FIELD DEPENDENCE OF GASEOUS ION MOBILITY: TEST OF APPROXIMATE FORMULAS*

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ABSTRACT

The accuracies of three approximate formulas due to Wannier, Frost, and Patterson are tested by comparison with special cases for which accurate results can be found. The Wannier free-flight theory is superior, and can be extended to yield a formula without further adjustable constants that gives an exact result at low electric fields and good results at medium and high fields, applicable for any ion-neutral force law and mass ratio.

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I. INTRODUCTION

It is well known that the drift velocity of an ion in a neutral gas depends on field strength. No general expression for the field dependence is known, although several approximate formulas have been suggested. The purpose of this paper is to test these approximate formulas by comparison with several accurate results for special cases, and to suggest an improved connection formula that can be used at all fields. The most critical test occurs for the case of light ions and heavy neutrals (Lorentzian mixture), for which the drift velocity can be found at all fields by numerical integration.

Dimensional arguments suffice to show that the drift velocity v_d depends on the elctric field strength E and on the number density of the gas N only through the ratio E/N. At low fields v_d is directly proportional to E/N for all ion-neutral interactions, and is given accurately by the Chapman-Enskog kinetic theory. At high fields the nature of the ion-neutral interaction determines the dependence of v_d on E/N; for example, it is known that v_d varies directly as E/N for an r^{-4} interaction potential and as $(E/N)^{1/2}$ for a rigid-sphere interaction.[1,2]

II. APPROXIMATE FORMULAS

In this section we briefly outline three formulas which give v_d as a function of E/N.

A. Wannier Free-Flight Theory

In 1953 Wannier[2] indicated how to obtain a simple interpolation formula for v_d; since his result has been almost universally overlooked, we indicate the line of arguments leading to it. An ion of mass m and charge e undergoes an acceleration eE/m between collisions. If the ion lost all its momentum on every collision, the drift velocity would be $(eE/m)\tau$, where τ is the mean time between collisions. But the ion loses only a fraction of its momentum on each collision. The mass dependence of the momentum loss on collision can be calculated from the equations of momentum and energy conservation; if we average this momentum loss over all collisions and ignore subtleties about the average of a product and the product of the averages, we obtain

$$v_d = \xi\left(1 + \frac{m}{M}\right)\left(\frac{eE}{m}\right)\tau \quad , \tag{1}$$

where M is the mass of a neutral molecule and ξ is a factor of order unity that depends in a complicated way on the ion-neutral force law and the masses m and M. The mean free time is given by

$$\tau = 1/N\bar{v}_r Q \quad , \tag{2}$$

where $\bar{v}_r$ is the mean relative speed of ions and neutrals and Q is the average momentum-transfer cross section. It is reasonable to take $\bar{v}_r$ as the root-mean-square speed,

$$\bar{v}_r = \left(\overline{v^2} + \overline{V^2}\right)^{1/2} \quad , \tag{3}$$

where $\overline{v^2}$ is the mean square ion velocity and $\overline{V^2}$ the mean square neutral velocity. For the latter quantity energy equipartition gives

$$\frac{1}{2} M\overline{V^2} = \frac{3}{2} kT \quad . \tag{4}$$

The only remaining problem is to find $\overline{v^2}$, which has both thermal and field components. At low fields $\overline{v^2}$ is entirely thermal, but at high fields it has a negligible thermal component. Wannier[1,2] has shown that if τ is constant, then the thermal and field energies of the ions are additive, and that the field energy is exhibited partly as a drift motion and partly as a random motion,

$$\frac{1}{2} m\overline{v^2} = \frac{3}{2} kT + \frac{1}{2} mv_d^{\,2} + \frac{1}{2} Mv_d^{\,2} \quad , \tag{5}$$

where $\frac{3}{2} kT$ is the thermal energy, $\frac{1}{2} mv_d^{\,2}$ is the drift energy, and $\frac{1}{2} Mv_d^{\,2}$ is the random part of the field energy.

