluated. When this was used in place of $I_{1,0}(\eta^{(k)}, \omega_k)$ in (5.34) the value obtained for δ_1 became $.311\ \omega_o \sigma_B$ instead the value given by the first of (6.14); the new δ_2 was substantially the same as the old. These computations were carried out using a desk calculator.

This second method was also applied to the nine-degree of freedom. To evaluate the effectiveness of the correction, a four sample systems ensemble was selected so that Prob. $(B = \pm .1265, \pm .06325) = 1/4$, which gives a variance equal to that in our original calculations, $\sigma_B^2 = .01$. Applying the correction to $I_{1,0}(\eta^{(k)}, \omega_k)$, we computed the δ_k , $k = 1, 2, \ldots, 9$.

Table 7.2 shows the average of the Monte Carlo results shown in Table 6.2; the original estimates for the δ_k ; and the corrected estimates just obtained.

Table 7.2

Evaluations of Correction Terms

k	Original Estimated δ_k	Average Monte Carlo	Corrected Estimates
1	.0199	.0037	.00215
2	.0287	.0120	.0115
3	0	0	0
4	.00882	.0024	.00145
5	.0214	.015	.0147
6	0	0	0
7	.00817	.0031	.00300
8	0	0	0
9	.00716	.0028	.00270

The improvement of the estimates both for low and high mode numbers is striking, being within 5% of the Monte Carlo estimates in most cases. The corrected estimates are now slightly below those obtained from the Monte Carlo simulation. In Tables 7.1 and 7.2, zeros have been used for mode numbers 3, 6 and 8 to indicate that all had exponents of 10^{-14} or less.

It should be noted that this improvement is obtained at the cost of solving m , the number of sample systems in the ensemble, additional nth order eigenvalue problems for eigenvalues and vectors.

SUMMARY

Summary of formulas needed to obtain estimates of the standard deviation, δ_k, of the kth natural frequency of a disordered system.

The differential equations describing the system are to be transformed into the form

$$2T = A_{ij}\dot{q}_i\dot{q}_j \quad ; \quad 2V = C_{ij}q_iq_j$$

where the $A_{ij} = \delta_{ij} + B_{ij}$.

The B_{ij} represent the normalized disorder associated with the system's mass matrix. The standard deviation of the kth natural frequency is estimated by

$$\delta_k \approx \frac{1}{2\omega_k}\sqrt{I_{1,0}(\eta^{(k)}, \omega_k)}$$

where

ω_k is the circular natural frequency of the kth mode of the mean system, $\eta^{(k)}$ is the normalized mode shape corresponding to the kth natural frequency, $I_{1,0}(\eta^{(k)}, \omega_k)$ is a measure of disorder defined by equation (4.9)

$$I_{1,0}(\eta^{(k)}, \omega_k) = \{\mathrm{Cov}[C_{ij}C_{i\ell}] - \omega_k^2(\mathrm{Cov}[A_{ij}C_{i\ell}] + \mathrm{Cov}[C_{ij}A_{i\ell}]) + \omega_k^4\,\mathrm{Cov}[A_{ij}A_{i\ell}]\}\,\eta_j^{(k)}\eta_\ell^{(k)}$$

where

$\eta^{(k)}$ is the eigenvector associate with the kth mode of the mean system in transformed coordinates.

The evaluation of $I_{1,0}(\eta^{(k)}, \omega_k)$ requires knowledge of the natural frequency, ω_k, and associated mode shape, $\eta^{(k)}$, obtained by solving the eigenvalue problem for the mean system for the kth mode. Evaluation of the covariance of the C's, A's and CA's requires knowledge of the covariances of the member disorders. If the disordered stiffness and masses are represented by $K_i = k_i(1 + B_i)$ and $M_i = m_i(1 + D_i)$ respectively, then

$\mathrm{Cov}[B_iB_j]$, $\mathrm{Cov}[D_iD_j]$, and $\mathrm{Cov}[D_iB_j]$ are required.

A more refined estimate of δ_k can be obtained by substracting the following correction from $I_{1,0}(\eta^{(k)}, \omega_k)$

$$\sum_{\sigma=1}^{m} p_\sigma \sum_{r=1}^{N}{}' (\omega_{r,\sigma}^2 - \omega_k^2)^2 \nu_{r,\sigma}'^2$$

where

m is a small number of sample systems used to obtain the correction

p_σ is the probability associated with each sample system

$\omega_{r,\sigma}$ is the rth natural frequency associated with the σth sample system

$\nu'_{r,\sigma} = \sum \eta_{j,\sigma}^{(r)} \eta_j^{(r)}$

$\eta_{j,\sigma}^{(r)}$ is the eigenvector associated with the rth mode of the σth sample system.

The parameter values and probability, p_σ , associated with each sample system should be selected so that the statistics of the ensemble match those of the original system. Evaluation of $\omega_{r,\sigma}$ and the $\nu'_{r,\sigma}$ requires the solution of an eigenvalue problem for each σ . The δ_k obtained using this correction will not necessarily be upper bounds to the corresponding standard deviations.

CONCLUSION

A method has been presented for estimating the variability of a system's natural frequencies arising from the variability of the system's parameters. The only information required to obtain the estimates is the member variability, in the form of second order properties, and the natural frequencies and mode shapes of the mean system. It has also been established for the systems studied by means of Monte Carlo estimates that the specification of second order properties in an adequate description of member variability. The method is structured in a way which facilitates the use of computers for obtaining estimates for complex systems. The variability in any natural frequency caused by each structural element can be obtained. These sensitivity coefficients can be used to establish bounds on member variability to secure a specified variability of a system's natural frequencies. From another point of view this information can be used to indicate which structural elements will have the greatest effect on a given natural frequency. Finally, by additional effort the estimates can be improved to make them less conservative.

To implement these methods information is now required on the variability of structural members.

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