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ANNUAL TECHNICAL REPORT

28 June 1970 to 15 January 1972

Contract NAS2-5062

THE INVESTIGATION OF VERTEBRAL INJURY SUSTAINED DURING AIRCREW EJECTION

National Aeronautics and Space Administration
Ames Research Center
Moffett Field, California

8531 N. NEW BRAUNFELS AVE. • SAN ANTONIO, TEXAS 78217

TECHNOLOGY INCORPORATED
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SAN ANTONIO, TEXAS

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THE INVESTIGATION OF VERTEBRAL INJURY SUSTAINED DURING AIRCREW EJECTION

PREPARED BY:

James V. Benedict, Ph. D.
Principal Investigator

APPROVED BY:

Ralph G. Allen, Ph. D.
General Manager

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NOMENCLATURE

A	Area of cross section
C_1	Constant of integration
E	Young's modulus
G	Shear modulus
I_1	Moment of inertia of cross section about y axis
K_1	Ratio of average shear stress to maximum shear stress
P	Longitudinal force, defined in equation (4)
P_b	Vertical acceleration force applied to the column base
S	Shear force on a cross section of column
S_b	Transverse acceleration force applied to column base
k	Timoshenko shear coefficient
k'	Huffington shear coefficient
m_h	Rigid mass attached to column at z=1
r	Displacement of center of mass from centroid of column
t	Time variable
u	Transverse displacement function
u_o	Initial value of u
w	Displacement of centroidal axis in z direction
x, y, z	Inertial reference (Sec. 1)
ρ	Mass density of column material
σ	Stress
ϕ	Angle of rotation of cross section due to bending

SUBSCRIPTS:

i	Refers to discrete variables in the z direction
n	Refers to discrete variables in the time domain

SUPERSCRIPTS

$*$	Refers to variables which will be iterated

1. INTRODUCTION

This report describes research conducted by the Life Sciences Division of Technology Incorporated to further describe and understand the mechanism of vertebral injury resulting from vertical accelerations of the spine. The work reported was performed during the period 28 June 1970 to 15 January 1972, and represents the concluding effort of a three year program.

Two significant results were accomplished during the reporting period. The dynamic and static properties of the human spine in flexure were determined and a complex continuum mathematical model describing the dynamic response of the human spine was formulated, solved and verified experimentally.

Details of these efforts are presented in a series of discussions, figures, tables and equations that follow.

2. FLEXURAL PROPERTIES OF THE HUMAN SPINE

A series of tests have been performed on excised human vertebral segments to determine the static and dynamic response of the thoraco-lumbar spine when loaded in flexion. A total of fifteen tests were performed on eleven specimens. Specimens were obtained from male donors ranging in age from 34 to 60 years. Demographic data pertinent to each specimen and the elapsed time between death of the donor and testing of each corresponding specimen are presented ~~in Table I~~. Only spinal segments comprised of lower thoracic and upper lumbar vertebrae were tested because in aircraft ejection injuries clinical complications in this anatomical region predominate.~~(1)~~

2.1 Specimen Preparation

Specimens were obtained at autopsy and consisted of eight to ten vertebrae, generally in the T7-L3 region. Although the transverse processes were transected to remove the ribs, care was exercised to keep the ligaments intact since they appear to contribute significantly to the bending stiffness of the spine. The ligamenta flava, the interspinous, supraspinous, anterior and posterior longitudinal ligaments were retained intact.

Specimens were moistened with physiological saline and stored in tightly sealed plastic bags at 5°C from the time of removal until approximately 24 hours prior to testing, when they were allowed to

TABLE I

Demographic Data on Specimens for Bending Tests

Specimen	Age	Sex/Race	Time from Death Until Testing (days)	Cause of Death
A111(1)	34	M/C	15	Pneumonia
A111(2)	34	M/C	15	Pneumonia
A111(3)	34	M/C	15	Pneumonia
A112	42	M/C	14	Chronic Alcoholism
A114	82	M/C	11	Cardiac Arrest
A116	48	M/N	18	Cancer of the Stomach
A117	57	M/C	11	Nephritic Carcinoma
A119	72	M/N	15	Acute Renal Failure
A121	60	M/N	15	Intra-Cranial Hemorrhage
A122	59	M/C	15	Carcinoma of the Lung
A123	36	M/N	12	Diabetes Mellitus
A126	50	M/N	13	Chronic Alcoholism
A127(1)	60	M/C	17	Intra-Cerebral Hemorrhage
A127(2)	60	M/C	17	Intra-Cerebral Hemorrhage
A127(3)	60	M/C	17	Intra-Cerebral Hemorrhage

equilibrate at room temperature, 22°C. Upon equilibration with room temperature, curved metal plates, for use as load-bearing surfaces, were bonded on the anterior surface of two vertebrae of each spinal segment. These plates were applied as shown in Figure 1.

Precise determination of the anatomical levels and initial curvature of each specimen were determined from radiographs taken in both the anterio-posterior (A-P) direction and laterally.

2.2 Test Equipment and Methods

The test equipment used to perform the bending tests is that which was described in detail in our annual technical report for Contract No. NAS2-5062 of 28 June 1969 - 27 June 1970.[2] It was noted in that report that the predominant source of deformation during a test is attributable to the bending stresses and that the shear deflections can be ignored. We therefore concentrated on the pure bending tests with the objective of defining quantitatively the bending stiffness, EI.

The test fixture used is shown in the photographs of Figures 2 and 3 and schematically in Figure 4. As noted in Figure 4, the loading fixture for the bending test does not necessarily result in, and the analysis is not dependent upon, a symmetrical loading pattern. The asymmetrical loading results because the reactionary loads at points B and C must be placed on vertebral bodies, which are generally not

Fig. 1 Positioning of load distributing clamps on a segment of fresh, unembalmed, human vertebral column.

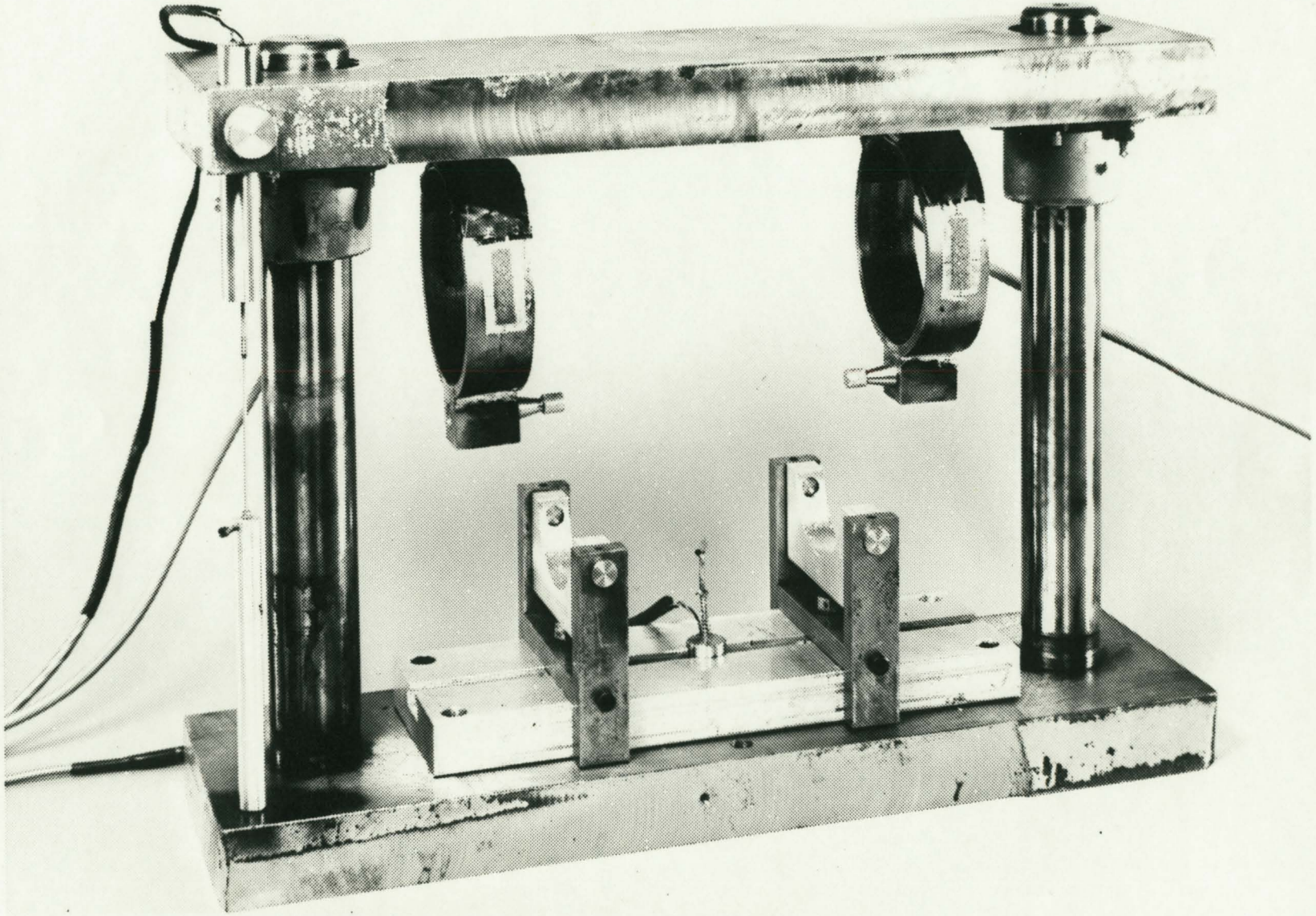

Fig. 2 Test fixture for bending experiments showing force transducers and variable differential transformers.

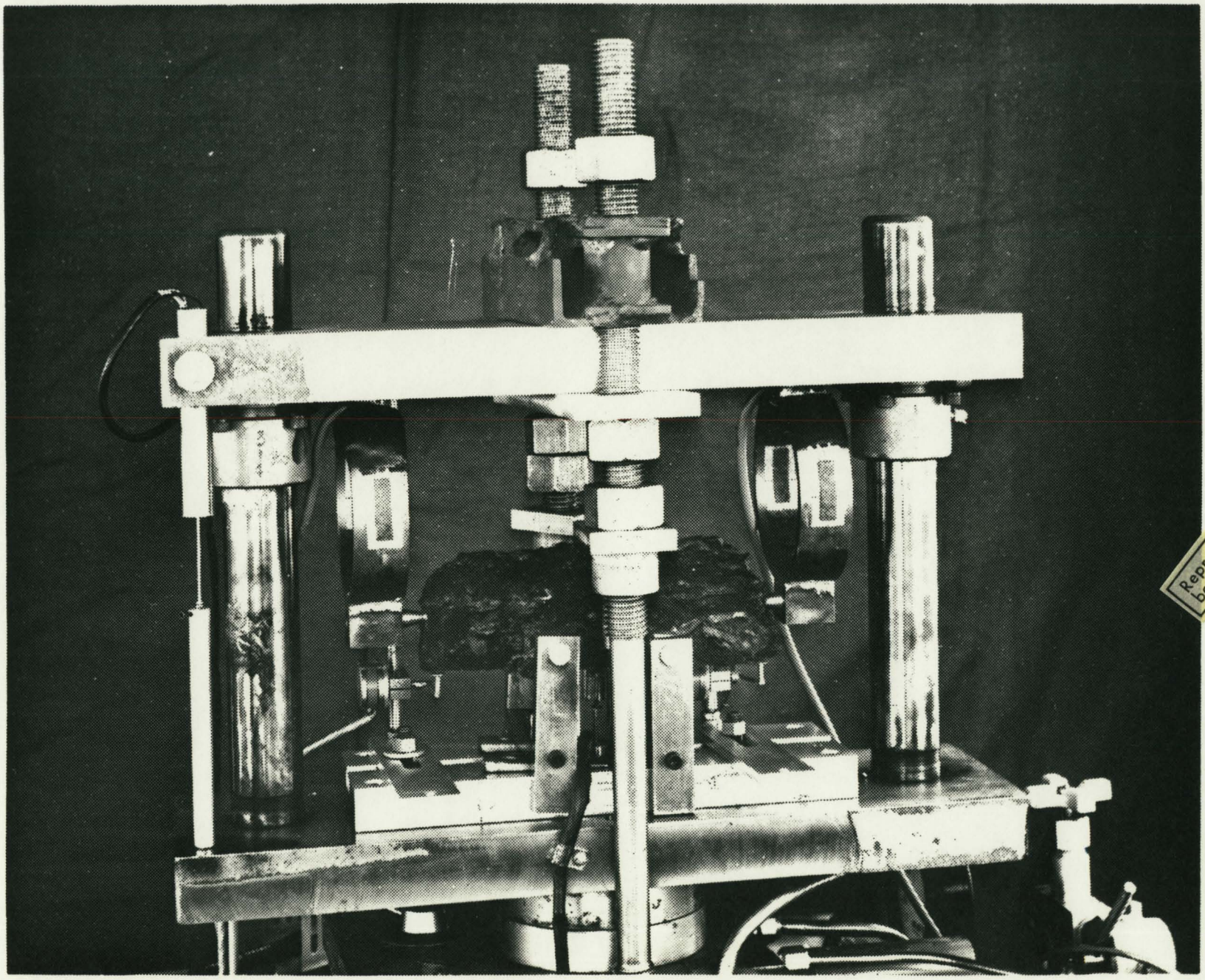

Fig. 3 Detroit testing machine with test fixture and specimen positioned for bending test.

