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ATTITUDE STABILITY OF SPINNING FLEXIBLE SPACECRAFT

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Final Report

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ABSTRACT

The study defined under Contract NAS8-26214 required two parallel efforts: 1) Determination of literal (non-numerical) attitude stability criteria for idealized spinning flexible spacecraft, and 2) Analytical support of a Marshall Space Flight Center computational study of the dynamics of Skylab B. The latter study is continuing, while the former has been the basis of the Ph. D. dissertation of Frank J. Barbera, which constitutes the present report. The scope of this report is described as follows.

The stability of spinning flexible satellites in a force-free environment is analyzed. The satellite is modeled as a rigid core having attached to it a flexible appendage idealized as a collection of particles (point masses) interconnected by springs. Both Liapunov and Routh-Hurwitz stability procedures are used where in the former the Hamiltonian of the system, constrained through the angular momentum integral so as to admit complete damping, is used as a testing function. Equations of motion are written using the hybrid coordinate formulation, which readily accepts a modal coordinate transformation ultimately allowing truncation to a level amenable to literal stability analysis. Closed form stability criteria are generated for the first mode of a restricted appendage model lying in a plane containing the system center of mass and orthogonal to the spin axis.

The effects of spin on flexible bodies are discussed by considering a very elementary particle model. It is shown how the linearized equations of motion for a non-spinning flexible appendage are modified by spin. In particular, it is shown how appendage mounting can greatly influence the natural frequencies of the structure. Stability criteria for the simple particle model are used to duplicate the results of various published studies. Results of analysis of this same particle model are used to draw engineering judgments on a very complex spacecraft, Skylab B.

Control of passively unstable spacecraft is briefly considered. It is shown how a simple rigid rotor directed along the spin axis, or a proportional controller representative of control moment gyros (in the first approximation), can enhance stability.

CONCLUSIONS AND RECOMMENDATIONS

The literal stability criteria developed in this report represent unanticipated progress in the search for closed-form conditions for attitude stability of spinning flexible spacecraft. These criteria have a substantial utility in preliminary design of flexible appendages to be attached to spinning spacecraft; they can be used for example to provide a lower bound on stiffness requirements for the boom structure on which masses must be deployed from Skylab B in order to shift the vehicle principal axis of maximum inertia into orthogonality with the solar panels. For a much simpler spacecraft (such as the crossed-dipole configuration of Explorer XX), the multiple-mode stability criteria available from this report may represent a definitive conclusion, but for spacecraft as complex as Skylab these criteria are merely preliminary. Yet it seems that the practical limits of algebraic complexity have been reached in this study, so that further progress in stability analysis must be accomplished with numerical methods.

Results of significance comparable to the stability criteria lie in the interpretations of the influence of spin on the modal vibrations of appendages of various structural configurations. Careful physical interpretation of the behavior of highly idealized appendages suggests for example, that the solar panels on the Orbital Workshop of Skylab B would be less deleterious to stability if they were unfurled radially from the vehicle spin axis, rather than tangentially from circles of varying diameters centered on the spin axis. This conclusion comes too late to influence the Skylab B solar panel configuration, but it may serve as a practical guide in the design of future spacecraft. Furthermore, these conceptual conclusions may help in the evaluation of numerical results being generated for realistic models of solar panels on Skylab B or subsequent spacecraft designs.

The methods employed in this report are limited primarily by their reliance on vehicle mathematical models consisting of rigid bodies with flexible appendages. The primary recommendation for further work is the proposal that a more general formulation be developed, permitting the modeling of a spacecraft as a collection of arbitrarily interconnected elastic substructures.

CONTENTS

FIGURES

TABLES

NOMENCLATURE

CM	~	center of mass
N	~	location of system CM when steadily spinning
O	~	location of system CM when non-spinning
Body B	~	portion of system identified as rigid (core)
Body A	~	portion of system identified as flexible (appendage)
$\{\underline{b}\}$	~	vector basis fixed in Body B identified as $(\hat{\underline{b}}_1, \hat{\underline{b}}_2, \hat{\underline{b}}_3)^T = (\hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}})^T$
$\{\underline{n}\}$	~	inertial vector basis
Q	~	point at interface between Body A and Body B
$\underline{c}$	~	vector from CM to N
$\underline{c}'$	~	vector from N to O
$\underline{R}'$	~	vector from O to Q
$\underline{R}$	$\equiv$	$\underline{c}' + \underline{R}'$
$\underline{r}^{i'}$	~	vector from Q to location of i^{th} mass when system is not spinning
$\underline{\Delta}^i$	~	displacement vector of i^{th} mass induced by steady spin
$\underline{r}^i$	=	$\underline{r}^{i'} + \underline{\Delta}^i$
$\underline{u}^i, u^i$	~	deformation vector and its representation in $\{\underline{b}\}$
$\underline{\Gamma}^i, \Gamma^i$	~	undeformed location of i^{th} particle from point N, and its representation in $\{\underline{b}\}$. Note: $\underline{\Gamma}^i = \underline{R} + \underline{r}^i$
$\underline{\rho}$	~	body generic position vector
$\underline{a}$	~	inertial generic position vector
dm	~	differential mass element
H	~	Hamiltonian
T	~	kinetic energy of complete system
V	~	potential energy of complete system
$\underline{\omega}, \omega$	~	inertial angular velocity vector of $\{\underline{b}\}$ and its representation in $\{\underline{b}\}$

$\underline{w}$	=	$\underline{\omega} + \underline{\Omega}$
Θ	~	orthogonal transformation relating $\{\underline{b}\}$ to $\{\underline{n}\}$
$\theta_1, \theta_2, \theta_3$	~	Euler (attitude) angles
$\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$	~	unloaded natural frequencies of simple particle model
$\sigma_x, \sigma_y, \sigma_z$	~	loaded natural frequencies of simple particle model
$\omega_x, \omega_y, \omega_z$	~	loaded natural frequencies of simple particle model including centripetal acceleration terms
ω_i	~	loaded natural frequency of i^{th} modal coordinate
S	~	Laplacian operator
$\underline{\underline{I}}^N, I^N$	~	inertia dyadic of complete system about point N and its inertia matrix representation in $\{\underline{b}\}$
$\underline{\underline{I}}_B^N, I_B^N$	~	inertia dyadic of core about point N and its inertia matrix representation in $\{\underline{b}\}$
$\underline{\underline{I}}_A^N, I_A^N$	~	inertia dyadic of appendage about point N and its inertia matrix representation in $\{\underline{b}\}$
$\underline{\underline{I}}_{A'}^N, I_{A'}^N$	~	inertia dyadic of undeformed appendage contribution about point N and its inertia matrix representation in $\{\underline{b}\}$
$I_{A_1}^N$	~	inertia matrix about point N consisting of first order appendage terms
$I_{A_2}^N$	~	inertia matrix about point N consisting of second order appendage terms
I_O^N	~	$I_{A'}^N + I_B^N$ having principal elements A, B, and C
A', B', C'	~	core inertias of simple particle model
m_i	~	mass of i^{th} element
$\mathcal{M}$	~	mass of complete system

$\underline{h}$, ${}^{b}h$, h		angular momentum vector, its representation in $\{\underline{b}\}$, and its magnitude
$\underline{\underline{E}}$, E		identity dyadic and its corresponding identity matrix
M	~	mass matrix excluding CM shifts
M'	~	mass matrix including CM shifts
G	~	skew symmetric Coriolis coupling matrix
$\hat{K}$	~	non-spinning stiffness matrix
K	~	spinning stiffness matrix
q	~	deformation column matrix
$\underline{\Omega}$, Ω	~	spin vector and its representation in vector basis $\{\underline{b}\}$ assumed to be solely directed along $\hat{b}_3$
F_t	~	spin force in simple particle models
$F_{I, II, III}$	~	preload force on examples I, II, and III, respectively
η	~	column matrix of modal coordinates
Φ	~	transformation matrix relating η to q
Γ_o	~	location of particle (in simple particle model) prior to spin
ζ	~	damping ratio
$\underline{E}$	~	N by 1 column matrix with all elements unity
Γ_x	~	N by N diagonal matrix with Γ_x^i (i = 1 to N) as its elements
Γ_y	~	N by N diagonal matrix with Γ_y^i (i = 1 to N) as its elements
δ_x	=	$\Phi^T M \Gamma_x \underline{E}$ an N by 1 column matrix
δ_y	=	$\Phi^T M \Gamma_y \underline{E}$ an N by 1 column matrix
$\bar{\Gamma}_x, \bar{\Gamma}_y, \bar{\Gamma}_z$	~	location of particle (in simple particle model) subsequent to spin

CHAPTER 1

STATEMENT OF THE PROBLEM

The concern with flexibility on spinning spacraft came into prominence with the conclusion that flexible whip antennas on Explorer I were responsible for its unexpected dynamic behavior (See Reference 1; within the period of approximately 90 minutes Explorer I departed from a motion approximating spin about an inertially fixed axis to a motion of free precession with half cone angle approaching 60 degrees.) Ever since the anomalistic behavior of Explorer I the subject of flexibility effects on spinning spacecraft has been the basis of numerous technical papers as well as internal company reports. The results of these studies have proved beneficial from both an educational and a practical point of view.

Prior to the flight of Explorer I it had been generally accepted that satellites would exhibit stable free rotation in inertial space if the angular velocity vector was directed parallel to a principal axis of either a maximum or minimum moment of inertia, as predicted by rigid body stability analysis. Explorer I conformed to this rule in that its spin axis was the axis of minimum moment of inertia, and in fact the spacecraft was inertially symmetric about the axis of spin, exhibiting a longitudinal to transverse inertia ratio of approximately 70 to 1.

The analysis following Explorer I (Reference 1 and many subsequent studies) led to the general conclusion that for a flexible spinning satellite to exhibit stable free motion its axis of spin must be restricted to that of the principal axis of maximum inertia; this proposition is sometimes referred to as "the greatest moment of inertia rule." The "energy sink" method was the dominating analytical tool utilized in developing this result. The method is simply to model the spacecraft, as a rigid body having attached an idealized non-moveable mechanical energy dissipator, i.e. an energy sink. Analysis based upon energy sink methods resulted in the conclusion that the motion must ultimately be characterized by spinning in its minimum kinetic energy state. This minimum energy state corresponds to spin about the principal axis of maximum moment of inertia. Although the energy sink method is not analytically rigorous the approach is generally accepted by the engineering community since its results conform with the behavior observed in actual flight. Hence the only stable rotational motion for free quasirigid* dissipative bodies is widely understood to be spin about an inertially fixed principal axis of maximum moment of inertia. Any formal, rigorous analytical technique used to establish stability criteria for flexible spinning spacecraft must be expected to produce this conclusion as a necessary condition.

The requirement of maximum inertia axis spin for attitude stability could have significant impact on spacecraft design even though the spacecraft in its nominal on-orbit mode of operation is not spinning. For many designs it is desirable to spin the spacecraft throughout its transfer ellipse (generally a Hohmann transfer) to preclude the necessity of active control during the period of high torques imparted to the spacecraft by burn of the apogee motor. Moreover booster designs generally favor elongated spacecraft, so that it is more convenient to spin about the axis of <u>least</u> inertia. These considerations dictate the need for an estimate of energy dissipation to determine if the coning angle growth is accepted over the interval of spin. Energy sink methods have helped in these estimates as well.

* The term "quasirigid" means nonrigid but subject only to small relative motions; this term therefore excludes rotors from the vehicle, for example.

Clearly the analyses conducted since Explorer I have proved beneficial to spacecraft design, but the fact is that the ghost of Explorer I still remains to haunt us. Since Explorer I there have been at least four other spinning (or partially spinning) satellites that in some way or another have exhibited degraded performance attributed directly to flexibility effects – the latest of which is ATS 5 (Advanced Technology Satellite). (See Reference 2.)

It is clear that flexible spinning spacecraft will continue to be designed and flown in the future. It is expected that many of these satellites will exhibit large flexible appendages such as antenna arrays or solar panels. Skylab B is a prime example of the latter. The knowledge of previous studies coupled with the behavior of flown spinning satellites will influence these designs. The dominant influence will of course be "the greater moment of inertia rule." However, the question remains as to whether satisfaction of this rule is sufficient to assure stability of such spacecraft, or whether there are other more demanding criteria required to be satisfied. The search for and development of these latter criteria constitutes the focal point of this dissertation.

That other criteria exist in addition to the requirement for maximum inertia axis spin, is not at all surprising, in that one would expect the stability of spinning spacecraft with very large flexible appendages to be more precarious than spacecraft only slightly non-rigid. Moreover we would expect criteria to emerge which involve the modes of vibration and the natural frequencies of the structure.

Probably the earliest attempt to explore the dynamics of elastic spinning spacecraft was that by F. Buckens in 1963, Reference (3), and again in mid 1967, Reference (4). His approach in investigating the consequences of spin on flexible satellites was to analyze equations of motion linearized about a constant spin. In the first of these papers Buckens shows how the elastic modes couple with the normal rigid body modes. Moreover he shows how the system natural frequencies (coupled modes) may, in the presence of flexibility, be lower than the rigid body modes, and thus suggest a corresponding loss in stability. However, it was not until Reference (4) that Buckens directly considered the question of stability. Although he developed stability criteria for undamped motion, his analysis with damping present was limited to evaluating the frequency of the structure at the verge of instability. He ultimately concludes. . . . "that the nutational modes are also damped when damping exists in the elastic deformation modes, but the corresponding damping is very low when one of these frequencies becomes very small, which brings the system at the verge of instability." However in all of Bucken's work it is assumed that the spin frequency be very much smaller than the lowest natural frequency at the elastic structure — a severe limitation. Nevertheless his studies appear to be the analytical forerunner in the area of spinning flexible spacecraft. In fact Buckens extended his work to consider the effects of external torques including those induced by interaction with the earth's magnetic and gravitational fields. (No consideration of this subject is attempted in this dissertation.)

In August 1969 a NASA technical note appeared, Reference (5). This study by T. W. Flatley was directed toward the stability of a spacecraft idealized as a rigid body having attached to it four symmetrically mounted flexible booms, commonly referred to as a crossed-dipole configuration. The idealization was (and is) directly applicable to a number of satellites, e.g., Alouette I and II. The results of this study led to useful stability criteria descriptive of the wobble motion of spacecraft identified by the cited idealization.

Flatley's approach to the problem was based upon energy considerations. He considered a rotation where the spacecraft is forced to spin about an axis skewed from the nominal spin axis with the booms contributing to the total energy through strain energy storage, i.e., potential energy. He compared this state with a nominal spin state having an equivalent angular momentum and no potential energy. Flatley then developed stability criteria by suggesting that if the nominal state is stable then the energy associated with that state must be less than the energy associated with the forced state. The potential energy is derived using beam deflection theory and inertias of both states are developed by integrating over the structure. Then with the aid of a digital computer stability boundaries were generated and presented in the form of both curves and tables. The results clearly show how stability is degraded as the ratio of spin to radial boom stiffness increases.

In March of 1970 F.R. Vigneron presented a paper, Reference (6), considering the same model as Flatley, i.e., a crossed-dipole configuration. However his approach to the problem was completely different. Vigneron linearized the equations of motion and then simplified the set by noting that the equations descriptive of the nutational behavior (wobble motion) separate from the equations descriptive of motion about the spin axis. The former set, after simplification of the vibrational equations to a single mode, were then analyzed with the aid of a digital computer using Routh-Hurwitz stability criteria. The results are presented in the form of graphs delineating regions of stable and unstable motions. Moreover, he observed that the boundaries were separated by a relatively simple expression involving system inertias, the natural frequency of the truncated mode, and the magnitude of spin. Vigneron's work was a clear presentation on the subject leaving little doubt in the minds of the reader concerning the validity of the results.

The subject matter was again studied by J. E. Rakowski and M. L. Renard, presented at the Astrodynamics Conference in August of 1970, Reference (7). Similarly to the work of Flatley, Rakowski and Renard studied the nutational behavior of a torque free satellite through energy considerations. As with their predecessors they also directed their efforts toward a crossed-dipole configuration, and with the aid of a digital computer presented stability boundaries. However in order to develop these results they resorted to actually solving the equations of motion retaining all nonlinear terms – a needless effort. Probably their greatest contribution is in stressing the fact that rotation must be considered in developing the modes of a vibrating appendage. They rightfully point out that the modes normally associated with a non-rotating contelevered boom are inapplicable and, especially for large ratios of spin to stiffness, the non-rotating modes may be totally erroneous.

In the same conference in which Rakowski and Renard presented their results another paper on the subject appeared, Reference (8). This paper by L. Meirovitch presented a procedure for developing stability criteria for flexible spinning satellites by using the Hamilitonian as a Liapunov function. The paper was directed toward the exposition of a general procedure for obtaining sufficient conditions for stability and specific stability results were generated for a rigid structure having attached to it a pair of symmetrical booms directed along the spin axis. The results of this paper laid the framework for Meirovitch's more general study in which he, in conjunction with R. A. Calico, presented a procedure for generating sharper conditions for stability; this paper was presented at the

Astrodynamics Specialist Conference exactly one year later (August 1971), Reference (9). In this paper stability criteria are developed for a mathematical idealization consisting of three pairs of rigidly attached flexible rods, one pair along the spin axis and the other two pair in a crossed-dipole configuration. Thus the model of Reference (9) is a combination of the model considered in Reference (8) and the model considered by both Flatley and Vigneron. Using the Hamiltonian as a Liapunov function Calico and Meirovitch proceeded to develop stability criteria of this three-pair boom model.

In addition to the references cited above the very recent work by P. Y. Willems, Reference (10) must be mentioned. In this paper, the author considers the attitude stability of a deformable gyrostat (a dual spin satellite idealization). The effect of dissipation in either section of the system is discussed and a rigorous method permitting stability determination is set forth wherein the Hamiltonian is used as a Liapunov function. (It is precisely this method which forms the analytical foundation of our Chapter 3.) To preclude algebraic difficulties Willems limits his internal coordinates descriptive of deformations to a single modal coordinate variable which is not uniquely defined, but belongs to a prescribed class. Procedures for selecting a specific modal coordinate for a specific system are not discussed, this being set aside as a separate problem underlying the stability analysis presented in Reference (10). Willems directed his attention primarily to that portion of the stability criteria descriptive of inertia constraints associated with dual spin spacecraft, but he generates a condition on the internal stiffness properties as well. As Willems did not extend his stiffness dependent criteria to specific geometrically identifiable terms (these depending on the choice of modal coordinate) it is difficult to compare his results with ours, although conceptual similarities are identifiable.

To date then a number of studies, have been directed toward the subject of spinning flexible spacecraft. Moreover the developed stability criteria provide useful data to both the theoretician and the practicing engineer. What then can we offer to either enhance or augment these results? First of all it is clear that the cited studies (except for the early work of Buckens and the recent work of Willems) have been limited to pairs of booms. In this study we shall try to develop stability criteria for a general flexible appendage. Although we will succeed in developing a Liapunov function for the general problem, in testing that function we will be forced to specialize to the more restrictive case wherein the appendage lies in a plane containing the center of mass and orthogonal to the spin axis. We will examine this problem using both Liapunov and Routh-Hurwitz stability procedures. As in References (8), (9), and (10), the Hamiltonian is used as a Liapunov function: Chapters 3 and 4 are directed toward this effort. However, even more than this, the major contribution may very well be the analysis delineated in Chapter 2. Here we examine the effects of spin and in particular point out the significance of appendage mounting; this is a point which can be completely overlooked by restricting analysis to radial booms. Moreover if one tries to generalize the radial boom results to more complicated structures the conclusions may be totally erroneous, and in fact could lead to disastrous results in practice. Also in Chapter 2 we analyze a very elementary particle model and through suitable interpretation we show how this simple model can be used to obtain literal (nonnumerical) results which essentially duplicate the results of Flatley, Vigneron, Rokowski and Renard, and Calico and Meirovitch.* Thus these studies are tied together

*As Calico and Meirovitch considered the out-of-plane as well as the in-plane problem only a portion of their results is duplicated. We consider the out-of-plane problem in Chapter 4 only superficially.

through the analysis of a simple particle model. The simplicity of the model allows easy visualization and yet retains all the pertinent features of spin-flexibility interaction.

As the order of business is the stability analysis of rotating flexible spacecraft, we should at the outset investigate the concept of stability (or probably more significantly the concept of instability) as pertaining to the motions of interest. If one considers the consequences of a small perturbation applied to a rigid body initially rotating about a principal axis (major or minor) colinear with its inertial angular momentum vector, the resulting motion is characterized by a rotation of its spin axis about a new inertially directed angular momentum vector. (The new angular momentum vector is of course the vector sum of the initial angular momentum vector and the momentum vector introduced by the perturbation.) The spacecraft is said to "wobble"* and the angle between the spin axis and the momentum vector after perturbation is called "the wobble angle." Because subsequent to sufficiently small perturbation this wobble angle remains smaller than any preassigned value, the motion is said to be <u>Liapunov stable</u>.

It is predicted by heuristic energy sink methods of analysis that in the presence of a hypothetical non-moveable energy dissipator the spin axis will converge to the new angular momentum vector**

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*Sometimes called precession, nutation, or even coning; although the latter is usually restricted to the descriptive motion of axis-symmetric spacecraft.

**That the spin axis after perturbation converges to the new angular momentum vector (as apposed to the momentum vector prior to perturbation) assures that the system is not completely damped when the motion is described in terms of coordinates measuring the deviation from the inertial orientation existing prior to perturbation. As discussed in Chapter 3 this observation dictates significant constraints on the usage of the Hamiltonian as a Liapunov function.

if the spin axis is the axis of maximum moment of inertia, i.e., the wobble angle decreases with time. Conversely, if the spin axis is the axis of minimum moment of inertia the wobble angle will increase with time. In the former case the motion is said to be asymptotically stable whereas in the latter case the motion is said to be unstable. However, this type of instability is acceptable for many applications, i.e., spacecraft are still allowed to spin about their axis or least moment of inertia for short periods of time, such as during transfer ellipse. This of course requires the estimation of the energy dissipation in the structure (a black art indeed, especially when the spacecraft contains either fuel tanks or heat pipes) to determine or at least bound the amount of wobble angle increase over the time interval of spin. Thus this type of instability may very well be acceptable. In contrast, consider for example the type of instability associated with spin about the principal axis of intermediate moment of inertia. Here the motion after perturbation is characterized by a relatively violent departure of the spin axis from its initial state, more appropriately described as tumbling, and unstable by any definition. The influence of a flexible (and dissipative) appendage on a spinning spacecraft might cause either kind of instability, or might alternatively be manifested only as violent and potentially destructive oscillations of a small structural component, without significant influence on vehicle rotations. In this dissertation only local stability characteristics are examined; in order to explore the nature of instabilities detected here, nonlinear simulations would appear to be necessary.

The equations of motion analyzed in the sequel are a subset of the more general set developed in Reference (11) identified as a hybrid-coordinate formulation; this leads to equations of motion expressed in terms of a combination of discrete coordinates, describing the arbitrary rotational motions of the rigid bodies, and distributed or modal coordinates describing the small, time varying deformations of the flexible appendage; thus truncation of the normal mode equations can be accomplished to the level required for any particular application. In the text we shall on occassion truncate these equations to a single mode to reduce the number of coordinates in the system to a level amenable to literal stability analysis. In our formulation we restrict the flexible appendange equation to consist solely of a collection of particles, as opposed to a collection of bodies (and particles) or even the more general finite element approach. This idealization allows simplicity in both nomenclature and formulation, thus permitting concentration on the primary purpose of the text, i.e., spin-flexibility interaction. Although it should be pointed out that even the most general finite element approach results in equations of motion having a similar form, see Reference (12). The equations of motion analyzed in the sequel are derived in Appendix I.

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CHAPTER 2

SIMPLE PARTICLE MODELS

As the purpose of this dissertation is the investigation of spinning flexible spacecraft, in particular their stability behavior, it is desirable at the outset to examine the effects of spin-flexibility interaction. As such, much of the material in this chapter will be directed toward an elementary model.... elementary enough to provide both visibility and insight, yet with surprising sophistication allowing meaningful conclusions. The point to be made is simply that the influence of spin on flexible spacecraft may have a significant effect and that generalizations from what has been developed in the literature may be misleading. Indeed, for the simple model to be considered it will be shown that, depending upon the configuration (orientation of the flexible appendage with respect to the rigid core), spin effects may be either stabilizing or destabilizing. Much of the material presented in this chapter was developed jointly by the author and Mr. V. Baddeley and documented in a North American Rockwell internal report.*

EFFECTS OF SPIN

In Reference (11) it is shown that for non-spinning spacecraft, the linearized homogeneous matrix equation descriptive of node deformations in a vector basis fixed in the undeformed flexible appendage, may be written as

$$M \ddot{q} + \hat{K} q = 0 \qquad (1)$$

where M is the system mass matrix and $\hat{K}$ is the (non-spinning) stiffness matrix. The effect of spin will be to alter this equation in three ways:

*F. J. Barbera and V. Baddeley, "Effects of Solar Panel Flexibility," No. American Rockwell IL No. 192-405-70-058, Sept. 1970.

a) Preload Effect

b) Centripetal Acceleration

c) Coriolis Coupling

As will be shown the latter two effects will fall out naturally as a consequence of the dynamic formulation, and their effects on the natural frequencies of the system are predictable in a general fashion without regard to configuration. Conversely, the preload effect is configuration dependent, requiring a re-examination of structural properties.... it may or may not seriously modify the stiffness properties of the structure.

Centripetal acceleration will always tend to decrease the natural frequencies of the structure, i.e., a "softening" effect. On the other hand Coriolis coupling will actually increase some of the natural frequencies and simultaneously decrease others, independent of the configuration design, and as cited earlier, preload also may either soften or stiffen the structure; however, preload <u>is</u> configuration dependent.

The net effect of spin then is a three-fold modification of Equation (1) to the following:

$$M \ddot{q} + G \dot{q} + K q = 0 \qquad (2)$$

where the combination of preload and centripetal acceleration has altered the non-spinning stiffness matrix $\hat{K}$ to the spinning stiffness matrix K, and coriolis coupling has introduced the skew symmetric matrix G. Although herein the effect of the addition of G is considered only superficially its contribution should not be taken lightly. In fact modal analysis techniques, required to permit the <u>engineering necessity</u> of coordinate truncation, are significantly altered by the presence of G; these considerations are examined in depth in References (11) and (12).

In order to permit full appreciation of the consequences of these effects a simple model will be introduced and analyzed in detail. The simple model, shown in Figure (1), is described by a rigid core to which two particles are attached through springs. The principal axes $(\hat{x}, \hat{y}, \hat{z})$ of the system in its nominal state remain coincident with the principal axes of the core, as the particles, each of mass m, are symmetrically located a distance $\bar{r}_y$ along the $\hat{y}$ axis under steady spin. The particles are permitted to displace radially along the $\hat{y}$ axis, as well as rotationally through a two degree of freedom pivot at the attachment point allowing deformations in the $\hat{x}$ and $\hat{z}$ directions as well. (For easier visualization each particle is constrained to lie within a massless tube denoted by the dashed lines.) The non-spinning stiffness elements of the model are denoted by $\hat{k}_x$, $\hat{k}_y$, and $\hat{k}_z$ with corresponding nonspinning natural frequencies $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$. The spin, directed along the positive $\hat{z}$ axis, is denoted by Ω.

In the present discussion the simple particle model will be allowed significant configuration variations. In particular, to fully demonstrate the consequence of the effects discussed above, three examples will be considered, as shown in Figure 1A. In each case $\bar{r}_y$ denotes the steady spin particle location. To examine these effects we shall introduce deformations from steady spin for each of the three examples. Consider the structure of Example 1 both prior to and after a deformation u_z in the $\hat{z}$ direction (see Figure 1B).