Combining Eqs. (4) and (5), we obtain

$$\overline{v^2} + \overline{V^2} = 3kT\left(\frac{1}{m} + \frac{1}{M}\right) + v_d^{\,2}\left(1 + \frac{M}{m}\right) \quad , \tag{6}$$

and substituting back into Eqs. (1)-(3) we find

$$v_d = \xi\left(\frac{1}{m} + \frac{1}{M}\right)^{1/2} \frac{(eE/NQ)}{(3kT + Mv_d^{\,2})^{1/2}} \quad , \tag{7}$$

which is apparently a quadratic in the variable $v_d^{\,2}$,

$$(v_d^{\,2})^2 + \frac{3kT}{M}(v_d^{\,2}) - \frac{\xi^2}{M}\left(\frac{1}{m} + \frac{1}{M}\right)\left(\frac{eE}{NQ}\right)^2 = 0 \quad . \tag{8}$$

This quadratic dependence is only apparent, however, unless Q is a constant (rigid spheres). In general, Q depends on $\bar{v}_r$ in a manner determined by the ion-neutral force law. In any case, solution of Eq. (8) gives a reasonable result for v_d at all field strengths. At low fields we have $3kT \gg Mv_d^{\,2}$ and v_d is proportional to (eE/NQ); at high fields we have $Mv_d^{\,2} \gg 3kT$ and

v_d is proportional to $(eE/NQ)^{1/2}$. At low fields the results can be compared with the accurate Chapman-Enskog kinetic theory formula for the diffusion coefficient. All the dimensional factors in Eq. (7) are found to be correct, provided we identify the average cross section Q with a collision integral for diffusion, which we can write as[3,4]

$$\bar{\Omega}^{(1,1)}(T) = \frac{1}{2(kT)^3} \int_0^\infty e^{-\varepsilon/kT} Q^{(1)}(\varepsilon)\varepsilon^2 d\varepsilon \quad , \qquad (9)$$

where $\varepsilon = \frac{1}{2}\mu v_r^2$ is the relative energy of a colliding ion-neutral pair, $\mu = mM/(m + M)$ is the reduced mass, and $Q^{(1)}(\varepsilon)$ is a diffusion or momentum-transfer cross section,

$$Q^{(1)}(\varepsilon) = 2\pi \int_0^\infty (1 - \cos\theta) I(\theta) \sin\theta d\theta \quad . \qquad (10)$$

We have here chosen the normalization factors in Eqs. (9) and (10) so that both $\bar{\Omega}^{(1,1)}$ and $Q^{(1)}$ are equal to πd^2 for the collision of classical rigid spheres of diameter d. The value of ξ in Eq. (7) is still at our disposal; we choose it to give agreement with the Chapman-Enskog results,

$$\xi = \frac{3}{16} \frac{(6\pi)^{1/2}}{1 - \Delta} = \frac{0.814}{1 - \Delta} \quad , \qquad (11)$$

where Δ is a correction term incorporating higher Chapman-Enskog approximations and given by[3,4]

$$\Delta = \frac{M^2(6C^* - 5)^2}{30m^2 + 10M^2 + 16mMA^*} + \text{higher terms} \quad , \qquad (12)$$

in which A^* and C^* are dimensionless ratios of collision integrals.[4]

B. Kihara Medium-Field Expansion

Kihara[5] has shown how the kinetic-theory results based on the Boltzmann equation can be extended to higher fields, by avoiding the Chapman-Enskog assumption that the ion velocity distribution function differs only slightly from the Maxwelliam form. The result is an expansion for v_d in powers of the quantity $(E/N)^2$. Depending on the particular approximation procedure used to solve an infinite set of moment equations, the expansion can be written either as[6]

$$v_d = v_d(0)[1 + \alpha_1(E/N)^2 + \alpha_2(E/N)^4 + \ldots] \quad , \tag{13}$$

or as[5]

$$v_d = \frac{v_d(0)}{1 + \beta_1(E/N)^2 + \beta_2(E/N)^4 + \ldots} \quad , \tag{14}$$

where $v_d(0)$ is the low-field limit of v_d and is itself proportional to E/N. The coefficients α_i and β_i are complicated functions of the masses m and M, as well as of the ion-neutral force law. The form of the expansion obviously limits its validity to medium fields. Such an expansion in powers of $(E/N)^2$ can be obtained from Eq. (7) of the free-flight theory by expanding the denominator of Eq. (7) in powers of the small quantity $Mv_d^2/3kT$ and solving iteratively for v_d, but the values of α_1 and β_1 so obtained are not in general correct.

C. Frost and Patterson Interpolation Formulas

Knowing that v_d varies as E/N at low fields and as $(E/N)^{1/2}$ for rigid spheres at high fields, Frost[7] proposed the formula

$$v_d = A(E/N)[1 + a(E/N)]^{-1/2} \quad , \tag{15}$$

where A and <u>a</u> are constants that are different for every system. The form of this expression can be obtained from Eq. (7) by replacing the value of v_d^2 in the denominator of the right-hand side of Eq. (7) by its high-field value.