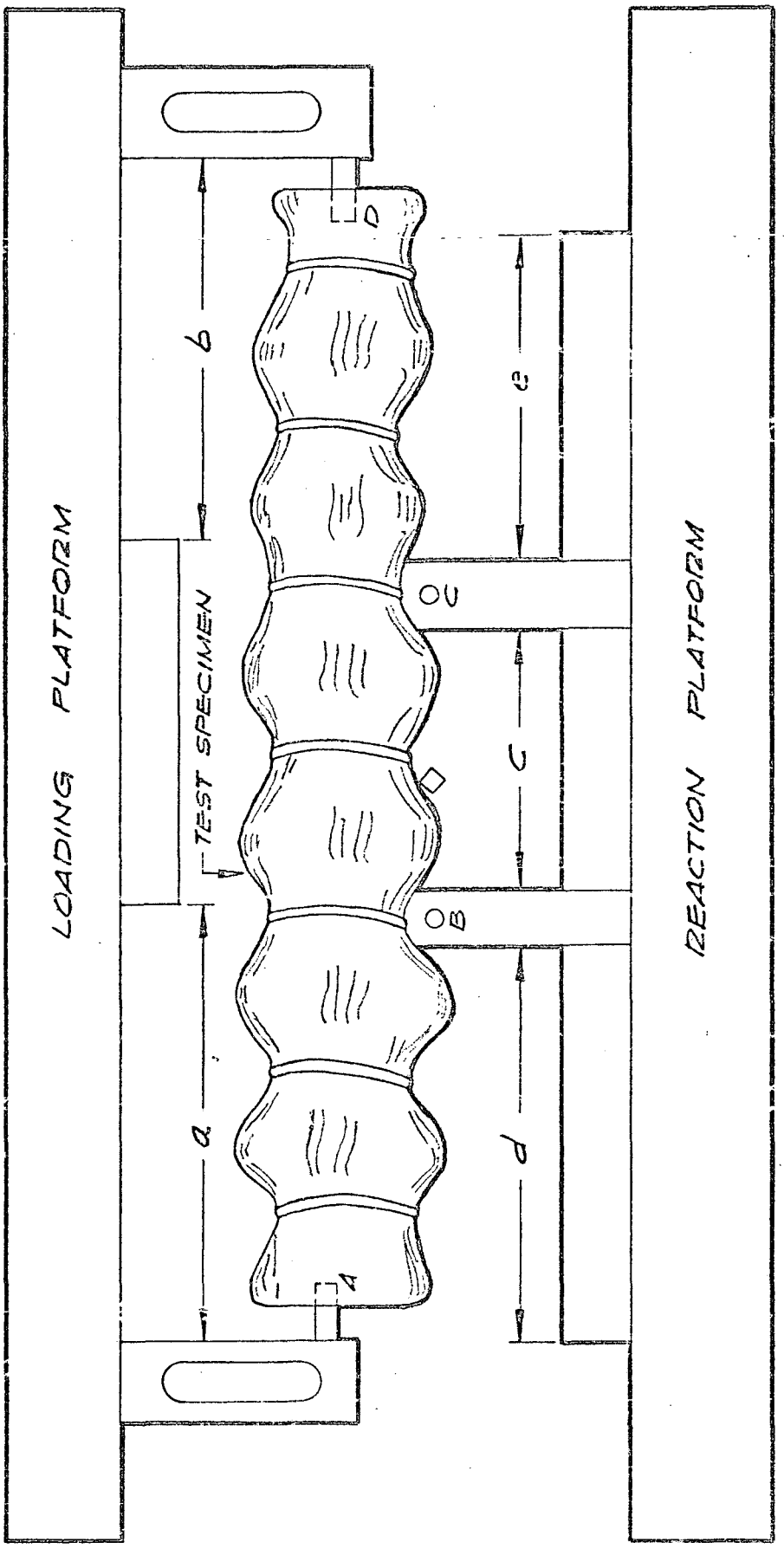

Figure 4. Schematic diagram of the Bending Test Fixture with a Specimen in place.

symmetrically placed with respect to the end loading points, A and D. If the loading pattern was purely symmetrical then the central portion of the specimen between B and C would be subjected to pure bending and no shear load. The small asymmetry of loading does therefore induce some shear loading in the center span. This force, however, has been calculated to be insignificant for the test conducted.

2.3 Results

The results of the bending tests are shown in Figure 5, in which the bending moment is plotted as a function of the change in curvature. More detailed information is presented in Table II which summarizes the specimen number, the anatomical level and the average value of the bending stiffness. An overall average value for EI of 8000 lb/in^2 was obtained from these tests.

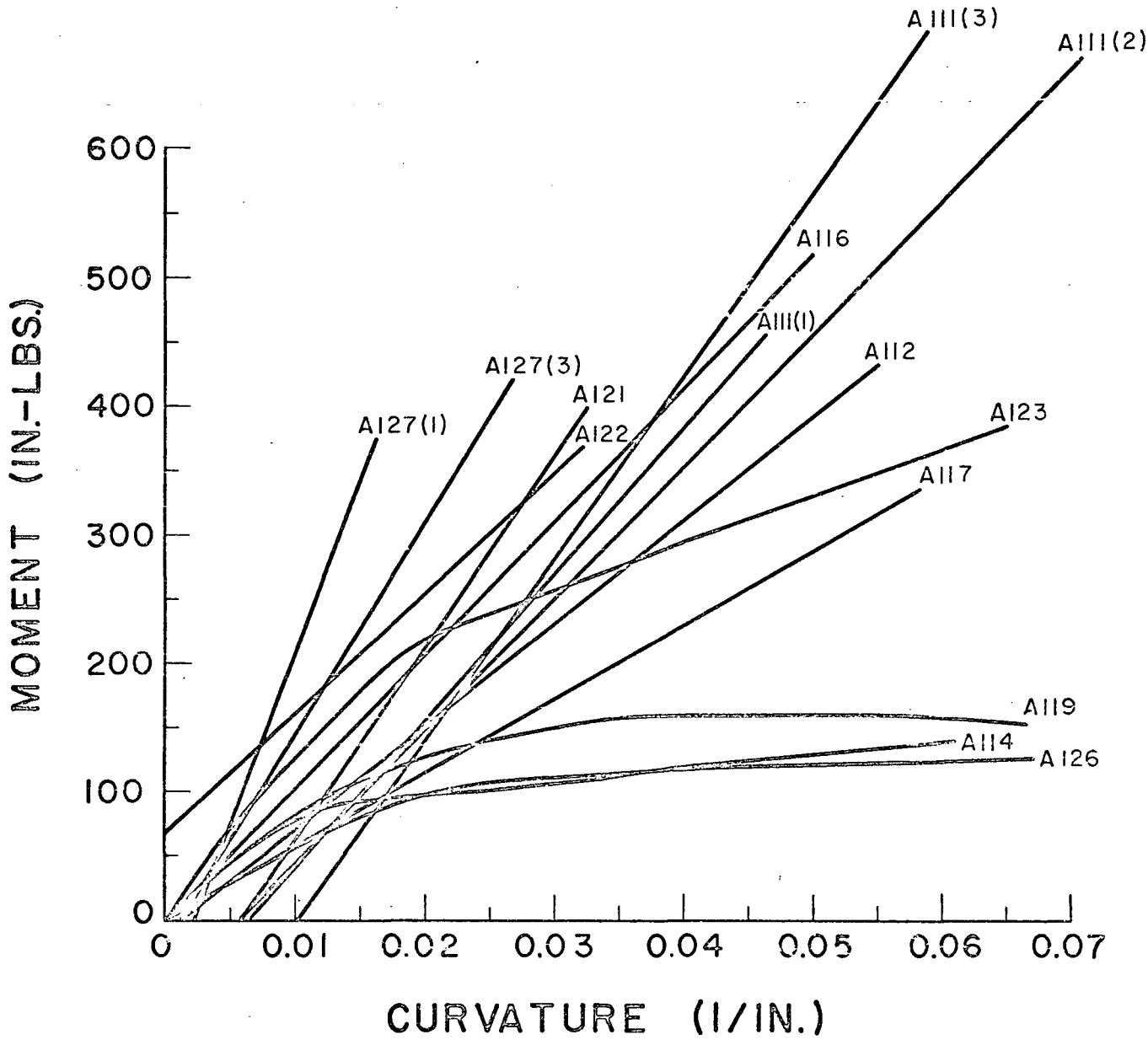

Figure 5. Bending Moment versus Curvature Change in Unembalmed Human Spine Segments

TABLE II

Bending Stiffness of Human Spine Segments, in Flexion

Specimen	Anatomical Level	Flexural Rigidity EI (lb/in^2)
11 run 1	T12-L5	9,930
11 run 2	T12-L5	10,580
11 run 3	T12-L5	12,460
12	T6 -L3	6,760
14	T7 -L3	4,950
16	T7 -L3	9,920
17	T7 -L3	5,800
19	T7 -L3	4,980
21	T7 -L3	13,380
22	T7 -L3	9,580
23	T7 -L3	5,660
26	T7 -L3	4,850
27 run 1	T9 -L2	21,000
27 run 2	T9 -L2	17,500
27 run 3	T9 -L2	13,600
All Specimens		8,000

3. MATHEMATICAL MODEL

The principal effort during the reporting period has been directed toward obtaining a stable, accurate solution to the differential equations that describe the dynamic response of the spine to impact loads. The fundamental assumptions underlying the derivation of the equations will be reviewed prior to a presentation of the solution. The terms used in the equations that follow are defined in the nomenclature section of this report.

3.1 Derivation of the Differential Equation

In the derivation of the differential equations, the spine is assumed to be a homogeneous, isotropic, linearly elastic, tapered beam column having an initial curvature in a plane. The mass center of each cross section is assumed not to coincide with the centroid of that cross section. Force is applied to one end of the beam column and a concentrated mass is assumed pinned to the opposite end. Deflections due to shear, axial compresion and bending are considered. The axial, rotary and transverse inertia of each element is incorporated. Gravity effects are neglected.

If motion is restricted to the plane of the initial curvature, and if the beam column is initially unstrained, applying Hamilton's principle results in the following set of coupled, non-linear, partial differential

equations and corresponding boundary conditions.

$$\frac{\partial P}{\partial z} = \rho A \frac{\partial^2 \bar{w}}{\partial t^2} + \rho A r \frac{\partial^2 \phi}{\partial t^2} \tag{1}$$

$$\frac{\partial}{\partial z}\left(EI_1 \frac{\partial \phi}{\partial z}\right) + kAG\left(\frac{\partial u}{\partial z} - \frac{\partial u_o}{\partial z} - \phi\right) = \rho I_2 \frac{\partial^2 \phi}{\partial t^2} + \rho A r \frac{\partial^2 \bar{w}}{\partial t^2} \tag{2}$$

$$\frac{\partial}{\partial z}\left[kAG\left(\frac{\partial u}{\partial z} - \frac{\partial u_o}{\partial z} - \phi\right) + P\frac{\partial u}{\partial z}\right] = \rho A \frac{\partial^2 u}{\partial t^2} \tag{3}$$

$$P = EA\left[\frac{\partial \bar{w}}{\partial z} + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial u_o}{\partial z}\right)^2\right] \tag{4}$$

The boundary conditions are:

AT $z = 0$: $M = 0$ (5a)

$$S_b = \frac{k}{k'} S + P \frac{\partial u}{\partial z} \tag{5b}$$

$$P = P_b \tag{5c}$$

AT $z = \ell$: $M = 0$ (5d)

$$\frac{k}{k'} S = -P \frac{\partial u}{\partial z} - m_h \frac{\partial^2 u}{\partial t^2} \tag{5e}$$

$$P = -m_h \frac{\partial^2 \bar{w}}{\partial t^2} \tag{5f}$$

The initial conditions are:

$$AT \quad t=0 \quad u = u_0 \tag{6}$$

$$\frac{\partial u}{\partial t} = 0 \tag{6i}$$

$$\bar{w} = 0 \tag{6c}$$

$$\frac{\partial \bar{w}}{\partial t} = 0 \tag{6c}$$

$$\phi = \phi_0 \tag{6e}$$

$$\frac{\partial \phi}{\partial t} = 0 \tag{6f}$$

The complete derivation of the differential equations and boundry conditions was presented in Quarterly Progress Report No. 2, Contract NAS2-5062.[1]

3.2 Nondimensional Form of the Equations

The equations presented in section 3.1 were simplified and made applicable to a wide range of problems by nondimensionalizing according to the following relationships,

$$P = \frac{P'}{EA} \quad , \qquad t = \frac{t'C}{\ell} \quad ,$$

$$S = \frac{S'}{kAG} \quad , \qquad z = \frac{Z}{\ell} \quad ,$$

$$u_o = \frac{u_o'}{\ell} \quad , \qquad u = \frac{u'}{\ell} \quad ,$$

$$w = \frac{w'}{\ell} \quad ,$$

where the primed variables are in dimensional form and the unprimed variables are nondimensional. Using the preceding relationships in equations (1) through (4) results in the set of nondimensionalized equations presented below:

$$\frac{\partial P}{\partial z} - \frac{\partial^2 W}{\partial t^2} - \left(\frac{r}{\ell}\right) \frac{\partial^2 \phi}{\partial t^2} = 0 \, , \tag{7}$$

$$\frac{\partial^2 \phi}{\partial z^2} + \frac{1}{EI_1} \frac{\partial (EI_1)}{\partial z} \frac{\partial \phi}{\partial z} + \left(\frac{kG}{E}\right)\left(\frac{A\ell^2}{I_1}\right)\left(\frac{\partial u}{\partial z} - \frac{\partial u_o}{\partial z} - \phi\right) - \frac{I_2}{I_1} \frac{\partial^2 \phi}{\partial t^2}$$

$$- \left(\frac{r}{\ell}\right)\left(\frac{A\ell^2}{I_1}\right) \frac{\partial^2 W}{\partial t^2} = 0 \, , \tag{8}$$

$$\left(\frac{kG}{E} + P\right) \frac{\partial^2 u}{\partial z^2} + \left[\frac{1}{EA} \frac{\partial (kAG)}{\partial z} + \frac{P}{EA} \frac{\partial (EA)}{\partial z} + \frac{\partial P}{\partial z}\right] \frac{\partial u}{\partial z}$$

$$- \frac{\partial^2 u}{\partial t^2} - \frac{kG}{E} \frac{\partial \phi}{\partial z} - \frac{1}{EA} \frac{\partial (kAG)}{\partial z} \phi - \frac{kG}{E} \frac{\partial^2 u_o}{\partial z} - \frac{1}{EA} \frac{\partial (kAG)}{\partial z} \frac{\partial u_o}{\partial z} = 0 \, , \tag{9}$$

and

$$P = \frac{\partial W}{\partial z} + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial u_0}{\partial z}\right)^2 \tag{10}$$

The boundary conditions become

$$z = 0: \qquad M = 0 \tag{11a}$$

$$S_b = \frac{k}{k'} S + \left(\frac{E}{kG}\right) P \frac{\partial u}{\partial z} \tag{11b}$$

$$P = P_b \tag{11c}$$

$$z = l \qquad M = 0 \tag{11d}$$

$$\frac{k}{k'} S = \frac{E}{kG} P \frac{\partial u}{\partial z} - \frac{m_h c^2}{(kAG)l} \frac{\partial^2 u}{\partial t^2} \tag{11e}$$

$$P = -\frac{m_h c^2}{EAl} \frac{\partial^2 W}{\partial t^2} \tag{11f}$$

These equations were solved using finite difference techniques described in the next section.