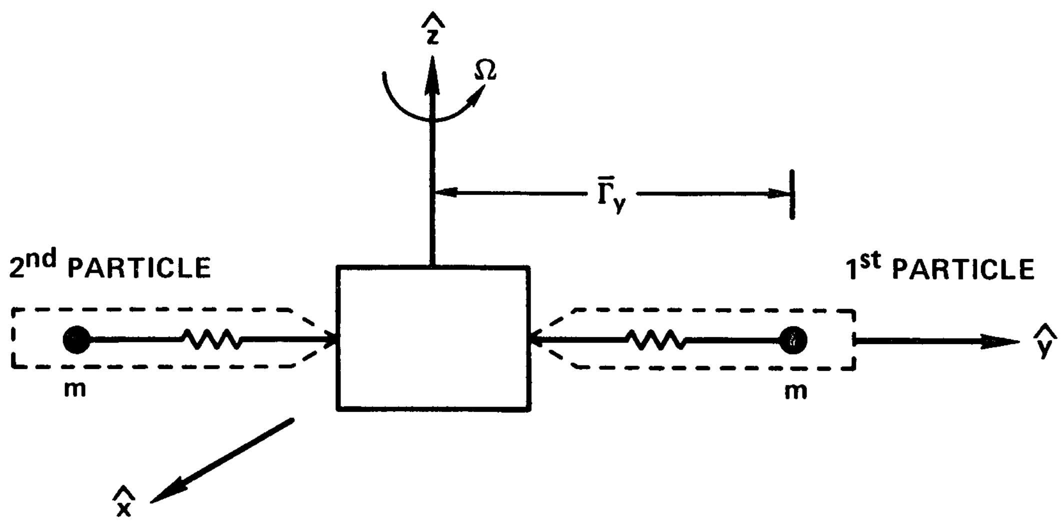


Figure 1. Simple Particle Model

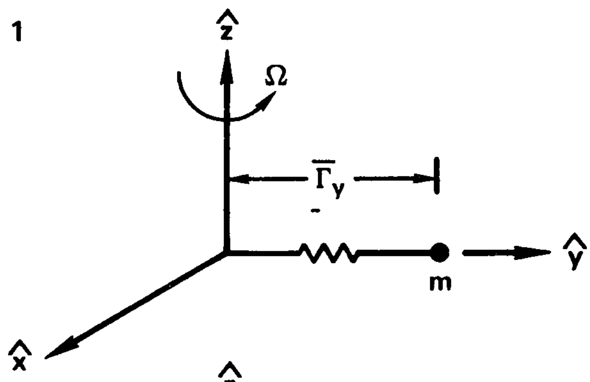
$\bar{r}_y$ ~ steady state particle locations

Ω ~ spin rate

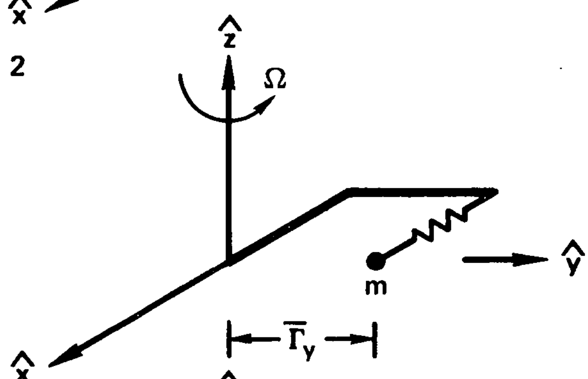
$\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ ~ natural frequencies without spin

$\hat{k}_x, \hat{k}_y, \hat{k}_z$ ~ non-spinning stiffness elements of K

EXAMPLE 1



EXAMPLE 2



EXAMPLE 3

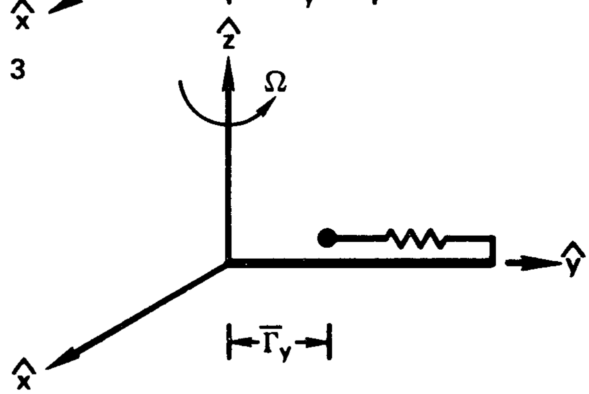


Figure 1A. Alternative Configurations

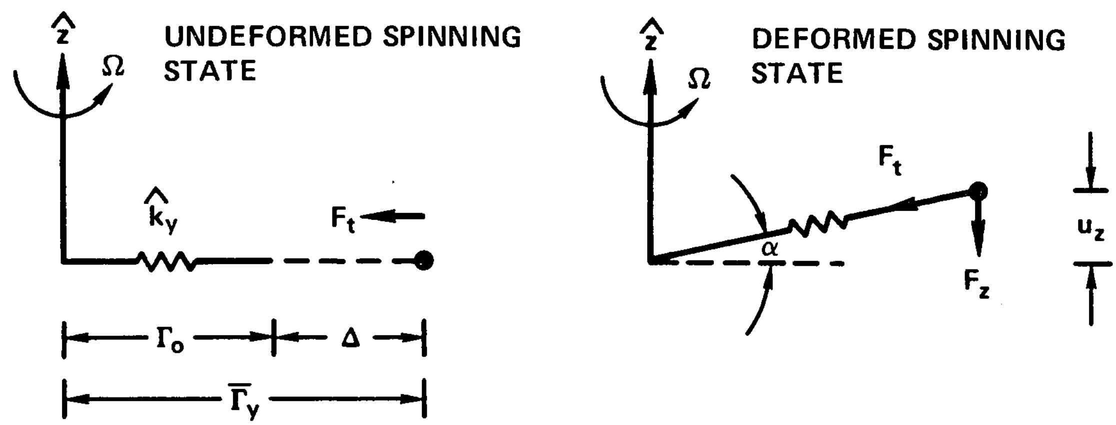


Figure 1B. Example I

As $\overline{\Gamma}_y$ is representative of the particle location under steady spin, it is comprised of a non-spinning contribution Γ_o dictated by the configuration, and a (stretch) contribution Δ induced by spin. Following Newton's Law the force on the particle under steady spin $-\hat{k}_y \Delta$ must be identical to the product of mass and acceleration $-m\Omega^2(\Gamma_o + \Delta)$, permitting the evaluation of Δ as

$$\Delta = \frac{m \Gamma_o \Omega^2}{\hat{k}_y - m \Omega^2}$$

Moreover the tension in the span F_t is precisely $m\Omega^2(\Gamma_o + \Delta)$. Clearly for a finite distortion due to spin the inequality $\hat{k}_y - m\Omega^2 > 0$ must be satisfied.

When the particle is displaced an amount u_z, the spring force $-\hat{k}_z u_z$ in the $\hat{z}$ direction is augmented by a force F_z in the negative $\hat{z}$ direction, as given by

$$F_z \approx F_t \sin\alpha \approx F_t \frac{u_z}{\overline{\Gamma}_y} = m\Omega^2 u_z$$

Similar reasoning shows that a u_x deformation results in a force in the $\hat{x}$ direction given by $-\hat{k}_x u_x - m\Omega^2 u_x$.

The total force on the particle is then

$$F_I = -\begin{pmatrix} \hat{k}_x + m\Omega^2 & 0 & 0 \\ 0 & \hat{k}_y & 0 \\ 0 & 0 & \hat{k}_z + m\Omega^2 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} - \begin{pmatrix} 0 \\ m\bar{\Gamma}_y \Omega^2 \\ 0 \end{pmatrix} .$$

Clearly for particles configured as in Example I the effect of preload is to stiffen the structure in the $\hat{x}$ and $\hat{z}$ directions. A practical example is offered by a helicopter blade which essentially is free of stiffness in the direction of spin yet when rotating is observed to vibrate at the spin frequency (i. e., the square root of the augmented stiffness $m\Omega^2$ divided by the mass).

In the above example it is shown how preload augments the stiffness properties; however, not always are the natural frequencies increased when the structure spins. To demonstrate this consider Example III both prior to and after a u_z deformation.

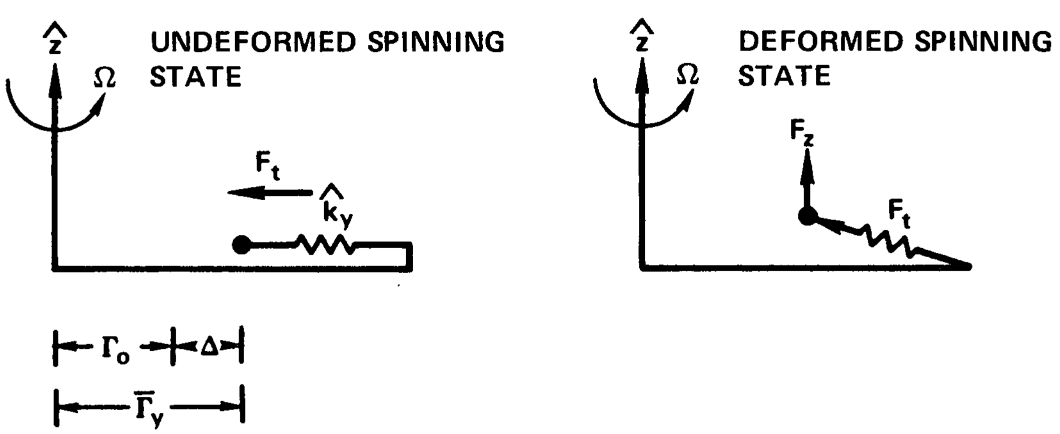


Figure 1C. Example III

As in Example I the spin distortion Δ is prescribed by

$$\Delta = \frac{m\,\Gamma_o \Omega^2}{\hat{k}_y - m\Omega^2}$$

and the force in the span F_t remains $m\,\bar{\Gamma}_y\Omega^2$. Again the influence of spin introduces the force $m\Omega^2 u_z$ equivalent in magnitude to that observed in Example I but now however directed in the <u>positive</u> $\hat{z}$ direction. When combined with the unloaded stiffness $\hat{k}_z$ the total stiffness impeding u_z deformations is then $\hat{k}_z - m\Omega^2$. A similar effect is observed when u_x deformations are introduced, permitting the total force on the particle to be written as

$$F_{III} = -\begin{pmatrix} \hat{k}_x - m\Omega^2 & 0 & 0 \\ 0 & \hat{k}_y & 0 \\ 0 & 0 & \hat{k}_z - m\Omega^2 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} - \begin{pmatrix} 0 \\ m\bar{\Gamma}_y\Omega^2 \\ 0 \end{pmatrix}$$

In contrast to Example I, the effect of spin preload in Example III is to decrease the stiffness properties (and hence the natural frequencies) in directions $\hat{x}$ and $\hat{z}$.

Clearly the cantilevered particle of Example I and the anti-cantilevered particle of Example III exhibit grossly different stiffness properties, demonstrating the configuration-dependent effect of preload. As might be expected Example II exhibits a preload effect intermediate from Examples I and III, i.e., its stiffness matrix is unaltered by spin, to wit

$$F_{II} = -\begin{pmatrix} \hat{k}_x & 0 & 0 \\ 0 & \hat{k}_y & 0 \\ 0 & 0 & \hat{k}_z \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ y_z \end{pmatrix} - \begin{pmatrix} 0 \\ m\bar{\Gamma}_y\Omega^2 \\ 0 \end{pmatrix}$$

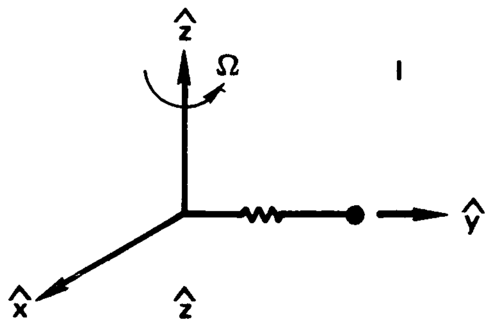
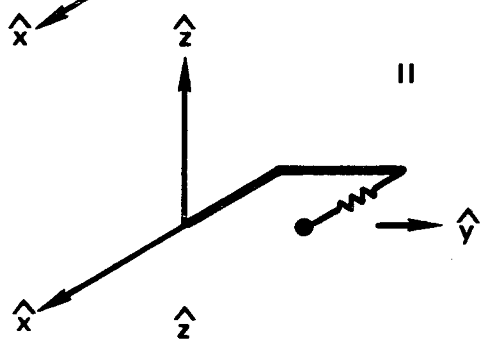
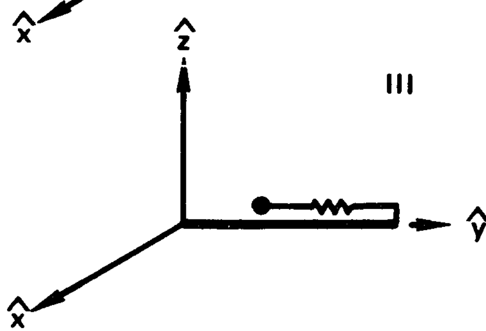
For Example I it has been shown that stiffness elements in the presence of spin were increased by $m\Omega^2$; in Example II the stiffness elements were unaltered; and in Example III stiffness elements were decreased by $m\Omega^2$. One may conclude that the effect of preload is highly configuration dependent.

The above observations allow us to speak of an unloaded stiffness matrix defined by the non-spinning stiffness properties, and a loaded stiffness matrix accounting for preload, which in this application is induced by spin. Accompanying these definitions one may also speak of loaded and unloaded natural frequencies, dictated by the frequencies observed with and without spin, respectively. As the unloaded stiffness matrix elements are identified as $\hat{k}_x$, $\hat{k}_y$, and $\hat{k}_z$, we shall similarly identify the loaded stiffness matrix elements as k_x, k_y, and k_z. The natural frequencies will be defined similarly, i.e., $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ correspond to unloaded frequencies whereas σ_x, σ_y, and σ_z correspond to their loaded counterparts. Table 1 summarizes the effects of preload on the three particle model examples. The matrices in the first column of this Table, when multiplied by the column matrix of deformations u_x, u_y, u_z from the steady state, describe the actual contact <u>forces</u> applied to the particle; changes in the effective stiffness matrix due explicitly to accelerations (rather than forces) have yet to be considered.

In addition to preload the stiffness elements are further altered by centripetal acceleration terms induced by spin. These terms, independent of configuration, are introduced about each of the two axes orthogonal to spin simply as a consequence of the dynamic formulation. Centripetal acceleration terms which are proportional to deformations act to add negatively to elements of the stiffness matrix. Hence the effect "softens" the stiffness elements, resulting in a decrease in system natural frequencies. Centripetal acceleration

Table 1

Effect of Preload

Example	Loaded Stiffness Elements	Loaded Natural Frequencies
I 	$\begin{pmatrix} \hat{k}_x + m\Omega^2 & 0 & 0 \\ 0 & \hat{k}_y & 0 \\ 0 & 0 & \hat{k}_z + m\Omega^2 \end{pmatrix}$	$\begin{pmatrix} \hat{\sigma}_x^2 + \Omega^2 & 0 & 0 \\ 0 & \hat{\sigma}_y^2 & 0 \\ 0 & 0 & \hat{\sigma}_z^2 + \Omega^2 \end{pmatrix}$
II 	$\begin{pmatrix} \hat{k}_x & 0 & 0 \\ 0 & \hat{k}_y & 0 \\ 0 & 0 & \hat{k}_z \end{pmatrix}$	$\begin{pmatrix} \hat{\sigma}_x^2 & 0 & 0 \\ 0 & \hat{\sigma}_y^2 & 0 \\ 0 & 0 & \hat{\sigma}_z^2 \end{pmatrix}$
III 	$\begin{pmatrix} \hat{k}_x - m\Omega^2 & 0 & 0 \\ 0 & \hat{k}_y & 0 \\ 0 & 0 & \hat{k}_z - m\Omega^2 \end{pmatrix}$	$\begin{pmatrix} \hat{\sigma}_x^2 - \Omega^2 & 0 & 0 \\ 0 & \hat{\sigma}_y^2 & 0 \\ 0 & 0 & \hat{\sigma}_z^2 - \Omega^2 \end{pmatrix}$

terms arise from expressions of the form ($\underline{\omega} \times \underline{\omega} \times \underline{\rho}$). In matrix notation centripetal acceleration is expressed as $\tilde{\Omega}\,\tilde{\Omega}\,\rho$. Here ρ is a three by one column matrix representing the generic position vector $\underline{\rho}$ in the body fixed vector basis ($\hat{x}, \hat{y}, \hat{z}$). In general the tilde operator prescribed over the column matrix representation of any vector $\underline{c}$ is defined as

$$\tilde{c} \triangleq \begin{pmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_x \\ -c_y & c_x & 0 \end{pmatrix}$$

In this text however the spin will always be directed along the $\hat{z}$ axis (i.e., $\underline{\Omega} = \Omega\hat{z}$) so that $\tilde{\Omega}$ is simply given by

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These definitions permit the centripetal acceleration when pre-multiplied by the mass matrix M to be written as shown below for the simple models under consideration.

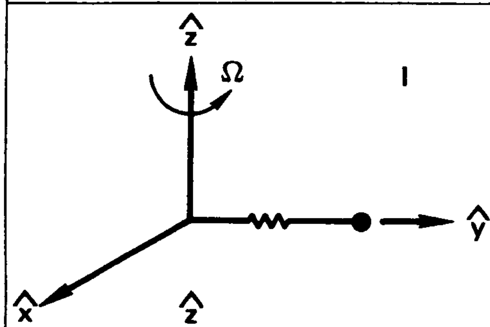
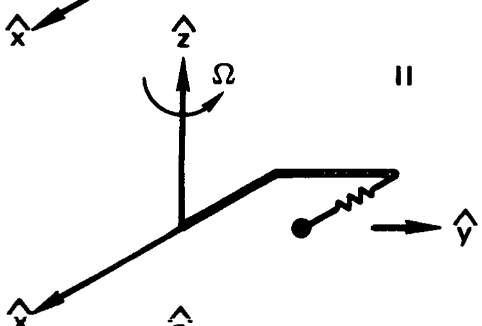
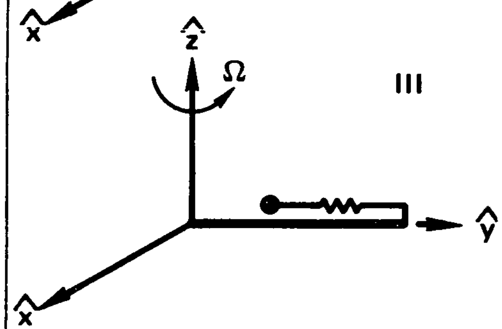
$$M\tilde{\Omega}\tilde{\Omega}\rho = -\begin{pmatrix} m\Omega^2 & 0 & 0 \\ 0 & m\Omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} - \begin{pmatrix} 0 \\ \overline{\Gamma}_y \Omega^2 \\ 0 \end{pmatrix}$$

The modified stiffness matrices for the three examples, obtained by combining the effects of both preload and centripetal acceleration, are summarized in Table 2. The corresponding natural frequencies are also given, where $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ are representative of non-spinning natural frequencies about $\hat{x}, \hat{y}$, and $\hat{z}$, respectively.

Table 2 clearly shows how the combination of preload and centripetal acceleration has altered the stiffness matrices and hence the natural frequencies of the simple examples. In particular

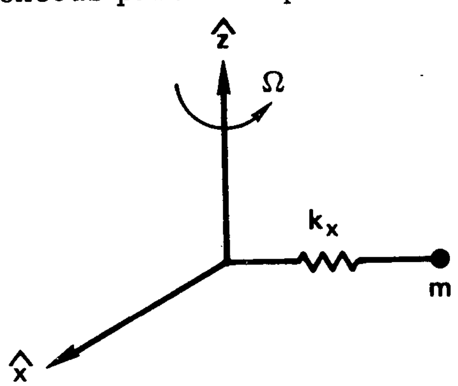
Table 2

Combined Effect of Preload and Centripetal Acceleration

Example	Modified Effective Stiffness	Modified Effective Natural Frequencies
I 	$\begin{pmatrix} \hat{k}_x & 0 & 0 \\ 0 & \hat{k}_y - m\Omega^2 & 0 \\ 0 & 0 & \hat{k}_z + m\Omega^2 \end{pmatrix}$	$\begin{pmatrix} \hat{\sigma}_x^2 & 0 & 0 \\ 0 & \hat{\sigma}_y^2 - \Omega^2 & 0 \\ 0 & 0 & \hat{\sigma}_z^2 + \Omega^2 \end{pmatrix}$
II 	$\begin{pmatrix} \hat{k}_x - m\Omega^2 & 0 & 0 \\ 0 & \hat{k}_y - m\Omega^2 & 0 \\ 0 & 0 & \hat{k}_z \end{pmatrix}$	$\begin{pmatrix} \hat{\sigma}_x^2 - \Omega^2 & 0 & 0 \\ 0 & \hat{\sigma}_y^2 - \Omega^2 & 0 \\ 0 & 0 & \hat{\sigma}_z^2 \end{pmatrix}$
III 	$\begin{pmatrix} \hat{k}_x - 2m\Omega^2 & 0 & 0 \\ 0 & \hat{k}_y - m\Omega^2 & 0 \\ 0 & 0 & \hat{k}_z - m\Omega^2 \end{pmatrix}$	$\begin{pmatrix} \hat{\sigma}_x^2 - 2\Omega^2 & 0 & 0 \\ 0 & \hat{\sigma}_y^2 - \Omega^2 & 0 \\ \hat{0} & 0 & \hat{\sigma}_z^2 - \Omega^2 \end{pmatrix}$

observe the influence spin has on the natural frequencies at Example I: one frequency was unaltered, one was decreased, and one was increased.

We have yet to discuss the effect of Coriolis coupling which, unlike preload and centripetal acceleration, has no influence on the stiffness properties....yet greatly influences the system natural frequencies. Its effect is to introduce the skew symmetric matrix G and hence couple the equations orthogonal to the spin axis through terms proportional to deformation derivatives. For the simple model of Figure (1), restricted to a single particle, as shown below, the homogeneous particle equations orthogonal to spin are given by:



$$m\ddot{u}_x - 2m\Omega\dot{u}_y + (k_x - m\Omega^2)u_x = 0$$

$$m\ddot{u}_y + 2m\Omega\dot{u}_x + (k_y - m\Omega^2)u_y = 0$$

where k_x and k_y are the loaded stiffness elements.

Define the natural frequencies of the system, accounting for all effects, as ω_x, ω_y, and ω_z, i.e.,

Unloaded	+ loaded		+ centripetal
$\hat{\sigma}_x^2$	$\Rightarrow \sigma_x^2 = \hat{\sigma}_x^2 + \Omega^2$	$\Rightarrow$	$\omega_x^2 = \sigma_x^2 - \Omega^2 = \hat{\sigma}_x^2$
$\hat{\sigma}_y^2$	$\Rightarrow \sigma_y^2 = \hat{\sigma}_y^2$	$\Rightarrow$	$\omega_y^2 = \sigma_y^2 - \Omega^2 = \hat{\sigma}_y^2 - \Omega^2$
$\hat{\sigma}_z^2$	$\Rightarrow \sigma_z^2 = \hat{\sigma}_z^2 + \Omega^2$	$\Rightarrow$	$\omega_z^2 = \sigma_z^2 = \hat{\sigma}_z^2 + \Omega^2$

Hence the homogeneous equations descriptive of particle motion orthogonal to the spin can be written as

$$\ddot{u}_x - 2\Omega\dot{u}_y + \omega_x^2 u_x = 0$$

$$\ddot{u}_y + 2\Omega\dot{u}_x + \omega_y^2 u_y = 0$$

Dividing through by m and introducing the Laplacian operator S, we find that the characteristic equation of the above set may be written as

$$\left(S^2 + \omega_x^2\right)\left(S^2 + \omega_y^2\right) + 4\Omega^2 S^2 = 0 \tag{3}$$

In equation (3) the effect of spin other than Coriolis coupling has been accounted for in the frequencies ω_x and ω_y (actually for the cantilevered model $\omega_x = \hat{\sigma}_x$) so that the term involving Ω^2 is solely due to Coriolis coupling. To permit the observation of the effect of Coriolis coupling, equation (3) is put in root locus form by dividing through by $\left(S^2 + \omega_x^2\right)\left(S^2 + \omega_y^2\right)$ and plotting the roots of the characteristic equation as Ω^2 is increased from zero to infinity. Figure (2) demonstrates these results.

In Figure 2 the poles (crosses) represent the location of the roots of the characteristic equation for $\Omega^2 = 0$, i.e., no Coriolis coupling. Conversely the zeros are representative of the roots for infinite spin. The vertical axis is representative of the complex component of the roots whereas the horizontal axis represents the real component. As no damping* is present in the problem under discussion the roots must always lie on the vertical axis. The poles closest to the origin in Figure (2) represent the lower of the two natural frequencies ω_x or ω_y. As Ω^2 is increased the roots follow the delineated arrows showing that the effect of Coriolis coupling is to decrease the smaller of the two <u>loaded</u> frequencies and increase the higher. The effect then is a <u>separation</u> of the two loaded frequencies.

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*Although Coriolis coupling introduces terms proportional to deformation derivatives the skew symmetric property of G assures that these are not damping terms. Clearly spin itself cannot dissipate energy.

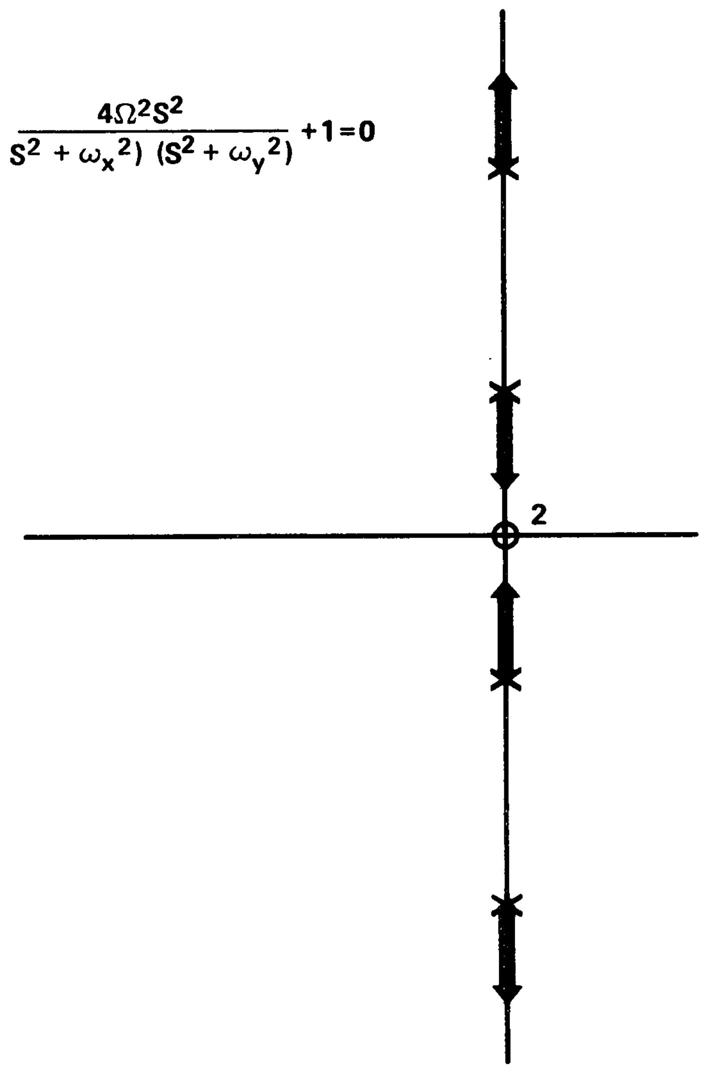


Figure 2. Effect of Coriolis Coupling

Although the discussion of this chapter has been limited to elementary models, two interesting conclusions concerning the effects of spin can be drawn:

1) The stiffness matrix is modified in such a way that natural frequencies of deformation parallel to the spin axis may increase or decrease depending upon the configuration.
2) A skew symmetric matrix is introduced which further separates the two natural frequencies orthogonal to the spin axis.

Simple Particle Stability Analysis

The above results and supporting analysis have shown the significance of spin on simple particle models; as yet however the subject of stability has not been introduced. Although the subsequent two chapters are directed toward this subject, stability criteria for the simple model will be developed here. Moreover, with proper interpretation, it will be shown how the developed stability criteria for the simple model can be used to duplicate results obtained in the literature. In particular the works of Flatley, Vigneron, Rakowski and Renard, and Meirovitch and Calico will be discussed. (References 5-9.)

From equation (I-8) of Appendix I the equations descriptive of the wobble motion for the simple particle model of Figure (1) are given as

$$A\dot{w}_x - (B - C)\,\Omega w_y + m\bar{\Gamma}_y\,(\ddot{\mu} + \Omega^2\mu) = 0$$

$$B\dot{w}_y - (C - A)\,\Omega w_x = 0$$

$$2\bar{\Gamma}_y\,(\dot{w}_x + \Omega w_y) + \ddot{\mu} + 2\,\zeta\sigma_z\dot{\mu} + \sigma_z^2\mu = 0$$

where A, B, anc C are the total system inertias about $\hat{x}, \hat{y}$, and $\hat{z}$, respectively, μ is the difference between the two $\hat{z}$ axis deformations of the particles (asymmetrical mode), and w_x, w_y, and w_z are angular velocity variational coordinates from the nominal spin. With S representative of the Laplacian operator the characteristic equation can be expressed as

$$\begin{vmatrix} AS & -(B-C)\Omega & m\overline{\Gamma}_y\left(S^2+\Omega^2\right) \\ -(C-A)\Omega & BS & 0 \\ 2\overline{\Gamma}_y S & 2\overline{\Gamma}_y\Omega & S^2+2\zeta\sigma_z S+\sigma_z^2 \end{vmatrix} = 0$$

For the simple particle model under consideration the total system inertias can easily be written in terms of the core inertias $A^1_;$ $B^1_,$ C^1:

$$A = A' + 2m\overline{\Gamma}_y^2$$

$$B = B'$$

$$C = C' + 2m\overline{\Gamma}_y^2$$

Defining terms α^2, β^2 and K' as

$$\alpha^2 = \Omega^2 \frac{(C' - A')}{B'}$$

$$\beta^2 = \frac{(C' - B')}{A'}$$

$$K' = \frac{2m\overline{\Gamma}_y^2}{A'}$$

permits us to write the characteristic equation as

$$S^4 + \left(2\zeta\sigma_z + 2\zeta\sigma_z K'\right) S^3 + \left[\alpha^2\beta^2 + \sigma_z^2 + K'\left(\sigma_z^2 - \Omega^2\right)\right] S^2$$

$$+ \left(2\zeta\sigma_z\alpha^2\beta^2 + 2\zeta\sigma_z K'\alpha^2\right) S + \left[\sigma_z^2\alpha^2\beta^2 + K'\alpha^2\left(\sigma_z^2 - \Omega^2\right)\right] = 0$$

which is of the form

$$S^4 + p_3 S^3 + p_2 S^2 + p_1 S + p_o = 0$$

Using Routh-Hurwitz criteria the conditions both necessary and sufficient* for asymptotic stability can be found from any one of a number of references. One form of these conditions, as delineated in Reference (13), is prescribed as

$$p_3 > 0$$

$$p_1 > 0$$

$$p_1 \, p_2 \, p_3 - p_1^2 - p_o \, p_3^2 > 0$$

$$p_o > 0$$

The inequality $p_3 > 0$ simply dictates the uninteresting condition that the damping be positive, i.e.,

$$p_3 > 0$$

$$\Rightarrow 2\zeta\sigma_z \, (1 + K') > 0$$

$$\Rightarrow 2\zeta\sigma_z \left(1 + \frac{2m\overline{\Gamma}_y}{A'}\right)$$

$$\Rightarrow 2\zeta\sigma_z \frac{A}{A'} > 0$$

$$\Rightarrow \zeta > 0$$

The second condition yields the familiar <u>rigid</u> body stability criteria, to wit

---

*Necessary and sufficient for that portion of the system descriptive of the wobble motions. In terms of the complete system these conditions can only be classified as necessary.

$$p_1 > 0$$

$$\Rightarrow 2\zeta\sigma_z\alpha^2\left(\beta^2 + K'\right) > 0$$

$$\Rightarrow 2\zeta\sigma_z\Omega^2 \frac{(C' - A')}{B'} \left[\frac{(C' - B')}{A'} + \frac{2m\bar{\Gamma}_y^2}{A'}\right] > 0$$

$$\Rightarrow 2\zeta\sigma_z\Omega^2 \frac{(C - A)}{B'} \frac{(C - B)}{A'} > 0$$

$$\Rightarrow (C - A)\ (C - B) > 0$$

Or simply that the spin axis must either be an axis of major or minor moment of inertia.