Patterson[8] incorporated the medium-field expansion of Kihara into a somewhat more elaborate interpolation formula,

$$v_d = A(E/N)[1 + b(E/N)^2 + c(E/N)^4]^{-1/8} , \tag{16}$$

where A, b, and c are constants. This preserves the high-field variation of rigid spheres, and at medium fields mimics the expansions of Eqs. (13) and (14) with $b = \frac{1}{8}\beta_1$.

III. ACCURATE RESULTS

Only a few accurate results are available for testing the foregoing formulas. An r^{-4} potential (Maxwell model) can be treated accurately at all fields for all ion-neutral mass ratios. The result is that the low-field expression for v_d is valid at all fields, which is not very interesting or even physically realistic. Other known special cases are as follows.

A. High Fields

If the ions are either much heavier or much lighter than the neutrals, then v_d can be found for any ion-neutral interaction. If the ions and neutrals have equal masses, then v_d is known only for a rigid-sphere interaction. For $m \gg M$ the result is[2,9]

$$v_d = \left[\frac{eE}{MNQ^{(1)}(v_d)}\right]^{1/2} , \tag{17}$$

where the momentum-transfer cross section is evaluated at v_d. For $m \ll M$ the result is given as integrals,[10,11]

$$v_d = \frac{4\pi}{3}\left(\frac{eE}{m}\right)\int_0^\infty f^{(0)}\phi v^4 dv, \tag{18}$$

where $f^{(0)}$ is the isotropic part of the ion distribution function and ϕ the directional part,

$$\ln f^{(0)} = \ln B - 3\left(\frac{m}{M}\right)\left(\frac{mN}{eE}\right)^2 \int [Q^{(1)}(v)]^2 v^3 dv , \tag{19}$$

$$\phi = \frac{3}{N}\left(\frac{m}{M}\right)\left(\frac{mN}{eE}\right)^2 Q^{(1)}(v) , \tag{20}$$

in which B is a normalization constant. Numerical integration is required unless the velocity dependence of $Q^{(1)}(v)$ is simple. For $m = M$ the value of v_d has been calculated for rigid spheres by a method involving a trial distribution function judiciously selected to satisfy the first few moment equations; the result is[2]

$$v_d = 1.1467\left(\frac{eE}{mNQ^{(1)}}\right)^{1/2} , \qquad (21)$$

in which $Q^{(1)} = \pi d^2$ is a constant for rigid spheres of mutual collision diameter d.

A comparison of the foregoing accurate results with Eq. (7) of the free-flight theory is simple for the case of an inverse-power ion-neutral potential,

$$V(r) = C/r^n , \qquad (22)$$

where C and n are constants. The momentum-transfer cross section for this potential is[3,4]

$$Q^{(1)}(v) = 2\pi\left(\frac{2nC}{\mu v^2}\right)^{2/n} A^{(1)}(n) , \qquad (23)$$

where the $A^{(1)}(n)$ are pure numbers that are evaluated by numerical integration. An extensive tabulation of $A^{(1)}(n)$ has been given by Higgins and Smith.[12] To use Eq. (23) with the free-flight results, we note that the ion energy at high fields is given by

$$mv^2 = mv_d^2 + Mv_d^2 ,$$

from which it follows that

$$\mu v^2 = Mv_d^2 . \qquad (24)$$

With the energy dependence of $Q^{(1)}(v)$ as given by Eq. (23), the integrals of Eqs. (18) and (19) can be evaluated to yield the exact result for $m << M$,

$$v_d = \left(\frac{4n-8}{3n}\right)^{\frac{3n-4}{4n-8}} \frac{\Gamma\left(\frac{3n-2}{2n-4}\right)}{\Gamma\left(\frac{3n}{4n-8}\right)} \left[\left(\frac{M}{m}\right)^{1/2}\left(\frac{eE}{MN}\right)\left(\frac{M}{2nC}\right)^{2/n} \frac{1}{2\pi A^{(1)}(n)}\right]^{\frac{n}{2n-4}} . \qquad (25)$$

Similarly, the energy dependence of $Q^{(1)}(v)$ can be substituted into Eq. (17), which can then be solved to yield an accurate result for $m >> M$,

$$v_d = \left[\left(\frac{eE}{MN}\right)\left(\frac{M}{2nC}\right)^{2/n} \frac{1}{2\pi A^{(1)}(n)}\right]^{\frac{n}{2n-4}} . \qquad (26)$$

Comparison of these accurate values with the free-flight formula shows that the latter has all the dimensional factors correct. The numerical accuracy is shown in Table I for a number of values of n. Even though ξ was chosen to fit only the low-field results, the agreement at high fields is quite reasonable, the largest deviations being less than 20%. No comparison with the other formulas can be made—the Kihara expansion breaks down at high fields, and the Frost and Patterson formulas are valid only for rigid spheres.