3.3 Solution of the Differential Equations

The principal dependent variables of interest are u, w and ϕ. Equation (10) was therefore substituted into equation (7) and (9) which reduces the set of equations to be solved from four to three, as follows:

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} - \frac{\partial u_o}{\partial z}\frac{\partial^2 u_o}{\partial z^2} - \frac{\partial^2 w}{\partial t^2} - \left(\frac{r}{\ell}\right)\frac{\partial^2 \phi}{\partial t^2} = 0 , \tag{12}$$

$$\frac{\partial^2 \phi}{\partial z^2} + \frac{1}{EI_1}\frac{dEI_1}{dz}\frac{\partial \phi}{dz} + \left(\frac{kG}{E}\right)\left(\frac{A\ell^2}{I_1}\right)\frac{\partial u}{dz} - \left(\frac{kG}{E}\right)\left(\frac{A\ell^2}{I_1}\right)\frac{\partial u_o}{\partial z} - \left(\frac{kG}{E}\right)\left(\frac{A\ell^2}{I_1}\right)\phi$$

$$-\frac{I_2}{I_1}\frac{\partial^2 \phi}{\partial t^2} - \left(\frac{r}{\ell}\right)\left(\frac{A\ell^2}{I_1}\right)\frac{\partial^2 w}{\partial t^2} = 0 , \tag{13}$$

$$\left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial^2 u}{\partial z^2}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 u}{\partial z^2} - \frac{1}{2}\left(\frac{\partial u_o}{\partial z}\right)^2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{EA}\frac{d(EA)}{dz}\frac{\partial w}{\partial z}\frac{\partial u}{\partial z}$$

$$+\frac{1}{2}\frac{1}{EA}\frac{d(EA)}{dz}\left(\frac{\partial u}{\partial z}\right)^3 - \frac{1}{2}\frac{1}{EA}\frac{d(EA)}{dz}\left(\frac{\partial u_o}{\partial z}\right)^2 \frac{\partial u}{\partial z} + \frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z} + \left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 u}{\partial z^2}$$

$$-\frac{\partial u_o}{\partial z}\frac{\partial^2 u_o}{\partial z^2}\frac{\partial u}{\partial z} + \frac{1}{EA}\frac{d(kAG)}{dz}\frac{\partial u}{\partial z} - \frac{1}{EA}\frac{d(kAG)}{dz}\frac{\partial u_o}{\partial z} - \frac{1}{EA}\frac{d(kAG)}{dz}\phi$$

$$+\frac{kG}{E}\frac{\partial^2 u}{\partial z^2} - \frac{kG}{E}\frac{\partial^2 u_o}{\partial z^2} - \frac{kG}{E}\frac{\partial \phi}{\partial z} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad (14)$$

The equations were then written with terms combined for a particular variable or combination of variables. Coefficients are enclosed in brackets and variables in parentheses.

$$\left(\frac{\partial^2 w}{\partial z^2}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial^2 u}{\partial z^2}\right) - \left(\frac{\partial^2 w}{\partial t^2}\right) - \left[\frac{r}{\ell}\right]\left(\frac{\partial^2 \phi}{\partial t^2}\right) - \left[\frac{\partial u_o}{\partial z}\frac{\partial^2 u_o}{\partial z^2}\right] = 0, \qquad (15)$$

$$\left(\frac{\partial^2 \phi}{\partial z^2}\right) + \left[\frac{1}{EI_1}\frac{dEI_1}{dz}\right]\left(\frac{\partial \phi}{\partial z}\right) + \left[\frac{kG}{E}\frac{A\ell^2}{I_1}\right]\left(\frac{\partial u}{\partial z}\right) - \left[\frac{kG}{E}\frac{A\ell^2}{I_1}\right](\phi)$$

$$-\left[\frac{I_2}{I_1}\right]\left(\frac{\partial^3 \phi}{\partial t^3}\right) - \left[\frac{r}{\ell}\frac{A\ell^2}{I_1}\right]\left(\frac{\partial^2 w}{\partial t^2}\right) - \left[\frac{kG}{E}\frac{A\ell^2}{I_1}\frac{\partial u_o}{\partial z}\right] = 0, \qquad (16)$$

and

$$\left[\frac{kG}{E} - \frac{1}{2}\left(\frac{\partial u_o}{dz}\right)^2\right]\left(\frac{\partial^2 u}{\partial z^2}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial^2 u}{\partial z^2}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial^2 w}{\partial z^2}\right) + \left[\frac{1}{EA}\frac{dEA}{dz}\right]\left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial u}{\partial z}\right)$$

$$+\left[\frac{3}{2}\right]\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial^2 u}{\partial z^2}\right) + \left[\frac{1}{2}\frac{1}{EA}\frac{dEA}{dz}\right]\left(\frac{\partial u}{\partial z}\right)^3 + \left[\frac{1}{EA}\frac{dkAG}{dz} - \frac{1}{2}\frac{1}{EA}\frac{dEA}{dz}\left(\frac{\partial u_o}{\partial z}\right)^2\right.$$

$$\left.-\frac{\partial u_o}{\partial z}\frac{\partial^2 u_o}{\partial z^2}\right]\left(\frac{\partial u}{\partial z}\right) - \left[\frac{kG}{E}\right]\left(\frac{\partial \phi}{\partial z}\right) - \left[\frac{1}{EA}\frac{dkAG}{dz}\right](\phi) - \left(\frac{\partial^2 u}{\partial t^2}\right)$$

$$-\left[\frac{1}{EA}\frac{dkAG}{dz}\frac{\partial u_o}{\partial z} + \frac{kG}{E}\frac{\partial^2 u_o}{\partial z^2}\right] = 0. \qquad (17)$$

The boundary conditions become

$$z = 0: \quad \frac{\partial \phi}{\partial z} = 0, \tag{18a}$$

$$P_b \frac{\partial u}{\partial z} = \frac{kG}{E} \left(S_b - \frac{\partial u}{\partial z} + \frac{\partial u_o}{\partial z} + \phi \right), \tag{18b}$$

and

$$\frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial u_o}{\partial z}\right)^2 = P_b. \tag{18c}$$

$$z = 1: \quad \frac{\partial \phi}{\partial z} = 0, \tag{18d}$$

$$\frac{\partial^2 u}{\partial t^2} = -\left(\frac{EA\ell}{m_h c^2}\right)\left[\frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial u_o}{\partial z}\right)^2 + \left(\frac{kG}{E}\right)\left(\frac{\partial u}{\partial z}\right)\right.$$

$$\left. - \left(\frac{kG}{E}\right)\left(\frac{\partial u_o}{\partial z}\right) - \frac{kG}{E}\phi\right], \tag{18e}$$

and

$$\frac{\partial^2 w}{\partial t^2} = -\left(\frac{EA\ell}{m_h c^2}\right)\left[\left(\frac{\partial w}{\partial z}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial u_o}{\partial z}\right)^2\right] \tag{18f}$$

To simplify the form and to reduce the order of the original equations the following new variables were introduced:

$$V^{(1)} = \frac{\partial \phi}{\partial t} \qquad V^{(4)} = \frac{\partial \phi}{\partial z}$$

$$V^{(2)} = \frac{\partial w}{\partial t} \qquad V^{(5)} = \frac{\partial w}{\partial z} \tag{19}$$

$$V^{(3)} = \frac{\partial u}{\partial t} \qquad V^{(6)} = \frac{\partial u}{\partial z}$$

There are now a total of six variables. Originally, there were only three equations: Three new equations must therefore be derived. The three new equations are

$$\frac{\partial V^{(1)}}{\partial z} = \frac{\partial V^{(4)}}{\partial t} , \tag{20}$$

$$\frac{\partial V^{(2)}}{\partial z} = \frac{\partial V^{(5)}}{\partial t} , \tag{21}$$

and

$$\frac{\partial V^{(3)}}{\partial z} = \frac{\partial V^{(6)}}{\partial t} . \tag{22}$$

Using the variables defined in equations (19), the equations were rearranged such that equations 17, 16, and 15 become, respectively:

$$\left[\frac{kG}{E} - \frac{1}{2}\left(\frac{\partial u_o}{\partial z}\right)^2\right]\frac{\partial v^{(6)}}{\partial z} + \left(\frac{\partial v^{(6)}}{\partial z}\right) v^{(5)} + v^{(6)}\frac{\partial v^{(5)}}{\partial z} + \left[\frac{1}{EA}\frac{dEA}{dz}\right] v^{(6)} v^{(5)}$$

$$+\left[\frac{3}{2}\right]\left(v^{(6)}\right)^2\frac{\partial v^{(6)}}{\partial z} + \left[\frac{1}{2}\frac{1}{EA}\frac{dEA}{dz}\right]\left(v^{(6)}\right)^3 + \left[\frac{1}{EA}\frac{dkAG}{dz} - \frac{1}{2}\frac{1}{EA}\frac{dEA}{dz}\left(\frac{\partial u_o}{\partial z}\right)^2\right.$$

$$\left.-\left(\frac{\partial u_o}{\partial z}\right)\frac{\partial^2 u_o}{\partial z^2}\right] v^{(6)} - \left[\frac{kG}{E}\right] v^{(4)} - \left[\frac{1}{EA}\frac{dkAG}{dz}\right]\phi - \frac{\partial v^{(3)}}{\partial t}$$

$$-\left[\frac{1}{EA}\frac{dkAG}{dz}\frac{\partial u_o}{\partial z} + \frac{kG}{E}\frac{\partial^2 u_o}{\partial z^2}\right] = 0 , \qquad (23)$$

$$\frac{\partial v^{(4)}}{\partial z} + \left[\frac{1}{EI_1}\frac{dEI_1}{dz}\right] v^{(4)} + \left[\frac{kG}{E}\frac{A\ell^2}{I_1}\right] v^{(6)} - \left[\frac{kG}{E}\frac{A\ell^2}{I_1}\right]\phi$$

$$-\frac{I_2}{I_1}\frac{\partial v^{(1)}}{\partial t} - \left[\frac{r}{\ell}\frac{A\ell^2}{I_1}\right]\frac{\partial v^{(2)}}{\partial t} - \left[\frac{kG}{E}\frac{A\ell^2}{I_1}\frac{\partial u_o}{\partial z}\right] = 0 , \qquad (24)$$

and

$$\frac{\partial v^{(5)}}{\partial z} + v^{(6)}\frac{\partial v^{(6)}}{\partial z} - \frac{\partial v^{(2)}}{\partial t} - \left[\frac{r}{\ell}\right]\frac{\partial v^{(1)}}{\partial t} - \left[\frac{\partial u_o}{\partial z}\frac{\partial^2 u_o}{\partial z^2}\right] = 0 . \qquad (25)$$

The corresponding boundary conditions are

$$z = 0: \quad V^{(4)} = 0, \tag{26a}$$

$$\left[P_b + \frac{kG}{E}\right] V^{(6)} - \left[\frac{kG}{E}\right] \phi - \left[\frac{kG}{E}\left(S_b + \frac{\partial u_0}{\partial z}\right)\right] = 0, \tag{26b}$$

and

$$V^{(5)} + \frac{1}{2}\left(V^{(6)}\right)^2 - \left[\frac{1}{2}\left(\frac{\partial u_0}{\partial z}\right)^2 + P_b\right] = 0 \tag{26c}$$

$$z = 1: \quad V^{(4)} = 0 \tag{26d}$$

$$\frac{\partial V^{(3)}}{\partial t} + \left[\frac{EA\ell}{m_h c^2}\right] V^{(5)} + \left[\frac{1}{2}\frac{EA\ell}{m_h c^2}\right]\left(V^{(6)}\right)^2 + \left[\frac{EA\ell}{m_h c^2}\frac{kG}{E}\right] V^{(6)}$$

$$- \left[\frac{EA\ell}{m_h c^2}\frac{kG}{E}\right]\phi - \left[\frac{EA\ell}{m_h c^2}\left(\frac{1}{2}\right)\left(\frac{\partial u_0}{\partial z}\right)^2 + \frac{EA\ell}{m_h c^2}\frac{kG}{E}\frac{\partial u_0}{\partial z}\right] = 0, \tag{26e}$$

and

$$\frac{\partial V^{(2)}}{\partial t} + \left[\frac{EA\ell}{m_h c^2}\right] V^{(6)} + \left[\frac{1}{2}\frac{EA\ell}{m_h c^2}\right]\left(V^{(6)}\right)^2 - \left[\frac{1}{2}\frac{EA\ell}{m_h c^2}\left(\frac{\partial u_0}{\partial z}\right)^2\right] = 0 \tag{26f}$$

At

$$t = 0 \qquad V^{(1)} = 0 \tag{27a}$$

$$V^{(2)} = 0 \tag{27b}$$

$$V^{(3)} = 0 \qquad (27c)$$

$$V^{(4)} = \frac{\partial \phi_0}{\partial z} \qquad (27d)$$

$$V^{(5)} = 0 \qquad (27e)$$

$$V^{(6)} = \frac{\partial u_0}{\partial z} \qquad (27f)$$

Before proceeding, it is useful to examine the form of the boundary conditions and to determine where each variable appears.

At $z = 0$:

from (26a) $V^{(4)}$ is known

from (26b) $V^{(6)}$ is given in terms of ϕ, and hence in terms of $V^{(1)}$ (refer to equation 19).

from (26c) $V^{(5)}$ may be expressed in terms of $V^{(6)}$ and hence in terms of $V^{(1)}$, from above

At $z = 1$:

from (26d) $V^{(4)}$ is known and from equation (20) we may determine $V^{(1)}$ at $z = 1$.

from (26e) $V^{(3)}$ may be expressed in terms of $V^{(5)}$, $V^{(6)}$ and ϕ and hence in terms of $V^{(5)}$, $V^{(6)}$ and $V^{(1)}$

from (26f) $V^{(2)}$ is given in terms of $V^{(5)}$ and $V^{(6)}$

Thus, in effect, $V^{(4)}$, $V^{(5)}$ and $V^{(6)}$ are known at $z = 0$ and $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$ are known at $z = 1$

Because the preceding is the form of the boundary conditions, the following numbering system will be used

At $z = 0$ $\quad i = 1$ for $V^{(4)}$, $V^{(5)}$ and $V^{(6)}$

$\quad i = 2$ for $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$.

Since ϕ, w and u are computed from $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$, they will follow the same numbering system.

A general equation with this numbering system will therefore contain levels i - 1 and i of $V^{(4)}$, $V^{(5)}$ and $V^{(6)}$ and levels i and i + 1 of $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$.

In the approach to the solution, an inspection of the six equations to be solved [equations (20 - 25) and boundary conditions (26a - f),] reveals that all non-linearities are directly attributable to the term $V^{(6)}$. This, therefore, is the only variable that needs to be used at the value of the

previous iterate.

Next, let $F = \phi$,

Y = the value of $V^{(6)}$ at the previous iterate

and

X = value of $V^{(6)}$ at $t = 0$.

Furthermore, if we call LM the total number of increments, then $LM + 1 = L$ is the total number of points, and $L + 1 = LP$, by definition.

If we consider $V^{(4)}$, $V^{(5)}$ and $V^{(6)}$ to be the Group 1 variables and $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, u, w, and F to be the Group 2 variables, then the number system to be used in deriving the finite difference equations is as shown in Figure 6 on the following page. The spatial variables are denoted by the subscript, i, whereas the temporal variables are denoted by the subscript, n .

If we let $\Delta t = \Delta z$ and center the equations at the midpoint as shown in Figure 6, then iet us write the finite difference equations for the general index where $3 \leq i \leq LM$.