The third condition, the most difficult to show, is automatically satisfied through constraints of inertia properties if C is the maximum moment of inertia, and never satisfied if C is the minimum.

$$p_1\, p_2\, p_3 - p_1^2 - p_o\, p_3^2 > 0$$

$$\Rightarrow \left(2\zeta\sigma_z\alpha^2\beta^2 + 2\zeta\sigma_z K'\alpha^2\right)\left[\alpha^2\beta^2 + \sigma_z^2 + K'\left(\sigma_z^2 - \Omega^2\right)\right]\left(2\zeta\sigma_z + 2\zeta\sigma_z K'\right)$$

$$> \left(2\zeta\sigma_z\alpha^2\beta^2 + 2\zeta\sigma_z\alpha^2\right)^2 + \left[\sigma_z^2\alpha^2\beta^2 + K'\alpha^2\left(\sigma_z^2 - \Omega^2\right)\right]\left(2\zeta\sigma_z + 2\zeta\sigma_z K'\right)^2$$

$$\Rightarrow \left[\alpha^2\beta^2 + \sigma_z^2 + K'\left(\sigma_z^2 - \Omega^2\right)\right] > \frac{\left(2\zeta\sigma_z\,\alpha^2\beta^2 + 2\zeta\sigma_z K'\,\alpha^2\right)}{(2\zeta\sigma_z + 2\zeta\sigma_z K')}$$

$$+ \frac{\left[\sigma_z^2\alpha^2\beta^2 + K'\alpha^2\left(\sigma_z^2 - \Omega^2\right)\right]\left(2\zeta\sigma_z + 2\zeta\sigma_z K'\right)}{\left(2\zeta\sigma_z\alpha^2\beta^2 + 2\zeta\sigma_z K'\alpha^2\right)}$$

$$\Rightarrow \left[\alpha^2\beta^2 + \sigma_z^2 + K'\left(\sigma_z^2 - \Omega^2\right)\right] > \frac{\left(\alpha^2\beta^2 + K'\alpha^2\right)}{(1 + K')} + \frac{(1 + K')\left[\sigma_z^2\beta^2 + K'\left(\sigma_z^2 - \Omega^2\right)\right]}{(\beta^2 + K')}$$

$$\Rightarrow \left[\alpha^2\beta^2 + \sigma_z^2 + K'\left(\sigma_z^2 - \Omega^2\right)\right] > \frac{\alpha^2\left(\beta^2 + K'\right)}{(1 + K')} + \frac{(1 + K')\sigma_z^2\beta^2}{\left(\beta^2 + K'\right)} + \frac{(1 + K')K'\left(\sigma_z^2 - \Omega^2\right)}{\left(\beta^2 + K'\right)}$$

$$\Rightarrow \left(\alpha^2\beta^2 - K'\Omega^2\right) + \sigma_z^2 (1+K') > \alpha^2 \frac{\left(\beta^2+K'\right)}{(1+K')} + \frac{\sigma_z^2}{\left(\beta^2+K'\right)} \left[\beta^2(1+K') + K'(1+K')\right] - \frac{K'\Omega^2(1+K')}{\left(\beta^2+K'\right)}$$

$$\Rightarrow \left(\alpha^2\beta^2 - K'\Omega^2\right) > \alpha^2 \frac{\left(\beta^2+K'\right)}{(1+K')} - K' \frac{\Omega^2(1+K')}{\left(\beta^2+K'\right)}$$

$$\Omega^2\left(\frac{\alpha^2}{\Omega^2}\beta^2 - K'\right) > \Omega^2\left(\frac{\alpha^2}{\Omega^2}\right)\frac{\left(\beta^2+K'\right)}{(1+K')} - K' \frac{\Omega^2(1+K')}{\left(\beta^2+K'\right)}$$

$$\Rightarrow \left(\frac{\alpha^2}{\Omega^2}\right)\left[\beta^2 - \frac{\left(\beta^2+K'\right)}{(1+K')}\right] > K' - K' \frac{(1+K')}{\left(\beta^2+K'\right)}$$

$$\Rightarrow \left(\frac{\alpha^2}{\Omega^2}\right) K' \frac{\left(\beta^2-1\right)}{(1+K')} > K' \frac{\left(\beta^2-1\right)}{\left(\beta^2+K'\right)}$$

But $\beta^2 \triangleq \frac{C'-B'}{A'}$ is always less than 1

Hence $\beta^2 - 1 < 0$

$$\Rightarrow \left(\frac{\alpha^2}{\Omega^2}\right) < \frac{(1+K')}{\left(\beta^2+K'\right)}$$

$$\Rightarrow \frac{(C'-A')}{B'} < \frac{A' + 2m\bar{\bar{\Gamma}}_y^2}{C'-B'+2m\bar{\bar{\Gamma}}_y^2}$$

$$\Rightarrow \frac{C-A}{B} < \frac{A}{C-B}$$

which is always true for C > B and never true for C < B. Hence, the combination of conditions two and three dictates that the spin axis be the axis of maximum moment of inertia, i.e.,

$C > A$

$C > B$

Condition four leads to a stability criterion which bounds the loaded natural frequency in terms of the spin frequency and the system inertia properties:

$$p_o > 0$$

$$\Rightarrow \left[\sigma_z^2 \alpha^2 \beta^2 + K' \alpha^2 \left(\sigma_z^2 - \Omega^2\right)\right] > 0$$

$$\Rightarrow \alpha^2 \left[\sigma_z^2 \beta^2 + K' \left(\sigma_z^2 - \Omega^2\right)\right] > 0$$

$$\Rightarrow \Omega^2 \frac{(C' - A')}{B'} \left[\sigma_z^2 \frac{(C' - B')}{A'} + \frac{2m\overline{\Gamma}_y^2}{A'} \left(\sigma_z^2 - \Omega^2\right)\right] > 0$$

However from condition three we observe that $C > A$ which implies $C' > A'$ so that the above leads to:

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{2m\overline{\Gamma}_y^2}{2m\overline{\Gamma}_y^2 + C' - B'}$$

In summary, for the system characterized as a simple particle model, the conditions both necessary and sufficient for wobble motion asymptotic stability are:

$\zeta > 0$ (Positive Damping)

$C > A$ and $C > B$ (Spin about axis of maximum moment of inertia)

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{2m\overline{\Gamma}_y^2}{2m\overline{\Gamma}_y^2 + (C' - B')} \qquad (4a)$$

with $C = 2m\overline{\Gamma}_y^2 + C'$ and $B = B'$ the last of these conditions takes the form

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{2m\bar{\Gamma}_y^2}{C-B} \qquad (4b)$$

The significance of loaded natural frequency with respect to stability is now readily apparent from inequalities (4a) and (4b). It is clear that when the natural frequencies are augmented positively so that the unloaded natural frequencies are increased due to spin, the margin of stability is enhanced. Moreover, if the flexible structure is radially mounted as in Example I, and if the core inertia about the spin axis exceeds its transverse inertias, then stability is assured. To observe this, substitute in equation (4a) the results for σ_z^2 shown in Table 2, i.e., $\sigma_z^2 = \hat{\sigma}_z^2 + \Omega^2$. The result yields the following

$$\left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 + 1 > \frac{2m\bar{\Gamma}_y^2}{2m\bar{\Gamma}_y^2 + (C' - B')}$$

As the right hand side of the above expression is always less than unity for $C' > B'$ then this condition is always statisfied..... independent of the amount of flexibility or the magnitude of spin (Meirovitch and Calico have observed this phenomena in connection with a structure idealized as a rigid core having attached to it flexible rods. Quoting from Reference (9): "..... any stable satellite possessing axial rods alone will remain stable with the addition of radial rods.")

Conversely, if the flexible structure is mounted anti-cantilevered as in Example III) or possibly orthogonally mounted as in Example II, then stability may be seriously degraded by spin even if $C' > B'$. To emphasize the effect of preload stability criteria for the three examples are re-examined for the special case where the core possesses inertial symmetry about the $\hat{x}$ axis so that $C' = B'$. The stability criteria for equation (4) then reduces to

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > 1$$

The resulting stability criteria for each of the three examples are summarized in Table 3. Clearly, the structure configuration is as important as the natural frequencies themselves (if not more so).

Well aware of the pitfalls of generalizations we can, however, apply the above limited analysis to a practical application, namely, Skylab B shown in Figure 3. The purpose is not to examine in detail the stability of such a complex spacecraft, but rather to make educated guesses based solely upon analysis of the simple particle model. Clearly much more elaborate analytical techniques, including simulation studies, are required to gain the confidence necessary prior to flight.

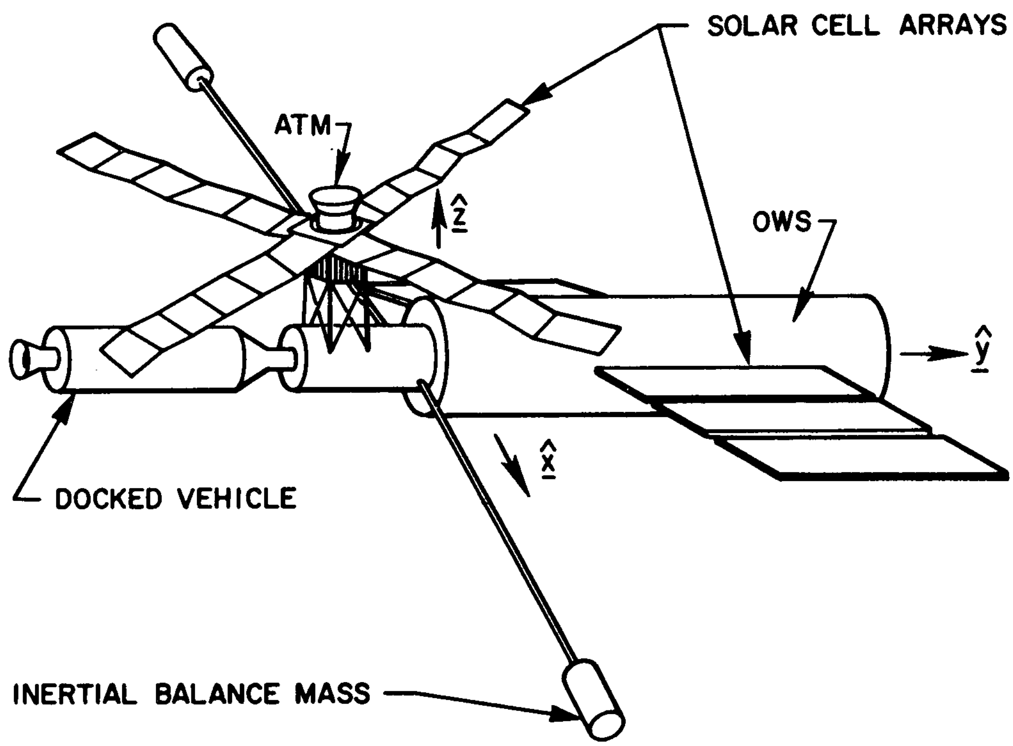
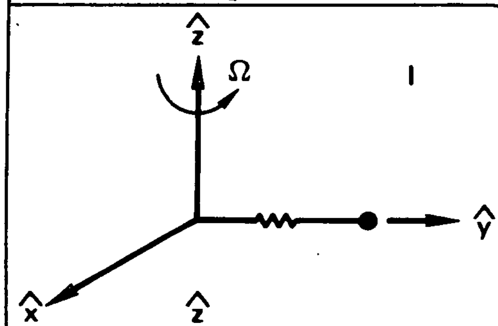
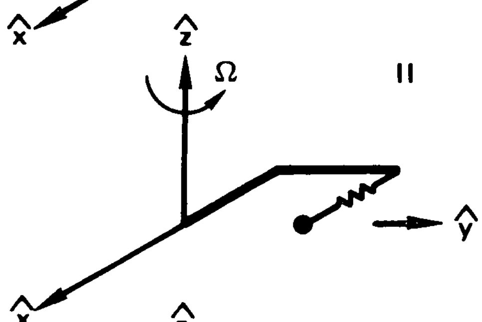
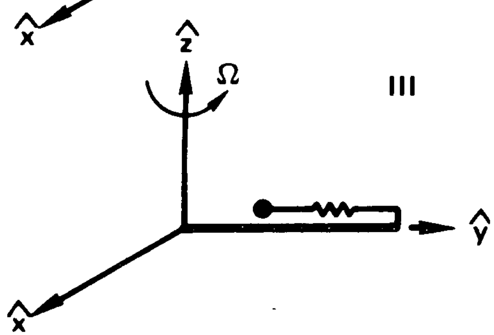


Figure 3. Skylab B Representation

Table 3

STABILITY CRITERIA FOR SIMPLE PARTICLE MODEL FOR THE SPECIALIZATION C' = B'

Example	Loaded Natural Frequency	Stability Criteria
I 	$\sigma_z^2 = \hat{\sigma}_z^2 + \Omega^2$	$\left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 > 0$ always satisfied
II 	$\sigma_z^2 = \hat{\sigma}_z^2$	$\left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 > 1$
III 	$\sigma_z^2 = \hat{\sigma}_z^2 - \Omega^2$	$\left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 > 2$

As shown, Skylab B will undergo a steady spin about the indicated $\hat{z}$ axis as an experiment to provide artificial gravity through centripetal acceleration. The structure contains flexible solor panels distributed about the (assumedly rigid) core in a fashion which exhibits similarity to each of the simple particle model examples considered in this chapter. In particular, the two solar panels attached to the Orbital Workshop (OWS) are configured much like Example II; whereas of the four panels attached to the Apollo Telescope Mount (ATM) the two directed in the negative $\hat{y}$ direction exhibit a structure similar to that of Example I and the two directed in the positive $\hat{y}$ direction can be associated with Example III. We would require then that the lowest unloaded natural frequency of each of the two OWS mounted panels be at least as high as the spin frequency. Also, we would require that the two panels mounted on the ATM directed in the positive $\hat{y}$ direction exhibit an unloaded lowest natural frequency significantly higher than the spin (as much as two times). On the other hand, we would expect the two remaining ATM mounted panels to be almost incidental to the question of stability, and in fact, they may safely exhibit unloaded natural frequencies lower than the spin frequency.

Although the cited associations are at most remote, nevertheless the similarities are persuasive enough to allow engineering judgments, as well as insight. Perhaps a more direct association is provided by the same spacecraft. In the shown configuration the principal axis of maximum moment of inertia is not colinear with the desired direction of spin (normal to the solar panels). To overcome this deficiency cables with large tip masses were suggested as a means of effecting a more satisfying mass distribution. We can now apply our newly gained knowledge to analyze this proposal. Recall that our simple particle model stability criteria can be written as

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{2m\bar{\Gamma}_y^2}{2m\bar{\Gamma}_y^2 + (C' - B')}$$

where C' and B' are representative of inertias of the rigid core about the $\hat{z}$ and $\hat{y}$ axis, respectively. The mass distribution of Skylab B is such that C' < B', thus assuring that the quantity on the right hand side of the above expression is greater than unity. In terms of the unloaded natural frequency the stability criteria for this example can then be written as

$$\left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 + 1 > N$$

where N is some number greater than 1. For cables $\hat{\sigma}_z$ is identically zero clearly violating the required stability conditions. Thus we conclude unequivocally that cables would not suffice, requiring perhaps radial booms having stiffness properties more amenable to the mass distribution. Then, too, we must re-analyze the problem to assure sufficient stability margin.

Comparison With Previous Studies

In this chapter we have tried to provide some insight into the effects of spin with heavy concentration on a simple particle model. Stability criteria for that elementary model were developed showing the stability dependence of spin, structural frequencies, and mass properties. Some of these ideas were then applied to a very complicated practical example to provide some understanding of its stability requirements. Throughout, the dependence of configuration on structural properties and, hence, stability was emphasized. The practical example considered, Skylab B, reflected this dependency as it will be mounted with flexible appendages having both favorable and unfavorable characteristics. Fortunately, most spinning flexible spacecraft are not as complicated as Skylab B. Indeed a large class of spacecraft exhibit solely favorable flexible characteristics having booms radially mounted outward from a rigid core. It is in fact, this class of spacecraft which has received most of the attention in the literature. This chapter concludes with a comparison of previous work and that of the author using the results of the simple particle model (Example I), which is not too unlike a boom configuration. Stability criteria for a spacecraft containing a more general* flexible appendage are developed in the succeeding chapter.

Since the literature is directed toward spacecraft having flexible radial booms, it is required that the simple particle model material be reinterpreted accordingly. We shall start with the works of Rakowski and Renard, Reference 7. They presented a series of curves relating normalized inertia properties (${}^{R}\Gamma$ vs. ${}^{R}K_{p}$) for different values of a parameter termed ${}^{R}\bar{\lambda}$, which is a measure of the ratio of centrifugal to elastic "forces", i.e., the amount of spin. (The superscript R has been added to identify these parameters as those defined by Rakowski and Renard.)

*The development of the next chapter requires some restrictions and as such is not entirely general. However, radially mounted booms are a clear subset of that development.

$$R_{\Gamma} = \frac{\rho \ell^3}{I_{hz}}$$

$$R_{K_p} = \frac{I_{hz}}{I_{hp}}$$

$$R_{\bar{\lambda}} = \frac{\rho \ell^4 \omega_z^2}{EI}$$

Here ρ is the mass per unit length of each rod, ω_z is the spin frequency, I_{hz} is the core inertia about the spin axis, and I_{hp} is the core transverse inertia. (The core was assumed symmetrical about the spin axis.) Figure 4 shows their results.*

In terms of the variables used in this paper the above defined parameters become:

$$R_{\Gamma} = \frac{\rho \ell^3}{C'}$$

$$R_{K_p} = \frac{C'}{B'}$$

$$R_{\bar{\lambda}} = \frac{\Omega^2}{EI/\rho \ell^4} = \frac{\rho \ell^4 \Omega^2}{EI}$$

The inertia of a uniformly distributed beam about the core defined as I_B is

$$I_B = \frac{1}{3}\, m\ell^2 = \frac{1}{3}\, \rho \ell^3$$

which allows us to write R_{Γ} as

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*This figure was not taken from the cited reference; however by permission of Dr. Rakowski it was traced from his dissertation, which provided the foundation for that reference.

$$R_\Gamma = \frac{3\, I_B}{C'}$$

In the simple particle model the inertia of each particle about the core is $m\bar{\Gamma}_y^2$. If we identify I_B with $m\bar{\Gamma}_y^2$ then the simple particle model analogy to R_Γ is:

$$R_\Gamma \sim \frac{3m\bar{\Gamma}_y^2}{C'}$$

In Appendix II the loaded natural frequency of a massless beam with a point mass is developed and found to be:

$$\sigma^2 = \hat{\sigma}^2 + 1.2\Omega^2$$

At first glance a massless beam with a point mass may appear grossly different from that of a uniformly distributed beam. If one however, used the latter, then as cited in Reference 6, the coefficient of Ω^2 would be 1.193 instead of 1.2. As a matter of interest let us digress slightly and compare the approximate loaded natural frequencies for a particle mass, a uniform beam, and a massless beam with a point mass.

Table 4

COMPARISON OF LOADED NATURAL FREQUENCIES FOR BOOMS AND PARTICLE MODELS

Model	Loaded Frequency
Particle mass	$\sigma^2 = \hat{\sigma}^2 + \Omega^2$
Uniform beam	$\sigma^2 = \hat{\sigma}^2 + 1.193\,\Omega^2$
Massless beam with point mass	$\sigma^2 = \hat{\sigma}^2 + 1.2\,\Omega^2$

One observes that although the models are significantly different the loaded frequencies are not.

Using the loaded frequency for a massless beam with a point mass, and substituting the expression for the unloaded lowest frequency for a uniform beam, recognized as $\hat{\sigma}_z^2 = (3.515)^2 \, EI/\rho\ell^4$, one finds

$$\sigma_z^2 = 12.4 \frac{EI}{\rho\ell^4} + 1.2\,\Omega^2$$

$$= 12.4 \frac{\Omega^2}{R_{\bar{\lambda}}} + 1.2\,\Omega^2$$

which after dividing through by Ω^2 can be written as

$$\left(\frac{\sigma_z}{\Omega}\right)^2 = \frac{12.4}{R_{\bar{\lambda}}} + 1.2$$

For the simple particle model the stability criterion has been shown to be

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{2m\bar{\Gamma}_y^2}{2\,m\,\bar{\Gamma}_y^2 + C' - B'}$$

which in terms of the above definitions becomes:

$$\frac{12.4}{R_{\bar{\lambda}}} + 1.2 > \frac{\frac{2}{3}\,R_{\Gamma}}{\frac{2}{3}\,R_{\Gamma} + 1 - \frac{1}{R_{K_p}}}$$

The above expression is shown as dots on the curves of Figure 4. Clearly, the stability boundaries compare well. The results on Figure 4 were generated through computer simulation by actually solving the complete equations of motion in all their nonlinear splendor. By contrast, our data points are the results of closed

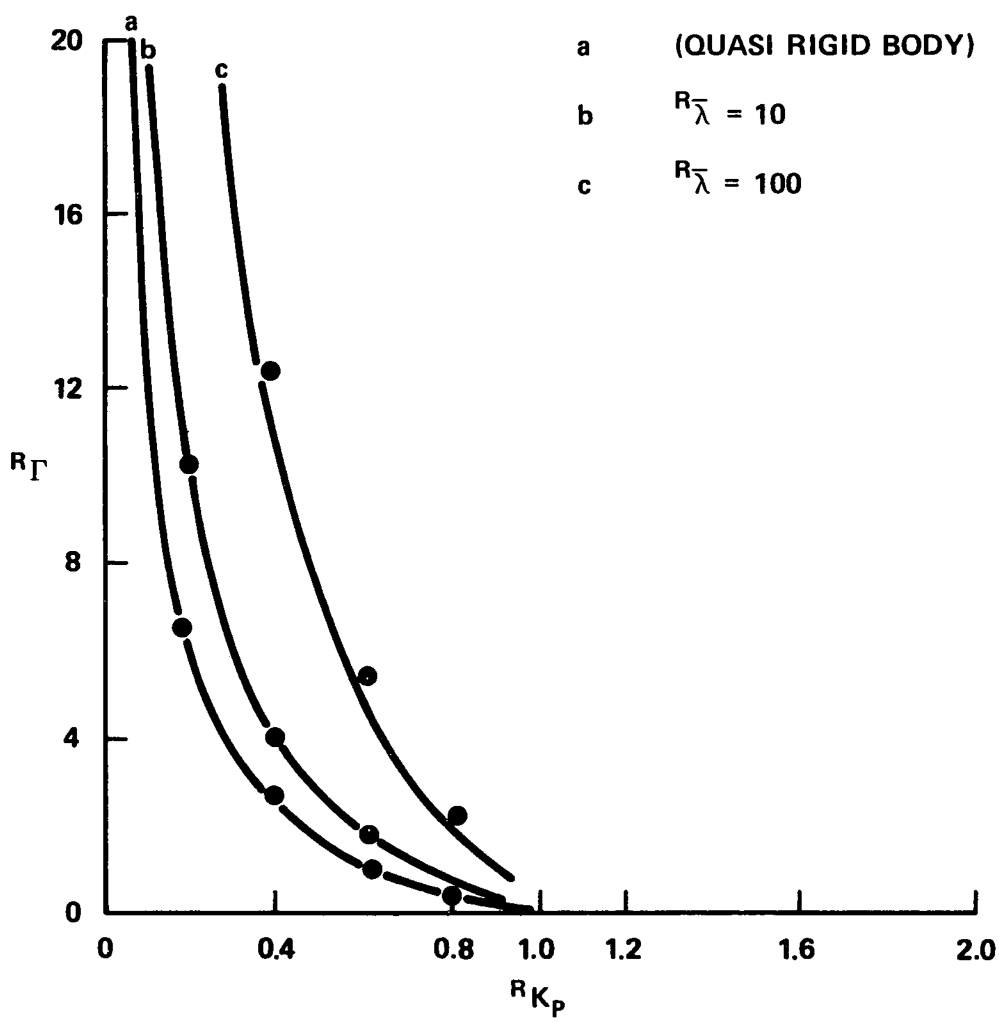

Figure 4. Correspondence with Results of Rakowski and Renard

form stability criteria developed for a simple particle model. In light of this the closeness of the comparison on Figure 4 is truly remarkable (and quite surprising). What is termed by Rakowski and Renard, on Figure 4, as the "Quasi Rigid Body" case ($R_{\bar{\lambda}} = 0$), is simply the major moment of inertia rule dictated by energy sink methods. In terms of the above inequality this condition reduces to

$$\frac{2}{3} R_{\Gamma} + 1 - \frac{1}{R_{K_p}} > 0$$

The approach taken by Vigneron, Reference 6, was to linearize the wobble equations of motion and, with the aid of a digital computer, apply Routh-Hurwitz stability criteria. His results are given in the form of plots in inertia space. He observed that the stability boundary (his expression (21)) when re-interpreted into nomenclature used herein, is prescribed by

$$\left(\frac{C-B}{A}\right)\left(\frac{\hat{\sigma}_z^2}{\Omega^2} + 1.193\right) > 1 - \frac{A'}{A}$$

where, as cited earlier, the constant 1.193 results from approximating the loaded frequency of a uniformly distributed beam. The above expression can be re-written as

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{A-A'}{C-B}$$

which is precisely the stability criteria generated herein for the simple particle model (recall $A - A' = 2m\bar{\Gamma}_y^2$).

Meirovitch and Calico, Reference 9, also developed stability criteria for a spacecraft characterized by a rigid core having attached to it flexible booms. However, unlike all the other presentations, they accounted for booms extended along the spin axis as well as normal to the spin. Their results do, however, separate, allowing stability criteria (their expression (43b)) for the radial rods to be given by

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{\text{(Inertia Of The Rods)}}{(C - A)}$$

Once again the result is analogous to the criterion generated from analysis of the simple particle model.

Our results are now compared with the work of Flatley, Reference 5. He too examined the radial rod problem (although his efforts preceded the others) and presented results in the form of both curves and tables. Flatley plotted a normalized core inertia difference $(B' - C')/\rho\ell^3$ vs a term $F_k \equiv \rho\ell^4\Omega^2/2EI$ which is a measure of centrifugal to elastic forces. (Note the latter term is identical to $R_{\bar{\lambda}}/2.$) With $\rho\ell^3$ recognized as $3I_B$ and I_B identified as $m\bar{r}_y^2$ we can write $(B' - C')/\rho\ell^3$ as $(B' - C')/3m\bar{r}_y^2$. The first unloaded natural frequency of a uniform beam $\hat{\sigma}_z = 3.52\,(EI/\rho\ell^4)^{1/2}$ allows

$$\left(\frac{\sigma_z}{\Omega}\right)^2 = \frac{6.2}{F_k} + 1.2$$

where the coefficient 1.2 results from assuming that the loaded natural frequency is identical to that of a massless cantilevered beam with a point mass. If for the moment we assume that coefficient to be a variable identified as ξ then the above expression becomes

$$\left(\frac{\sigma_z}{\Omega}\right)^2 = \frac{6.2}{F_k} + \xi$$

The simple particle model stability criteria developed earlier can be written as

$$\left(\frac{\sigma_z}{\Omega}\right)^2 > \frac{1}{1 + \dfrac{C' - B'}{2 m \bar{\Gamma}_y^2}} = \frac{1}{1 - \dfrac{3}{2} \dfrac{(C' - B')}{\rho \ell^3}}$$

Combining the above two expressions yields

$$\frac{B' - C'}{\rho \ell^3} < \frac{2}{3} \frac{\left(\dfrac{6.2}{F_k} + \xi - 1\right)}{\left(\dfrac{6.2}{F_k} + \xi\right)}$$

the right hand side of which is delineated in Table 5 as a function of F_k for $\xi = 1.2$ and $\xi = 1.193$, along with the results of Flatley.