B. Intermediate Fields

Only for $m << M$ is an exact result known for arbitrary field strengths. The ion distribution function is given by[10,11]

$$\ell n f^{(0)} = \ell n B - \int \frac{mv^3 dv}{kTv^2 + \frac{1}{3} M[eE/mNQ^{(1)}]^2} , \qquad (27)$$

$$\phi = \frac{mv/NQ^{(1)}}{kTv^2 + \frac{1}{3} M[eE/mNQ^{(1)}]^2} , \qquad (28)$$

for which Eqs. (19) and (20) are the high-field limits. Given the energy dependence of $Q^{(1)}$, the integral in Eq. (27) can be evaluated, after which v_d can be found by the integration in Eq. (18). In order to test the approximate formulas, we have carried through the integrations for rigid spheres. The integration in Eq. (27) can be performed analytically, but the final integration for v_d must still be done numerically. We can consolidate the temperature and field dependence of v_d by defining the dimensionless quantities,

$$v_d^* = \left(\frac{M}{2kT}\right)^{1/2} v_d \quad , \tag{29}$$

$$\mathcal{E}^* = \frac{3\pi^{1/2}}{16kT}\left(\frac{m + M}{m}\right)^{1/2} \frac{eE}{NQ^{(1)}} \quad . \tag{30}$$

The dimensional argument underlying these choices can perhaps be seen most easily from the free-flight result of Eq. (8), which becomes

$$(v_d^*)^4 + \frac{3}{2}(v_d^*)^2 - \frac{3}{2(1 - \Delta)^2}(\mathcal{E}^*)^2 = 0 \ , \tag{31}$$

in which we have substituted the value of ξ from Eq. (11). No assumption of $m << M$ has been made in obtaining Eq. (31). The exact equations become, with $m << M$,

$$v_d^* = \frac{16}{9\pi^{1/2}}\mathcal{E}^* \frac{I_1(\gamma)}{I_2(\gamma)} \ , \tag{32}$$

where

$$I_1(\gamma) = \int_0^\infty x^2(x + \gamma)^{\gamma-1} e^{-x}dx \quad , \tag{33}$$

$$I_2(\gamma) = \int_0^\infty x^{1/2}(x + \gamma)^{\gamma} e^{-x}dx \quad , \tag{34}$$

$$x = mv^2/2kT \ , \qquad \gamma = 128(\mathcal{E}^*)^2/27\pi.$$

Equations (33) and (34) were evaluated by numerical integration using Simpson's rule. The results are given in Table II, and may be used as a convenient test case for any proposed theory of the dependence of v_d on E/N.

A comparison of the free-flight Eq. (31) and the exact Eq. (32) is shown in Fig. 1. In the free-flight calculations we have used the exact value[13] of $1 - \Delta = 9\pi/32$ in order to make the two results agree at low fields. The agreement is remarkably good over the whole range, the worst disagreements being about 8% at intermediate fields and about 6% at high fields (as shown in

Table I). The Frost interpolation formula of Eq. (15) is also shown, the constants A and a being chosen to secure agreement at both low and high fields. The agreement with the exact result is no better than that for the free-flight result, except at high fields. Even the Patterson interpolation formula of Eq. (16) produces very little improvement, despite the use of an additional parameter.

The medium-field expansions of Eqs. (13) and (14) are compared with the exact results in Fig. 2. Because of the difficulty of computing higher terms in the expansions as well as accurate values of the expansion coefficients, we have stopped with the following approximations:

$$v_d^* = \frac{32}{9\pi} \mathcal{E}^* \left[1 - \frac{11}{42} (\mathcal{E}^*)^2 + \frac{247}{1260} (\mathcal{E}^*)^4 + \ldots \right] , \qquad (35)$$

$$v_d^* = \frac{32}{9\pi} \mathcal{E}^* \left[1 + \frac{1}{3} (\mathcal{E}^*)^2 - \frac{1}{18} (\mathcal{E}^*)^4 + \ldots \right]^{-1} , \qquad (36)$$

which can be obtained from the results in refs. 5 and 6. The numerical constants in these two equations are not yet mutually consistent in this order of approximation. It is clear that these expansions give a good representation only at fairly low fields, and are not to be trusted when the deviations from the zero-field asymptote are larger than about 10%. Equation (36) is somewhat better than Eq. (35).