Equation (20) becomes

$$\frac{1}{2}\left[\frac{V^{(1)}_{i+1,n+1} - V^{(1)}_{i,n+1}}{\Delta z} + \frac{V^{(1)}_{i+1,n} - V^{(1)}_{i,n}}{\Delta z}\right] =$$

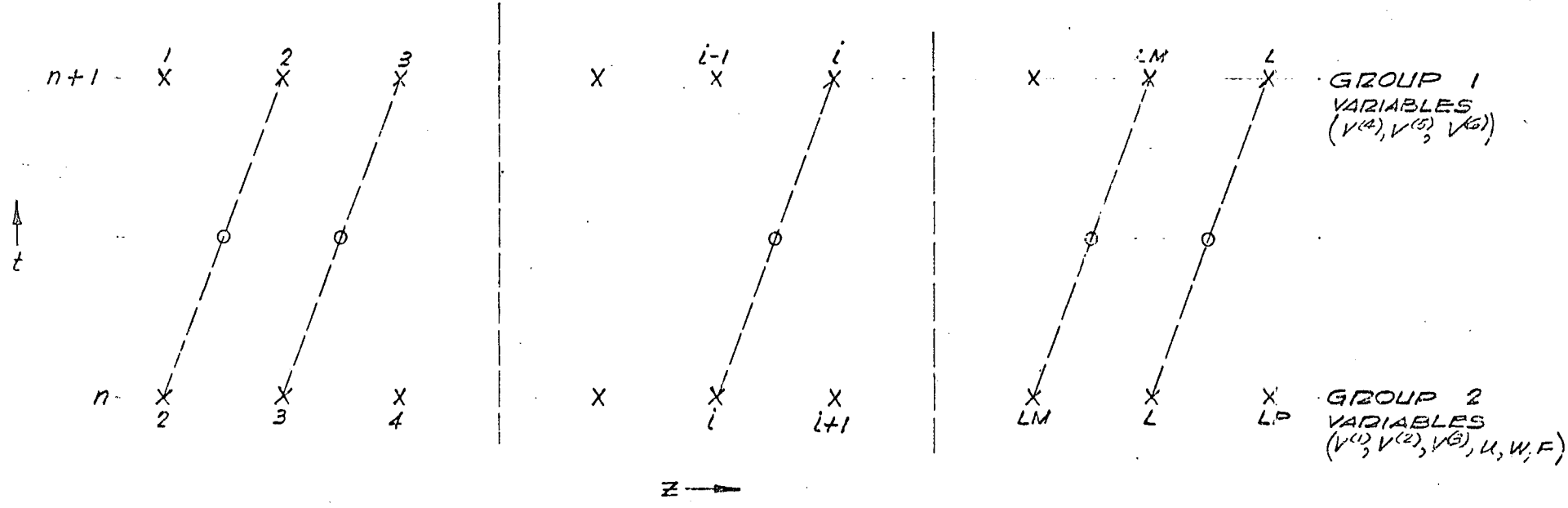

Figure 6. Numbering system for the finite difference equations.

$$\frac{1}{2}\left[\frac{V^{(4)}_{i,n+1} - V^{(4)}_{i,n}}{\Delta Z} + \frac{V^{(4)}_{i-1,n+1} - V^{(4)}_{i-1,n}}{\Delta Z}\right]$$

Now, since $\Delta Z = \Delta t$, this expression becomes

$$-V^{(1)}_{i,n+1} + V^{(1)}_{i+1,n+1} - V^{(4)}_{i-1,n+1} - V^{(4)}_{i,n+1} = -V^{(1)}_{i,n} - V^{(1)}_{i+1,n} - V^{(4)}_{i-1,n} - V^{(4)}_{i,n}$$

Changing signs and reordering the equation results in

$$V^{(4)}_{i-1,n+1} + V^{(1)}_{i,n+1} + V^{(4)}_{i,n+1} - V^{(1)}_{i+1,n+1} = V^{(1)}_{i,n} + V^{(1)}_{i+1,n} + V^{(4)}_{i-1,n} + V^{(4)}_{i,n} \qquad (28$$

Similarly, we have, for equations (21) and (22)

$$V^{(5)}_{i-1,n+1} + V^{(2)}_{i,n+1} + V^{(5)}_{i,n+1} - V^{(2)}_{i+1,n+1} = -V^{(2)}_{i,n} + V^{(2)}_{i+1,n} + V^{(5)}_{i-1,n} + V^{(5)}_{i,n} \qquad (29$$

and

$$V^{(6)}_{i-1,n+1} + V^{(3)}_{i,n+1} + V^{(6)}_{i,n+1} - V^{(3)}_{i+1,n+1} = -V^{(3)}_{i,n} + V^{(3)}_{i+1,n} + V^{(6)}_{i-1,n} + V^{(6)}_{i,n} \qquad (30$$

Now, for equation (25)

$$\frac{1}{2}\left[\frac{V^{(5)}_{i,n+1} - V^{(5)}_{i-1,n+1}}{\Delta Z} + \frac{V^{(5)}_{i,n} - V^{(5)}_{i-1,n}}{\Delta Z}\right] + \left[\frac{1}{4}\left(Y_i + Y_{i-1} + V^{(6)}_{i,n} + V^{(6)}_{i-1,n}\right)\right]$$

$$\left[\frac{1}{2}\left(\frac{V^{(6)}_{i,n+1} - V^{(6)}_{i-1,n+1}}{\Delta Z} + \frac{V^{(6)}_{i,n} - V^{(6)}_{i-1,n}}{\Delta Z}\right)\right] - \frac{1}{2}\left[\frac{V^{(2)}_{i+1,n+1} - V^{(2)}_{i+1,n}}{\Delta t} + \frac{V^{(2)}_{i,n+1} - V^{(2)}_{i,n}}{\Delta t}\right]$$

$$-\left[\frac{r}{\ell}\right]\frac{1}{2}\left[\frac{V^{(1)}_{i+1,n+1} - V^{(1)}_{i+1,n}}{\Delta t} + \frac{V^{(1)}_{i,n+1} - V^{(1)}_{i,n}}{\Delta t}\right] - \left[\frac{1}{2}\left(X_i + X_{i-1}\right)\left(\frac{X_i - X_{i-1}}{\Delta Z}\right)\right] = 0$$

Note, on the non-linear term, $\left(V^{(6)}_{i,n+1} + V^{(6)}_{i-1,n+1}\right)$ is used in place of $\left(Y_i + Y_{i-1}\right)$ as the coefficient of $\frac{1}{\Delta z}\left(V^{(6)}_{i,n} - V^{(6)}_{i-1,n}\right)$. The new iterate of $V^{(6)}_{i,n+1}$ is thus introduced more often, yet the equation remains linear. Thus, the term becomes.

$$\frac{1}{4}\left(Y_i + Y_{i-1} + V^{(6)}_{i,n} + V^{(6)}_{i-1,n}\right)\left(V^{(6)}_{i,n+1} - V^{(6)}_{i-1,n+1}\right) + \frac{1}{4}\left(V^{(6)}_{i,n+1} + V^{(6)}_{i-1,n+1} + V^{(6)}_{i,n} + V^{(6)}_{i-1,n}\right)\left(V^{(6)}_{i,n} - V^{(6)}_{i-1,n}\right).$$

Then,

$$-V^{(5)}_{i-1,n+1} + \left[-\frac{1}{4}\left(Y_i + Y_{i-1} + V^{(6)}_{i,n} + V^{(6)}_{i-1,n}\right) + \frac{1}{4}\left(V^{(6)}_{i,n} - V^{(6)}_{i-1,n}\right)\right] V^{(6)}_{i-1,n+1} - \left[\frac{r}{\ell}\right]$$

$$V^{(1)}_{i,n+1} - V^{(2)}_{i,n+1} + V^{(5)}_{i,n+1} + \left[\frac{1}{4}\left(Y_i + Y_{i+1} + V^{(6)}_{i,n} + V^{(6)}_{i-1,n}\right) + \frac{1}{4}\left(V^{(6)}_{i,n} - V^{(6)}_{i-1,n}\right)\right]$$

$$V^{(6)}_{i,n+1} - \left[\frac{r}{\ell}\right] V^{(1)}_{i+1,n+1} - V^{(2)}_{i+1,n+1} = -\left(V^{(5)}_{i,n} - V^{(5)}_{i-1,n}\right) - \frac{1}{4}\left(V^{(6)}_{i,n} + V^{(6)}_{i-1,n}\right)$$

$$\left(V^{(6)}_{i,n} - V^{(6)}_{i-1,n}\right) + \left(-V^{(2)}_{i+1,n} - V^{(2)}_{i,n}\right) + \frac{r}{\ell}$$

$$\left(-V^{(1)}_{i+1,n} - V^{(1)}_{i,n}\right) + \left(X_i + X_{i-1}\right)\left(X_i - X_{i-1}\right).$$

Now, change signs and combine the terms.

$$V^{(5)}_{i-1,n+1} + \left[\frac{1}{4}\left(Y_i + Y_{i-1} + 2V^{(6)}_{i-1,n}\right)\right] V^{(6)}_{i-1,n+1} + \left[\left(\frac{r}{\ell}\right)_i\right] V^{(1)}_{i,n+1} + V^{(2)}_{i,n+1}$$

$$-V^{(5)}_{i,n+1} - \left[\frac{1}{4}\left(Y_i + Y_{i+1} + 2V^{(6)}_{i,n}\right)\right] V^{(6)}_{i,n+1} + \left[\left(\frac{r}{\ell}\right)_i\right] V^{(1)}_{i+1,n+1} + V^{(2)}_{i+1,n+1} =$$

$$\left(V_{i,n}^{(5)} - V_{i-1,n}^{(5)}\right) + \frac{1}{4}\left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(6)} - V_{i-1,n}^{(6)}\right) + \left(V_{i+1,n}^{(2)} + V_{i,n}^{(2)}\right)$$

$$+ \frac{r}{\ell}\left(V_{i+1,n}^{(1)} + V_{i,n}^{(1)}\right) - \left(X_i + X_{i-1}\right)\left(X_i - X_{i-1}\right) \qquad (31$$

Now, for equation (24) which is linear. Note, though, that

$$\phi_{i,n+1} = \left(\phi_{i,n} + \frac{\Delta t}{2} V_{i,n}^{(1)}\right) + \frac{\Delta t}{2} V_{i,n+1}^{(1)}$$

Similarly,

$$\phi_{i+1,n+1} = \left(\phi_{i+1} + \frac{\Delta t}{2} V_{i+1,n}^{(1)}\right) + \frac{\Delta t}{2} V_{i+1,n+1}^{(1)}$$

Thus,

$$\phi_{i+1/2,n+1/2} \approx \frac{1}{4}\left[\phi_{i,n} + \phi_{i+1,n} + \phi_{i,n+1} + \phi_{i+1,n+1}\right]$$

$$= \frac{1}{4}\left[2\left(\phi_{i,n} + \phi_{i+1,n}\right) + \frac{\Delta t}{2}\left(V_{i,n}^{(1)} + V_{i+1,n}^{(1)}\right) + \frac{\Delta t}{2}\left(V_{i,n+1}^{(1)} + V_{i+1,n+1}^{(1)}\right)\right]$$

$$\phi_{2,n+1} = \phi_{3,n} + \frac{\Delta t}{2}\left(V_{3n}^{(1)} + V_{2,n+1}^{(1)}\right)$$

$$\phi_{LP,n+1} = \phi_{LP,n} + \frac{\Delta t}{2}\left(V_{LP,n}^{(1)} + V_{LP,n+1}^{(1)}\right)$$

Then,

$$\frac{1}{2}\left[\frac{V_{i,n+1}^{(4)} - V_{i-1,n+1}^{(4)}}{\Delta Z} + \frac{V_{i,n}^{(4)} - V_{i-1,n}^{(4)}}{\Delta Z}\right] + \left[\left(\frac{1}{EI}\right)_i \left(\frac{dEI}{dZ}\right)_i\right] \frac{1}{4}\left(V_{i,n+1}^{(4)} + V_{i-1,n+1}^{(4)} + V_{i,n}^{(4)}\right.$$

$$\left. + V_{i-1,n}^{(4)}\right) + \left[\left(\frac{kG}{E}\right)_i \left(\frac{A\ell^2}{I}\right)_i\right] \frac{1}{4}\left(V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)$$

$$-\left[\left(\frac{kG}{E}\right)_i\left(\frac{A\ell^2}{I_1}\right)_i\right]\frac{1}{4}\left[2\left(\phi_{i,n}+\phi_{i+1,n}\right)+\frac{\Delta t}{2}\left(V_{i,n}^{(1)}+V_{i+1,n}^{(1)}\right)\right.$$

$$\left.+\frac{\Delta t}{2}\left(V_{i,n+1}^{(1)}+V_{i+1,n+1}^{(1)}\right)\right]-\left(\frac{I_2}{I_1}\right)\frac{1}{2}\left[\frac{V_{i+1,n+1}^{(1)}-V_{i+1,n}^{(1)}}{\Delta t}+\frac{V_{i,n+1}^{(1)}-V_{i,n}^{(1)}}{\Delta t}\right]$$

$$-\left[\left(\frac{r}{\ell}\right)_i\left(\frac{A\ell^2}{I_1}\right)_i\right]\frac{1}{2}\left[\frac{V_{i+1,n+1}^{(2)}-V_{i+1,n}^{(2)}}{\Delta t}+\frac{V_{i,n+1}^{(2)}-V_{i,n}}{\Delta t}\right]-\left[\left(\frac{kG}{E}\right)_i\right.$$

$$\left.\left(\frac{A\ell^2}{I_1}\right)_i\frac{1}{2}\left(X_i+X_{i-1}\right)\right] \qquad = 0$$

Combining terms yields,

$$\left[-1+\frac{\Delta z}{2}\left(\frac{1}{EI}\right)_i\left(\frac{dEI}{dz}\right)_i\right]V_{i-1,n+1}^{(4)}+\left[\frac{\Delta z}{2}\left(\frac{kG}{E}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]V_{i-1,n+1}^{(6)}$$

$$+\left[-\frac{I_2}{I_1}-\frac{(\Delta z)^2}{4}\left(\frac{kG}{E}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]V_{i,n+1}^{(1)}+\left[-\left(\frac{r}{\ell}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]V_{i,n+1}^{(2)}$$

$$+\left[1+\frac{\Delta z}{2}\left(\frac{1}{EI}\right)_i\left(\frac{dEI}{dz}\right)_i\right]V_{i,n+1}^{(4)}+\left[\frac{\Delta z}{2}\left(\frac{kG}{E}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]V_{i,n+1}^{(6)}$$

$$+\left[-\frac{I_2}{I_1}-\frac{(\Delta z)^2}{4}\left(\frac{kG}{E}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]V_{i+1,n+1}^{(1)}+\left[-\left(\frac{r}{\ell}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]V_{i+1,n+1}^{(2)}$$

$$=\left[1-\frac{\Delta z}{2}\left(\frac{1}{EI}\right)_i\left(\frac{dEI}{dz}\right)_i\right]V_{i-1,n}^{(4)}+\left[-1-\frac{\Delta z}{2}\left(\frac{1}{EI}\right)_i\left(\frac{dEI}{dz}\right)_i\right]$$

$$V_{i,n}^{(4)}-\left[\frac{\Delta z}{2}\left(\frac{kG}{E}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]\left[V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right]+\left[-\frac{I_2}{I_1}+\frac{(\Delta z)^2}{4}\left(\frac{kG}{E}\right)_i\right.$$

$$\left.\left(\frac{A\ell^2}{I}\right)_i\right]\left[V_{i,n}^{(1)}+V_{i+1,n}^{(1)}\right]+\left[\Delta z\left(\frac{kG}{E}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]\left[\phi_{i,n}+\phi_{i+1,n}\right]$$

$$-\left[\left(\frac{r}{\ell}\right)_i\left(\frac{A\ell^2}{I}\right)_i\right]\left[V_{i,n}^{(2)}+V_{i+1,n}^{(2)}\right]+\left[\Delta Z\left(\frac{KG}{E}\right)_i\left(\frac{A\ell^2}{I}\right)_i\left(X_i+X_{i+1}\right)\right] \tag{32}$$