Table 5

COMPARISON WITH THE WORK OF FLATLEY

F_k	$\dfrac{\frac{2}{3}\left(\frac{6.2}{F_k} + \xi - 1\right)}{\left(\frac{6.2}{F_k} + \xi\right)}$		
$\left(\dfrac{\rho \ell^4 \Omega^2}{2\,EI}\right)$	$\xi = 1.2$	$\xi = 1.193$	Flatley's Results
0	.6667	.6667	.6667
.1	.6561	.6561	.6564
1	.5766	.5765	.5787
10	.3004	.2990	.3023
100	.1384	.1355	.0997
∞	.1111	.1079	.0000

The results are essentially identical for low spin to stiffness ratios (i.e., $F_k < 10$). However, for high values of F_k a substantial difference arises. In particular, for $F_k = \infty$ our results yield a value of approximately .11 whereas Flatley showed it to be identically zero. The discrepancy arises solely in approximating the loaded natural frequency by

$$\left(\frac{\sigma_z}{\Omega}\right)^2 = \left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 + \xi$$

as compared to the true value for a massless beam with a tip mass as shown in Appendix II to be:

$$\left(\frac{\sigma_z}{\Omega}\right)^2 = \frac{1}{\left[1 - \dfrac{\tanh\left(\sqrt{3}\,\dfrac{\Omega}{\hat{\sigma}_z}\right)}{\left(\sqrt{3}\,\dfrac{\Omega}{\hat{\sigma}_z}\right)}\right]}$$

If the true value is used then in the limit as $\frac{\Omega}{\hat{\sigma}_z}$ approaches ∞ we find that $\frac{\sigma_z}{\Omega}$ approaches unity, i.e.,

$$\lim_{\frac{\Omega}{\hat{\sigma}_z} \to \infty} \frac{\sigma_z}{\Omega} = 1$$

so that in the limit the true value of ξ is 1 (as opposed to 1.2) resulting in

$$\frac{2}{3}\,\frac{\left(\dfrac{6.2}{F_k} + \xi - 1\right)}{\left(\dfrac{6.2}{F_k} + \xi\right)}\Bigg|_{F_k \to \infty} = 0$$

as it should be.

In the above we have tried to show how the analysis of a simple particle model can be used to duplicate the results obtained in the literature on the subject of spinning flexible spacecraft. The purpose is not in any way to demean the cited references but rather to verify our results. Also no attempt is made to expound on all the details discussed in these references, for each substantially enhanced the general knowledge of spinning flexible spacecraft. However, one glaring fact remains: All the cited references limited their analysis to flexible appendages idealized as booms, except for the very recent work of Willems. Admittedly, this idealization encompasses a large class of spacecraft; however, the question of stability of spacecraft having attached a general flexible appendage is unanswered. A step in that direction is provided by the analysis of the remaining two chapters.

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CHAPTER 3

THE HAMILTONIAN AS A LIAPUNOV FUNCTION

In this chapter we shall endeavor to establish stability criteria for a spacecraft characterized as a rigid body having attached a flexible appendage idealized as a collection of spring-connected particles. We shall at the outset permit the flexible appendage to be configured with respect to the rigid body in a general fashion. As we proceed the structure will be somewhat specialized to allow formal closed form stability criteria without the aid of computer simulation. We shall invoke the well known stability procedures offered by Liapunov's direct (second) method. Among the many theorems on stability and instability of the class developed by Liapunov two have direct relevance to the development here. Described in detail by Pringle, Reference (14), the theorems of interest are paraphrased below.

Theorem 1: The null solution $X(t) = 0$ of the differential equation $\dot{X} = F(X)$ is asymptotically stable if there exists a function $L(X)$ in a region around the origin both positive definite and strictly decreasing for all solutions in that region except for $X \equiv 0$.

Theorem 2: The null solution $X(t) = 0$ of the differential equation $\dot{X} = F(X)$ is unstable if there exists a function $L(X)$ in a region around the origin both negative definite (or sign variable) and strictly decreasing for all solutions in that region except for $X \equiv 0$.

Although the implementation of Liapunov's direct method is impeded by the lack of a general formal procedure for the generation of a testing function, the Hamiltonian serves this purpose for a wide class of dynamical systems. Specifically, if the total energy of the system is free of explicit time dependence then for completely damped systems the Hamiltonian,* given the symbol H, is a suitable testing function for asymptotic stability and instability. For our purpose the concept "complete damping" requires that energy be dissipated for any possible motion other than the nominal motion in the neighborhood of the nominal motion in the coordinate space adopted. However, the damping of a freely spinning body with internal energy dissipation is not complete in terms of inertial attitude angles which are zero prior to perturbation, since after perturbation the vehicle cannot return to its original state. Thus, for such systems the Hamiltonian is not strictly decreasing in the neighborhood of the null solution, and therefore asymptotic stability cannot be proclaimed as a consequence of the positive definiteness of H. In 1969 R. Pringle, Reference (14), provided a method to circumvent this problem. The method is to constrain the attitude angles through the angular momentum integral in such a fashion that they represent deviations from an inertial direction, $\hat{\underline{n}}_3$, which is colinear with the instantaneous angular momentum vector $\underline{h}$ after perturbation from its nominal inertial orientation. The resulting attitude angles (defined as θ_1 and θ_2 in the sequel) will in the case of a stable vehicle ultimately reduce to zero after initial perturbation, thus assuring complete damping and asymptotic stability.

The remaining portion of this chapter is devoted to the development of the Hamiltonian for a rigid body having attached a general flexible appendage (idealized as a collection of point masses),

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*The term Hamiltonian is here applied to the function $H \equiv \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$, where L is the Lagrangian and q_i $(i=1, \ldots, n)$ is a generalized coordinate of the system. This usage is not universal.

and then the determination of stability criteria from the results. The procedure is compounded by algebraic complexity but the results are rewarding. Figure (5) identifies the basic nomenclature used throughout.

The kinetic energy of the system, given the symbol T, can be derived from the general expression

$$2T = \int_{A,B} \dot{\underline{a}} \cdot \dot{\underline{a}} \, dm$$

where $\underline{a}$ is an inertial generic position vector and the capital letters A and B denote that the integration is carried out over bodies A and B. The dot over a vector denotes time differentiation of that vector with respect to an inertial frame. Since the system is assumed unforced, the mass center is inertially fixed, and $\underline{a}$ can be written as the sum $\underline{c} + \underline{\rho}$ where $\underline{c}$ is a position vector directed from the system center of mass CM to a point N fixed in B, and $\underline{\rho}$ is a generic position vector directed from N; moreover, N is selected so as to be coincident with CM when the structure is steadily spinning, and hence, elastically distorted through forces induced by spin, but otherwise undeformed, to wit

$$2T = \int_{A,B} (\dot{\underline{c}} + \dot{\underline{\rho}}) \cdot (\dot{\underline{c}} + \dot{\underline{\rho}}) dm$$

$$= \underline{c} \cdot \int_{A,B} (\dot{\underline{c}} + \dot{\underline{\rho}}) dm + \int_{A,B} \dot{\underline{\rho}} \cdot (\dot{\underline{c}} + \dot{\underline{\rho}}) dm \quad .$$

The first term vanishes by definition of mass center, i.e.,

$$\int_{A,B} (\underline{c} + \underline{\rho}) \, dm \equiv 0 \tag{5}$$

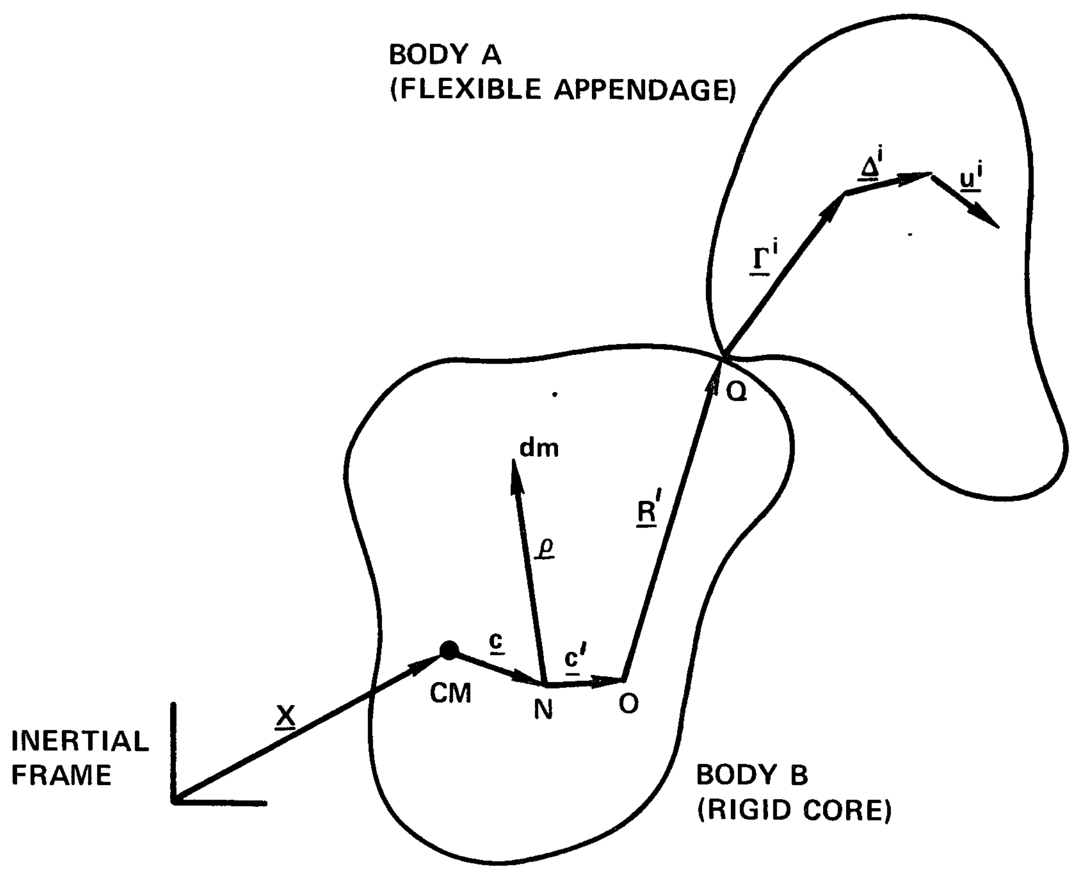

Figure 5. Mathematical Model

allowing the kinetic energy of the system to be written as

$$2T = \underline{\dot{c}} \cdot \int_{A,B} \underline{\dot{\rho}} \, dm + \int_{A,B} \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm \quad . \tag{6}$$

With use of Equation (5) the first term of Equation (6) may be simplified by

$$\underline{\dot{c}} \cdot \int_{A,B} \underline{\dot{\rho}} \, dm = - \underline{\dot{c}} \cdot \int_{A,B} \underline{\dot{c}} \, dm = -\mathcal{M} \underline{\dot{c}} \cdot \underline{\dot{c}} \tag{7}$$

where $\mathcal{M} \equiv \int_{A,B} dm$ is the total system mass; the second term expands to provide the more useful relationship

$$\int_{A,B} \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm = \int_A \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm + \int_B \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm$$

$$= \int_A \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm + \int_B (\underline{\omega} \times \underline{\rho}) \cdot (\underline{\omega} \times \underline{\rho}) dm$$

$$= \int_A \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm + \underline{\omega} \times \int_B \underline{\rho} \cdot (\underline{\omega} \times \underline{\rho}) dm$$

$$= \int_A \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm + \underline{\omega} \cdot \int_B \underline{\rho} \times (\underline{\omega} \times \underline{\rho}) dm$$

$$= \int_A \underline{\dot{\rho}} \cdot \underline{\dot{\rho}} \, dm + \underline{\omega} \cdot \underline{\underline{I}}_B^N \cdot \underline{\omega} \tag{8}$$

where $\underline{\omega}$ is the inertial angular velocity vector of vector basis $\{\underline{b}\}$ fixed in B and $\underline{\underline{I}}_B^N$ is the inertia dyadic of body B about point N. The combination of (6), (7) and (8) allows the system kinetic energy to take the form

$$2T = \underline{\omega} \cdot \underline{\underline{I}}_B^N \cdot \underline{\omega} + \int_A \dot{\underline{\rho}} \cdot \dot{\underline{\rho}} \, dm - \mathscr{M} \dot{\underline{c}} \cdot \dot{\underline{c}} \tag{9}$$

As written, the term in Equation (9) containing the integral includes the complete kinetic energy of the appendage, not just the energy contribution due to appendage deformations, i.e., a contribution of the integral persists even when the system is undeformed from its steady state shape. Conversely, the last term in Equation (9) vanishes when the system is undeformed from its steady state configuration. (Note the term "deformation" is descriptive of particle perturbations from their steady state spinning location.)

The three terms in Equation (9) can be expanded and written in matrix notation with respect to vector basis $\{\underline{b}\}$ fixed in body B, as follows:

$$\begin{aligned}
\mathscr{M} \dot{\underline{c}} \cdot \dot{\underline{c}} &= \mathscr{M} [\mathring{\underline{c}} + (\underline{\omega} \times \underline{c})] \cdot [\mathring{\underline{c}} + (\underline{\omega} \times \underline{c})] \\
&= \mathscr{M} [\mathring{\underline{c}} \cdot \mathring{\underline{c}} + 2\, \mathring{\underline{c}} \cdot (\underline{\omega} \times \underline{c}) + (\underline{\omega} \times \underline{c}) \cdot (\underline{\omega} \times \underline{c})] \\
&= \mathscr{M} [\dot{c}^T \dot{c} - 2 \dot{c}^T \tilde{c} \omega + \omega^T \tilde{c}^T \tilde{c} \omega]
\end{aligned}$$

where $\mathring{\underline{c}}$ implies time differentiation with respect to the vector basis $\{\underline{b}\}$ and the tilde ($\sim$) operation as cited earlier is defined as

$$\tilde{c} \triangleq \begin{pmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_x \\ -c_y & c_x & 0 \end{pmatrix} .$$

(These definitions are of course applicable to any vector.)

$$\underline{\omega} \cdot \underline{\underline{I}}_B^N \cdot \underline{\omega} = \omega^T I_B^N \omega$$

$$\int_A \dot{\underline{\rho}} \cdot \dot{\underline{\rho}}\, dm = \sum_i^N m_i\, \dot{\underline{\rho}}^i \cdot \dot{\underline{\rho}}^i = \sum_i^N m_i (\dot{\underline{u}}^i + \dot{\underline{\Gamma}}^i) \cdot (\dot{\underline{u}}^i + \dot{\underline{\Gamma}}^i)$$

$$= \sum_i^N m_i [\overset{\circ}{\underline{u}}{}^i + \underline{\omega} \times (\underline{u}^i + \underline{\Gamma}^i)] \cdot [\overset{\circ}{\underline{u}}{}^i + \underline{\omega} \times (\underline{u}^i + \underline{\Gamma}^i)]$$

$$= \sum_i^N m_i \Big\{ \dot{u}^{iT} \dot{u}^i + 2\, \dot{u}^{iT} \tilde{\omega} (u^i + \Gamma^i) + [\tilde{\omega}(u^i + \Gamma^i)]^T [\tilde{\omega}(u^i + \Gamma^i)] \Big\}$$

$$= \sum_i^N m_i \Big\{ \dot{u}^{iT} \dot{u}^i - 2\, \dot{u}^{iT} \tilde{u}^i \omega - 2\, \dot{u}^{iT} \tilde{\Gamma}^i \omega + \omega^T \Big[\tilde{\Gamma}^{iT} \tilde{\Gamma}^i + \left(\tilde{\Gamma}^{iT} \tilde{u}^i + \tilde{u}^{iT} \tilde{\Gamma}^i \right) + \tilde{u}^{iT} \tilde{u}^i \Big] \omega \Big\}$$

Here $\underline{\Gamma}^i$ is defined as the sum of $\underline{c}'$, the position vector from N to point O which coincides with the CM when neither spinning nor deformed (i.e., at rest), $\underline{R}'$, the position vector from O to point Q prescribed as the appendage-core interface, $\underline{r}^{i'}$, the position vector from Q to the location of the i^{th} particle at rest, and $\underline{\Delta}^i$ the displacement of the i^{th} particle due solely to forces induced by spin, to wit

$$\underline{\Gamma}^i = \underline{c}' + \underline{R}' + \underline{r}^{i'} + \underline{\Delta}^i \quad .$$

The combination of the above expansions permits Equation (9) to be written as

$$T = \frac{1}{2}\ \omega^T \left(I_B^N + \sum m_i\ \tilde{\Gamma}^{iT} \tilde{\Gamma}^i \right) \omega + \frac{1}{2} \sum m_i\ \dot{u}^{iT} \dot{u}^i$$

$$+ \frac{1}{2}\ \omega^T \left[\sum m_i \left(\tilde{\Gamma}^{iT} \tilde{u}^i + \tilde{u}^{iT} \tilde{\Gamma}^i \right) \right] \omega + \frac{1}{2}\ \omega^T \left[\sum m_i \tilde{u}^{iT} \tilde{u}^i \right] \omega$$

$$- \sum m_i\ \dot{u}^{iT} \left(\tilde{\Gamma}^i + \tilde{u}^i \right) \omega - \frac{\mathcal{M}}{2} \left[\dot{c}^T \dot{c} - 2\dot{c}^T \tilde{c} \omega + \omega^T \tilde{c}^T \tilde{c}\, \omega \right]$$

(For notation simplicity the symbol $\sum\limits_{i}^{N}$ has been replaced by $\sum$.)

As the appendage is restricted to be a collection of particles the term $\sum m_i \tilde{\Gamma}^{iT} \tilde{\Gamma}^i$ is the inertia of the undeformed appendage about point N. Its combination with I_B^N precisely defines the inertia of the system about point N when undeformed:

$$I_O^N \triangleq I_B^N + \sum m_i\ \tilde{\Gamma}^{iT} \tilde{\Gamma}^i$$

defined to be diagonal since $\hat{\underline{b}}_1$, $\hat{\underline{b}}_2$, and $\hat{\underline{b}}_3$ are assumed parallel to principal axes, i.e.,

$$I_O^N = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

As the system kinetic energy consists solely of terms second order in generalized velocities* the Hamiltonian of the system is simply the sum of the potential and kinetic energies. With the potential energy given the symbol V plus a constant C_o the system Hamiltonian can be written as

*Although the inertial angular velocity components are themselves not generalized coordinates, they can be expressed as linear combinations of derivatives of generalized coordinates.

$$H = \frac{1}{2}\omega^T I_O^N \omega + \frac{1}{2}\sum m_i \dot{u}^{iT}\dot{u}^i + \frac{1}{2}\omega^T\left[\sum m_i\left(\tilde{r}^{iT}\tilde{u}^i + \tilde{u}^{iT}\tilde{r}^i\right)\right]\omega$$

$$+ \frac{1}{2}\omega^T\left[\sum m_i \tilde{u}^{iT}\tilde{u}^i\right]\omega - \sum m_i \dot{u}^{iT}\left(\tilde{r}^i + \tilde{u}^i\right)\omega$$

$$- \frac{\mathcal{M}}{2}\left[\dot{c}^T\dot{c} - 2\dot{c}^T\tilde{c}\,\omega + \omega^T\tilde{c}^T\tilde{c}\,\omega\right] + V + C_o \qquad (10)$$

The system angular momentum may be derived from the general expression

$$\underline{h} = \int_{A,B} (\underline{c} + \underline{\rho}) \times (\dot{\underline{c}} + \dot{\underline{\rho}})\, dm$$

$$= \int_{A,B} (\underline{c} + \underline{\rho}) \times (\overset{\circ}{\underline{c}} + \overset{\circ}{\underline{\rho}})\, dm + \int_{A,B} (\underline{c} + \underline{\rho}) \times [\underline{\omega} \times (\underline{c} + \underline{\rho})]\, dm$$

$$= \underline{c} \times \int_{A,B} (\overset{\circ}{\underline{c}} + \overset{\circ}{\underline{\rho}})\, dm - \overset{\circ}{\underline{c}} \times \int_{A,B} \underline{\rho}\, dm + \int_{A,B} \underline{\rho} \times \overset{\circ}{\underline{\rho}}\, dm$$

$$+ \underline{c} \times \left[\underline{\omega} \times \int_{A,B} (\underline{c} + \underline{\rho})\, dm\right] + \int_{A,B} \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\, dm - (\underline{\omega} \times \underline{c}) \times \int_{A,B} \underline{\rho}\, dm$$

The first and fourth terms vanish by mass center definition, Equation (5), the second and sixth terms combine, and the fifth is recognized as the dot product of $\underline{\omega}$ and the inertia dyadic of the complete system about point N, $\underline{\underline{I}}^N$, so that

$$\underline{h} = \underline{\underline{I}}^N \cdot \underline{\omega} + \mathcal{M}\, \dot{\underline{c}} \times \underline{c} + \int_A \underline{\rho} \times \overset{\circ}{\underline{\rho}}\, dm \quad . \qquad (11)$$

The integral is representative of motion relative to the vector basis $\{\underline{b}\}$; as body B is assumed to be rigid the only contribution is from body A.

The latter two terms in Equation (11) may be written in the vector basis $\{\underline{b}\}$ as

$$\mathcal{M}\,\dot{\underline{c}} \times \underline{c} = \mathcal{M}[\,\mathring{\underline{c}} + (\underline{\omega} \times \underline{c})] \times \underline{c} = \{\underline{b}\}^T \mathcal{M}(\tilde{\mathring{c}}\, c + \tilde{c}\,\tilde{c}\,\omega)$$

$$\int_A \underline{\rho} \times \mathring{\underline{\rho}}\, dm = \sum m_i \left(\underline{\Gamma}^i + \underline{u}^i\right) \times \mathring{\underline{u}}^i = \{\underline{b}\}^T \sum m_i \left(\tilde{\Gamma}^i + \tilde{u}^i\right) \dot{u}^i$$

which allows the matrix representation of $\underline{h}$ in $\{\underline{b}\}$:

$${}^b h = I^N \omega + \sum m_i \left(\tilde{\Gamma}^i + \tilde{u}^i\right) \dot{u}^i + \mathcal{M}\left(\tilde{\mathring{c}}\, c + \tilde{c}\,\tilde{c}\,\omega\right) \qquad (12)$$

As described earlier the procedure to follow is to constrain the attitude angles through the angular momentum integral so that they represent deviations from an inertial direction $\hat{\underline{n}}_3$ which is colinear with the instantaneous angular momentum vector $\underline{h}$ after perturbation from its nominal inertial orientation. To proceed, define an inertial vector basis $\{\underline{n}\}$ and its corresponding transformation with respect to $\{\underline{b}\}$ as Θ, i.e.,

$$\{\underline{n}\} = \Theta\{\underline{b}\} \quad .$$

In particular let Θ be representative of a 3-1-2 attitude angle sequence with θ_3 the first rotation about $\hat{\underline{n}}_3$, θ_1 the second rotation about the displaced $\hat{\underline{n}}_1$ axis, and θ_2 the third rotation about $\hat{\underline{b}}_2$. The resulting transformation matrix is then

$$\Theta = \begin{pmatrix} \cos\theta_2 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3 & -\cos\theta_1 \sin\theta_3 & \sin\theta_2 \cos\theta_3 + \cos\theta_2 \sin\theta_1 \sin\theta_3 \\ \cos\theta_2 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3 & \cos\theta_1 \cos\theta_3 & \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_2 \cos\theta_3 \\ -\sin\theta_2 \cos\theta_1 & \sin\theta_1 & \cos\theta_1 \cos\theta_2 \end{pmatrix}$$

The unit vector $\hat{\underline{n}}_3$ is so defined that the angular momentum vector is colinear with $\hat{\underline{n}}_3$ and remains so because the system is torque free; i. e.,

$$\underline{h} \cdot \hat{\underline{n}}_1 = 0$$

$$\underline{h} \cdot \hat{\underline{n}}_2 = 0$$

$$\underline{h} \cdot \hat{\underline{n}}_3 = h$$

With $\underline{h}$ written in vector basis $\{\underline{b}\}$ as prescribed by (12) the above constraint equations become

$$\Theta\left[I^N\omega + \sum m_i\left(\tilde{\Gamma}^i + \tilde{u}^i\right)\dot{u}^i + \mathscr{M}\left(\tilde{\dot{c}}\, c + \tilde{c}\,\tilde{c}\,\omega\right)\right] = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

which allows solutions of the angular velocity components of ω in terms of the deformation coordinates, their derivatives and the attitude angles, i. e.,

$$\omega = \left[I^N + \mathscr{M}\tilde{c}\,\tilde{c}\right]^{-1}\left\{\Theta^T\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \sum m_i\left(\tilde{\Gamma}^i + \tilde{u}^i\right)\dot{u}^i - \mathscr{M}\tilde{\dot{c}}\, c\right\}$$

The 3-1-2 choice for the sequential rotation allows the matrix product given by the first term in the bracket to be simply

$$\Theta^T\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} = \begin{pmatrix} -\sin\theta_2\cos\theta_1 \\ \sin\theta_1 \\ \cos\theta_1\cos\theta_2 \end{pmatrix} h$$

permitting the approximate expression for $\omega = (\omega_x, \omega_y, \omega_z)^T$ to be written as:

$$\omega \approx \left[I^N + \mathscr{M} \tilde{c}\, \tilde{c} \right]^{-1} \left\{ \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} + \left[\begin{pmatrix} -h\theta_2 \\ h\theta_1 \\ 0 \end{pmatrix} - \sum m_i \tilde{\Gamma}^i \dot{u}^i \right] + \left[\begin{pmatrix} 0 \\ 0 \\ -\frac{h}{2}(\theta_1^2 + \theta_2^2) \end{pmatrix} - \sum m_i \tilde{u}^i \dot{u}^i - \mathscr{M} \tilde{\dot{c}}\, c \right] \right\} \qquad (13)$$

where terms in the bracket higher than second order have been dropped since the quadratic approximation of the Hamiltonian (and hence the Liapunov function) is an acceptable indicator of its sign character in the neighborhood of the origin as long as all coordinates are included.

The solution of (13) for the angular velocity components, when substituted into Equation (10) and approximated by terms no higher than second order, will yield a Liapunov function whose sign character is a test of the stability of the system under consideration. As the algebraic complexity of the solution of (13) is compounded by the second order terms containing c, they will initially be ignored and accounted for later.