C. Resonant Charge Transfer

If resonant charge transfer is possible, than each collision converts a fast ion and a nearly stationary neutral into a fast neutral and a nearly stationary ion. Thus the ion may be regarded as coming essentially to rest after each collision, and the kinetic-theory problem becomes simple. Solutions have been obtained by Fahr and Müller[14] and by Smirnov.[15] If the charge-transfer cross section Q_T is independent of velocity, the low-field result is

$$v_d(0) = \frac{A'}{(mkT)^{1/2}} \frac{eE}{NQ_T} \quad . \tag{37}$$

Fahr and Müller find $A' = 0.330$ and Smirnov finds $A' = 0.341$. At high fields both obtain

$$v_d(\infty) = \left(\frac{2}{\pi} \frac{eE}{mNQ_T}\right)^{1/2} \quad . \tag{38}$$

It is interesting to compare these with the previous results for rigid spheres. When charge transfer is the dominant process in collisions, an accurate relation is[16,17]

$$Q^{(1)} = 2Q_T \quad . \tag{39}$$

With this expression, Eq. (37) is the same as the Chapman-Enskog result with the constant $A' = 3\pi^{1/2}/16(1-\Delta) = 0.338$, a value in good agreement with Fahr and Müller and Smirnov. At high fields Eq. (38) may be compared with Wannier's rigid-sphere result given in Eq. (21). The form of the two results is the same; the numerical constant from Eq. (38) is $(2/\pi)^{1/2} = 0.798$, and from Eq. (21) is $1.1467/2^{1/2} = 0.811$, in good agreement.

Thus the interpolation formulas we have tested should apply also to mobility with charge transfer, provided Eq. (39) holds.

IV. EXTENDED INTERPOLATION FORMULAS

On the basis of the comparisons in the preceding section, it is easy to extend the formulas giving the field dependence of the mobility. The only serious candidate is Wannier's free-flight theory, since the other formulas are restricted to rigid-sphere interactions. The three main defects of the free-flight theory are as follows. First, the averaging of the momentum-transfer cross section is too crude. At low fields it is averaged over a velocity distribution to yield the temperature-dependent collision integral $\bar{\Omega}^{(1,1)}$ of Eq. (9), but at high fields it corresponds to the drift energy according to Eq. (24). Second, the free-flight theory predicts that the initial deviations of v_d from linearity in E/N are always negative; that is, that α_1 in Eq. (13) is negative or that β_1 in Eq. (14) is positive. Actually, the sign of the deviation depends on the ion-neutral force law in a sensitive way, and is potentially a valuable source of information on ion-neutral interactions.[6] Third, the limiting behavior at high fields can be incorrect in magnitude by as much as 20%, according to Table I.

The first defect is easily remedied. Referring to Eq. (9), we see that $Q^{(1)}(\varepsilon)$ is averaged over relative energies $\varepsilon = \frac{1}{2}\mu v_r^2$ with a weight factor $\varepsilon^2 e^{-\varepsilon/kT}$ corresponding to the spread in thermal energies. A plausible procedure is to imagine that the same weight factor is appropriate at all fields, but is centered about the drift energy, so that ε in Eq. (9) is replaced by an energy ε', given by

$$\varepsilon' = \varepsilon + \varepsilon(\text{field}) = \varepsilon + \frac{1}{2}Mv_d^2 , \qquad (40)$$

the last step following from Eq. (24). Thus Eq. (9) is replaced by

$$\bar{\Omega}^{(1,1)}(T,v_d) = F\int_0^\infty e^{-(\varepsilon+\frac{1}{2}Mv_d^2)/kT} Q^{(1)}(\varepsilon+\tfrac{1}{2}Mv_d^2)(\varepsilon+\tfrac{1}{2}Mv_d^2)^2 d\varepsilon \quad ,$$

$$= F\int_{\frac{1}{2}Mv_d^2}^\infty e^{-\varepsilon'/kT} Q^{(1)}(\varepsilon')(\varepsilon')^2 d\varepsilon' \quad , \tag{41}$$

where F is a normalization factor,

$$F^{-1} = \int_{\frac{1}{2}Mv_d^2}^\infty e^{-\varepsilon'/kT}(\varepsilon')^2 d\varepsilon' \quad . \tag{42}$$

For $\frac{1}{2}Mv_d^2 << kT$, Eq. (41) reduces to Eq. (9), and for $\frac{1}{2}Mv_d^2 >> kT$ it reduces to $Q^{(1)}(\frac{1}{2}Mv_d^2)$. However, Eq. (41) has no theoretical status other than a reasonable interpolation formula.