Now, in considering Equation (23), examine the non-linear terms first.

$$\left[\left(\frac{\partial V^{(6)}}{\partial Z}\right)\cdot V^{(5)}\right]_{i-1/2,n+1/2} \approx \frac{1}{2}\left[\frac{V_{i,n+1}^{(6)}-V_{i-1,n+1}^{(6)}}{\Delta Z}+\frac{V_{i,n}^{(6)}-V_{i-1,n}^{(6)}}{\Delta Z}\right]$$

$$\left[\frac{1}{4}\left(V_{i,n+1}^{(5)}+V_{i,n}^{(5)}+V_{i-1,n+1}^{(5)}+V_{i-1,n+1}\right)\right] \approx \frac{1}{8}\frac{1}{\Delta Z}\left[\left(Y_i-Y_{i-1}\right.\right.$$

$$\left.+V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\left(V_{i,n+1}^{(5)}+V_{i-1,n+1}^{(5)}\right)+\left(V_{i,n}^{(5)}+V_{i-1,n}^{(5)}\right)\left(V_{i,n+1}^{(6)}-V_{i-1,n+1}^{(6)}\right)$$

$$\left.+\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\left(V_{i-1,n}^{(5)}+V_{i,n}^{(5)}\right)\right].$$

$$\left[V^{(6)}\frac{\partial V^{(5)}}{\partial Z}\right]_{i-1/2,n+1/2} \approx \frac{1}{4}\left[V_{i,n+1}^{(6)}+V_{i-1,n+1}^{(6)}+V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right]$$

$$\frac{1}{2}\left[\frac{V_{i,n+1}^{(5)}-V_{i-1,n+1}^{(5)}}{\Delta Z}+\frac{V_{i,n}^{(5)}-V_{i-1,n}^{(5)}}{\Delta Z}\right] \approx \frac{1}{8}\frac{1}{\Delta Z}\left[\left(Y_i+Y_{i-1}+V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)\right.$$

$$\left(V_{i,n+1}^{(5)}-V_{i-1,n+1}^{(5)}\right)+\left(V_{i,n}^{(5)}-V_{i-1,n}^{(5)}\right)\left(V_{i,n+1}^{(6)}+V_{i-1,n+1}^{(6)}\right)+\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)\left.\left(V_{i,n}^{(5)}-V_{i-1,n}^{(5)}\right)\right].$$

$$\left[\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dZ}\right)_i\right]\left[V^{(6)}V^{(5)}\right]_{i-1/2,n+1/2} \approx \frac{1}{4}\left[V_{i,n+1}^{(6)}+V_{i-1,n+1}^{(6)}+V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right]$$

$$\frac{1}{4}\left[V_{i,n+1}^{(5)}+V_{i-1,n+1}^{(5)}+V_{i,n}^{(5)}+V_{i-1,n}^{(5)}\right] \qquad \left[\left(\frac{1}{EA}\right)_i\left[\left(\frac{dEA}{dZ}\right)_i\right]\right] \approx \left[\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dZ}\right)_i\right]$$

$$\frac{1}{16}\left[\left(Y_i + Y_{i-1} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)\left(V_{i,n+1}^{(5)} + V_{i-1,n+1}^{(5)}\right) + \left(V_{i,n}^{(5)} + V_{i-1,n}^{(5)}\right)\right.$$

$$\left.\left(V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)}\right) + \left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(5)} + V_{i-1,n}^{(5)}\right)\right] .$$

$$\left[\left(V^{(6)}\right)^2\left(\frac{\partial V^{(6)}}{\partial Z}\right)\right]_{i-1/2,\,n+1/2} \approx \frac{1}{4}\left[V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right]$$

$$\frac{1}{4}\left[V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right]\frac{1}{2}\left[\frac{V_{i,n+1}^{(6)} - V_{i-1,n+1}^{(6)}}{\Delta Z} + \frac{V_{i,n}^{(6)} - V_{i-1,n}^{(6)}}{\Delta Z}\right]$$

$$\left[\left(V^{(6)}\right)^2\left(\frac{\partial V^{(6)}}{\partial Z}\right)\right]_{i-1/2,\,n+1/2} \approx \frac{1}{32}\,\frac{1}{\Delta Z}\left[\left(Y_i + Y_{i-1} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)^2\right.$$

$$\left(V_{i,n+1}^{(6)} - V_{i-1,n+1}^{(6)}\right) + 2\left(V_{i,n}^{(6)} - V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)\left(V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)}\right)$$

$$\left.+\left(V_{i,n}^{(6)} - V_{i-1,n}^{(6)}\right)\cdot\left(Y_i + Y_{i-1}\right)\left(V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)}\right)^2 + \left(V_{i,n}^{(6)} - V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)^2\right] .$$

$$\left[\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dZ}\right)_i\right]\left(V^{(6)}\right)^3_{i-1/2,\,n+1/2} \approx \frac{1}{4}\left[V_{i,n+1}^{(6)} + V_{i,n}^{(6)} + V_{i-1,n+1}^{(6)} + V_{i-1,n}\right]$$

$$\frac{1}{4}\left[V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right]\cdot\frac{1}{4}\left[V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right]$$

$$\approx \frac{1}{64}\left[\left(Y_i + Y_{i-1} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)^2\left(V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)}\right) + 2\left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)^2\right.$$

$$\left.\left(V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)}\right) + \left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)\left(Y_i + Y_{i-1}\right)\left(V_{i,n+1}^{(6)} + V_{i-1,n+1}^{(6)}\right) + \left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)^3\right]$$

$$\approx \frac{1}{64}\left[\left(Y_i+Y_{i-1}\right)^2\left(V^{(6)}_{i,n+1}+V^{(6)}_{i-1,n+1}\right)+3\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)\left(Y_i+Y_{i-1}\right)\right.$$

$$\left.\left(V^{(6)}_{i,n+1}+V^{(6)}_{i-1,n+1}\right)+3\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)^2\left(V^{(6)}_{i,n+1}+V^{(6)}_{i-1,n+1}\right)+\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)^3\right]\approx$$

$$\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dZ}\right)_i\frac{1}{64}\left\{\left[\left(Y_i+Y_{i-1}\right)^2+3\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)\left(Y_i+Y_{i+1}\right)+3\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)^2\right]\right.$$

$$\left.\left(V^{(6)}_{i,n+1}+V^{(6)}_{i-1,n+1}\right)+\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)^3\right\}.$$

Simplifying

$$\left[\left(V^{(6)}\right)^2\frac{\partial V^{(6)}}{\partial Z}\right]_{i-1/2,n+1/2}\approx\frac{1}{32}\frac{1}{\Delta Z}\left\{\left[\left(Y_i+Y_{i-1}\right)^2+2\left(Y_i+Y_{i-1}\right)\right.\right.$$

$$\left.\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)+\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)^2\right]\cdot\left(V^{(6)}_{i,n+1}-V^{(6)}_{i-1,n+1}\right)+\left[2\left(V^{(6)}_{i,n}-V^{(6)}_{i-1,n}\right)\right.$$

$$\left.\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)+\left(V^{(6)}_{i,n}-V^{(6)}_{i-1,n}\right)\left(Y_i+Y_{i-1}\right)\right]\left(V^{(6)}_{i,n+1}+V^{(6)}_{i-1,n+1}\right)+\left(V^{(6)}_{i,n}-V^{(6)}_{i-1,n}\right)$$

$$\left.\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)^2\right\},$$

$$\left[\left(V^{(6)}\right)^2\frac{\partial V^{(6)}}{\partial Z}\right]_{i-1/2,n+1/2}\approx\frac{1}{32}\frac{1}{\Delta Z}\left\{\left[\left(Y_i+Y_{i-1}\right)^2+2\left(Y_i+Y_{i-1}\right)\right.\right.$$

$$\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)+\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)^2+2\left(V^{(6)}_{i,n}-V^{(6)}_{i-1,n}\right)\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)$$

$$\left.+\left(V^{(6)}_{i,n}-V^{(6)}_{i-1,n}\right)\left(Y_i+Y_{i-1}\right)\right]V^{(6)}_{i,n+1}+\left[-\left(Y_i+Y_{i-1}\right)^2-2\left(Y_i+Y_{i-1}\right)\left(V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right)\right.$$

$$-\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)^2+2\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)+\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)$$

$$\left(Y_i+Y_{i-1}\right)\Big]\ V_{i-1,n+1}^{(6)}+\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)^2\Big\}$$

$$\frac{1}{32}\ \frac{1}{\Delta Z}\Big\{\Big[\left(Y_i+Y_{i-1}\right)^2+\left(Y_i+Y_{i-1}\right)\left(2V_{i,n}^{(6)}+2V_{i-1,n}^{(6)}+V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)$$

$$+\left(V_{i,n}^{(6)2}+2V_{i,n}^{(6)}V_{i-1,n}^{(6)}+V_{i-1,n}^{(6)2}+2V_{i,n}^{(6)2}-2V_{i-1,n}^{(6)2}\right)\Big]\ V_{i,n+1}^{(6)}+\Big[-\left(Y_i+Y_{i-1}\right)^2$$

$$+\left(Y_i+Y_{i-1}\right)\left(-2V_{i,n}^{(6)}-2V_{i-1,n}^{(6)}+V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)+\left(-V_{i,n}^{(6)2}-2V_{i,n}^{(6)}V_{i-1,n}^{(6)}\right.$$

$$\left.-V_{i-1,n}^{(6)2}+2V_{i,n}^{(6)2}-2V_{i-1,n}^{(6)2}\right)\Big]\ V_{i-1,n+1}^{(6)}+\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)^2\Big\}$$

$$=\left[\frac{3}{2}\right]\ \frac{1}{32}\ \frac{1}{\Delta Z}\Big\{\Big[\left(Y_i+Y_{i-1}\right)^2+\left(Y_i+Y_{i-1}\right)\left(3V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)+V_{i,n}^{(6)}\left(3V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)$$

$$+V_{i-1,n}^{(6)}\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\Big]\ V_{i,n+1}^{(6)}+\Big[-\left(Y_i+Y_{i-1}\right)^2-\left(Y_i+Y_{i-1}\right)\left(V_{i,n}^{(6)}+3V_{i-1,n}^{(6)}\right)$$

$$+V_{i,n}^{(6)}\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)-V_{i-1,n}^{(6)}\left(V_{i,n}+3V_{i-1,n}^{(6)}\right)\Big]\ V_{i-1,n+1}^{(6)}+\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)$$

$$\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)^2\Big\}.$$

$$\left[\frac{kG}{E}-\frac{1}{2}\left(\frac{\partial\psi_6}{\partial Z}\right)^2\right]\left(\frac{\partial V^{(6)}}{\partial Z}\right)\approx\frac{1}{2}\left[\frac{kG}{E}-\frac{1}{2}\left(\frac{X_i+X_{i-1}}{2}\right)^2\right]\left[\frac{V_{i,n+1}^{(6)}-V_{i-1,n+1}^{(6)}}{\Delta Z}+\frac{V_{i,n}^{(6)}-V_{i-1,n}^{(6)}}{\Delta Z}\right].$$

$$\left[\frac{1}{EA}\frac{dkAG}{dz}-\frac{1}{2}\frac{1}{EA}\frac{dEA}{dz}\left(\frac{\partial u_o}{\partial z}\right)^2-\left(\frac{\partial u_o}{\partial z}\right)\left(\frac{\partial^2 u_o}{\partial z^2}\right)\right]V^{(6)} \approx \left[\left(\frac{1}{EA}\right)_i\right.$$

$$\left.\left(\frac{dkAG}{dz}\right)_i-\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\left(\frac{X_i+X_{i-1}}{2}\right)^2-\left(\frac{X_i+X_{i-1}}{2}\right)\left(\frac{X_i-X_{i-1}}{\Delta z}\right)\right]$$

$$\frac{1}{4}\left(V^{(6)}_{i,n+1}+V^{(6)}_{i-1,n+1}+V^{(6)}_{i,n}+V^{(6)}_{i-1,n}\right).$$

$$-\left[\frac{kG}{E}\right]V^{(4)} \approx -\left[\left(\frac{kG}{E}\right)_i\right]\frac{1}{4}\left(V^{(4)}_{i,n+1}+V^{(4)}_{i-1,n+1}+V^{(4)}_{i,n}+V^{(4)}_{i-1,n}\right)$$

$$\left[\frac{1}{EA}\frac{dkAG}{dz}\right]\phi \approx -\left[\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i\right]\left[\frac{1}{2}\left(\phi_{i,n}+\phi_{i+1,n}\right)\right.$$

$$\left.+\frac{\Delta t}{8}\left(V^{(1)}_{i,n}+V^{(1)}_{i+1,n}\right)+\frac{\Delta t}{8}\left(V^{(4)}_{i,n+1}+V^{(4)}_{i+1,n+1}\right)\right]$$

$$-\frac{\partial V^{(3)}}{\partial t} \approx -\frac{1}{2}\left[\frac{V^{(3)}_{i,n+1}-V^{(3)}_{i,n}}{\Delta t}+\frac{V^{(3)}_{i+1,n+1}-V^{(3)}_{i+1,n}}{\Delta t}\right].$$

$$-\left[\frac{1}{EA}\frac{dkAG}{dz}\frac{\partial u_o}{\partial z}+\frac{kG}{E}\frac{\partial^2 u_o}{\partial z^2}\right] \approx -\left[\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i\right.$$

$$\left.\left(\frac{X_i+X_{i-1}}{2}\right)+\left(\frac{kG}{E}\right)_i\left(\frac{X_i-X_{i-1}}{\Delta z}\right)\right]$$

Now, to get equation (23) in finite difference form, combine these terms.