The inertia dyadic about point N of the complete system, $\underline{\underline{I}}^N$, consists of contributions from both the appendage and the core. The core contribution is simply defined as $\underline{\underline{I}}_B^N$ having an inertia matrix I_B^N in the vector basis $\{\underline{b}\}$ and consisting solely of non-varying elements; whereas the appendage contribution, $\underline{\underline{I}}_A^N$, must account for deformations as well as elements descriptive of the undeformed state, to wit

$$\underline{\underline{I}}_A^N = \sum m_i \left[\left(\underline{\Gamma}^i + \underline{u}^i \right) \cdot \left(\underline{\Gamma}^i + \underline{u}^i \right) \underline{\underline{E}} - \left(\underline{\Gamma}^i + \underline{u}^i \right) \left(\underline{\Gamma}^i + \underline{u}^i \right) \right]$$

$$= \underline{\underline{I}}_{A'}^N + \sum m_i \left[2 \underline{u}^i \cdot \underline{\Gamma}^i \underline{\underline{E}} - \left(\underline{\Gamma}^i \underline{u}^i + \underline{u}^i \underline{\Gamma}^i \right) \right]$$

$$+ \sum m_i \left[\underline{u}^i \cdot \underline{u}^i \underline{\underline{E}} - \underline{u}^i \underline{u}^i \right]$$

where $\underline{\underline{E}}$ is defined as the identity dyadic with the corresponding identity matrix E in vector basis $\{\underline{b}\}$, and $\underline{\underline{I}}_{A'}^N$ is the undeformed particle contribution, i.e.,

$$\underline{\underline{I}}_{A'}^N = \sum m_i \left[\underline{\Gamma}^i \cdot \underline{\Gamma}^i \underline{\underline{E}} - \underline{\Gamma}^i \underline{\Gamma}^i \right] = \{\underline{b}\}^T \sum m_i \tilde{\Gamma}^{iT} \tilde{\Gamma}^i \{\underline{b}\}$$

The inertia dyadic of the complete system is then

$$\underline{\underline{I}}^N = \underline{\underline{I}}_B^N + \underline{\underline{I}}_{A'}^N + \{\underline{b}\}^T \left[\sum m_i \left(2\Gamma^{iT} u^i E - u^i \Gamma^{iT} - \Gamma^i u^{iT} \right) \right] \{\underline{b}\}$$

$$+ \{\underline{b}\}^T \sum m_i \left(u^{iT} u^i E - u^i u^{iT} \right) \{\underline{b}\}$$

In vector basis $\{\underline{b}\}$ the inertia matrix of the complete system I^N consists as shown of terms independent of deformation variables u as well as terms both linear and second order in these variables

$$I^N = I_B^N + I_{A'}^N + I_{A_1}^N + I_{A_2}^N$$

where

$$I_{A_1}^N = 2 \sum m_i \Gamma^{iT} u^i E - \sum m_i \left(u^i \Gamma^{iT} + \Gamma^i u^{iT} \right)$$

$$I_{A_2}^N = \sum m_i u^{iT} u^i E - \sum m_i u^i u^{iT}$$

The undeformed particle contribution when summed with the core contribution has been earlier defined as I_O^N, prescribed to be principal and having diagonal elements A, B, and C. The expansions of $I_{A_1}^N$ and $I_{A_2}^N$ combine with I_O^N to form the system inertia matrix delineated below

$$
\Gamma^N = \begin{bmatrix}
A + \sum m_i\left(2u^i_y\Gamma^i_y + 2u^i_z\Gamma^i_z + u^{i2}_y + u^{i2}_z\right) & -\sum m_i\left(u^i_x\Gamma^i_y + u^i_y\Gamma^i_x + u^i_x u^i_y\right) & -\sum m_i\left(u^i_x\Gamma^i_z + u^i_z\Gamma^i_x + u^i_x u^i_z\right) \\
-\sum m_i\left(u^i_x\Gamma^i_y + u^i_y\Gamma^i_x + u^i_x u^i_y\right) & B + \sum m_i\left(2u^i_x\Gamma^i_x + 2u^i_z\Gamma^i_z + u^{i2}_x + u^{i2}_z\right) & -\sum m_i\left(u^i_y\Gamma^i_z + u^i_z\Gamma^i_y + u^i_y u^i_z\right) \\
-\sum m_i\left(u^i_x\Gamma^i_z + u^i_z\Gamma^i_x + u^i_x u^i_z\right) & -\sum m_i\left(u^i_y\Gamma^i_z + u^i_z\Gamma^i_y + u^i_y u^i_z\right) & C + \sum m_i\left(2u^i_x\Gamma^i_x + 2u^i_y\Gamma^i_y + u^{i2}_x + u^{i2}_y\right)
\end{bmatrix}
$$

The inverse of I^N is required to evaluate the angular velocity components in terms of the variables θ_1, θ_2, u^i, and $\dot{u}^i$. Identify I^N as

$$I^N = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}$$

having the determinant

$$\begin{aligned} \text{Det } I^N = & \; I_{11} I_{22} I_{33} + 2I_{12} I_{13} I_{23} \\ & - \left(I_{11} I_{23}^2 + I_{22} I_{13}^2 + I_{33} I_{12}^2 \right) . \end{aligned}$$

Since I_{ij} for $i \neq j$ consists solely of first and second order terms, and only second order terms and below are required, the second term above vanishes. Also the terms in the bracket yield no terms lower than second order. Evaluating Det I^N term by term yields

$$\begin{aligned} \text{Det } I^N \approx & \; ABC + AB \sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i\right) + AC \sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i\right) \\ & + BC \sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i\right) + AB \sum m_i \left(u_x^{i2} + u_y^{i2}\right) \\ & + AC \sum_i m_i \left(u_x^{i2} + u_z^{i2}\right) + BC \sum m_i \left(u_y^{i2} + u_z^{i2}\right) \\ & + A \left\{ \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i\right)\right] - \left[\sum m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i\right)\right]^2 \right\} \\ & + B \left\{ \left[\sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i\right)\right] - \left[\sum m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i\right)\right]^2 \right\} \\ & + C \left\{ \left[\sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i\right)\right] - \left[\sum m_i \left(u_x^i \Gamma_y^i + u_y^i \Gamma_x^i\right)\right]^2 \right\} \end{aligned}$$

In the evaluation of $(I^N)^{-1}$ the quadratic approximation to the inverse of Det I^N is required. If Det I^N is identified as

$$\text{Det } I^N = a_o + a_1 x + a_2 y^2$$

where x and y^2 represent first and second order terms respectively, then its inverse quadratic approximation is given by

$$\frac{1}{\text{Det } I^N} \approx \frac{1}{a_o}\left\{1 - \frac{a_1}{a_o}\, x - \frac{a_2}{a_o}\, y^2 + \left(\frac{a_1}{a_o}\right)^2 x^2\right\}$$

Or, more specifically

$$\frac{1}{\text{Det } I^N} \approx \frac{1}{ABC}\Bigg[1 - \frac{1}{C}\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right) - \frac{1}{B}\sum m_i\left(2u_x^i\Gamma_x^i + 2u_z^i\Gamma_z^i\right)$$

$$- \frac{1}{A}\sum m_i\left(2u_y^i\Gamma_y^i + 2u_z^i\Gamma_z^i\right) - \frac{1}{C}\sum m_i\left(u_x^{i\,2} + u_y^{i2}\right)$$

$$- \frac{1}{B}\sum m_i\left(u_x^{i2} + u_z^{i2}\right) - \frac{1}{A}\sum m_i\left(u_y^{i2} + u_z^{i2}\right)$$

$$- \frac{1}{BC}\left\{\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_z^i\Gamma_z^i\right)\right]\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right)\right] - \left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]^2\right\}$$

$$- \frac{1}{AC}\left\{\left[\sum m_i\left(2u_y^i\Gamma_y^i + 2u_z^i\Gamma_z^i\right)\right]\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right)\right] - \left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]^2\right\}$$

$$- \frac{1}{AB}\left\{\left[\sum m_i\left(2u_y^i\Gamma_y^i + 2u_z^i\Gamma_z^i\right)\right]\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_z^i\Gamma_z^i\right)\right] - \left[\sum m_i\left(u_x^i\Gamma_y^i + u_y^i\Gamma_x^i\right)\right]^2\right\}$$

$$+ \frac{1}{(ABC)^2}\Bigg\{AB\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right) + AC\sum m_i\left(2u_x^i\Gamma_x^i + 2u_z^i\Gamma_z^i\right)$$

$$+ BC\sum m_i\left(2u_y^i\Gamma_y^i + 2u_z^i\Gamma_z^i\right)\Bigg\}^2\Bigg]$$

With the adjoint of I^N identified as

$$\text{Adj } I^N = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Its elements excluding terms higher than second order are found to be

$$A_{11} = \Big\{ BC + B \sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i + u_x^{i2} + u_y^{i2}\right)$$

$$+ C \sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i + u_x^{i2} + u_z^{i2}\right)$$

$$+ \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i\right)\right]\Big\}$$

$$- \left[\sum m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i\right)\right]^2$$

$$A_{22} = \Big\{ AC + A \sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i + u_x^{i2} + u_y^{i2}\right)$$

$$+ C \sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i + u_y^{i2} + u_z^{i2}\right)$$

$$+ \left[\sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i\right)\right]\Big\}$$

$$- \left[\sum m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i\right)\right]^2$$

$$A_{33} = \Big\{ AB + A \sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i + u_x^{i2} + u_z^{i2}\right)$$

$$+ B \sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i + u_y^{i2} + u_z^{i2}\right)$$

$$+ \left[\sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i\right)\right]\Big\}$$

$$- \left[\sum m_i \left(u_x^i \Gamma_y^i + u_y^i \Gamma_x^i\right)\right]^2$$

$$A_{12} = \Big\{ C\left[\sum m_i \left(u_x^i \Gamma_y^i + u_y^i \Gamma_x^i + u_x^i u_y^i\right]\right.$$

$$+ \left[\sum m_i \left(u_x^i \Gamma_y^i + u_y^i \Gamma_x^i\right)\right] \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_y^i \Gamma_y^i\right)\right]$$

$$+ \left[\sum m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i\right)\right] \left[\sum m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i\right)\right]\Big\}$$

$$A_{13} = \left[\sum m_i \left(u_x^i \Gamma_y^i + u_y^i \Gamma_x^i\right)\right] \left[\sum m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i\right)\right]$$

$$+ B \sum m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i + u_x^i u_z^i\right)$$

$$+ \left[\sum m_i \left(2u_x^i \Gamma_x^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i\right)\right]$$

$$A_{23} = \Big\{ A \sum m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i + u_y^i u_z^i\right)$$

$$+ \left[\sum m_i \left(2u_y^i \Gamma_y^i + 2u_z^i \Gamma_z^i\right)\right] \left[\sum m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i\right)\right]$$

$$+ \left[\sum m_i \left(u_x^i \Gamma_y^i + u_y^i \Gamma_x^i\right)\right] \left[\sum m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i\right)\right]\Big\}$$

With the inverse of I^N identified as

$$\left(I^N\right)^{-1} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{pmatrix}$$

its elements are evaluated by

$$B_{ij} = \frac{A_{ij}}{\text{Det } I^N}$$

Observation of Equation (13) yields one further simplification: As the immediate goal is the derivation of expressions for the components of ω to terms at most second order, and the only contribution to the bracket in (13) which post multiplies $\left(I^N\right)^{-1}$ having terms independent of small quantities is $(0, 0, h)^T$, then only the elements of $\left(I^N\right)^{-1}$ identified as B_{i3} need be expanded to second order. The remaining three terms need only be evaluated to first order. A summary of these expansions is offered below.

$$B_{11}\Big|_{1^{st}} = \frac{1}{A} - \frac{2}{A^2}\left\{\sum m_i\left(u_y^i\Gamma_y^i + u_z^i\Gamma_z^i\right)\right\}$$

$$B_{12}\Big|_{1^{st}} = \frac{1}{AB}\left\{\sum m_i\left(u_x^i\Gamma_y^i + u_y^i\Gamma_x^i\right)\right\};\quad B_{22}\Big|_{1^{st}} = \frac{1}{B} - \frac{2}{B^2}\left\{\sum m_i\left(u_x^i\Gamma_x^i + u_z^i\Gamma_z^i\right)\right\}$$

$$B_{13}\Big|_{2^{nd}} = \frac{1}{AC}\left\{\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right\} + \frac{1}{AC}\sum m_i u_x^i u_z^i + \frac{1}{ABC}\left\{\left[\sum m_i\left(u_x^i\Gamma_y^i + u_y^i\Gamma_x^i\right)\right]\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]\right\}$$

$$- \frac{1}{AC^2}\left\{\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right)\right]\right\} - \frac{1}{A^2C}\left\{\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]\left[\sum m_i\left(2u_y^i\Gamma_y^i + 2u_z^i\Gamma_z^i\right)\right]\right\}$$

$$B_{23}\Big|_{2^{nd}} = \frac{1}{BC}\left\{\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right) + \frac{1}{BC}\sum m_i u_y^i u_z^i + \frac{1}{ABC}\left\{\left[\sum m_i\left(u_x^i\Gamma_y^i + u_y^i\Gamma_x^i\right)\right]\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]\right.\right.$$

$$- \frac{1}{BC^2}\left\{\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right)\right]\right\} - \frac{1}{B^2C}\left\{\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_z^i\Gamma_z^i\right)\right]\right\}$$

$$B_{33}\Big|_{2^{nd}} = \frac{1}{C} - \frac{2}{C^2}\left\{\sum m_i\left(u_x^i\Gamma_x^i + u_y^i\Gamma_y^i\right)\right\} - \frac{1}{C^2}\sum m_i\left(u_x^{i2} + u_y^{i2}\right) + \frac{1}{C^3}\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right)\right]^2$$

$$+ \frac{1}{BC^2}\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]^2 + \frac{1}{AC^2}\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]^2$$

The evaluation of the elements of $\left(I^N\right)^{-1}$, to the degree required to obtain the quadratic approximation of the Hamiltonian, permits the inertial angular velocity components of vector basis $\{\underline{b}\}$ (for c = 0) to be determined by expanding (13).

$$\omega = \left(I^N\right)^{-1}\left\{\begin{pmatrix}0\\0\\h\end{pmatrix} + \left[\begin{pmatrix}-h\,\theta_2\\h\,\theta_1\\0\end{pmatrix} + \begin{pmatrix}\sum m_i\left(\Gamma_z^i\,\dot{u}_y^i - \Gamma_y^i\,\dot{u}_z^i\right)\\\sum m_i\left(\Gamma_x^i\,\dot{u}_z^i - \Gamma_z^i\,\dot{u}_x^i\right)\\\sum m_i\left(\Gamma_y^i\,\dot{u}_x^i - \Gamma_x^i\,\dot{u}_y^i\right)\end{pmatrix}\right]\right.$$

$$\left. + \left[\begin{pmatrix}0\\0\\-\frac{h}{2}\left(\theta_1^2+\theta_2^2\right)\end{pmatrix} + \begin{pmatrix}\sum m_i\left(u_z^i\,\dot{u}_y^i - u_y^i\,\dot{u}_z^i\right)\\\sum m_i\left(u_x^i\,\dot{u}_z^i - u_z^i\,\dot{u}_x^i\right)\\\sum m_i\left(u_y^i\,\dot{u}_x^i - u_x^i\,\dot{u}_y^i\right)\end{pmatrix}\right]\right\}$$

By examining the character of $\left(I^N\right)^{-1}$ it is observed that only ω_z has a term independent of small quantities. Further, only second order terms need be retained in evaluating the quantities in the Hamiltonian which appear as products of either deformations, deformations and angular velocity components, or angular velocity components. These observations allow ω_x and ω_y to be expanded only to first order terms, whereas second order terms must be retained in evaluating ω_z:

$$\omega_x\Big|_{1^{st}} = -\frac{h\,\theta_2}{A} + \frac{1}{A}\sum m_i\left(\Gamma_z^i\dot{u}_y^i - \Gamma_y^i\,\dot{u}_z^i\right) + \frac{h}{AC}\sum m_i\left(u_x^i\,\Gamma_z^i + u_z^i\,\Gamma_x^i\right) \tag{14a}$$

$$\omega_y\Big|_{1^{st}} = \frac{h\,\theta_1}{B} + \frac{1}{B}\sum m_i\left(\Gamma_x^i\dot{u}_z^i - \Gamma_z^i\dot{u}_x^i\right) + \frac{h}{BC}\sum m_i\left(u_y^i\,\Gamma_z^i + u_z^i\,\Gamma_y^i\right) \tag{14b}$$

$$\omega_z\Big|_{2nd} = \frac{h}{C} - \frac{2h}{C^2}\left[\sum m_i\left(u_x^i\Gamma_x^i + u_y^i\Gamma_y^i\right)\right] + \frac{1}{C}\sum m_i\left(\Gamma_y^i\dot{u}_x^i - \Gamma_x^i\dot{u}_y^i\right)$$

$$- \frac{h\theta_2}{AC}\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right] + \frac{h\theta_1}{BC}\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]$$

$$+ \frac{1}{AC}\left[\sum m_i\left(\Gamma_z^i\dot{u}_y^i - \Gamma_y^i\dot{u}_z^i\right)\right]\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]$$

$$+ \frac{1}{BC}\left[\sum m_i\left(\Gamma_x^i\dot{u}_z^i - \Gamma_z^i\dot{u}_x^i\right)\right]\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]$$

$$- \frac{h}{C^2}\sum m_i\left(u_x^{i2} + u_y^{i2}\right) + \frac{h}{C^3}\left[\sum m_i\left(2u_x^i\Gamma_x^i + 2u_y^i\Gamma_y^i\right)\right]^2$$

$$+ \frac{h}{BC^2}\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]^2 + \frac{h}{AC^2}\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]^2$$

$$- \frac{2}{C^2}\left[\sum m_i\left(\Gamma_y^i\dot{u}_x^i - \Gamma_x^i\dot{u}_y^i\right)\right]\left[\sum m_i\left(u_x^i\Gamma_x^i + u_y^i\Gamma_y^i\right)\right]$$

$$- \frac{h}{2C}\left(\theta_1^2 + \theta_2^2\right) + \frac{1}{C}\sum m_i\left(u_y^i\dot{u}_x^i - u_x^i\dot{u}_y^i\right) \tag{14c}$$

Substitution of the set of Equations (14) into Equation (10) with c equal to zero yields the Hamiltonian in terms of the coordinates u_x^i, u_y^i, u_z^i, θ_1, and θ_2. Expanding (10) term by term and dropping components higher than second order yields the desired results:

$$H = \frac{1}{2}\frac{h^2\theta_2^2}{A} + \frac{1}{2}\frac{h^2\theta_1^2}{B} - \frac{h^2}{2C}\left(\theta_1^2 + \theta_2^2\right)$$

$$- \frac{1}{2A}\left[\sum m_i\left(\Gamma_z^i\dot{u}_y^i - \Gamma_y^i\dot{u}_z^i\right)\right]^2 - \frac{1}{2B}\left[\sum m_i\left(\Gamma_x^i\dot{u}_z^i - \Gamma_z^i\dot{u}_x^i\right)\right]^2$$

$$- \frac{1}{2C}\left[\sum m_i\left(\Gamma_y^i\dot{u}_x^i - \Gamma_x^i\dot{u}_y^i\right)\right]^2 + \frac{h^2}{2C}$$

$$+ \frac{1}{2}\frac{h^2}{AC^2}\left[\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right)\right]^2 + \frac{1}{2}\frac{h^2}{BC^2}\left[\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)\right]^2$$

$$+ \frac{2h^2}{C^3}\left[\sum m_i\left(u_x^i\Gamma_x^i + u_y^i\Gamma_y^i\right)\right]^2$$

$$- \frac{h^2}{C^2}\left[\sum m_i\left(u_x^i\Gamma_x^i + u_y^i\Gamma_y^i\right)\right] - \frac{h^2}{2C^2}\sum m_i\left(u_x^{i2} + u_y^{i2}\right)$$

$$- \frac{h^2\theta_2}{AC}\sum m_i\left(u_x^i\Gamma_z^i + u_z^i\Gamma_x^i\right) + \frac{h^2\theta_1}{BC}\sum m_i\left(u_y^i\Gamma_z^i + u_z^i\Gamma_y^i\right)$$

$$+ \frac{1}{2}\sum m_i\left(\dot{u}_x^{i2} + \dot{u}_y^{i2} + \dot{u}_z^{i2}\right) + V + C_o \qquad (15)$$

Prior to extracting stability criteria from (15) it is first modified to account for mass center shifts due to particle deformations. Thus we shall re-examine the development of the previous pages noting the required modifications which permit center of mass shifts. Recall first Equation (10), which is representative of H prior to the substitution for ω in terms of attitude angles. The mass center shift contribution is

$$- \frac{\mathcal{M}}{2}\left[\dot{c}^T\dot{c} - 2\dot{c}^T\tilde{c}\,\omega + \omega^T\tilde{c}^T\tilde{c}\,\omega\right]$$

where c is the representation of $\underline{c}$ in vector basis $\{\underline{b}\}$, and $\underline{c}$ is expressed solely as a linear sum of the deformation vectors

$\underline{u}^i$, i.e.,

$$\underline{c} = -\frac{1}{\mathcal{M}} \sum m_j \underline{u}^j$$

Hence

$$c = -\frac{1}{\mathcal{M}} \sum m_j u^j$$

Neglecting quantities higher than second order, then one finds through term by term expansion the following:

$$\dot{c}^T \dot{c} = \frac{1}{\mathcal{M}^2} \left[\left(\sum m_j \dot{u}^j_x \right)^2 + \left(\sum m_j \dot{u}^j_y \right)^2 + \left(\sum m_j \dot{u}^j_z \right)^2 \right]$$

$$-2\, \dot{c}^T \tilde{c}\, \omega \approx -\frac{2\omega_z}{\mathcal{M}^2} \left[\left(\sum m_j \dot{u}^j_x \right) \left(\sum m_j u^j_y \right) - \left(\sum m_j \dot{u}^j_y \right) \left(\sum m_j u^j_x \right) \right]$$

$$\omega^T \tilde{c}^T \tilde{c}\, \omega \approx \frac{\omega_z^2}{\mathcal{M}^2} \left[\left(\sum m_j u^j_x \right)^2 + \left(\sum m_j u^j_y \right)^2 \right]$$

Thus when mass center shifts are accounted for equation (10) expands to:

$$\begin{aligned}
H = {} & \frac{1}{2} \omega^T I^N_O \omega + \frac{1}{2} \sum m_i \dot{u}^{iT} \dot{u}^i \\
& + \frac{1}{2} \omega^T \left[\sum m_i \left(\tilde{\Gamma}^{iT} \tilde{u}^i + \tilde{u}^{iT} \tilde{\Gamma}^i \right) \right] \omega + \frac{1}{2} \omega^T \left[\sum m_i \tilde{u}^{iT} \tilde{u}^i \right] \omega \\
& - \sum m_i \dot{u}^{iT} \left(\tilde{\Gamma}^i + \tilde{u}^i \right) \omega + V + C_o \\
& - \frac{1}{2\mathcal{M}} \left[\left(\sum m_j \dot{u}^j_x \right)^2 + \left(\sum m_j \dot{u}^j_y \right)^2 + \left(\sum m_j \dot{u}^j_z \right)^2 \right] \\
& - \frac{\omega_z}{\mathcal{M}} \left[\left(\sum m_j \dot{u}^j_y \right) \left(\sum m_j u^j_x \right) - \left(\sum m_j \dot{u}^j_x \right) \left(\sum m_j u^j_y \right) \right] \\
& - \frac{\omega_z^2}{2\mathcal{M}} \left[\left(\sum m_j u^j_x \right)^2 + \left(\sum m_j u^j_y \right)^2 \right]
\end{aligned} \tag{16}$$

Due to the addition of the terms after V, all of which results from CM shifts, one has to modify the angular velocity components as well. Starting with Equation (13) we first have to calculate $(I^N + \mathcal{M}\tilde{c}\,\tilde{c})$ and its inverse for $c \neq o$.

$$\mathcal{M}\tilde{c}\,\tilde{c} = \mathcal{M}\begin{pmatrix} 0 & -c_z & c_y \\ c_y & 0 & -c_x \\ -c_y & c_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_x \\ -c_y & c_x & 0 \end{pmatrix} = \mathcal{M}\begin{pmatrix} -c_z^2 - c_y^2 & c_y c_x & c_z c_x \\ c_y c_x & -c_z^2 - c_x^2 & c_y c_z \\ c_z c_x & c_y c_z & -c_y^2 - c_x^2 \end{pmatrix}$$

$$= \frac{1}{\mathcal{M}}\begin{pmatrix} -(\sum m_j u_z^j)^2 - (\sum m_j u_y^j)^2 & (\sum m_j u_x^j)(\sum m_j u_y^j) & (\sum m_j u_x^j)(\sum m_j u_z^j) \\ (\sum m_j u_x^j)(\sum m_j u_y^j) & -(\sum m_j u_z^j)^2 - (\sum m_j u_x^j)^2 & (\sum m_j u_y^j)(\sum m_j u_z^j) \\ (\sum m_j u_x^j)(\sum m_j u_z^j) & (\sum m_j u_y^j)(\sum m_j u_z^j) & -(\sum m_j u_y^j)^2 - (\sum m_j u_x^j)^2 \end{pmatrix}$$

which must be added to the inertia matrix I^N. We now identify $(I^N + \mathcal{M}\tilde{c}\,\tilde{c})$ as $\hat{I}^N$, and its corresponding inverse elements as

$$\left(\hat{I}^N\right)^{-1} \triangleq \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{12} & \hat{B}_{22} & \hat{B}_{23} \\ \hat{B}_{13} & \hat{B}_{23} & \hat{B}_{33} \end{pmatrix}$$

By observation of the elements of I^N and the corresponding modifications $\mathcal{M}\tilde{c}\,\tilde{c}$ it is apparent that to account for the term $\mathcal{M}\cdot\tilde{c}\,\tilde{c}$ one must replace:

$$\sum m_i u_x^{i^2} \text{ by } \sum m_i u_x^{i^2} - \frac{1}{\mathscr{M}} \left(\sum m_j u_x^j\right)^2$$

$$\sum m_i u_y^{i^2} \text{ by } \sum m_i u_y^{i^2} - \frac{1}{\mathscr{M}} \left(\sum m_j u_y^j\right)^2$$

$$\sum m_i u_z^{i^2} \text{ by } \sum m_i u_z^{i^2} - \frac{1}{\mathscr{M}} \left(\sum m_j u_z^j\right)^2$$

and

$$\sum m_i u_x^i u_y^i \text{ by } \sum m_i u_x^i u_y^i - \frac{1}{\mathscr{M}} \left(\sum m_j u_x^j\right)\left(\sum m_j u_y^j\right)$$

$$\sum m_i u_x^i u_z^i \text{ by } \sum m_i u_x^i u_z^i - \frac{1}{\mathscr{M}} \left(\sum m_j u_x^j\right)\left(\sum m_j u_z^j\right)$$

$$\sum m_i u_y^i u_z^i \text{ by } \sum m_i u_y^i u_z^i - \frac{1}{\mathscr{M}} \left(\sum m_j u_y^j\right)\left(\sum m_j u_z^j\right)$$

The inverse elements follow immediately.

$$\hat{B}_{11} = B_{11} \; ; \; \hat{B}_{12} = B_{12} \; ; \; \hat{B}_{22} = B_{22}$$

$$\hat{B}_{13} = B_{13} - \frac{1}{\mathscr{M}AC} \left(\sum m_j u_x^j\right)\left(\sum m_j u_z^j\right)$$

$$\hat{B}_{23} = B_{23} - \frac{1}{\mathscr{M}BC} \left(\sum m_j u_y^j\right)\left(\sum m_j u_z^j\right)$$

$$\hat{B}_{33} = B_{33} + \frac{1}{\mathscr{M}C^2} \left(\sum m_j u_x^j\right)^2 + \frac{1}{\mathscr{M}C^2} \left(\sum m_j u_y^j\right)^2$$

Rewriting Equation (13) as

$$\omega \approx \left[I^N + \mathscr{M}\tilde{c}\tilde{c}\right]^{-1} \left\{ \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} + \left[\begin{pmatrix} -h\,\theta_2 \\ h\,\theta_1 \\ 0 \end{pmatrix} - \sum m_i \tilde{\Gamma}^i \dot{u}^i \right] + \left[\begin{pmatrix} 0 \\ 0 \\ -\frac{h}{2}\left(\theta_1^2 + \theta_2^2\right) \end{pmatrix} - \sum_i m_i \tilde{u}^i \dot{u}^i - \mathscr{M}\tilde{c}\; c \right] \right\} \qquad (13)$$

it is clear that all terms have been identified except the term $-\mathcal{M}\tilde{\dot{c}}\,c$, which is simply

$$-\mathcal{M}\tilde{\dot{c}}\,c = \frac{1}{\mathcal{M}}\begin{pmatrix} \left(\sum m_j u_y^j\right)\left(\sum m_j \dot{u}_z^j\right) - \left(\sum m_j u_z^j\right)\left(\sum m_j \dot{u}_y^j\right) \\ \left(\sum m_j u_z^j\right)\left(\sum m_j \dot{u}_x^j\right) - \left(\sum m_j u_x^j\right)\left(\sum m_j \dot{u}_z^j\right) \\ \left(\sum m_j u_x^j\right)\left(\sum m_j \dot{u}_y^j\right) - \left(\sum m_j u_y^j\right)\left(\sum m_j \dot{u}_x^j\right) \end{pmatrix}$$

These identifications now allow us to solve for ω_x, ω_y, and ω_z to the order necessary to assure that terms up to second order in the Hamiltonian are retained:

$$\omega_x\Big|_{1st} = \omega_x\Big|_{1st \text{ for } c = 0}$$

$$\omega_y\Big|_{1st} = \omega_y\Big|_{1st \text{ for } c = 0}$$

$$\omega_z\Big|_{2nd} = \omega_z\Big|_{2nd \text{ for } c = 0} + \frac{h}{\mathcal{M}C^2}\left(\sum m_j u_x^j\right)^2 + \frac{h}{\mathcal{M}C^2}\left(\sum m_j u_y^j\right)^2$$
$$+ \frac{1}{\mathcal{M}C}\left(\sum m_j u_x^j\right)\left(\sum m_j \dot{u}_y^j\right) - \frac{1}{\mathcal{M}C}\left(\sum m_j u_y^j\right)\left(\sum m_j \dot{u}_x^j\right)$$

Now consider the terms in (16) one by one to determine the alterations required to accommodate CM shifts. The term $\frac{1}{2}\omega^T I_o^N \omega$ is altered to:

$$\frac{1}{2} C\left(\frac{2h}{C}\right)\Big[\frac{h}{\mathcal{M}C^2}\left(\sum m_j u_x^j\right)^2 + \frac{h}{\mathcal{M}C^2}\left(\sum m_j u_y^j\right)^2$$
$$+ \frac{1}{\mathcal{M}C}\left(\sum m_j u_x^j\right)\left(\sum m_j \dot{u}_y^j\right) - \frac{1}{\mathcal{M}C}\left(\sum m_j u_y^j\right)\left(\sum m_j \dot{u}_x^j\right)\Big]$$
$$= \frac{h^2}{\mathcal{M}C}\left(\sum m_j u_x^j\right)^2 + \frac{h^2}{\mathcal{M}C^2}\left(\sum m_j u_y^j\right)^2$$
$$+ \frac{h}{\mathcal{M}C}\left(\sum m_j u_x^j\right)\left(\sum m_j \dot{u}_y^j\right) - \frac{h}{\mathcal{M}C}\left(\sum m_j u_y^j\right)\left(\sum m_j \dot{u}_x^j\right)$$

The remaining terms prior to V in (16) are unaltered. As the first term after V is independent of ω it also remains unaltered. The remaining two terms are expanded to second order by substituting $\frac{h}{C}$ for ω_z. Hence, the collection of terms required to account for CM shifts which must be added to Equation (15) are:

$$\frac{h^2}{\mathcal{M}C^2}\left[\left(\sum m_j u_x^j\right)^2 + \left(\sum m_j u_y^j\right)^2\right] + \frac{h}{\mathcal{M}C}\left[\left(\sum m_j u_x^j\right)\left(\sum m_j \dot{u}_y^j\right) - \left(\sum m_j u_y^j\right)\left(\sum m_j \dot{u}_x^j\right)\right]$$

$$- \frac{1}{2\mathcal{M}}\left[\left(\sum m_j \dot{u}_x^j\right)^2 + \left(\sum m_j \dot{u}_y^j\right)^2 + \left(\sum m_j \dot{u}_z^j\right)^2\right]$$

$$- \frac{h}{\mathcal{M}C}\left[\left(\sum m_j \dot{u}_y^j\right)\left(\sum m_j u_x^j\right) - \left(\sum m_j \dot{u}_x^j\right)\left(\sum m_j u_y^j\right)\right]$$

$$- \frac{h^2}{2\mathcal{M}C^2}\left[\left(\sum m_j u_x^j\right)^2 + \left(\sum m_j u_y^j\right)^2\right]$$

which simplifies and combines to

$$- \frac{1}{2\mathcal{M}}\left[\left(\sum m_j \dot{u}_x^j\right)^2 + \left(\sum m_j \dot{u}_y^j\right)^2 + \left(\sum m_j \dot{u}_z^j\right)^2\right]$$

$$+ \frac{h^2}{2\mathcal{M}C^2}\left[\left(\sum m_j u_x^j\right)^2 + \left(\sum m_j u_y^j\right)^2\right]$$

This result allows us to rewrite the Hamiltonian accommodating all terms including mass center shifts, to wit

$$
\begin{aligned}
H = {} & \frac{h^2\theta_2^2}{2A} + \frac{h^2\theta_1^2}{2B} - \frac{h^2}{2C}(\theta_1^2 + \theta_2^2) \\
& - \frac{h^2\theta_2}{AC} \sum m_i\left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i\right) + \frac{h^2\theta_1}{BC} \sum m_i\left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i\right) \\
& + \frac{h^2}{2AC^2}\left[\sum m_i\left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i\right)\right]^2 + \frac{h^2}{2BC^2}\left[\sum m_i\left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i\right)\right]^2 \\
& + \frac{2h^2}{C^3}\left[\sum m_i\left(u_x^i \Gamma_x^i + u_y^i \Gamma_y^i\right)\right]^2 - \frac{h^2}{C^2}\left[\sum m_i\left(u_x^i \Gamma_x^i + u_y^i \Gamma_y^i\right)\right] \\
& - \frac{h^2}{2C^2} \sum m_i\left(u_x^{i2} + u_y^{i2}\right) + \frac{h^2}{2\mathcal{M}C^2}\left[\left(\sum m_j u_x^j\right)^2 + \left(\sum m_j u_y^j\right)^2\right] \\
& + V - \frac{1}{2C}\left[\sum m_i\left(\Gamma_y^i \dot{u}_x^i - \Gamma_x^i \dot{u}_y^i\right)\right]^2 \\
& - \frac{1}{2A}\left[\sum m_i\left(\Gamma_z^i \dot{u}_y^i - \Gamma_y^i \dot{u}_z^i\right)\right]^2 - \frac{1}{2B}\left[\sum m_i\left(\Gamma_x^i \dot{u}_z^i - \Gamma_z^i \dot{u}_x^i\right)\right]^2 \\
& + \frac{1}{2} \sum m_i\left(\dot{u}_x^{i2} + \dot{u}_y^{i2} + \dot{u}_z^{i2}\right) + \frac{h^2}{2C} \\
& - \frac{1}{2\mathcal{M}}\left[\left(\sum m_j \dot{u}_x^j\right)^2 + \left(\sum m_j \dot{u}_y^j\right)^2 + \left(\sum m_j \dot{u}_z^j\right)^2\right] + C_o \qquad (17)
\end{aligned}
$$

The potential energy term must accommodate the steady state deflection of the particles induced by the constant spin rate Ω. When expanded it consists of a term quadratic in the deformation variables, a term linear in the deformation variables, and a constant term. The constant term may be combined with C_o to form a new constant K_o, and the linear term cancels the linear term in H identified as

$$- \frac{h^2}{C^2} \left[\sum m_i \left(u_x^i \Gamma_x^i + u_y^i \Gamma_y^i \right) \right]$$

The quadratic term in V, defined as V_2 persists and can be combined with other terms in H quadratic in the deformation variables.