To alleviate the other two defects we have available only one plausible generalization in the derivation of the free-flight theory, namely the partitioning of the ion field energy into drift and random components according to Eq. (5). This partitioning of energy is strictly correct only for constant mean free time (Maxwell model); we can allow for other partitioning by introduction of another adjustable parameter, so that Eq. (5) becomes

$$\tfrac{1}{2}m\overline{v^2} = \tfrac{3}{2}mkT + \tfrac{1}{2}mv_d^2 + \tfrac{1}{2}(1+\delta)Mv_d^2 \quad . \tag{43}$$

The parameter δ can be adjusted to eliminate one of the defects, but not both. If we choose to obtain the correct initial field dependence, we can then use the high-field limit as an indication of the overall success of the interpolation formula, and vice versa. Picking δ to reproduce the correct value of β_1 in Eq. (14), we obtain the result

$$v_d = \frac{3}{16(1-\Delta)}\left(\frac{2\pi}{\mu kT}\right)^{1/2}[1 + (1-\Delta)\Delta'(Mv_d^2/kT)]^{-1/2}\frac{eE}{N\bar{\Omega}^{(1,1)}} \quad , \tag{44}$$

where $\bar{\Omega}^{(1,1)}$ is given by Eq. (41) and depends on T and v_d, Δ is given by Eq. (12), and Δ' is[5,6]

$$\Delta' = \frac{10}{3}(6C^*-5)\left(\frac{9m + 3MA^*}{5m + 3MA^*}\right)\frac{(m + M)^2}{30m^2 + 10M^2 + 16mMA^*} \quad . \tag{45}$$

We can now test Eq. (44) at high fields in the same way that Eq. (7) of the free-flight theory was tested, using the inverse-power potential given in Eq. (22). The results are shown in Table III; comparison with Table I suggests that the price of improved agreement at medium fields is not worthwhile unless the ion-neutral short-range repulsion is rather steep. Similarly, we can pick δ to reproduce the correct high-field value of v_d for the special cases shown in Tables I and III, and then compare values of the coefficient β_1. The results are shown in Table IV, and appear rather worse than those in Table III. Moreover, they depend strongly on the mass ratio m/M.

V. CONCLUSIONS

The free-flight theory indicated by Wannier provides a reliable interpolation formula for the mobility as a function of field strength. Choosing one adjustable parameter to force agreement with the Chapman-Enskog theory at low fields, and extending the definition of the average momentum-transfer cross section (or diffusion collision integral) according to Eq. (41), we obtain a formula useful for all fields and all ion-neutral force laws and mass ratios. An additional parameter can be introduced and adjusted to force agreement with the Kihara theory at medium fields according to Eq. (44), but the resulting agreement at high fields may be spoiled. Further tests of the formula with ion-neutral interactions containing both attractive and repulsive components would be interesting, but require extensive numerical integration.

Table I. Test of the Wannier approximate free-flight Eq. (7) for the drift velocity at high fields for the potential $V(r) = C/r^n$.

n	v_d(approx.)/v_d(accurate)		
	m>>M	m = M	m<<M
4	0.814	0.814	0.814
6	0.857		0.902
8	0.872		0.943
10	0.879		0.966
12	0.884		0.982
25	0.894		1.022
50	0.898		1.041
∞	0.902	0.944	1.060

Table II. Exact drift velocity as a function of field strength for rigid spheres with $m \ll M$.

$\mathcal{E}^*$	v_d^*
0.10	0.1122
0.12	0.1342
0.15	0.1667
0.20	0.2197
0.25	0.2709
0.30	0.3203
0.35	0.3679
0.40	0.4137
0.45	0.4578
0.5	0.5004
0.6	0.5811
0.7	0.6566
0.8	0.7274
0.9	0.7943
1.0	0.8576

$\mathcal{E}^*$	v_d^*
1.2	0.9753
1.5	1.134
2.0	1.364
2.5	1.564
3.0	1.743
3.5	1.906
4.0	2.057
4.5	2.198
5	2.331
6	2.576
7	2.801
8	3.008
9	3.203
10	3.386

Table III. Test of the extended free-flight Eq. (44) for the drift velocity at high fields for the potential $V(r) = C/r^n$.

n	v_d(approx.)/v_d(accurate)		
	m>>M	m = M	m<<M
8	1.302		1.368
10	1.210		1.147
12	1.164		1.124
25	1.077		1.093
50	1.049		1.089
∞	1.025	0.932	1.088