Equation (23) then becomes

$$-\left[\frac{1}{4}\left(\frac{kG}{E}\right)_i\right] V_{i-1,n+1}^{(4)} + \left\{\frac{1}{8(\Delta z)}\left(Y_i - Y_{i-1} + V_{i,n}^{(6)} - V_{i-1,n}^{(6)}\right) - \frac{1}{8(\Delta z)}\right.$$

$$\left.\left(Y_i + Y_{i-1} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right) + \frac{1}{16}\left[\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]\left(Y_i + Y_{i-1} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)\right\}$$

$$V_{i-1,n+1}^{(5)} + \left\{-\frac{1}{8\Delta z}\left(V_{i,n}^{(5)} + V_{i-1,n}^{(5)}\right) + \frac{1}{8\Delta z}\left(V_{i,n}^{(5)} - V_{i-1,n}^{(5)}\right) + \left[\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]\right.$$

$$\frac{1}{16}\left(V_{i,n}^{(5)} + V_{i-1,n}^{(5)}\right) + \frac{1}{64}\left[\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]\left[\left(Y_i + Y_{i-1}\right)^2 + 3\left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)\right.$$

$$\left.\left(Y_i + Y_{i+1}\right) + 3\left(V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right)^2\right] + \frac{3}{64\Delta z}\left[-\left(Y_i + Y_{i-1}\right)^2 - \left(Y_i + Y_{i-1}\right)\right.$$

$$\left.\left(V_{i,n}^{(6)} + 3V_{i-1,n}^{(6)}\right) + V_{i,n}^{(6)}\left(V_{i,n}^{(6)} - V_{i-1,n}^{(6)}\right) - V_{i-1,n}^{(6)}\left(V_{i,n}^{(6)} + 3V_{i-1,n}^{(6)}\right)\right] - \frac{1}{2\Delta z}$$

$$\left[\left(\frac{kG}{E}\right)_i - \frac{1}{2}\left(\frac{X_i + X_{i-1}}{2}\right)^2\right] + \frac{1}{4}\left[\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i - \frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\left(\frac{X_i + X_{i-1}}{2}\right)^2\right.$$

$$\left.\left.-\left(\frac{X_i + X_{i-1}}{2}\right)\left(\frac{X_i - X_{i-1}}{\Delta z}\right)\right]\right\} V_{i-1,n+1}^{(6)} + \left[-\frac{\Delta t}{8}\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i\right] V_{i,n+1}^{(1)} + \left[0\right] V_{i,n+1}^{(2)}$$

$$+\left[-\frac{1}{2\Delta t}\right] V_{i,n+1}^{(3)} + \left[-\frac{1}{4}\left(\frac{kG}{E}\right)_i\right] V_{i,n+1}^{(4)} + \left\{\frac{1}{8\Delta z}\left(Y_i - Y_{i-1} + V_{i,n}^{(6)} - V_{i-1,n}^{(6)}\right)\right.$$

$$+\frac{1}{8\Delta z}\left(Y_i + Y_{i-1} + V_{i,n}^{(6)} + V_{i-1,n}^{(6)}\right) + \frac{1}{16}\left[\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]\left(Y_i + Y_{i-1} + V_{i,n}^{(6)}\right.$$

$$\left.\left.+ V_{i-1,n}^{(6)}\right)\right\} V_{i,n+1}^{(5)}$$

$$+\left\{\frac{1}{8\Delta z}\left(V_{i,n}^{(5)}+V_{i-1,n}^{(5)}\right)+\frac{1}{8\Delta z}\left(V_{i,n}^{(5)}-V_{i-1,n}^{(5)}\right)+\frac{1}{16}\left[\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]\right.$$

$$\left(V_{i,n}^{(5)}+V_{i-1,n}^{(5)}\right)+\frac{1}{64}\left[\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]\left[\left(Y_i+Y_{i-1}\right)^2+3\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)\right.$$

$$\left(Y_i+Y_{i-1}\right)+3\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)^2\Big]+\frac{3}{64\Delta z}\left[\left(Y_i+Y_{i-1}\right)^2+\left(Y_i+Y_{i-1}\right)\left(3V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)\right.$$

$$+V_{i,n}^{(6)}\left(3V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)+V_{i-1,n}^{(6)}\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\Big]+\frac{1}{2\Delta z}\left[\left(\frac{KG}{E}\right)_i\right.$$

$$\left.-\frac{1}{2}\left(\frac{X_i+X_{i-1}}{2}\right)^2\right]+\frac{1}{4}\left[\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i-\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\left(\frac{X_i+X_{i-1}}{2}\right)^2\right.$$

$$\left.\left.-\left(\frac{X_i+X_{i-1}}{2}\right)\left(\frac{X_i-X_{i-1}}{\Delta z}\right)\right]\right\}V_{i,n+1}^{(6)}$$

$$+\left[-\frac{\Delta t}{8}\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i\right]V_{i+1,n+1}^{(1)}+\left[0\right]V_{i+1,n+1}^{(2)}+\left[-\frac{1}{2\Delta t}\right]V_{i+1,n+1}^{(3)}$$

$$=-\frac{1}{8(\Delta z)}\left[\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\left(V_{i-1,n}^{(5)}+V_{i,n}^{(5)}\right)\right]-\frac{1}{8(\Delta z)}\left[\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)\right.$$

$$\left.\left(V_{i,n}^{(5)}-V_{i-1,n}^{(5)}\right)\right]$$

$$-\frac{1}{16}\left[\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]\left[\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(5)}+V_{i-1,n}^{(5)}\right)\right]-\left[\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\right]$$

$$\left[\left(\frac{1}{64}\right)\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)^3\right]-\frac{3}{64(\Delta z}\left[\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)^2\right]$$

$$-\frac{1}{2(\Delta z)}\left[\left(\frac{kG}{E}\right)_i-\frac{1}{2}\left(\frac{X_i+X_{i-1}}{2}\right)^2\right]\left[\left(V_{i,n}^{(6)}-V_{i-1,n}^{(6)}\right)\right]-\frac{1}{4}\left[\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i\right.$$

$$\left.-\frac{1}{2}\left(\frac{1}{EA}\right)_i\left(\frac{dEA}{dz}\right)_i\left(\frac{X_i+X_{i-1}}{2}\right)^2-\left(\frac{X_i+X_{i-1}}{2}\right)\left(\frac{X_i-X_{i-1}}{\Delta z}\right)\right]\left[\left(V_{i,n}^{(6)}+V_{i-1,n}^{(6)}\right)\right]$$

$$+\left[\frac{1}{4}\left(\frac{kG}{E}\right)_i\right]\left[\left(V_{i,n}^{(4)}+V_{i-1,n}^{(4)}\right)\right]+\left[\left(\frac{1}{EA}\right)_i\left(\frac{dkG}{dz}\right)_i\right]\left[\frac{1}{2}\left(\phi_{i,n}+\phi_{i+1,n}\right)\right.$$

$$\left.+\frac{\Delta t}{8}\left(V_{i,n}^{(1)}+V_{i+1,n}^{(1)}\right)\right]+\left[-\frac{1}{2\Delta t}\right]\left[V_{i,n}^{(3)}+V_{i+1,n}^{(3)}\right]+\left[\left(\frac{1}{EA}\right)_i\left(\frac{dkAG}{dz}\right)_i\right.$$

$$\left.\left(\frac{X_i+X_{i-1}}{2}\right)+\left(\frac{kG}{E}\right)_i\left(\frac{X_i-X_{i-1}}{\Delta z}\right)\right]. \qquad (33$$

If the boundary conditions are now discretized utilizing the same considerations as employed above, we have the following

$$A + Z = 0: \qquad V_{1,n+1}^{(4)} = 0, \tag{34a}$$

$$V_{1,n+1}^{(6)} = \frac{1}{P_b + \left(\frac{KG}{E}\right)_1}\left[\left(\frac{KG}{E}\right)_1\right]\left[\left(S_b + X_1\right) + \phi_{2,n} + \frac{\Delta t}{2} V_{2,n}^{(1)}\right]$$

$$+\left[\frac{(KG/E)_1}{P_b + \left(\frac{KG}{E}\right)_1}\right]\left[\frac{\Delta t}{2}\right] V_{2,n+1}^{(1)} \tag{34b}$$

and

$$V_{1,n+1}^{(5)} = \left[\frac{1}{2} X_1^2 + P_b\right] - \frac{1}{2} V_{1,n+1}^{(6)} \cdot V_{1,n+1}^{(6)}$$

Using (34b) we obtain

$$V_{1,n+1}^{(5)} = \left\{\left[\frac{1}{2} X_1^2 + P_b\right] - \frac{1}{2}\left[\frac{(KG/E)_1}{P_b + \left(\frac{KG}{E}\right)_1}\right]^2 \left[\left(S_b + X_1 + \phi_{2,n} + \frac{\Delta t}{2} V_{2,n}^{(1)}\right)^2\right]\right\}$$

$$-\left\{\frac{1}{2}\left[\frac{(KG/E)_1}{P_b + \left(\frac{KG}{E}\right)_1}\right]\left[\frac{\Delta t}{2}\right]\left[\frac{(KG/E)_1}{P_b + \left(\frac{KG}{E}\right)_1}\left(S_b + X_1 + \phi_{2,n} + \frac{\Delta t}{2V_{2,n}^{(1)}}\right) + Y_1\right]\right\} V_{2,n+1}^{(1)} \tag{34c}$$

At $Z = 1$: $\quad V^{(4)}_{L,n+1} = 0,$ (34c

$$V^{(3)}_{LP,n+1} = \left\{ V^{(3)}_{LP,n} + \Delta t \left[\left(\frac{EA\ell}{m_h c^2} \right)_{LP} \frac{1}{2} X_L^2 + \left(\frac{EA\ell}{m_h c^2} \right)_{LP} \left(\frac{kG}{E} \right)_{LP} X_L \right] \right.$$

$$\left. + \Delta t \left[\left(\frac{EA\ell}{m_h c^2} \right)_{LP} \left(\frac{kG}{E} \right)_{LP} \right] \left[\phi_{LP,n} + \frac{\Delta t}{2} V^{(1)}_{LP,n} \right] \right\} + \left[(-\Delta t) \left(\frac{EA\ell}{m_h c^2} \right)_{LP} \right]$$

$$V^{(5)}_{L,n+1} + \left[\left(-\frac{\Delta t}{2} \right) \left(\frac{EA\ell}{m_h c^2} \right)_{LP} \left(Y_L + 2 \left(\frac{kG}{E} \right)_{LP} \right) \right] V_{L,n+1} + \left[\frac{(\Delta t)^2}{2} \left(\frac{EA\ell}{m_h c^2} \right)_{LP} \left(\frac{kG}{E} \right)_{LP} \right] V^{(1)}_{LP,n+1}, \quad (34$$

and,

$$V^{(2)}_{LP,n+1} = \left\{ V^{(2)}_{LP,n} + \frac{\Delta t}{2} \left(\frac{EA\ell}{m_h c^2} \right)_{LP} X_L^2 \right\} + \left[-\Delta t \left(\frac{EA\ell}{m_h c^2} \right)_{LP} \right] V^{(5)}_{L,n+1}$$

$$+ \left[-\frac{\Delta t}{2} \left(\frac{EA\ell}{m_h c^2} \right)_{LP} Y_L \right] V^{(6)}_{L,n+1} . \quad (34$$

A table is constructed on the following page which presents the variables present at the n+1 time level for each of the six equations, [(28)(33)]. The reason for presenting the equations in this order (rather than the order of their appearance in the report) will be readily apparent.

TABLE III:

Variables appearing in each equation at the n+1 time step.

Equation	i - 1	i	i + 1
28	$V^{(4)}$	$V^{(1)}, V^{(4)}$	$V^{(1)}$
29	$V^{(5)}$	$V^{(2)}, V^{(5)}$	$V^{(2)}$
33	$V^{(4)}, V^{(5)}, V^{(6)}$	$V^{(1)}, V^{(3)}, V^{(4)}, V^{(5)}, V^{(6)}$	$V^{(1)}, V^{(3)}$
32	$V^{(4)}, V^{(6)}$	$V^{(1)}, V^{(2)}, V^{(4)}, V^{(6)}$	$V^{(1)}, V^{(2)}$
31	$V^{(5)}, V^{(6)}$	$V^{(1)}, V^{(2)}, V^{(5)}, V^{(6)}$	$V^{(1)}, V^{(2)}$
30	$V^{(6)}$	$V^{(3)}, V^{(6)}$	$V^{(3)}$

Let us now arrange these variables and equations in the order shown in Figure 7 on the following page, where a zero (0) means that the coefficient of a particular variable is equal to zero and a one (1) means that the coefficient of a variable is not equal to zero.

Thus the band width is thirteen. This is consistent since the nature of equations (28), (29) and (30) fixes the minimum permissible band width at thirteen.

Furthermore, Figure 3 presents the band and shows schematically those coefficients which do not change with either iteration or time (C), those coefficients which change only with iteration (L) and those which change with both iteration and time step (T_i).