To cement these ideas we shall temporarily digress and reconsider the simple particle model discussed in Chapter 2, re-sketched below as Figure 6.

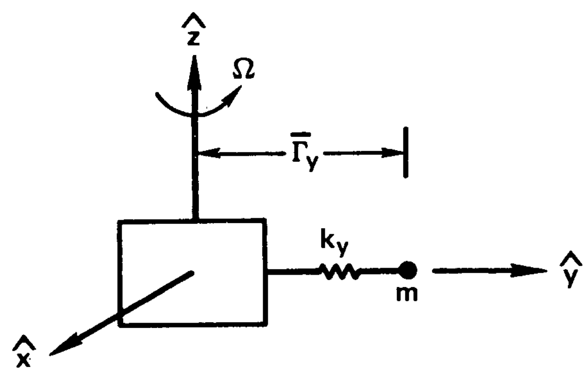

Figure 6. Simple Particle Model

From Equation (17) the linear term is shown to be:

$$- \frac{h^2}{C^2} m \, u_y \bar{\Gamma}_y \tag{18}$$

The potential energy is clearly

$$V = \frac{1}{2} k_y (u_y + \Delta)^2$$

$$= \frac{1}{2} k_y u_y^2 + k_y \Delta u_y + \frac{1}{2} k_y \Delta^2$$

The first and last terms are recognized as the aforementioned quadratic and constant terms, respectively. The linear term

$k_y \Delta u_y$ is expanded as follows: From Chapter 2 the steady state deflection Δ is shown to be

$$\Delta = \frac{m \Gamma_o \Omega^2}{k_y - m\Omega^2}$$

where $\quad \Gamma_o = \bar{\Gamma}_y - \Delta \; .$

Thus,

$$\Delta = \frac{m(\bar{\Gamma}_y - \Delta)\, \Omega^2}{k_y - m\Omega^2} \Rightarrow \Delta k_y \equiv m \bar{\Gamma}_y \Omega^2$$

Hence, the linear term in V expands to

$$k_y \Delta u_y = m \bar{\Gamma}_y \Omega^2 u_y \tag{19}$$

Recognizing that to the first approximation $h \approx C\Omega$, it is clear that the sum of (18) and (19) reduces to zero.

If in the general expression for the Hamiltonian, Equation (17), we expand V, and cancel the linear terms, the results, after identifying K_o as $-\frac{h^2}{2C}$ and rearranging terms, can be written as:

$$
\begin{aligned}
H = {} & \frac{1}{2} h^2 \theta_2^2 \frac{(C-A)}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{(C-B)}{BC} \\
& - \frac{h^2 \theta_2}{AC} \sum_i^N m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i \right) + \frac{h^2 \theta_1}{BC} \sum_i^N m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i \right) \\
& + \frac{h^2}{2AC^2} \left[\sum_i^N m_i \left(u_x^i \Gamma_z^i + u_z^i \Gamma_x^i \right) \right]^2 + \frac{h^2}{2BC^2} \left[\sum_i^N m_i \left(u_y^i \Gamma_z^i + u_z^i \Gamma_y^i \right) \right]^2 \\
& + \frac{2h^2}{C^3} \left[\sum_i^N m_i \left(u_x^i \Gamma_x^i + u_y^i \Gamma_y^i \right) \right]^2 - \frac{h^2}{2C^2} \sum_i^N m_i \left(u_x^{i2} + u_y^{i2} \right) \\
& + \frac{h^2}{2\mathcal{M} C^2} \left[\left(\sum_j^N m_j u_x^j \right)^2 + \left(\sum_j^N m_j u_y^j \right)^2 \right] - \frac{1}{2C} \left[\sum_i^N m_i \left(\Gamma_y^i \dot{u}_x^i - \Gamma_x^i \dot{u}_y^i \right) \right]^2 \\
& - \frac{1}{2A} \left[\sum_i^N m_i \left(\Gamma_z^i \dot{u}_y^i - \Gamma_y^i \dot{u}_z^i \right) \right]^2 - \frac{1}{2B} \left[\sum_i^N m_i \left(\Gamma_x^i \dot{u}_z^i - \Gamma_z^i \dot{u}_x^i \right) \right]^2 \\
& + \frac{1}{2} \sum_i^N m_i \left(\dot{u}_x^{i2} + \dot{u}_y^{i2} + \dot{u}_z^{i2} \right) - \frac{1}{2\mathcal{M}} \left[\left(\sum_j^N m_j \dot{u}_y^j \right)^2 + \left(\sum_j^N m_j \dot{u}_y^j \right)^2 + \left(\sum_j^N m_j \dot{u}_z^j \right)^2 \right] \\
& + \frac{1}{2} \sum_\ell^{3N} \sum_m^{3N} k_{\ell m} u^\ell u^m \qquad (20)
\end{aligned}
$$

Where V_2 is identified as the indicated double summation over the 3N deformation coordinates. (Note that this is in contrast to the other summations which are to be carried out only over the N particles.)

Equation (20) is the Hamiltonian of a rigid body having attached a general flexible appendage. We have not, as of yet, specialized in any way; except, of course, within the bounds of our math model, i. e., the appendage idealization as a collection of particles. As $\dot{H}$ in the presence of damping is negative definite then by Theorem 1 the system is asymptotically stable for H positive definite (or by Theorem 2 unstable for H either negative definite or sign variable). Since for asymptotic stability the complete function must be positive definite then it is clear that the following must be satisfied

$$\frac{1}{2} h^2 \theta_2^2 \frac{(C - A)}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{(C - B)}{BC} > 0$$

which leads to the familiar necessary stability criterion predicted by energy sink methods for spinning bodies having an internal energy dissipator, i.e.,

$$C > A \quad \text{and} \quad C > B \quad .$$

Thus by inspection of the Hamiltonian we can formally conclude, that in the presence of damping, the spin axis must be the axis of maximum moment of inertia. Note that for a freely spinning rigid body $\dot{H} = 0$ allowing, by virtue of a stability theorem similar to Theorem 1, Liapunov stability (as opposed to asymptotic stability) for either major or minor axis spin (since in the latter case we can use $-H$ as a Liapunov function). These results are, of course, expected; and anything short of them would be cause for alarm. However, our endeavor is to extract additional stability criteria (if any exist), and this requires the determination from Equation (20) of conditions for positive definite H. Although such conditions could be established in any given specific case by means of numerical procedures, in order to obtain literal closed-form stability criteria we are forced to restrict our flexible appendage model to lie in a plane containing the CM and normal to the spin axis ($\Gamma_z^i \equiv 0$, $i=1,\ldots,N$), see Figure 7.

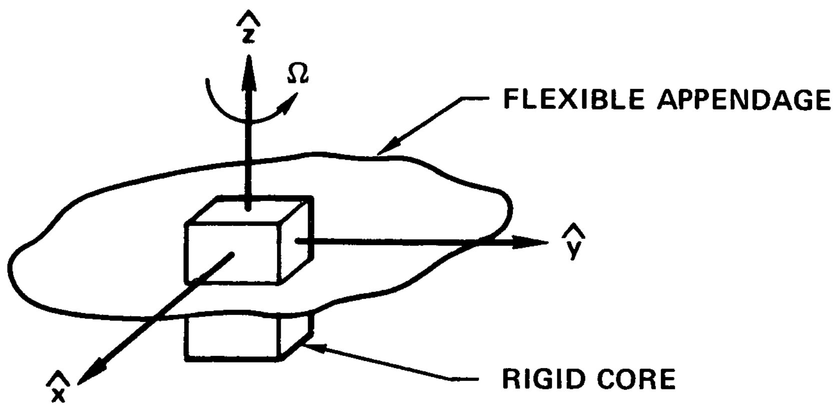

Figure 7. Restricted Appendage Model

Moreoever the stiffness elements orthogonal to the spin axis are assumed infinitely large ($u_x^i = u_y^i \equiv 0$) so that the structure is allowed to vibrate only in the $\pm \hat{z}$ direction. At first glance this latter restriction may seem overly severe. However, this is not true since the former restriction $\Gamma_z^i = 0$ separates that portion of the Hamiltonian descriptive of wobble motion from the remaining portion descriptive of the spinning motion. Accordingly we have to assume also that the stiffness elements are uncoupled. Thus, the restriction $\Gamma_z^i = 0$ separates the Hamiltonian, and hence, the stability conditions into two parts:

$$H_1 = \frac{1}{2} h^2 \theta_2^2 \frac{(C-A)}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{(C-B)}{BC}$$

$$- \frac{h^2 \theta_2}{AC} \sum_i^N m_i u_z^i \Gamma_x^i + \frac{h^2 \theta_1}{BC} \sum_i^N m_i u_z^i \Gamma_y^i$$

$$+ \frac{h^2}{2AC^2} \left[\sum_i^N m_i \left(u_z^i \Gamma_x^i \right) \right]^2 + \frac{h^2}{2BC^2} \left[\sum_i^N m_i \left(u_z^i \Gamma_y^i \right) \right]^2$$

$$- \frac{1}{2A} \left[\sum_i^N m_i \Gamma_y^i \dot{u}_z^i \right]^2 - \frac{1}{2B} \left[\sum_i^N m_i \Gamma_x^i \dot{u}_z^i \right]^2$$

$$+ \frac{1}{2} \sum_i^N m_i \dot{u}_z^{i2} - \frac{1}{2\mathcal{M}} \left[\sum_j^N m_j \dot{u}_z^j \right]^2 + \frac{1}{2} \sum_\ell^N \sum_m^N k_{\ell m} u_z^\ell u_z^m$$

$$H_2 = \frac{2h^2}{C^3} \left[\sum_i^N m_i \left(u_x^i \Gamma_x^i + u_y^i \Gamma_y^i \right) \right]^2 - \frac{h^2}{2C^2} \sum_i^N m_i \left(u_x^{i2} + u_y^{i2} \right)$$

$$+ \frac{h^2}{2\mathcal{M}C^2} \left[\left(\sum_j^N m_j u_x^j \right)^2 + \left(\sum_j^N m_j u_y^j \right)^2 \right] - \frac{1}{2C} \left[\sum_i^N m_i \left(\Gamma_y^i \dot{u}_x^i - \Gamma_x^i \dot{u}_y^i \right) \right]^2$$

$$+ \frac{1}{2} \sum_i^N m_i \left(\dot{u}_y^{i2} + \dot{u}_x^{i2} \right) - \frac{1}{2\mathcal{M}} \left[\left(\sum_j^N m_j \dot{u}_x^j \right)^2 + \left(\sum_j^N m_j \dot{u}_y^j \right)^2 \right]$$

$$+ \frac{1}{2} \sum_\ell^{2N} \sum_m^{2N} k_{\ell m} u_{x,y}^\ell u_{x,y}^m$$

The restriction $u_x^i = u_y^i \equiv 0$ reduces the total Hamiltonian to H_1 allowing stability criteria extracted from H_1 to be <u>both</u> necessary and sufficient. We shall in the following assume that these restrictions apply, i.e.,

$$H = H_1$$

To formulate the restricted Hamiltonian in a more useful form define the following matrices.

$$M \triangleq \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_N \end{pmatrix}_{N \text{ by } N} ; \quad q \triangleq \begin{pmatrix} u_z^1 \\ u_z^2 \\ \vdots \\ u_z^N \end{pmatrix}_{N \text{ by } 1} ; \quad \underline{E} \triangleq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{N \text{ by } 1}$$

$$\Gamma_x \triangleq \begin{pmatrix} \Gamma_x^1 & & & \\ & \Gamma_x^2 & & \\ & & \ddots & \\ & & & \Gamma_x^N \end{pmatrix}_{N \text{ by } N} ; \quad \Gamma_y \triangleq \begin{pmatrix} \Gamma_y^1 & & & \\ & \Gamma_y^2 & & \\ & & \ddots & \\ & & & \Gamma_y^N \end{pmatrix}_{N \text{ by } N}$$

where M, Γ_x and Γ_y are all N by N diagonal matrices, q and $\underline{E}$ are N by 1 column matrices with the elements of the latter all unity. These definitions allow the following identifications.

$$\sum m_i \left(u_z^i \Gamma_x^i\right) = \left(M \Gamma_x \underline{E}\right)^T q$$

$$\sum m_i \left(u_z^i \Gamma_y^i\right) = \left(M \Gamma_y \underline{E}\right)^T q$$

$$\left[\sum m_i \left(u_z^i \Gamma_x^i\right)\right]^2 = \left(M \Gamma_x \underline{E}\right)^T q \left(M \Gamma_x \underline{E}\right)^T q = q^T \left(M \Gamma_x \underline{E}\right)\left(M \Gamma_x \underline{E}\right)^T q$$

$$\left[\sum m_i \left(u_z^i \Gamma_y^i\right)\right]^2 = q^T \left(M \Gamma_y \underline{E}\right)\left(M \Gamma_y \underline{E}\right)^T q \ .$$

With K defined as the structural stiffness matrix, the potential energy takes the form $\frac{1}{2} q^T K q$ allowing the restricted Hamiltonian to be written as:

$$
\begin{aligned}
H = {} & \frac{1}{2} h^2 \theta_2^2 \frac{(C-A)}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{(C-B)}{BC} \\
& - \frac{h^2 \theta_2}{AC} (M \Gamma_x \underline{E})^T q + \frac{h^2 \theta_1}{BC} (M \Gamma_y \underline{E})^T q + \frac{1}{2} q^T K q \\
& + \frac{h^2}{2AC^2} q^T (M \Gamma_x \underline{E})(M \Gamma_x \underline{E})^T q + \frac{h^2}{2BC^2} q^T (M \Gamma_y \underline{E})(M \Gamma_y \underline{E})^T q \\
& + \frac{1}{2} \dot{q}^T M' \dot{q} - \frac{1}{2A} \dot{q}^T (M \Gamma_y \underline{E})(M \Gamma_y \underline{E})^T \dot{q} - \frac{1}{2B} \dot{q}^T (M \Gamma_x \underline{E})(M \Gamma_x \underline{E})^T \dot{q}
\end{aligned}
\tag{21}
$$

Where the symmetric N by N matrix M' is defined as

$$
M' = \begin{pmatrix}
m_1 - \frac{m_1^2}{\mathcal{M}} & -\frac{m_1 m_2}{\mathcal{M}} & -\frac{m_1 m_3}{\mathcal{M}} & \cdots\cdots & -\frac{m_1 m_N}{\mathcal{M}} \\
-\frac{m_1 m_2}{\mathcal{M}} & m_2 - \frac{m_2^2}{\mathcal{M}} & -\frac{m_2 m_3}{\mathcal{M}} & \cdots\cdots & -\frac{m_2 m_N}{\mathcal{M}} \\
\vdots & & & & \\
-\frac{m_1 m_N}{\mathcal{M}} & -\frac{m_2 m_N}{\mathcal{M}} & -\frac{m_3 m_N}{\mathcal{M}} & \cdots\cdots & m_N - \frac{m_N^2}{\mathcal{M}}
\end{pmatrix}
$$

As written the scalar Equation (21) is descriptive of all modes of vibration having a total of N + 2 coordinates (N deformation variables and 2 attitude angles). Our quest is the development of closed form stability criteria; thus a reduction of coordinates is desired to reduce the complexity of the problem to a level amenable to mathematical analysis. We seek then a transformation which transforms the N discrete coordinates to (normal mode) coordinates which are totally uncoupled, thus permitting coordinate truncation with the assurance that the mathematical model is a complete representation of the selected modes. The "hybrid-coordinate" approach, described in detail in Reference 11, offers a practical means of approximating this goal for spacecraft wherein only the homogeneous matrix equation descriptive of appendage deformations is transformed to uncoupled modal equations. Although the equations remain coupled through forcing terms descriptive of spacecraft rotational coordinates, the resulting uncoupled homogeneous equations provide some justification for truncation (in an engineering sense). This approach has received acceptance in a growing number of aerospace corporations (e.g., Hughes, JPL, Lockheed, and North American Rockwell) and has in fact provided the mathematical foundation for digital simulation studies for a number of spacecraft. In this dissertation we shall adopt this technique as an acceptable mathematical tool. Introduce then the coordinate transformation $q = \Phi\eta$ which transforms the N by 1 deformation column matrix q to the N by 1 modal column matrix η. Here Φ is the N by N matrix of eigenvectors normally associated with matrix modal analysis, see for example Reference 15. Moreover, let Φ be suitably normalized so that the following matrix equalities are satisfied

$$\Phi^T M' \Phi = E$$

$$\Phi^T K \Phi = \begin{pmatrix} \omega_1^2 & & & 0 \\ & \omega_2^2 & & \\ & & \ddots & \\ 0 & & & \omega_N^2 \end{pmatrix} \triangleq \omega^2$$

where the diagonal matrix ω^2 has as its nonzero elements the modal natural frequencies of the appendage restricted to vibrate orthogonal to the plane of the flexible structure. These frequencies are the loaded natural frequencies of the appendage accounting for preload (recall that centripetal effects are zero for vibrations in the $\hat{z}$ direction).

Define the N by 1 column matrices

$$\delta_x \triangleq \boldsymbol{\phi}^T M \Gamma_x \underline{E}$$

$$\delta_y \triangleq \boldsymbol{\phi}^T M \Gamma_y \underline{E}$$

and rewrite Equation (21) as

$$H = \frac{1}{2} h^2 \theta_2^2 \frac{(C-A)}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{(C-B)}{BC}$$

$$- \frac{h^2 \theta_2}{AC} \delta_x^T \eta + \frac{h^2 \theta_1}{BC} \delta_y^T \eta$$

$$+ \frac{1}{2} \eta^T \left[\omega^2 + \frac{h^2}{AC^2} \delta_x \delta_x^T + \frac{h^2}{BC^2} \delta_y \delta_y^T \right] \eta$$

$$+ \frac{1}{2} \dot{\eta}^T \left[E - \frac{1}{A} \delta_y \delta_y^T - \frac{1}{B} \delta_x \delta_x^T \right] \dot{\eta} \qquad (22)$$

Note that the identity matrix E in Equation (22) is of dimension N by N.

The N modal deformation coordinates of equation (22) are now truncated to a single mode, identified by index 1 (although not necessarily the mode having the lowest frequency); thus the total number of coordinates is reduced to three. Accordingly the N by 1 column matrices η, δ_x, and δ_y reduce to the scalars η_1, δ_{x1}, and δ_{y1}, respectively; and the N by N matrix of modal frequencies reduces to the scalar ω_1^2. Implementing this simplification allows us to write the stability condition (H positive definite) as

$$\frac{1}{2} h^2 \theta_2^2 \frac{(C-A)}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{(C-B)}{BC}$$

$$- \frac{h^2 \theta_2}{AC} \delta_{x1} \eta_1 + \frac{h^2 \theta_1}{BC} \delta_{y1} \eta_1$$

$$+ \frac{1}{2} \eta_1^2 \left(\omega_1^2 + \frac{h^2}{AC^2} \delta_{x1}^2 + \frac{h^2}{BC^2} \delta_{y1}^2 \right)$$

$$+ \frac{1}{2} \dot{\eta}_1^2 \left(AB - B \delta_{y1}^2 - A \delta_{x1}^2 \right) > 0 \qquad (23)$$

where it is to be understood that > 0 means positive for all values of θ_1, θ_2, η_1 and $\dot{\eta}_1$ in the neighborhood of the origin $\theta_1 = \theta_2 = \eta_1 = \dot{\eta}_1 = 0$, except equal to zero at the origin itself. Note that the last term in expression (23), and similarly its general counterpart in Equation (21), is uncoupled from the remaining terms; moreover it is a positive definite function. We shall demonstrate this by implementing interpretations set forth in Reference 11.

In general, the terms $\boldsymbol{\Phi}^T M \Gamma_x \underline{E}$ and $\boldsymbol{\Phi}^T M \Gamma_y \underline{E}$ are N by 1 matrices so that $\delta_{x1} = \boldsymbol{\Phi}_1^T M \Gamma_x \underline{E}$ and $\delta_{y1} = \boldsymbol{\Phi}_1^T M \Gamma_y \underline{E}$ are both scalars. The products

$$\delta_x^T \delta_x = \underline{E}^T \Gamma_x M \boldsymbol{\Phi} \boldsymbol{\Phi}^T M \Gamma_x \underline{E}$$

$$\delta_y^T \delta_y = \underline{E}^T \Gamma_y M \boldsymbol{\Phi} \boldsymbol{\Phi}^T M \Gamma_y \underline{E}$$

are identified in Reference 11 as the moment of inertia differences of the total structure and that of the rigid core about the body $\hat{y}$ and $\hat{x}$ axes, respectively. Moreover

$$\delta_x^T \delta_y = \delta_y^T \delta_x = \underline{E}^T \Gamma_x M \boldsymbol{\Phi} \boldsymbol{\Phi}^T M \Gamma_y \underline{E}$$

is identified as the cross products of inertia of the flexible appendage with respect to the $\hat{x}$ and $\hat{y}$ principal axes of the total structure.

With these interpretations consider the geometrical representation in Figure 8 and define:

$\hat{x}_c, \hat{y}_c, \hat{z}_c$ as principal axes of the rigid core;

$\hat{x}_a, \hat{y}_a, \hat{z}_a$ as principal axes of the flexible appendage;

and $\hat{x}, \hat{y}, \hat{z}$ as the total system principal axes

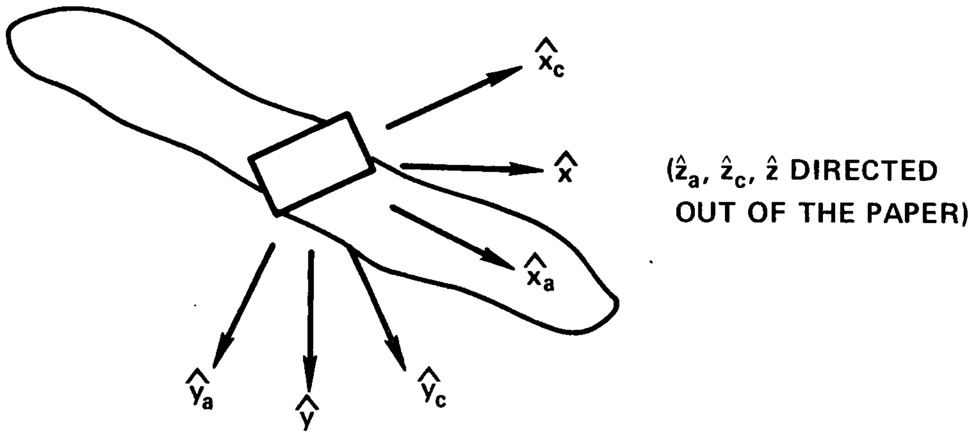

Figure 8. Principal Axes

Total System Inertia About $\hat{x}, \hat{y}, \hat{z}$	Rigid Core Inertia About $\hat{x}, \hat{y}, \hat{z}$	Flexible Appendage Inertia About $\hat{x}, \hat{y}, \hat{z}$

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} = \begin{bmatrix} A' & D & 0 \\ D & B' & 0 \\ 0 & 0 & C' \end{bmatrix} + \begin{bmatrix} \delta_y^T \delta_y & \delta_x^T \delta_y & 0 \\ \delta_x^T \delta_y & \delta_x^T \delta_x & 0 \\ 0 & 0 & C_a \end{bmatrix}$$

Note that $(\delta_x^T \delta_x)(\delta_y^T \delta_y) > (\delta_x^T \delta_y)(\delta_x^T \delta_y)$ and that $D = -\delta_x^T \delta_y$. Thus by properties of physical realizable inertia matrices (positive definite) we conclude that

$$A' B' > (\delta_x^T \delta_y)(\delta_x^T \delta_y)$$

And in particular for $\delta_x^T = \delta_{x1}$ and $\delta_y^T = \delta_{y1}$ we observe that

$$A'B' - (\delta_{x1}\,\delta_{y1})^2 > 0$$

With A' and B' identified as $(A - \delta_{y1}^2)$ and $(B - \delta_{x1}^2)$, respectively, it is clear by substitution that our immediate objective has been satisfied, i.e.,

$$A'B' - (\delta_{x1}\delta_{y1})^2 = (A - \delta_{y1}^2)(B - \delta_{x1}^2) - (\delta_{x1}\delta_{y1})^2$$

$$= AB - B\,\delta_{y1}^2 - A\,\delta_{x1}^2$$

At the risk of over-kill we shall demonstrate the satisfaction of $AB - B\,\delta_{y1}^2 - A\,\delta_{x1}^2 > 0$ by way of example. Thus consider a particle connected to a rigid core as in Figure 9.

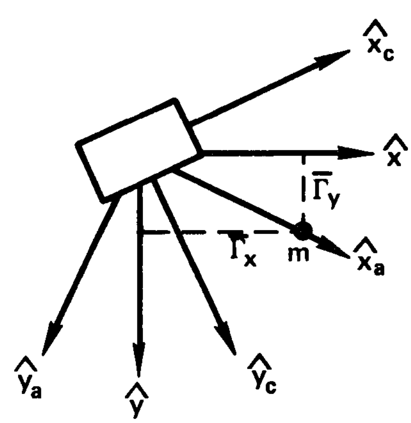

Figure 9. Principal Axes for Particle Appendage System

The inertia matrix of the particle with respect to the $\hat{x}$, $\hat{y}$, $\hat{z}$ axes is

$$\begin{bmatrix} m\bar{\Gamma}_y^2 & -m\bar{\Gamma}_x\bar{\Gamma}_y & 0 \\ -m\bar{\Gamma}_x\bar{\Gamma}_y & m\bar{\Gamma}_x^2 & 0 \\ 0 & 0 & m(\bar{\Gamma}_x^2 + \bar{\Gamma}_y^2) \end{bmatrix}$$

So that we may identify the following:

$$\delta_{y1}^2 = m\,\overline{\Gamma}_y^2$$

$$\delta_{x1}^2 = m\,\overline{\Gamma}_x^2$$

$$\delta_{x1}\delta_{y1} = m\,\overline{\Gamma}_x\overline{\Gamma}_y$$

If the inertia matrix of the core about the $\hat{x}$, $\hat{y}$, $\hat{z}$ axes is defined as

$$\begin{bmatrix} A' & D & 0 \\ D & B' & 0 \\ 0 & 0 & C \end{bmatrix}$$

then it is clear that D is precisely $m\overline{\Gamma}_x\overline{\Gamma}_y$. By virtue of positive definiteness of physically realizable bodies, we are assured that

$$A'B' - m^2\overline{\Gamma}_x^2\overline{\Gamma}_y^2 > 0$$

It is of interest to observe that the term we have been directing our attention to is the leading coefficient of the characteristic equation descriptive of wobble motion. In particular, the characteristic equation for the example at hand is given by

$$\begin{vmatrix} I_x S & \Omega(I_z - I_y) & m\overline{\Gamma}_y(\Omega^2 + S^2) \\ \Omega(I_x - I_z) & I_y S & -m\overline{\Gamma}_x(\Omega^2 + S^2) \\ \overline{\Gamma}_y S + \Omega\overline{\Gamma}_x & \Omega\overline{\Gamma}_y - \overline{\Gamma}_x S & S^2 + 2\zeta\omega S + \omega^2 \end{vmatrix} = 0$$

The leading coefficient (of S^4) is observed to be

$$I_x I_y - mI_x\overline{\Gamma}_x^2 - mI_y\overline{\Gamma}_y^2 \equiv A'B' - m^2\overline{\Gamma}_x^2\overline{\Gamma}_y^2$$

All this may in fact be superfluous to the reader in that after proper interpretation of terms it may be obvious that the expression

$(AB - B\delta_{y1}^2 - A\delta_{x1}^2)$ is indeed greater than zero. Moreover, its general counterpart

$$\left(E - \frac{1}{A} \delta_y \delta_y^T - \frac{1}{B} \delta_x \delta_x^T \right)$$

through properties of physically realizable structures must also be positive definite; a somewhat less obvious fact but nevertheless true. In any case no new stability criteria emerges from these considerations.