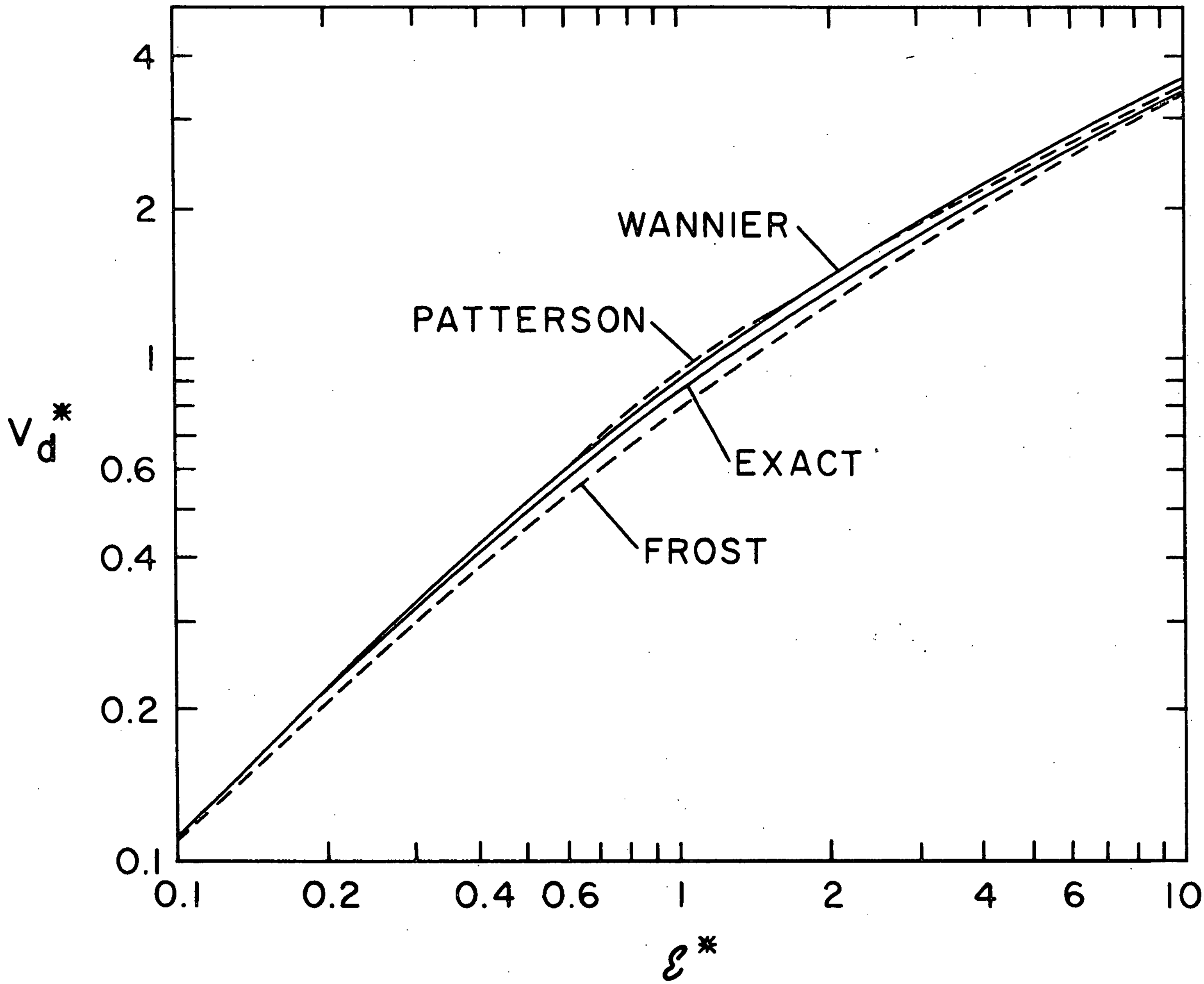
Table IV. Test of the extended free-flight theory for medium fields, when the high-field result is forced to be correct. The expansion coefficient β_1 appears in Eq. (14), and the potential is $V(r) = C/r^n$.

n	β_1(approx.)/β_1(accurate)		
	m>>M	m = M	m<<M
8	2.209		1.103
10	1.841		0.879
12	1.657		0.767
25	1.315		0.558
50	1.201		0.488
∞	1.104	0.742	0.430