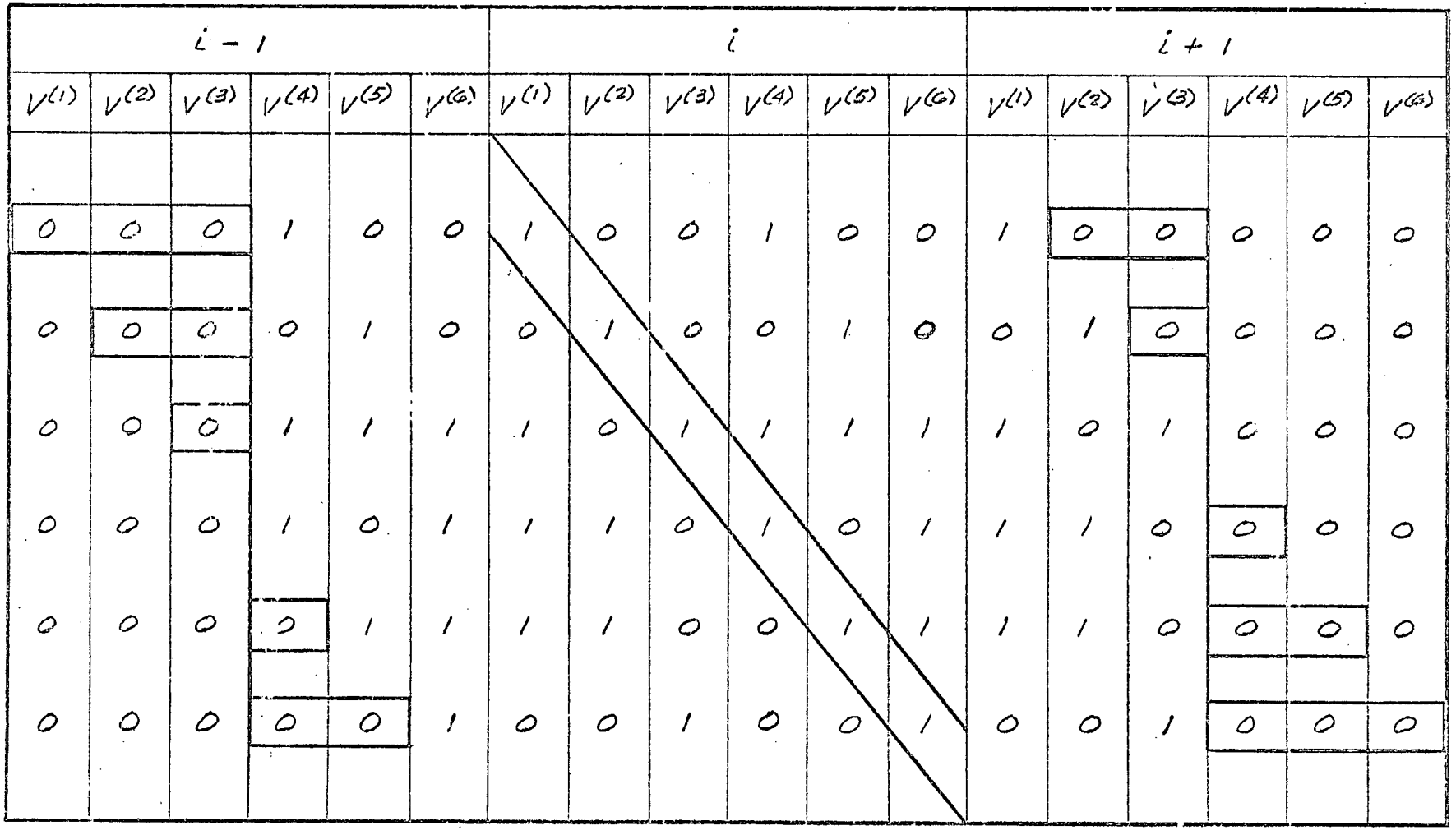

Figure 7. Band Width Schematic for the Finite Difference Equations

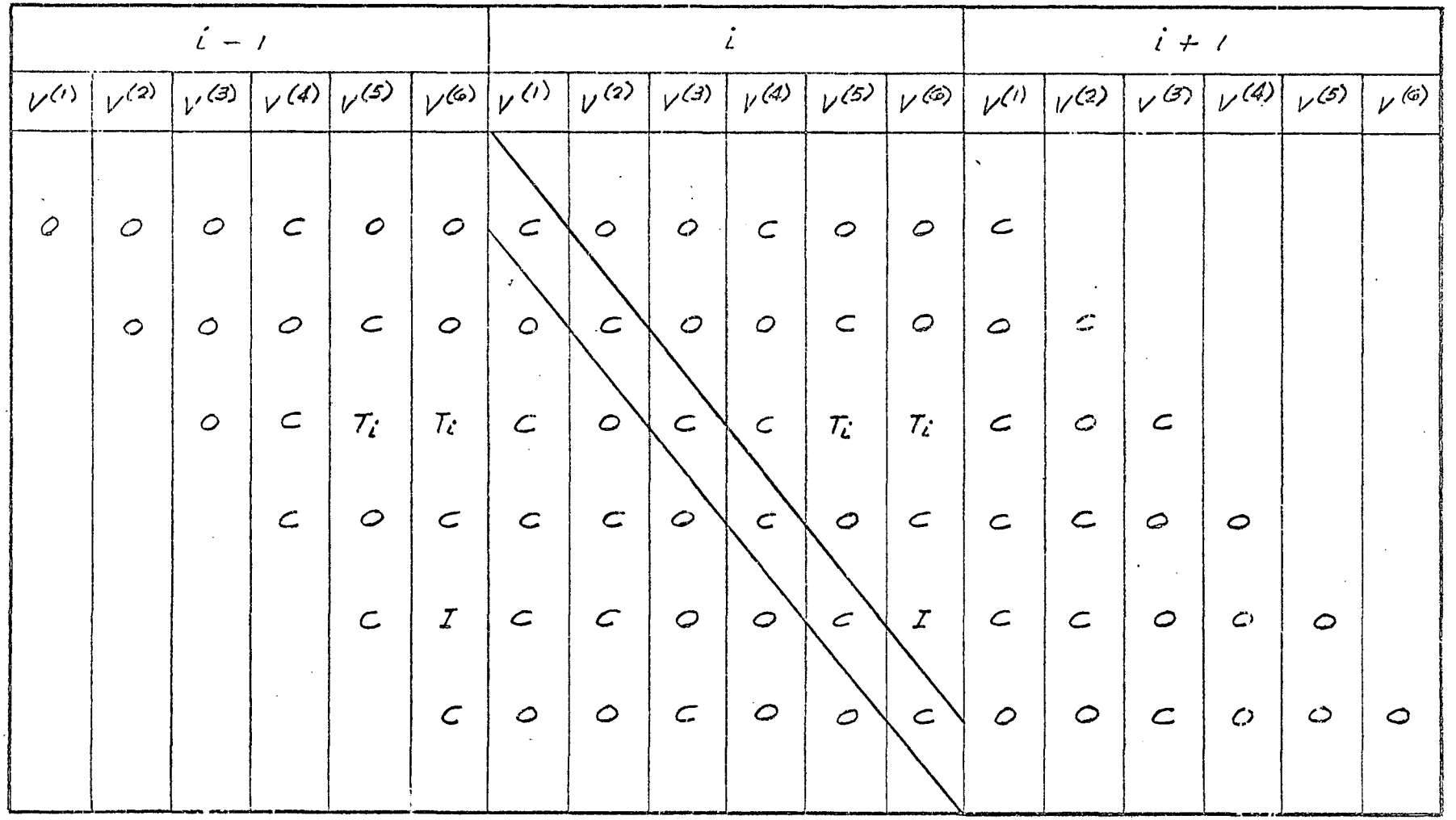

Figure 8. Schematic Representation of the Band Showing (1) Coefficients which are Constant for all Iterations and Time Steps (C), (2) Coefficients which Change with Iteration (I), and (3) Coefficients which Change with Iteration and Time Step (T_i).

The right hand side of each of the six equations changes with the time step but not with the iteration.

From an inspection of Figure 8 and by recalling that (1) at $z = 0$, $V^{(4)}$, $V^{(5)}$ and $V^{(6)}$ are known, and (2) at $z = 1$, $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$ are known, it is obvious that the finite difference are of a form that is capable of solution using an algorithm of the type discussed elsewhere. (3)

Using such an algorithm and iterating three times within each time space results in a solution which is stable and convergent to five significant figures.

The equations are solved for $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, $V^{(4)}$, $V^{(5)}$ and $V^{(6)}$. The variables of interest, namely u, w and ϕ may be obtained from $V^{(1)}$ - $V^{(6)}$ by using the relations established in Equations (19). Furthermore, all stresses and strains are calculated according to the relationship derived previously. (3)

The solutions obtained using the numerical techniques discussed above have been verified experimentally. These experiments are discussed in the next section.

3.4 Experimental Verification of the Solutions

An experimental program was designed to check the accuracy and adequacy of the solutions to Equations (1-4) and boundary Equations (5-6).

A beam of low density polyethylene was fabricated having the same taper size and initial curvature as that of a representative human spine.[4] The bottom end was mounted in a pin joint fixture such that the applied moment at the inferior end was zero.

The beam was instrumented with strain gages in the approximate anatomical levels of L3 and T7. Accelerometers were mounted on the base and at the superior end of the beam. The entire beam and support fixture were mounted on the acceleration carriage of the HYGE shock tester. The test arrangement with strain gages and accelerometers positioned is shown in Figure 9.

The output from the strain gages at L3 and T7, and from the accelerometers at the inferior and superior ends of the beam were compared with the theoretical predictions from the mathematical model of the same beam. Figure 10 shows the input acceleration applied to the base of the spine. Figures 11 - 14 present the comparison between experimental results and theoretical predictions

An inspection of Figure 11 - 14 demonstrates that the qualitative agreement between theory and experiment is quite good. Considering that the experimental program was performed very rapidly and that it was a secondary consideration to the overall objective of this project, the quantitative agreement is also quite good. To check the quantitative predictive capability of the model, it is

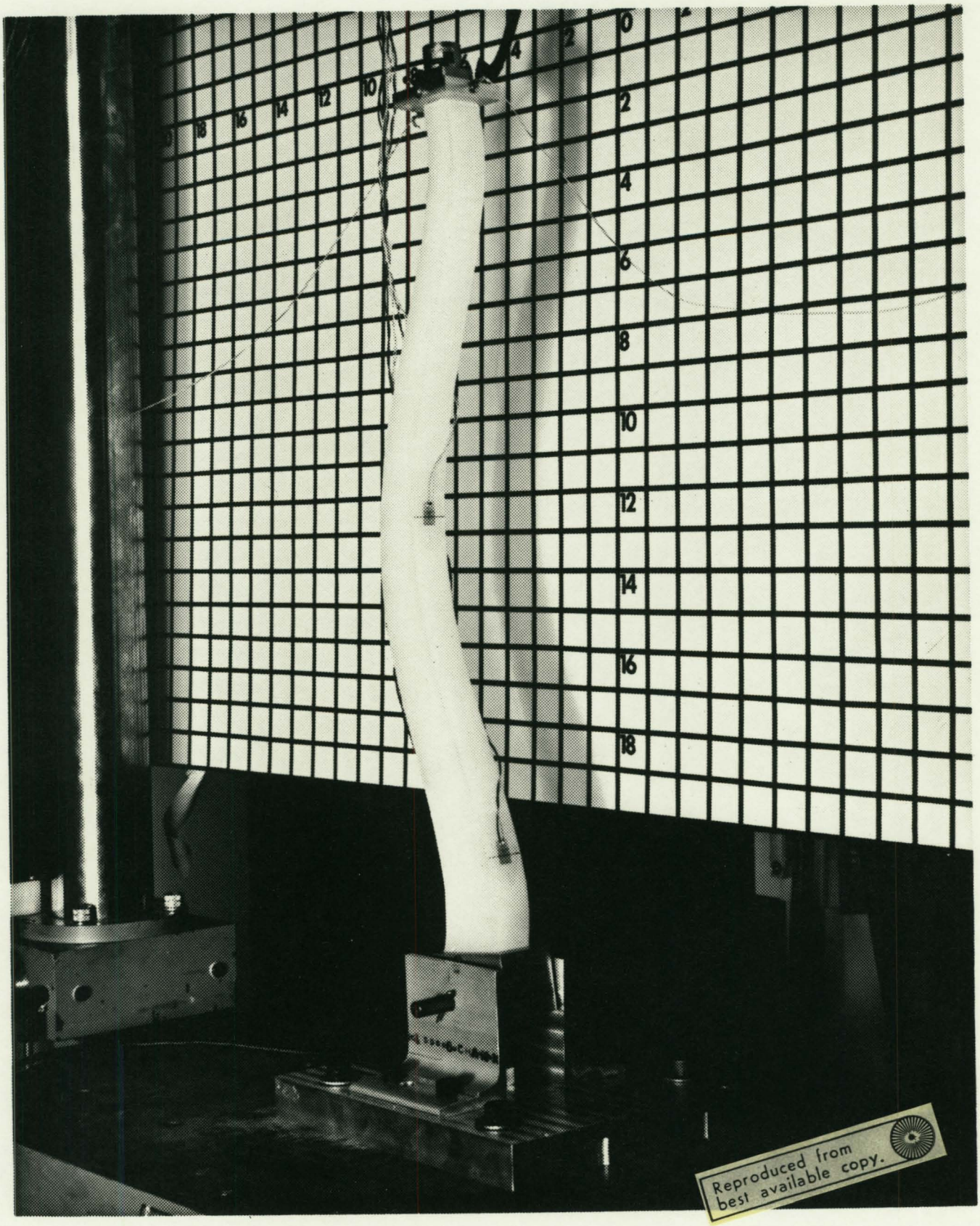

Figure 9. Test beam instrumented with strain gages and accelerometer.

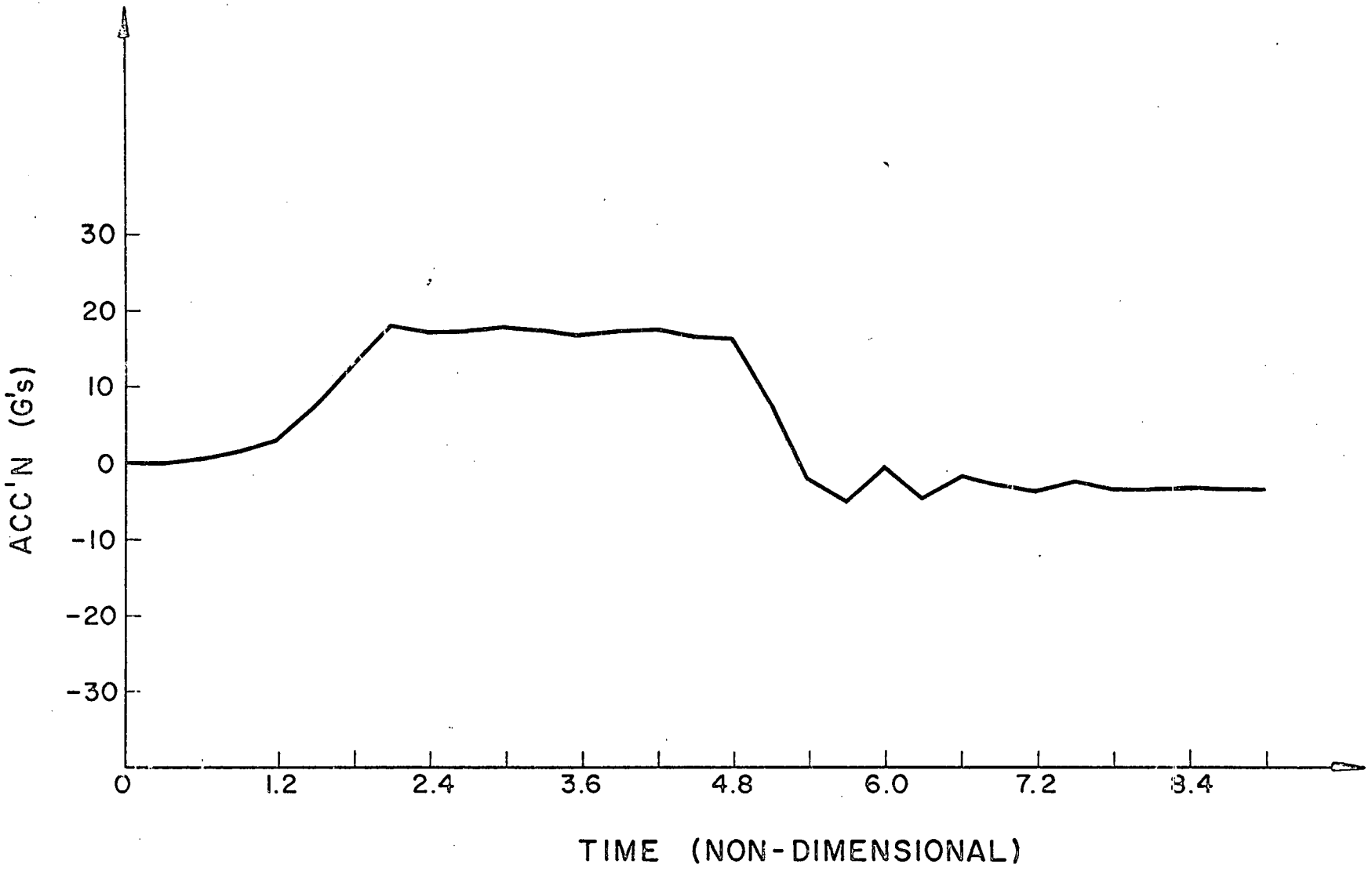

Figure 10. Input acceleration to the base of the beam-column.