Having established that for physically realizable inertia properties the coefficient of $\dot{\eta}_1^2$ is positive, we have left then to consider the condition

$$\frac{1}{2} h^2 \theta_2^2 \frac{(C-A)}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{(C-B)}{BC} - \frac{h^2\theta_2}{AC} \delta_{x1} \eta_1$$
$$+ \frac{h^2\theta_1\delta_{y1}}{BC} \eta_1 + \frac{1}{2} \eta_1^2 \left(\omega_1^2 + \frac{h^2}{AC^2} \delta_{x1}^2 + \frac{h^2}{BC^2} \delta_{y1}^2\right) > 0 \qquad (24)$$

which may alternatively be written as

$$(\theta_1 \theta_2 \eta_1) \begin{pmatrix} \frac{h^2}{2} \frac{(C-B)}{BC} & 0 & \frac{h^2}{2} \frac{\delta_{y1}}{BC} \\ 0 & \frac{h^2}{2} \frac{(C-A)}{AC} & -\frac{h^2}{2} \frac{\delta_{x1}}{AC} \\ \frac{h^2}{2} \frac{\delta_{y1}}{BC} & -\frac{h^2}{2} \frac{\delta_{x1}}{AC} & \frac{1}{2}\left(\omega_1^2 + \frac{h^2}{AC^2} \delta_{x1}^2 + \frac{h^2}{BC^2} \delta_{y1}^2\right) \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \eta_1 \end{pmatrix} > 0$$

The sign character of the above quadratic function is determined by testing the sign character of its corresponding symmetric matrix; and by Sylvester's Theorem (Reference 16) we are assured that for the cited matrix to be positive definite it is necessary and sufficient that all principal diagonal minors be simultaneously positive. If this test fails, H is either sign variable (implying instability) or positive semidefinite (but not positive definite). If we exclude this latter class of systems (thereby excluding limiting cases such as axi-symmetric vehicles with C = A or C = B), then conditions both necessary and sufficient for asymptotic stability of the restricted planar appendage model are:

$$\frac{h^2}{2}\,\frac{(C-B)}{BC} > 0$$

$$\frac{h^2}{2}\,\frac{(C-B)}{BC}\cdot\frac{h^2}{2}\,\frac{(C-A)}{AC} > 0$$

$$\frac{h^2}{2}\,\frac{(C-B)}{BC}\left\{\frac{h^2}{2}\,\frac{(C-A)}{AC}\left[\frac{\omega_1^2}{2}+\frac{h^2\delta_{x1}^2}{2AC^2}+\frac{h^2\delta_{y1}^2}{2BC^2}\right]-\frac{h^4\delta_{x1}^2}{4A^2C^2}\right\}$$
$$-\frac{h^2\delta_{y1}^2}{2BC}\left[\frac{h^4\delta_{y1}}{4BC}\,\frac{(C-A)}{AC}\right] > 0$$

The combination of the first two conditions, as predicted by energy sink methods, requires that the spin axis be the axis of maximum moment of inertia, i.e.,

$$C > A \quad \text{and} \quad C > B$$

This, of course, we have observed before. In addition however a new criterion emerges, requiring satisfaction of the third condition above. After expansion and combination of terms this additional criterion takes the form

$$\omega_1^2 > \frac{h^2}{C^2}\left[\frac{\delta_{x1}^2(C-B)+\delta_{y1}^2(C-A)}{(C-A)(C-B)}\right]$$

By replacing h by its zeroth order approximation $C\Omega$, where Ω is the nominal spin frequency, the above condition simplifies to the following:

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^2 (C-B) + \delta_{y1}^2 (C-A)}{(C-A)(C-B)} \tag{25}$$

Thus a stability criterion arises which explicitly bounds the first modal frequency of the N particle structure. By inspection one observes that this criterion is more stringent than the maximum moment of inertia rule. Although the terminology "first mode" has been used throughout this development it should be clear that the above condition is applicable to any mode.

Algebraic difficulty precludes the generation of explicit stability criteria for more than one mode; however, for reference, the general stability criterion is written in its simpler matrix form below. (We shall in the next Chapter extend stability criteria by implementing an observation brought to light by Routh-Hurwitz analysis.)

$$(\theta_1 \theta_2 \,\vdots\, \eta_1 \eta_2 \cdots) \begin{bmatrix} \frac{h^2(C-B)}{2CB} & 0 & \frac{h^2\delta_{y1}}{BC} & \frac{h^2\delta_{y2}}{BC} & \cdots \\ 0 & \frac{h^2(C-A)}{2AC} & -\frac{h^2\delta_{x1}}{AC} & -\frac{h^2\delta_{x2}}{AC} & \cdots \\ \hline \frac{h^2\delta_{y1}}{BC} & -\frac{h^2\delta_{x1}}{AC} & \frac{h^2\delta_{x1}^2}{2AC^2} + \frac{h^2\delta_{y1}^2}{2BC^2} + \frac{\omega_1^2}{2} & 0 & \cdots \\ \frac{h^2\delta_{y2}}{BC} & -\frac{h^2\delta_{x2}}{AC} & 0 & \frac{h^2\delta_{x2}^2}{2AC^2} + \frac{h^2\delta_{y2}^2}{2BC^2} + \frac{\omega_2^2}{2} & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \hline \eta_1 \\ \eta_2 \\ \vdots \end{pmatrix} > 0 \tag{26}$$

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CHAPTER 4

ROUTH-HURWITZ ANALYSIS

In the previous chapter stability criteria were derived for a structure having a general planar flexible appendage configured such that the appendage particles in their undeformed state lie in a plane perpendicular to the principal axis of spin ($\hat{\underline{z}}$ axis) and passing through the system center of a mass, (Figure 7). As the development is algebraically cumbersome it is expected that some difficulty in complete acceptance might lie in the minds of the reader. To restore confidence in these results we shall in this chapter rederive the stability conditions given by Equation (25) by linearizing the equations of motion and then applying the familiar Routh-Hurwitz criteria. We shall then use these results to derive stability criteria for some simple models; in particular, the simple particle model of Chapter 2 will be redeveloped.

In this chapter we shall also consider the case where the model in its deformed state has a displacement along the principal axis of spin (i.e., $\Gamma_z \neq 0$). Although the results for this model will be limited in that only a single particle will be considered, this case does provide some insight into the more general problem. We will find that for $\Gamma_z \neq 0$ stability will be degraded.

Finally, we conclude both this chapter and the dissertation by considering methods of enhancing stability by both a rigid rotor (momentum wheel) allowed to rotate in the direction of spin, and a controller implementing the use of an idealized proportional effector (as for example, control moment gyros).

Single Mode Stability Analysis

Equations of motion for the appendage are derived in Appendix I. In particular, Newton's Law for the i^{th} particle of the flexible appendage, Equation (I-3) is

$$F_d^i + F_k^i = m_i\left[\ddot{u}^i - \frac{1}{\mathcal{M}}\sum m_j \ddot{u}^j + \tilde{\dot{w}}\,\Gamma^i + 2\tilde{\Omega}\left(\dot{u}^i - \frac{1}{\mathcal{M}}\sum m_j \dot{u}^j\right)\right.$$
$$\left. + \left(\tilde{\Omega}\,\tilde{w} + \tilde{w}\,\tilde{\Omega}\right)\Gamma^i + \tilde{\Omega}\,\tilde{\Omega}\left(u^i - \frac{1}{\mathcal{M}}\sum m_j u^j\right)\right]$$

where F_d^i and F_k^i represent damping and stiffness forces, respectively.

Stability criteria for the simplification where CM shifts may be neglected are initially developed; and then as in Chapter 3 the criteria are redeveloped to include these effects. Implementing this simplification allows the above equation to be written as:

$$F_d^i + F_k^i = m_i\left[\ddot{u}^i + \tilde{\dot{w}}\Gamma^i + 2\tilde{\Omega}\dot{u}^i + \left(\tilde{\Omega}\,\tilde{w} + \tilde{w}\,\tilde{\Omega}\right)\Gamma^i + \tilde{\Omega}\,\tilde{\Omega}\,u^i\right]$$

With the spin directed along the $\hat{z}$ axis the above equations reduce to:

$$m_i\ddot{u}_x^i - 2m_i\Omega\dot{u}_y^i - m_i\Omega^2 u_x^i - \left(F_d^i + F_k^i\right)_x$$
$$= m_i\left(\dot{w}_z\Gamma_y^i - \dot{w}_y\Gamma_z^i\right) - m_i\left(-2\Omega w_z\Gamma_x^i + \Omega\, w_x\Gamma_z^i\right) \qquad (27a)$$

$$m_i\ddot{u}_y^i + 2m_i\Omega\dot{u}_x^i - m_i\Omega^2 u_y^i - \left(F_d^i + F_k^i\right)_y$$
$$= m_i\left(\dot{w}_x\Gamma_z^i - \dot{w}_z\Gamma_x^i\right) - m_i\left(-2\Omega w_z\Gamma_y^i + \Omega\, w_y\Gamma_z^i\right) \qquad (27b)$$

$$m_i\ddot{u}_z^i - \left(F_d^i + F_k^i\right)_z = m_i\left(\dot{w}_y\Gamma_x^i - \dot{w}_x\Gamma_y^i\right) - m_i\left(\Omega w_y\Gamma_y^i + \Omega w_x\Gamma_x^i\right) \qquad (27c)$$

where the subscripts on the damping and stiffness forces F_d^i and F_k^i, respectively, denote the corresponding component; for example $(F_d^i + F_k^i)_x$ represents the $\hat{\underline{x}}$ component of $(F_d^i + F_k^i)$.

The rotational equations of motion, equations (I-7) of Appendix I, are:

$$A\dot{w}_x - \Omega w_y(B - C) + \Omega^2 \sum m_i \left(\Gamma_y^i u_z^i + \Gamma_z^i u_y^i\right)$$
$$+ \sum m_i \left(\Gamma_y^i \ddot{u}_z^i - \Gamma_z^i \ddot{u}_y^i\right) - 2\Omega \sum m_i \Gamma_z^i \dot{u}_x^i = 0 \qquad (28a)$$

$$B\dot{w}_y + \Omega w_x(A - C) - \Omega^2 \sum m_i \left(\Gamma_x^i u_z^i + \Gamma_z^i u_x^i\right)$$
$$+ \sum m_i \left(\Gamma_z^i \ddot{u}_x^i - \Gamma_x^i \ddot{u}_z^i\right) - 2\Omega \sum m_i \Gamma_z^i \dot{u}_y^i = 0 \qquad (28b)$$

$$C\dot{w}_z + \sum m_i \left(\Gamma_x^i \ddot{u}_y^i - \Gamma_y^i \ddot{u}_x^i\right) + 2\Omega \sum m_i \left(\Gamma_x^i \dot{u}_x^i + \Gamma_y^i \dot{u}_y^i\right) = 0 \qquad (28c)$$

Assume the following:

1) $\left(F_d^i + F_k^i\right)_x$ and $\left(F_d^i + F_k^i\right)_y$ are independent of u_z and $\dot{u}_z$

2) $\left(F_d^i + F_k^i\right)_z$ is independent of u_x^i, u_y^i, $\dot{u}_x^i$ and $\dot{u}_y^i$

3) $\Gamma_z^i = 0$

These are precisely the assumptions employed in Chapter 3 which allowed us to isolate that portion of the Hamiltonian descriptive of wobble motion. With the equations of motion in front of us, this separation is more apparent. That is, Equations (27a), (27b), and (28c) separate from Equations (27c), (28a) and (28b). The latter three, the wobble equations, are rewritten below

$$m_i \ddot{u}_z^i - \left(F_d^i + F_k^i\right)_z = m_i\left(\dot{w}_y \Gamma_x^i - \dot{w}_x \Gamma_y^i\right) - m_i\left(\Omega w_y \Gamma_y^i + \Omega w_x \Gamma_x^i\right) \qquad (29a)$$

$$A\dot{w}_x - \Omega w_y(B - C) + \sum m_i\left(\Omega^2 \Gamma_y^i u_z^i + \Gamma_y^i \ddot{u}_z^i\right) = 0 \qquad (29b)$$

$$B\dot{w}_y + \Omega w_x(A - C) - \sum m_i \left(\Omega^2 \Gamma_x^i u_z^i + \Gamma_x^i \ddot{u}_z^i\right) = 0 \qquad (29c)$$

Equation (29a) may be written as the following composite appendage matrix equation:

$$M\ddot{q}+D\dot{q}+Kq = M\Gamma_x \underline{E}(\dot{w}_y - \Omega w_x) - M\Gamma_y \underline{E}(\dot{w}_x + \Omega w_y) \tag{30a}$$

where $(F_d^i + F_k^i)_z$ has been replaced by its matrix counterparts $-(D\dot{q} + Kq)$. All terms in Equation (30a) were previously identified in Chapter 3 except for the damping matrix D which is simply an N by N diagonal matrix consisting of elements representative of damping coefficients for each of the N particles, i.e.,

$$D = \begin{pmatrix} d_z^1 & 0 & 0 & . & . & . \\ 0 & d_z^2 & 0 & . & . & . \\ 0 & 0 & . & & & \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & d_z^N \end{pmatrix}_{N \text{ by } N}$$

Similarly, Equations (29b) and (29c) become:

$$A\dot{w}_x - \Omega w_y(B-C) + \Omega^2(M\Gamma_y \underline{E})^T q + (M\Gamma_y \underline{E})^T \ddot{q} = 0 \tag{30b}$$

$$B\dot{w}_y + \Omega w_x(A-C) - \Omega^2(M\Gamma_x \underline{E})^T q - (M\Gamma_x \underline{E})^T \ddot{q} = 0 \tag{30c}$$

where as before

$$(M\Gamma\underline{E})^T = (1, 1, \ldots 1) \begin{bmatrix} \Gamma^1 & \cdots & O \\ \vdots & \Gamma^3 & \vdots \\ \vdots & \ddots & \vdots \\ O & \cdots & \Gamma^N \end{bmatrix} \begin{bmatrix} m_1 & \cdots & O \\ \vdots & m_2 & \vdots \\ \vdots & \ddots & \vdots \\ O & \cdots & m_N \end{bmatrix}$$

$$= \left(\Gamma^1 m_1, \Gamma^2 m_2 \ldots . \Gamma^N m_N \right)$$

To proceed, we neglect damping and replace q by $\Phi\eta$, where Φ is the matrix of eigenvectors obtained from the eigenvalue problem associated with the homogeneous matrix equation

$$M\ddot{q} + Kq = 0$$

For complete details see Reference (11). As in the previous chapter we shall employ the simplification of Reference (11) wherein $\boldsymbol{\phi}$ is suitably normalized such that

$$\boldsymbol{\phi}^T M \boldsymbol{\phi} = E$$

$$\boldsymbol{\phi}^T K \boldsymbol{\phi} = \begin{pmatrix} \omega_1^2 & \cdots & 0 \\ \vdots & \omega_2^2 & \vdots \\ 0 & \cdots & \omega_3^2 \end{pmatrix}_{N \text{ by } N} \triangleq \omega^2$$

Thus, after neglecting damping and replacing q by $\boldsymbol{\phi}\eta$ in Equation (30a), and premultiplying by $\boldsymbol{\phi}^T$, one finds:

$$\ddot{\eta} + 2\zeta\omega\dot{\eta} + \omega^2\eta = \boldsymbol{\phi}^T M \Gamma_x \underline{E}(\dot{w}_y - \Omega w_x) - \boldsymbol{\phi}^T M \Gamma_y \underline{E}(\dot{w}_x + \Omega w_y) \quad (31)$$

Here modal damping has been arbitrarily added; the corresponding diagonal N by N damping matrix ζ is defined as

$$\zeta = \begin{pmatrix} \zeta_1 & \cdots & 0 \\ \vdots & \zeta_2 & \vdots \\ 0 & \cdots & \zeta_N \end{pmatrix}_{N \text{ by } N}$$

With $\delta_x \triangleq \boldsymbol{\phi}^T M \Gamma_x \underline{E}$ and $\delta_y \triangleq \boldsymbol{\phi}^T M \Gamma_y \underline{E}$ one finds that Equation (31) and the rotational Equations (30b) and (30c) can be written as

$$\ddot{\eta} + 2\zeta\omega\dot{\eta} + \omega^2\eta = \delta_x(\dot{w}_y - \Omega w_x) - \delta_y(\dot{w}_x + \Omega w_y) \quad (32a)$$

$$A\dot{w}_x - \Omega w_y(B - C) + \Omega^2 \delta_y^T \eta + \delta_y^T \ddot{\eta} = 0 \quad (32b)$$

$$B\dot{w}_y + \Omega w_x(A - C) - \Omega^2 \delta_x^T \eta - \delta_x^T \ddot{\eta} = 0 \quad (32c)$$

Truncate to one modal coordinate so that

$$\eta = \eta_1 \qquad \delta_x = \delta_{x1}$$

$$\omega^2 = \omega_1^2 \qquad \delta_y = \delta_{y1}$$

all of which are scalars. The set of Equation (32a-c) then reduces to:

$$\ddot{\eta}_1 + 2\zeta\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = \delta_{x1}(\dot{w}_y - \Omega w_x) - \delta_{y1}(\dot{w}_x + \Omega w_y) \tag{32d}$$

$$A\dot{w}_x - \Omega w_y(B-C) + \Omega^2\delta_{y1}\eta_1 + \delta_{y1}\ddot{\eta}_1 = 0 \tag{32e}$$

$$B\dot{w}_y + \Omega w_x(A-C) - \Omega^2\delta_{x1}\eta_1 - \delta_{x1}\ddot{\eta}_1 = 0 \tag{32f}$$

With S the Laplacian operator the characteristic equation can be written as:

$$\begin{vmatrix} A\,S & \Omega(C-B) & \delta_{y1}(\Omega^2 + S^2) \\ \Omega(A-C) & B\,S & -\delta_{x1}(\Omega^2 + S^2) \\ (\Omega\delta_{x1} + S\delta_{y1}) & (-\delta_{x1}S + \delta_{y1}\Omega) & S^2 + 2\zeta\omega_1 S + \omega_1^2 \end{vmatrix} = 0$$

In the following for notation simplicity the subscript unity has been dropped. With this simplification the above expands to:

$$\begin{aligned}
&S^4\left[AB - A\delta_x^2 - B\delta_y^2\right] \\
&+ S^3\left[2\zeta\omega AB + A\delta_x\delta_y\Omega - \delta_x\delta_y(C-B)\Omega - B\delta_x\delta_y\Omega + \delta_x\delta_y(C-A)\Omega\right] \\
&+ S^2\Big[\Omega^2(C-A)(C-B) + \omega^2 AB - A\delta_x^2\Omega^2 - \delta_x^2\Omega^2(C-B) \\
&\qquad - B\delta_y^2\Omega^2 - \delta_y^2\Omega^2(C-A)\Big] \\
&+ S\Big[2\zeta\omega\Omega^2(C-A)(C-B) + A\delta_x\delta_y\Omega^3 - \delta_x\delta_y(C-B)\Omega^3 \\
&\qquad - B\delta_x\delta_y\Omega^3 + \delta_x\delta_y(C-A)\Omega^3\Big] \\
&+ \omega^2\Omega^2(C-A)(C-B) - \delta_x^2(C-B)\Omega^4 - \delta_y^2(C-A)\Omega^4 = 0
\end{aligned}$$

Identify the coefficients of S^3, S^2, S^1, and S^0 as p_3, p_2, p_1 and p_0, respectively. From the previous chapter we are assured that

$$AB - A\,\delta_x^{\,2} - B\,\delta_y^{\,2} > 0$$

allowing us to divide through by this quantity. The conditions both necessary and sufficient* for asymptotic stability of the wobble equations are obtained by satisfying the following inequalities

$$p_3 > 0$$
$$p_1 > 0$$
$$p_1\, p_2\, p_3 - p_1^2 - p_0\, p_3^2 > 0$$
$$p_0 > 0$$

The condition $p_3 > 0 \Rightarrow$

$$2\zeta\omega\, AB + \delta_x\, \delta_y\, \Omega\, [A - C + B - B + C - A] > 0$$

$\Rightarrow \zeta > 0$ as in the simple particle model of Chapter 2. The condition $p_1 > 0 \Rightarrow$

$$2\zeta\omega\Omega^2(C - A)(C - B) + \delta_x\delta_y\,\Omega^3\,[A - C + B - B + C - A] > 0$$
$$\Rightarrow\ (C - A)(C - B) > 0$$

which is the familiar stability criterion for rigid spinning bodies. The condition $p_1\, p_2\, p_3 - p_1^2 - p_0\, p_3^2 > 0 \Rightarrow$

$$\left[2\zeta\omega\Omega^2(C-A)(C-B)\right] p_2\, (2\,\zeta\omega\, AB)$$
$$- (AB - A\,\delta_x^{\,2} - B\,\delta_y^{\,2})(2\zeta\omega)^2\,\Omega^4 (C - A)^2 (C - B)^2$$
$$- \left[\omega^2\Omega^2 (C - A)(C - B) - \delta_x^{\,2}(C - B)\Omega^4 - \delta_y^{\,2}(C - A)\Omega^4\right](2\zeta\omega AB)^2 > 0$$

---

*The qualification in the footnote on page 30 applies again here, and stands as an obstacle to rigorous determination of sufficient conditions for asymptotic stability; only in the special case adopted in the assumptions following Eq. (28) can it be shown that even the non-linear version of Eq. (28c) says $\dot{\omega}_z = 0$, permitting a rigorous argument to be established via Routh-Hurwitz also.

$$\Rightarrow AB\Omega^2(C-A)(C-B)p_2 - (AB - A\delta_x^2 - B\,\delta_y^2)\Omega^4(C-A)^2(C-B)^2$$

$$-(AB)^2\left[\omega^2\Omega^2(C-A)(C-B) - \delta_x^2(C-B)\Omega^4 - \delta_y^2(C-A)\Omega^4\right] > 0$$

$$\Rightarrow AB\Omega^2(C-A)(C-B)p_2 - AB\Omega^4(C-A)^2(C-B)^2$$

$$+ A\,\delta_x^2\Omega^4(C-A)^2(C-B)^2 + B\delta_y^2\Omega^4(C-A)^2(C-B)^2$$

$$-(AB)^2\omega^2\Omega^2(C-A)(C-B) + (AB)^2\delta_x^2\Omega^4(C-B)$$

$$+(AB)^2\delta_y^2\Omega^4(C-A) > 0$$

Substitution of

$$p_2 = \Omega^2(C-A)(C-B) + \omega^2 AB - \delta_x^2\Omega^2(C-B+A) - \delta_y^2\Omega^2(C-A+B)$$

provides :

$$AB\Omega^2(C-A)(C-B)\left[-\delta_x^2\Omega^2(C-B+A) - \delta_y^2\Omega^2(C-A+B)\right]$$

$$+ A\,\delta_x^2\Omega^4(C-A)^2(C-B)^2 + B\,\delta_y^2\Omega^4(C-A)^2(C-B)^2$$

$$+(AB)^2\,\delta_x^2\Omega^4(C-B) + (AB)^2\,\delta_y^2\Omega^4(C-A) > 0$$

$$\Rightarrow \delta_x^2\left[-AB(C-A)(C-B)(C-B+A) + A(C-A)^2(C-B)^2 + (AB)^2(C-B)\right]$$

$$+\delta_y^2\left[-AB(C-A)(C-B)(C-A+B) + B(C-A)^2(C-B)^2 + (AB)^2(C-A)\right] > 0$$

which requires that each bracket be greater than zero.

Note: 1^{st} term in each bracket is always negative

2^{nd} term in each bracket is always positive

3^{rd} term in each bracket is positive for C the maximum moment of inertia and negative for C the minimum moment of inertia.

Consider the requirement that the coefficient of δ_x^2 be > 0

$$\left[- AB(C-A)(C-B)(C-B+A) + A(C-A)^2(C-B)^2 + (AB)^2(C-B)\right] > 0$$

Dividing through by A(C- A)(C- B), which is always positive, gives

$$\Rightarrow \quad - B(C-B+A) + (C-A)(C-B) + \frac{AB^2}{C-A} > 0$$

$$\Rightarrow \quad \frac{AB^2}{(C-A)} - AB + (C-B)(C-A-B) > 0$$

$$\Rightarrow \quad AB\left(\frac{B-C+A}{C-A}\right) + (C-B)(C-A-B) > 0$$

$$\Rightarrow \quad \frac{AB}{(C-A)}(B+A-C) > (C-B)(B+A-C)$$

but B + A - C is always > 0, therefore

$$\frac{AB}{(C-A)} > C - B$$

If C is the maximum moment of inertia then AB > (C- A)(C- B).... an inequality always satisfied. If C is the minimum moment of inertia then AB < (C-A)(C- B).... an inequality never satisfied. The requirement that the coefficient of δ_y^2 be > 0 leads to the

$$\frac{AB}{(C-B)} > C - A$$

resulting in the same conclusion, namely

C > A and C > B

which is the familiar stability criterion predicted by energy sink methods for spinning bodies with energy dissipation. The condition $p_o > 0 \Rightarrow$

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^{\,2}(C-B) + \delta_{y1}^{\,2}(C-A)}{(C-A)(C-B)}$$

(Note that we have now returned to the more general nomenclature where the subscript unity is employed to denote the first mode).

These three inequalities are both necessary and sufficient for asymptotic stability of the null solution of the linearized wobble equations (32d) - (32f) for single mode vibration. Reversal of one of these equalities (changing > to <) is sufficient for instability of that null solution. The limitations of linearization as a method for stability analysis are such that in either of the cases just cited, the stability properties of the linearized equation belong also to the corresponding nonlinear equation; but if inspection shows that in the preceding inequalities the symbol > is violated and equality prevails, then the linearized equations are useless. Since this is possible only for vehicles belonging to certain limiting cases (e. g., with C = A or C = B), such vehicles will be excluded from consideration. Thus, with this provision, and with the acknowledgment of the previous footnote, we conclude that conditions both necessary and sufficient for asymptotic stability of spin as established by the equations of wobble motion of the given class of vehicle are:

$$C > A \quad \text{and} \quad C > B$$

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^2 (C-B) + \delta_{y1}^2 (C-A)}{(C-A)(C-B)} \tag{33}$$

which are identical to the criteria developed in Chapter 3, except that at this stage the mass matrix M, as opposed to the more general matrix M', was used in the modal analysis. It is shown below that this discrepancy vanishes by accounting for CM shifts.

One can account for CM shifts in a straight forward manner by incorporating the term $\sum \frac{m_j u^j}{\mathcal{M}}$. By inspection of Equation (I-3) of Appendix I it is clear that we must consider three additional terms:

$$\frac{\tilde{\Omega}\tilde{\Omega}}{\mathcal{M}} \sum m_j u^j$$

$$\frac{2\tilde{\Omega}}{\mathcal{M}} \sum m_j \dot{u}^j$$

$$\frac{m_i}{\mathcal{M}} \sum m_j \ddot{u}^j$$

Since Ω is directed solely about the $\hat{z}$ axis the first two terms do not influence deformations in the $\hat{z}$ direction, i. e.,

$$\tilde{\Omega}\tilde{\Omega} = \begin{pmatrix} -\Omega^2 & 0 & 0 \\ 0 & -\Omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\Omega}\,\dot{u}^j = \Omega \begin{pmatrix} \dot{u}^j_y \\ \dot{u}^j_x \\ 0 \end{pmatrix}$$

The remaining term when combined with $m_i \ddot{u}^i$ appears as:

$$m_i \left(\ddot{u}^i - \frac{1}{\mathcal{M}} \sum m_j \ddot{u}^j \right)$$

which expands to the following 3 by 1 matrix.

$$\begin{pmatrix} m_i\left[\ddot{u}^i_x - \frac{1}{\mathcal{M}} \left(m_1 \ddot{u}^1_x + m_2 \ddot{u}^2_x + \ldots\right)\right] \\ m_i\left[\ddot{u}^i_y - \frac{1}{\mathcal{M}} \left(m_1 \ddot{u}^1_y + m_2 \ddot{u}^2_y + \ldots\right)\right] \\ m_i\left[\ddot{u}^i_z - \frac{1}{\mathcal{M}} \left(m_1 \ddot{u}^1_z + m_2 \ddot{u}^2_z + \ldots\right)\right] \end{pmatrix}$$

The last equation in the above set influences the wobble equations and can be handled by modifying the stiffness matrix, to wit

$$
M' = \begin{pmatrix}
m_1 - \frac{m_1^2}{\mathcal{M}} & -\frac{m_1 m_2}{\mathcal{M}} & -\frac{m_1 m_3}{\mathcal{M}} & \cdots & -\frac{m_1 m_N}{\mathcal{M}} \\
-\frac{m_2 m_1}{\mathcal{M}} & m_2 - \frac{m_2^2}{\mathcal{M}} & -\frac{m_2 m_3}{\mathcal{M}} & \cdots & -\frac{m_2 m_N}{\mathcal{M}} \\
\vdots & & & & \\
-\frac{m_N m_1}{\mathcal{M}} & -\frac{m_N m_2}{\mathcal{M}} & \cdots & \cdots & m_N - \frac{m_N^2}{\mathcal{M}}
\end{pmatrix}
$$

Everything else remains unaltered. Thus the comparison between the analysis of this and the previous Chapter is complete.