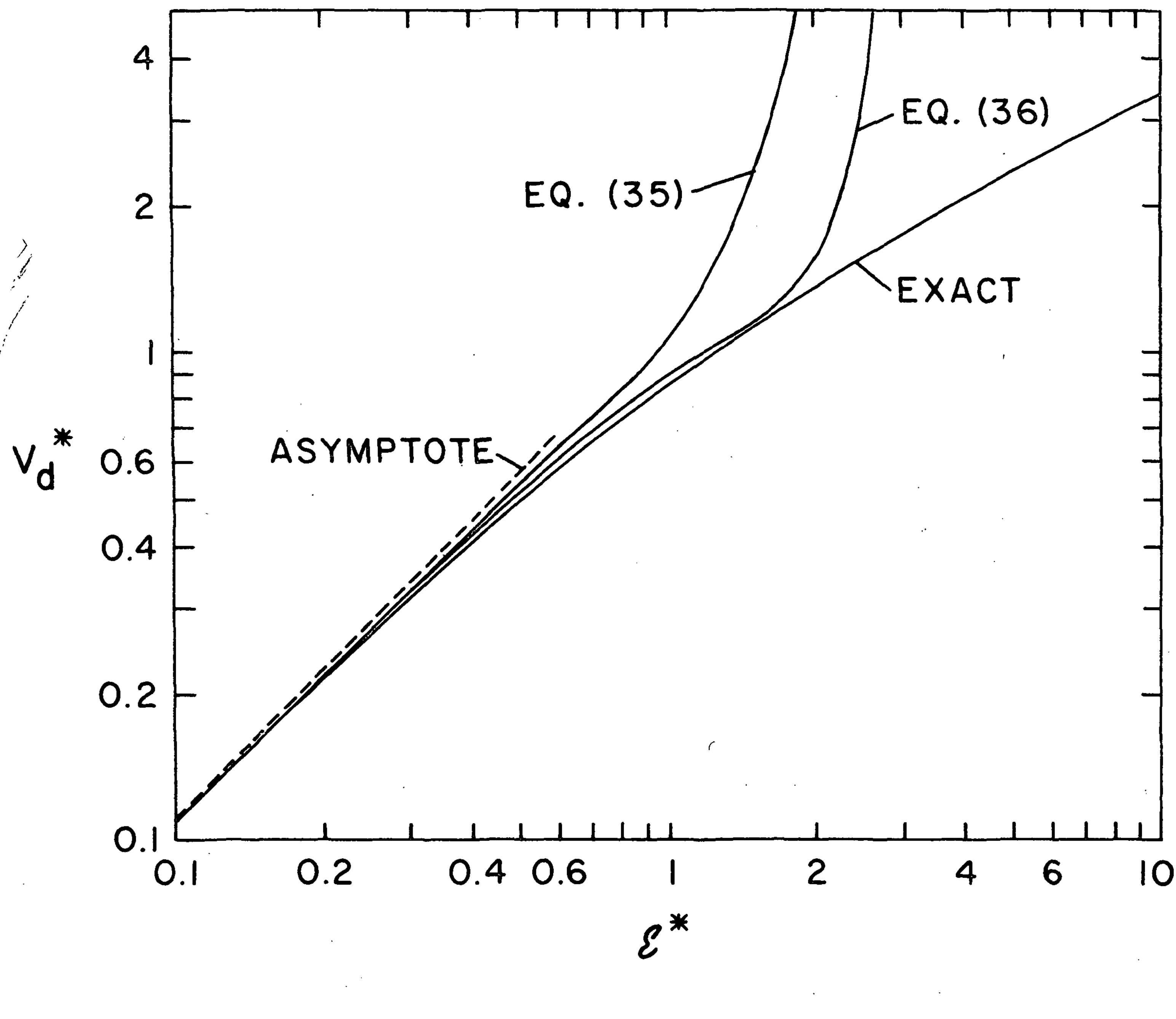
FIGURE CAPTIONS

Fig. 1. Reduced drift velocity as a function of reduced field strength for rigid spheres with $m \ll M$. The two solid curves are the exact numerical results (Table II) and Wannier's free-flight Eq. (7). The two dashed curves are the Frost and Patterson empirical formulas given by Eqs. (15) and (16), respectively.

Fig. 2. Reduced drift velocity as a function of reduced field strength for rigid spheres with $m \ll M$. The exact curves represents the numerical results of Table II, and the other two curves are the Kihara expansions in powers of $(E/N)^2$ as given by Eqs. (35) and (36).



WANNIER
PATTERSON
EXACT
FROST
4
2
1
0.6
0.4
0.2
0.1
V_d^*
0.1
0.2
0.4
0.6
1
2
4
6
10
$\mathcal{E}^*$



EQ. (35)
EQ. (36)
EXACT
ASYMPTOTE
V_d^*
4
2
1
0.6
0.4
0.2
0.1
0.1
0.2
0.4
0.6
1
2
4
6
10
$\mathcal{E}^*$

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EAM/sw