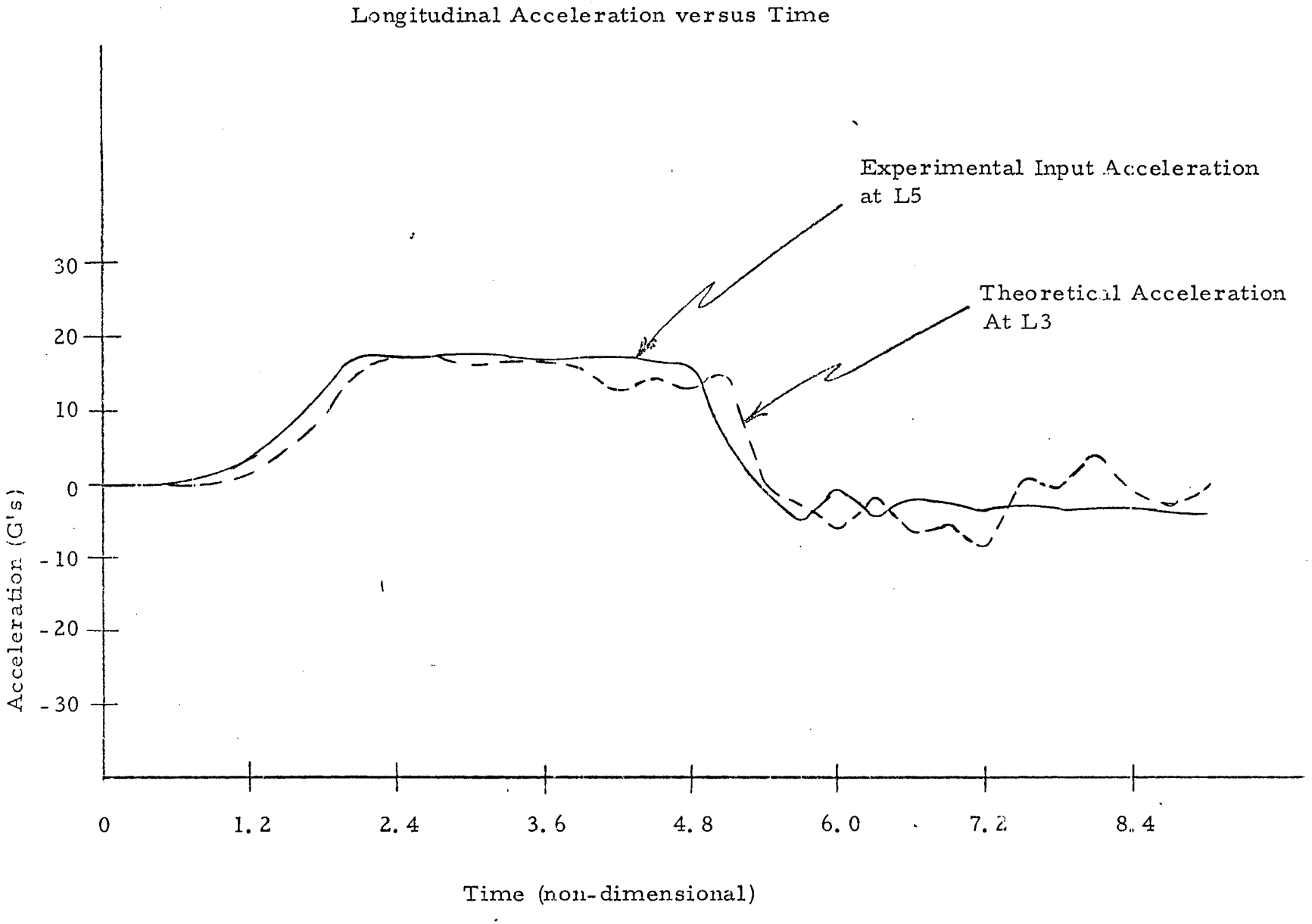

Figure 11. Comparison of theoretical predictions and experimental results of acceleration in the lower lumbar region.

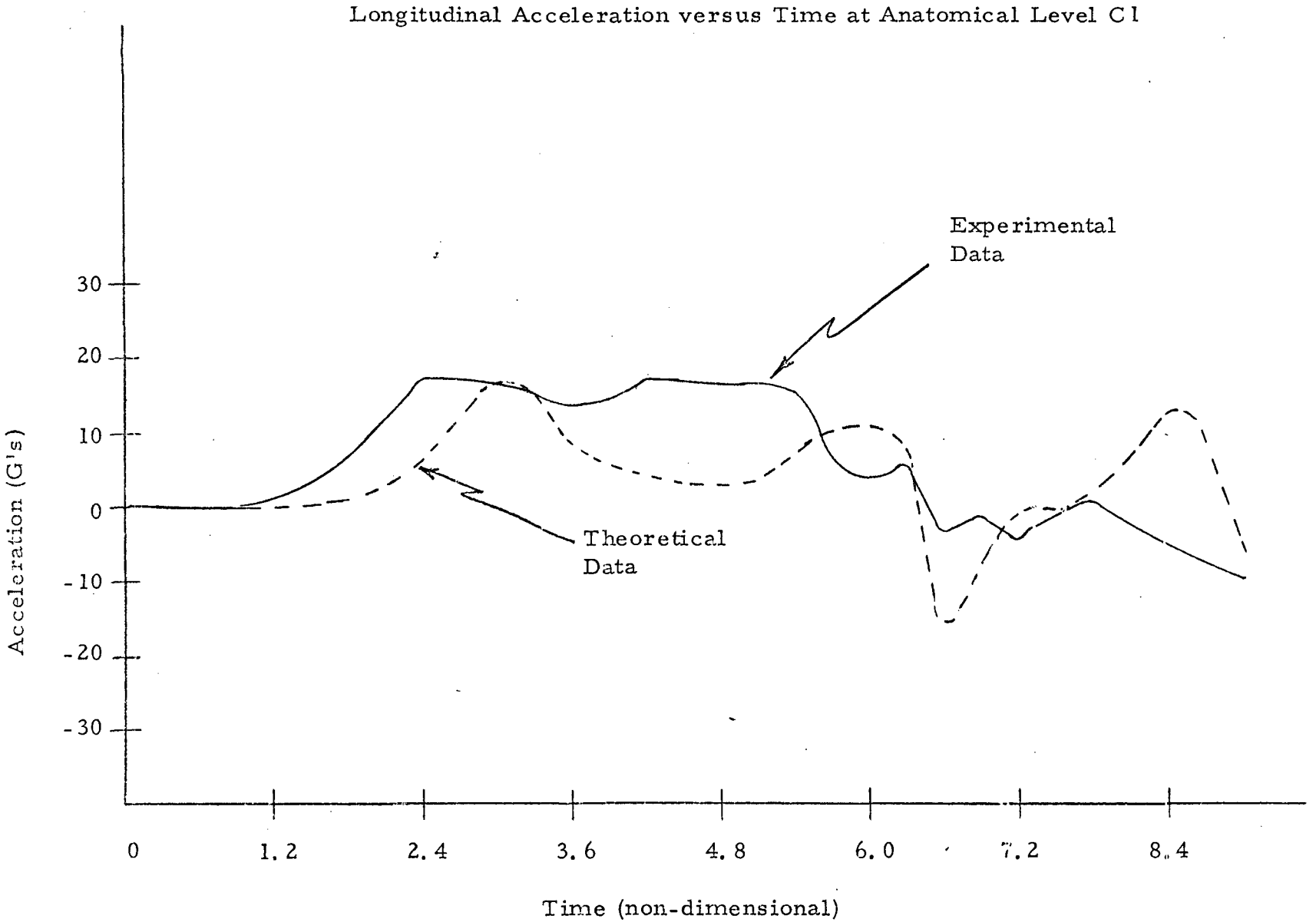

Figure 12. Comparison of theoretical predictions and experimental results for acceleration in the upper cervical region.

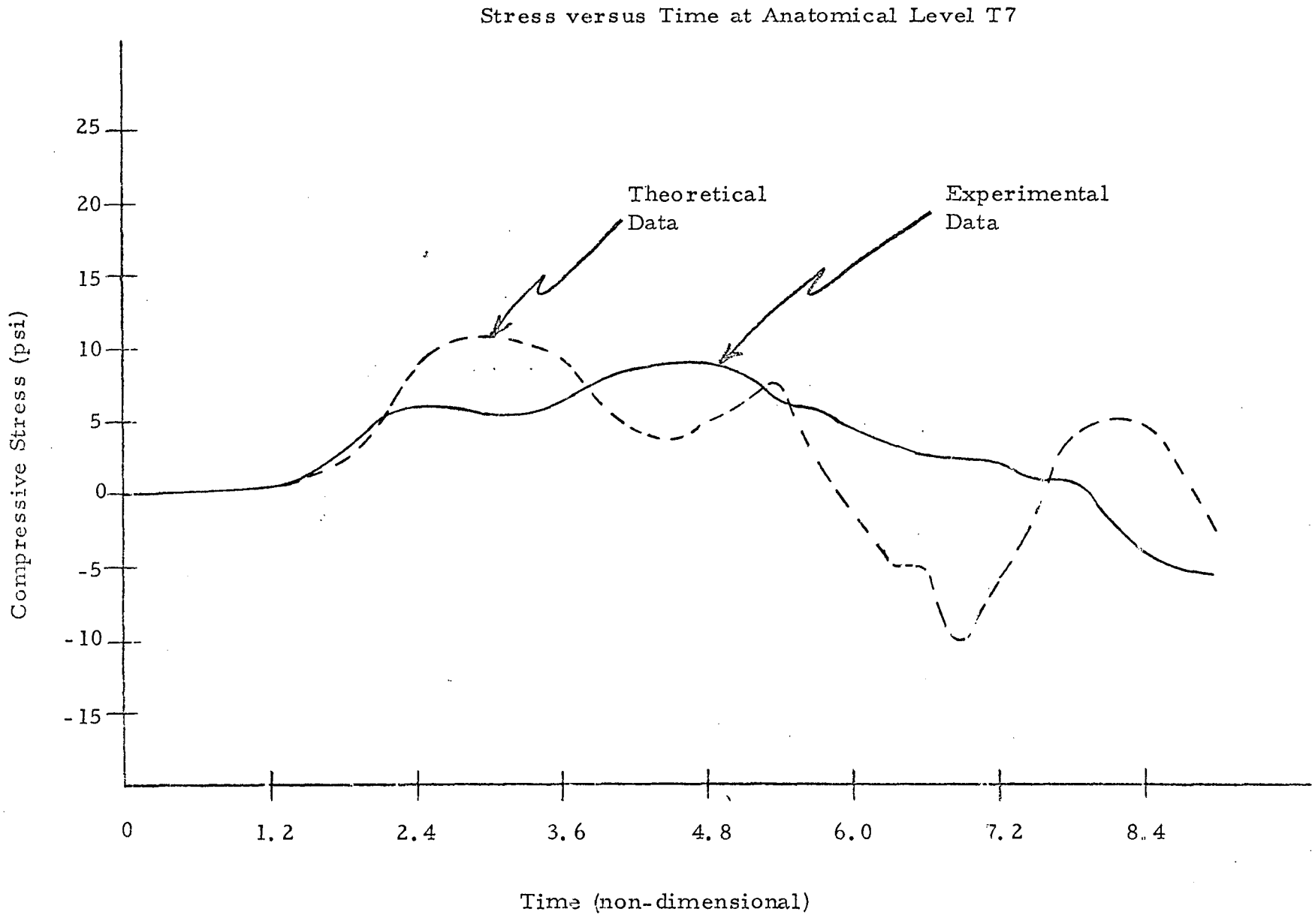

Figure 13. Comparison of theoretical predictions and experimental results for stress at Anatomical Level T7.

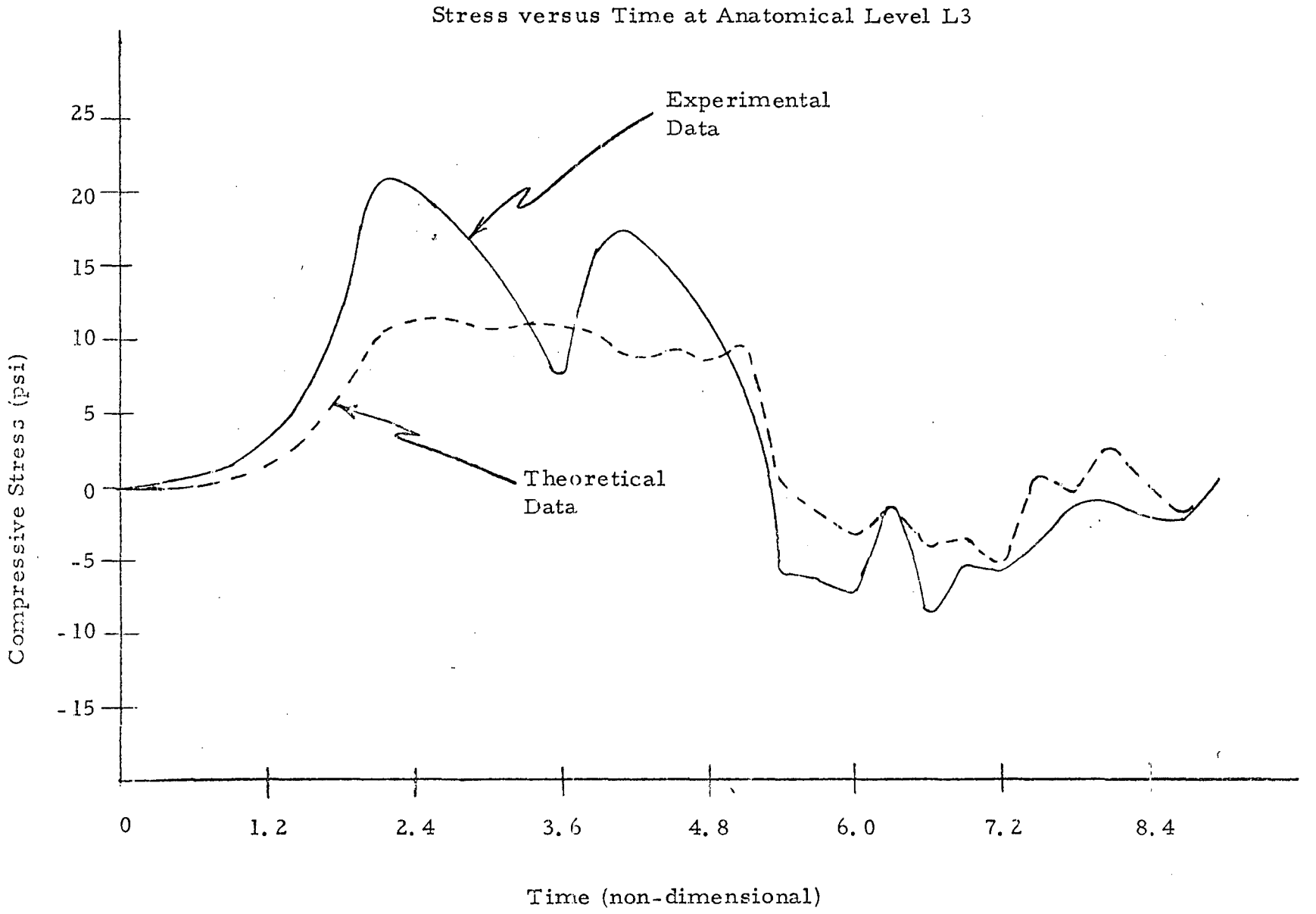

Figure 14. Comparison of theoretical predictions and experimental results for stress at Anatomical Level L3.

believed that a more precise, more complete experimental program should be considered.

Considering the constraints mentioned above, the experimental results offer excellent support to the theoretical predictions of the model.

3.5 Conclusions

When the results of section 3.4 are considered, it is concluded that the model derived and the solution effected are sufficient to predict the response of tapered, curved beam columns. Additional parameterization of the spinal tissue is probably required before the model can be applied with full confidence to predict injury of the spine. Experiments correlating model predictions with measured dynamic response of either primates or humans is now required to determine the limits of applicability of the model.

4. REFERENCES

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