Stability Criteria Extensions

Both here and in Chapter 3 we have developed asymptotic stability criteria for a spacecraft idealized as a rigid core having attached a flexible appendage. The appendage was assumed to lie in a plane containing the CM, normal to the spin axis. Furthermore, the criteria are, for the class of vehicle noted, both necessary and sufficient for stability of spin, as established by the wobble motion. We have however, restricted ourselves to truncation to a single mode. This may be sufficient for many applications, but nevertheless the question remains as to whether additional modes further degrade the stability boundary. For example suppose the lowest two modes exhibit natural frequencies in close proximity. Can we then assume that the satisfaction of Equation (33) for each of the modes individually assures stability for the modes jointly? Questions of this nature motivate us to develop stability criteria for more than one mode. Such procedures are compounded by algebraic complexity both in Liapunov and Routh-Hurwitz techniques. However, by observation of the analysis of this chapter one concludes that the critical stability criterion arises from satisfying the condition $p_o > 0$. With this observation we ask ourselves if it is possible to extract further criteria by simply satisfying this condition for multiple modes. The results can be considered formally only as necessary conditions for asymptotic stability of the null solution of the wobble equations; nevertheless, useful information is provided.

First consider the case where $\Gamma_x = 0$ (so $\delta_x = 0$). However we permit δ_y to be the general N by 1 matrix. To find p_o let all derivatives in Equation (32a-32c) equal zero.

$$\omega^2 \eta = - \Omega \, \delta_y w_y$$

$$- \Omega w_y (B - C) + \Omega^2 \delta_y^T \eta = 0$$

$$\Omega w_x (A - C) = 0$$

The last equation must persist since $\delta_x = 0$. The first equation can be written as:

$$\eta = - \Omega (\omega^2)^{-1} \delta_y w_y$$

which when substituted into the second equation above results in:

$$\Omega w_y (C - B) + \Omega^2 \delta_y^T \left[- \Omega (\omega^2)^{-1} \delta_y w_y \right] = 0$$

Then the condition $p_o > 0$ (for $\Gamma_x = 0$) can be written as

$$\left(\frac{1}{\Omega^2} \right) > \frac{\delta_y^T \left(\omega^2 \right)^{-1} \delta_y}{C - B} \tag{34}$$

where

$$(\omega^2)^{-1} = \begin{pmatrix} \frac{1}{\omega_1^2} & \cdots & & 0 \\ \vdots & \frac{1}{\omega_2^2} & & \vdots \\ & & \ddots & \\ 0 & \cdots & & \frac{1}{\omega_N^2} \end{pmatrix}$$

Similar results arise for the case $\Gamma_y = 0$, to wit

$$\left(\frac{1}{\Omega^2}\right) > \frac{\delta_x^T (\omega^2)^{-1} \delta_x}{C - A} \tag{35}$$

Satisfaction of the condition $p_o > 0$ for the case where neither Γ_x nor Γ_y equals zero requires considerably more algebraic complexity but nevertheless is manageable. To proceed let all derivatives in Equation (32a-c) equal zero:

$$\omega^2 \eta = - \Omega \, \delta_x w_x - \Omega \, \delta_y w_y$$

$$- \Omega \, w_y (B - C) + \Omega^2 \, \delta_y^T \eta = 0$$

$$\Omega \, w_x (A - C) - \Omega^2 \, \delta_x^T \eta = 0$$

The first of these is an N by 1 column matrix equation whereas the last two are scalars. Solving for η in the former and substitution into the remaining two equations provides the following:

3

$$\eta = -\Omega\left[\omega^2\right]^{-1}\left[\delta_x w_x + \delta_y w_y\right]$$

$$\Omega(C-B)w_y - \Omega^3 \delta_y^T\left[\omega^2\right]^{-1}\left[\delta_x w_x + \delta_y w_y\right] = 0$$

$$\Omega(C-A)w_x - \Omega^3 \delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x w_x + \delta_y w_y\right] = 0$$

Expanding the last two equations yields:

$$(C-B)w_y - \Omega^2 \delta_y^T\left[\omega^2\right]^{-1}\delta_x w_x - \Omega^2 \delta_y^T\left[\omega^2\right]^{-1}\delta_y w_y = 0$$

$$(C-A)w_x - \Omega^2 \delta_x^T\left[\omega^2\right]^{-1}\delta_x w_x - \Omega^2 \delta_x^T\left[\omega^2\right]^{-1}\delta_y w_y = 0$$

Therefore,

$$w_x = \frac{\Omega^2 \delta_x^T\left[\omega^2\right]^{-1}\delta_y w_y}{(C-A) - \Omega^2 \delta_x^T\left[\omega^2\right]^{-1}\delta_x}$$

$$(C-B)w_y - \Omega^2 \delta_y^T\left[\omega^2\right]^{-1}\delta_y w_y - \Omega^2 \delta_y^T\left[\omega^2\right]^{-1}\delta_x\left[\frac{\Omega^2 \delta_x^T\left[\omega^2\right]^{-1}\delta_y w_y}{(C-A) - \Omega^2 \delta_x^T\left[\omega^2\right]^{-1}\delta_x}\right] = 0$$

Thus, the term $p_o > 0$ can be written as:

$$(C-A)(C-B) - \Omega^2(C-A)\delta_y^T\left[\omega^2\right]^{-1}\delta_y - \Omega^2(C-B)\delta_x^T\left[\omega^2\right]^{-1}\delta_x$$
$$+\Omega^4\delta_x^T\left[\omega^2\right]^{-1}\delta_x\delta_y^T\left[\omega^2\right]^{-1}\delta_y - \Omega^4\delta_x^T\left[\omega^2\right]^{-1}\delta_y\delta_y^T\left[\omega^2\right]^{-1}\delta_x > 0$$

Note: $\delta_y^T \left[\omega^2\right]^{-1} \delta_x = \delta_x^T \left[\omega^2\right]^{-1} \delta_y$

Therefore

$$(C-A)(C-B) - \Omega^2 (C-A)\delta_y^T\left[\omega^2\right]^{-1}\delta_y - \Omega^2 (C-B)\delta_x^T\left[\omega^2\right]^{-1}\delta_x$$

$$+ \Omega^4 \delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_y > 0 \qquad (36)$$

In examining Equation (36) it is clear that for the special case where $\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right] = 0$ the stability criteria reduces to

$$\left(\frac{1}{\Omega}\right)^2 > \frac{(C-A)\delta_y^T\left[\omega^2\right]^{-1}\delta_y + (C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\delta_x}{(C-A)(C-B)} \qquad (37)$$

For the case where $\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right] \neq 0$ Equation (36) may be further simplified by recognizing that it can be written in the form

$$\Omega^4 a + \Omega^2 b + c > 0$$

where

$$a = \delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_y$$

$$b = -\left[(C-A)\delta_y^T\left[\omega^2\right]^{-1}\delta_y + (C-B)\delta_x^T\left[\omega^2\right]^{-1}\delta_x\right]$$

$$c = (C-A)(C-B)$$

The quadratic solutions are:

$$\Omega^2 + \frac{\left[(C-A)\,\delta_y^T\left[\omega^2\right]^{-1}\delta_y + (C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\delta_x\right]}{2\,\delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_x}$$

$$\pm \frac{\sqrt{\left[(C-A)\,\delta_y^T\left[\omega^2\right]^{-1}\delta_y + (C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\delta_x\right]^2 - 4(C-A)(C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_y}}{2\,\delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_x} > 0$$

which is of the form

$$(\Omega^2 + \alpha^2)\,(\Omega^2 + \beta^2) > 0$$

Thus either both terms must be positive or both terms must be negative. We disregard the possibility of both positive on the basis that it fails to simplify to previously established stability criteria. Thus for stability the following inequality must be satisfied.

$$\Omega^2 < \frac{\left[(C-A)\,\delta_y^T\left[\omega^2\right]^{-1}\delta_y + (C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\delta_x\right]}{2\,\delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_x}$$

$$\pm \frac{\sqrt{\left[(C-A)\,\delta_y^T\left[\omega^2\right]^{-1}\delta_y + (C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\delta_x\right]^2 - 4(C-A)(C-B)\,\delta_x^T\left[\omega^2\right]\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_y}}{2\,\delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T - \delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_x} \qquad (38)$$

By expanding terms under the radical the above reduces to:

$$\Omega^2 < \frac{\left[(C-A)\,\delta_y^T \left[\omega^2\right]^{-1} \delta_y + (C-B)\,\delta_x^T \left[\omega^2\right]^{-1} \delta_x\right]}{2\,\delta_x^T \left[\omega^2\right]^{-1} \left[\delta_x \delta_y^T - \delta_y \delta_x^T\right] \left[\omega^2\right]^{-1} \delta_x}$$

$$\pm \frac{\sqrt{\left[(C-A)\delta_y^T \left[\omega^2\right]^{-1} \delta_y - (C-B)\delta_x^T \left[\omega^2\right]^{-1} \delta_x\right]^2 + 4(C-A)(C-B)\delta_x^T \left[\omega^2\right]^{-1} \delta_y \delta_x^T \left[\omega^2\right]^{-1} \delta_y}}{2\,\delta_x^T \left[\omega^2\right]^{-1} \left[\delta_x \delta_y^T - \delta_y \delta_x^T\right] \left[\omega^2\right]^{-1} \delta_y} \quad (39)$$

For the appropriate cases these conditions simplify to previously established necessary asymptotic stability criteria, to wit

Case for single mode (note $\delta_x \delta_y^T - \delta_y \delta_x^T \equiv 0$):

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{(C-A)\,\delta_{y1}^2 + (C-B)\,\delta_{x1}^2}{(C-A)\,(C-B)} \quad (40)$$

Case where $\Gamma_x = 0$

$$\left(\frac{1}{\Omega}\right)^2 > \frac{\delta_y^T \left[\omega^2\right]^{-1} \delta_y}{(C-B)} \quad (41)$$

Case where $\Gamma_y = 0$

$$\left(\frac{1}{\Omega}\right)^2 > \frac{\delta_x^T \left[\omega^2\right]^{-1} \delta_x}{(C-A)} \quad (42)$$

Stability Criteria Applications

We shall now pause and apply the established stability criteria to two simple models, the first of which is the simple two-particle model of Figure 1, repeated here

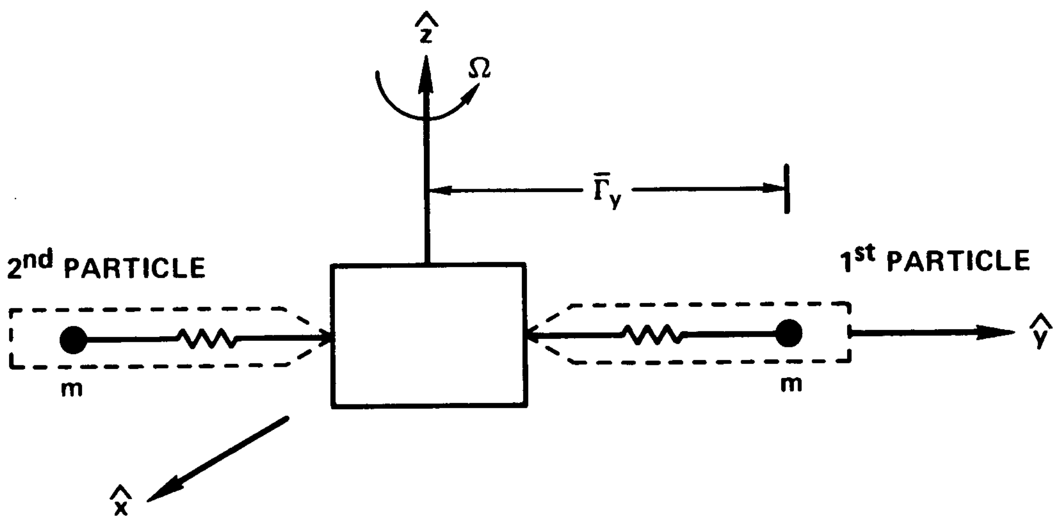

Figure 1. Simple Particle Model

$$M = m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad ; \; \Phi = \frac{\sqrt{2}}{2\sqrt{m}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

Note that $\Phi^T M \Phi = E$ as it should

$\Gamma_x = 0$ so that $\delta_x \equiv 0$

$$\Gamma_y = \begin{pmatrix}\bar{\Gamma}_y & 0\\ 0 & -\bar{\Gamma}_y\end{pmatrix}, \text{ denoting } \Gamma_y^1 = -\Gamma_y^2 = \bar{\Gamma}_y$$

$$\therefore \delta_y = \Phi^T m \Gamma_y \underline{E} = \frac{\sqrt{2}}{2\sqrt{m}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} m \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\bar{\Gamma}_y & 0\\ 0 & -\bar{\Gamma}_y\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$= \frac{\sqrt{2m}\,\bar{\Gamma}_y}{2}\begin{pmatrix}2\\ 0\end{pmatrix} = \sqrt{2m}\,\bar{\Gamma}_y\begin{pmatrix}1\\ 0\end{pmatrix}$$

For the problem under consideration we let ω_1 denote the asymmetric mode natural frequency and let ω_2 denote the (slightly higher) symmetric mode natural frequency, i.e.,

$$\omega^2 = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

Thus,

$$\left[\omega^2\right]^{-1} = \begin{pmatrix} \frac{1}{\omega_1^2} & 0 \\ 0 & \frac{1}{\omega_2^2} \end{pmatrix}$$

Substitution into Equation (41) allows the following:

$$\left(\frac{1}{\Omega}\right)^2 > \frac{\delta_y^T \left[\omega^2\right]^{-1} \delta_y}{(C - B)}$$

$$= \frac{\sqrt{2m}\ \bar{\Gamma}_y \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} \frac{1}{\omega_1^2} & 0 \\ 0 & \frac{1}{\omega_2^2} \end{pmatrix} \sqrt{2m}\ \bar{\Gamma}_y \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(C - B)}$$

$$= \frac{2m\ \bar{\Gamma}_y^2}{(\omega_1)^2\ (C - B)}$$

or as written in Chapter 2.

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{2m\ \bar{\Gamma}_y^2}{(C - B)}$$

Note that the symmetric mode of vibration has no influence on the attitude stability criterion. (This is evident as soon as δ_y is known.)

Now apply the stability criteria to a slightly more complicated model, sketched in Figure 10. The stability criteria of Equation (38) will be employed.

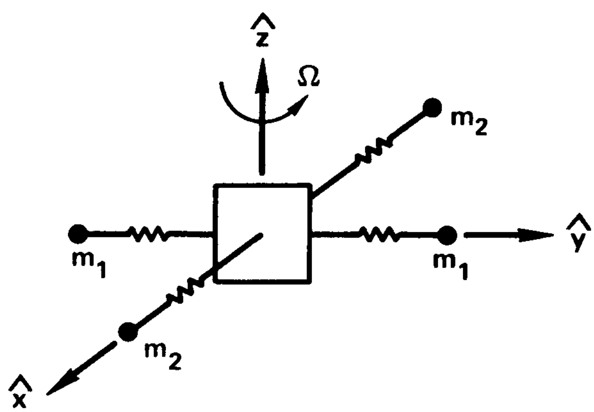

Figure 10. Simple Four-Particle Model

Define the loaded frequencies for the cantilever beams supporting m_1 and m_2 as ω_1 and σ_1, respectively. These frequencies will characterize the asymmetric modes of vibration, while the symmetric modes will be denoted ω_2 and σ_2.

The following matrices are applicable.

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}; \quad \Gamma_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ell & 0 \\ 0 & 0 & 0 & -\ell \end{bmatrix}; \quad \Gamma_y = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{m_1}} & \frac{1}{\sqrt{m_1}} & 0 & 0 \\ -\frac{1}{\sqrt{m_1}} & \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_2}} & \frac{1}{\sqrt{m_2}} \\ 0 & 0 & -\frac{1}{\sqrt{m_2}} & \frac{1}{\sqrt{m_2}} \end{bmatrix}$$

$$\delta_x = \boldsymbol{\Phi}^T M \Gamma_x \underline{E} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{m_1}} & -\frac{1}{\sqrt{m_1}} & 0 & 0 \\ \frac{1}{\sqrt{m_1}} & \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_2}} & -\frac{1}{\sqrt{m_2}} \\ 0 & 0 & \frac{1}{\sqrt{m_2}} & \frac{1}{\sqrt{m_2}} \end{bmatrix} m_2 \ell \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{m_2 \ell}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{m_1}} & -\frac{1}{\sqrt{m_1}} & 0 & 0 \\ \frac{1}{\sqrt{m_1}} & \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_2}} & -\frac{1}{\sqrt{m_2}} \\ 0 & 0 & \frac{1}{\sqrt{m_2}} & \frac{1}{\sqrt{m_2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{m_2 \ell}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{m_2}} \\ 0 \end{bmatrix} = \sqrt{2 m_2}\ \ell \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Similarly,

$$\delta_y = \sqrt{2m_1}\quad d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that only the asymmetric modes contribute to δ_x and δ_y.

$$\left[\omega^2\right] = \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & 0 \\ 0 & 0 & 0 & \sigma_2^2 \end{bmatrix} ; \quad \left[\omega^2\right]^{-1} = \begin{bmatrix} 1/\omega_1^2 & 0 & 0 & 0 \\ 0 & 1/\omega_2^2 & 0 & 0 \\ 0 & 0 & 1/\sigma_1^2 & 0 \\ 0 & 0 & 0 & 1/\sigma_2^2 \end{bmatrix}$$

Therefore

$$\delta_y^T \left[\omega^2\right]^{-1} \delta_y = \frac{2m_1\, d^2}{\omega_1^2}$$

$$\delta_x^T \left[\omega^2\right]^{-1} \delta_x = \frac{2m_2\, \ell^2}{\sigma_1^2}$$

$$\delta_x\delta_y^T - \delta_y\delta_x^T = \sqrt{2m_2}\ \ell - \sqrt{2m_1}\ d \left[\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} (1000) - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} (0010) \right]$$

$$= 2\,\ell\, d \sqrt{m_1 m_2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So that

$$\delta_x^T \left[\omega^2\right]^{-1} \left[\delta_x \delta_y^T - \delta_y \delta_x^T\right] \left[\omega^2\right]^{-1} \delta_y$$

$$= \delta_x^T \left[\omega^2\right]^{-1} 2\,\ell\, d \sqrt{m_1 m_2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\omega_1^2 & 0 & 0 & 0 \\ 0 & 1/\omega_2^2 & 0 & 0 \\ 0 & 0 & 1/\sigma_1^2 & 0 \\ 0 & 0 & 0 & 1/\sigma_2^2 \end{bmatrix} \sqrt{2m_1}\, d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \delta_x^T \left[\omega^2\right]^{-1} 2\,\ell\, d^2 m_1 \sqrt{2m_2} \begin{bmatrix} 0 \\ 0 \\ 1/\omega_1^2 \\ 0 \end{bmatrix}$$

Therefore

$$\delta_x^T \left[\omega^2\right]^{-1} \left[\delta_x \delta_y^T - \delta_y \delta_x^T\right] \left[\omega^2\right]^{-1} \delta_y$$

$$= \sqrt{2m_2}\ \ell\, (0\ 0\ 1\ 0) \begin{bmatrix} 1/\omega_1^2 & 0 & 0 & 0 \\ 0 & 1/\omega_2^2 & 0 & 0 \\ 0 & 0 & 1/\sigma_1^2 & 0 \\ 0 & 0 & 0 & 1/\sigma_2^2 \end{bmatrix} 2\,\ell\, d^2 m_1 \sqrt{2m_2} \begin{bmatrix} 0 \\ 0 \\ 1/\omega_1^2 \\ 0 \end{bmatrix}$$

$$= \frac{4 m_1 m_2 \ell^2 d^2}{\sigma_1^2 \omega_1^2}$$

Hence

$$2\,\delta_x^T \left[\omega^2\right]^{-1} \left[\delta_x \delta_y^T - \delta_y \delta_x^T\right] \left[\omega^2\right]^{-1} \delta_x = \frac{8 m_1 m_2 \ell^2 d^2}{\sigma_1^2 \omega_1^2}$$

One of the terms under the radical of Equation (38) expands as

$$4(C-A)(C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\left[\delta_x\delta_y^T-\delta_y\delta_x^T\right]\left[\omega^2\right]^{-1}\delta_y$$

$$= \frac{16\, m_1 m_2 \ell^2 d^2}{\sigma_1^2\,\omega_1^2}\,(C-A)(C-B)$$

The remaining term under the radical simplifies as follows:

$$\left[(C-A)\,\delta_y^T\left[\omega^2\right]^{-1}\delta_y+(C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\delta_x\right]^2$$

$$= \left[(C-A)\frac{2m_1 d^2}{\omega_1^2}+(C-B)\frac{2m_2\ell^2}{\sigma_1^2}\right]^2$$

$$= \frac{4m_1^2 d^4}{\omega_1^4}(C-A)^2+8(C-A)(C-B)\frac{m_1 m_2 d^2\ell^2}{\omega_1^2\,\sigma_1^2}+\frac{4m_2^2\ell^4(C-B)}{\sigma_1^4}$$

Thus the radical in Equation (38) becomes

$$\frac{4m_1^2 d^4(C-A)^2}{\omega_1^4}+\frac{4m_2^2\ell^4(C-B)}{\sigma_1^4}-\frac{8(C-A)(C-B)m_1 m_2\ell^2 d^2}{\omega_1^2\,\sigma_1^2}$$

$$= \left[(C-A)\frac{2m_1 d^2}{\omega_1^2}-(C-B)\frac{2m_2\ell^2}{\sigma_1^2}\right]^2$$

Substitution into Equation (38) provides two solutions:

$$\Omega^2 < \frac{\left[(C-A)\dfrac{2m_1 d^2}{\omega_1^2} + (C-B)\dfrac{2m_2 \ell^2}{\sigma_1^2}\right]}{\dfrac{8m_1 m_2 \ell^2 d^2}{\sigma_1^2 \omega_1^2}}$$

$$\pm \frac{\left[(C-A)\dfrac{2m_1 d^2}{\omega_1^2} - (C-B)\dfrac{2m_2 \ell^2}{\sigma_1^2}\right]}{\dfrac{8m_1 m_2 \ell^2 d^2}{\sigma_1^2 \omega_1^2}}$$

First solution is

$$\Omega^2 < \frac{\dfrac{4m_1 d^2 (C-A)}{\omega_1^2}}{\dfrac{8m_1 m_2 \ell^2 d^2}{\sigma_1^2 \omega_1^2}} = \frac{(C-A)\sigma_1^2}{2m_2 \ell^2}$$

Therefore

$$\left(\frac{\sigma_1}{\Omega}\right)^2 > \frac{2m_2 \ell^2}{(C-A)}$$

And the second solution leads to:

$$\Omega^2 < \frac{\dfrac{4m_2 \ell^2}{\sigma_1^2}(C-B)}{\dfrac{8m_1 m_2 \ell^2 d^2}{\sigma_1^2 \omega_1^2}} = \frac{(C-B)\omega_1^2}{2m_1 d^2}$$

Hence

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{2 m_1 d^2}{(C-B)}$$

It is of interest to observe that stability analysis of the crossed dipole configuration yields two uncoupled stability conditions; the same criteria would persist in considering two single dipole configurations one mounted along the $\hat{x}$ body axis and the other mounted along the $\hat{y}$ body axis.

It is hoped that the above examples will aid the reader in using the results of this dissertation. Clearly however for other than very simple models a modal analysis must precede the implementation of these results. Nevertheless, these simple models do provide insight into the more general problem. In this context a summary of stability criteria for simple particle models is delineated in Appendix III, along with the more pertinent results of this dissertation.

<u>Examination of the Case $\Gamma_z \neq 0$</u>

We shall now superficially examine the more general problem where $\Gamma_z \neq 0$. This generalization greatly compounds the analysis in that the spin and wobble equations no longer separate. Thus even for a single particle one has to contend with a ninth order characteristic equation. However, if we simply examine the necessary condition $p_o > 0$ some useful criteria emerge. To begin let us write the equations of motion for a spacecraft idealized as a rigid body having attached a single flexible particle of mass m with $\underline{\Gamma} = \overline{\Gamma}_x \hat{\underline{x}} + \overline{\Gamma}_y \hat{\underline{y}} + \overline{\Gamma}_z \hat{\underline{z}}$, to wit

$$\ddot{u}_x - 2\Omega \dot{u}_y - \Omega^2 u_x + \frac{k_x}{m} u_x$$
$$= \left(\dot{w}_z \overline{\Gamma}_y - \dot{w}_y \overline{\Gamma}_z\right) - \left(-2\Omega w_z \overline{\Gamma}_x + \Omega w_x \overline{\Gamma}_z\right)$$

$$\ddot{u}_y + 2\Omega \dot{u}_x - \Omega^2 u_y + \frac{k_y}{m} u_y$$
$$= \left(\dot{w}_x \overline{\Gamma}_z - \dot{w}_z \overline{\Gamma}_x\right) - \left(-2\Omega w_z \overline{\Gamma}_y + \Omega w_y \overline{\Gamma}_z\right)$$

$$\ddot{u}_z + \frac{k_z}{m} u_z = \left(\dot{w}_y \overline{\Gamma}_x - \dot{w}_x \overline{\Gamma}_y\right) - \left(\Omega w_y \overline{\Gamma}_y + \Omega w_x \overline{\Gamma}_x\right)$$

$$A \dot{w}_x - \Omega w_y (B - C) + \Omega^2 m \left(\overline{\Gamma}_y u_z + \overline{\Gamma}_z u_y\right)$$
$$+ m \left(\overline{\Gamma}_y \ddot{u}_z - \overline{\Gamma}_z \ddot{u}_y\right) - 2\Omega m \overline{\Gamma}_z \dot{u}_x = 0$$

$$B \dot{w}_y + \Omega w_x (A - C) - \Omega^2 m \left(\overline{\Gamma}_x u_z + \overline{\Gamma}_z u_x\right)$$
$$+ m \left(\overline{\Gamma}_z \ddot{u}_x - \overline{\Gamma}_x \ddot{u}_z\right) - 2\Omega m \overline{\Gamma}_z \dot{u}_y = 0$$

and

$$C \dot{w}_z + m \left(\overline{\Gamma}_x \ddot{u}_y - \overline{\Gamma}_y \ddot{u}_x\right) + 2\Omega m \left(\overline{\Gamma}_x \dot{u}_x + \overline{\Gamma}_y \dot{u}_y\right) = 0$$

or, upon integration and selection of appropriate constant of integration

$$C w_z + m \left(\overline{\Gamma}_x \dot{u}_y - \overline{\Gamma}_y \dot{u}_x\right) + 2\Omega m \left(\overline{\Gamma}_x u_x + \overline{\Gamma}_y u_y\right) = 0 \quad .$$

As before the loaded natural frequencies are identified as σ_x, σ_y, and σ_z, and their counterparts accounting for centripetal acceleration terms are identified as ω_x, ω_y, and ω_z (note $\sigma_z = \omega_z$). Thus, setting all derivatives equal to zero in the above set of equations and expanding the determinant results in the necessary condition for stability $p_o > 0$. The algebraic equations resulting from the preceding differential equations are

$$u_x \omega_x^2 = -\left(-2\Omega w_z \bar{\Gamma}_x + \Omega w_x \bar{\Gamma}_z\right)$$

$$u_y \omega_y^2 = -\left(-2\Omega w_z \bar{\Gamma}_y + \Omega w_y \bar{\Gamma}_z\right)$$

$$u_z \omega_z^2 = -\left(\Omega w_y \bar{\Gamma}_y + \Omega w_x \bar{\Gamma}_x\right)$$

$$-\Omega w_y (B - C) + \Omega^2 m \left(\bar{\Gamma}_y u_z + \bar{\Gamma}_z u_y\right) = 0$$

$$\Omega w_x (A - C) - \Omega^2 m \left(\bar{\Gamma}_x u_z + \bar{\Gamma}_z u_x\right) = 0$$

$$C w_z + 2\Omega m \left(\bar{\Gamma}_x u_x + \bar{\Gamma}_y u_y\right) = 0$$

Simply in order to reduce the determinant dimension from the intolerably tedious present value of six to the tolerably tedious value of five, we set one deformation component to zero. If we choose for example to constrain u_y to zero (so $k_y = \infty$), we have for $p_o > 0$ the expression

$$\begin{vmatrix} \Omega\bar{\Gamma}_z & 0 & -2\Omega\bar{\Gamma}_x & \omega_x^2 & 0 \\ \Omega\bar{\Gamma}_x & \Omega\bar{\Gamma}_y & 0 & 0 & \omega_z^2 \\ 0 & \Omega(C-B) & 0 & 0 & \Omega^2 m\bar{\Gamma}_y \\ \Omega(A-C) & 0 & 0 & -\Omega^2 m\bar{\Gamma}_z & -\Omega^2 m\bar{\Gamma}_x \\ 0 & 0 & C & 2\Omega m\bar{\Gamma}_x & 0 \end{vmatrix} > 0$$

Expanding the above results in:

$$-2\Omega\bar{\Gamma}_x \begin{vmatrix} \Omega\bar{\Gamma}_x & \Omega\bar{\Gamma}_y & 0 & \omega_z^2 \\ 0 & \Omega(C-B) & 0 & \Omega^2 m\bar{\Gamma}_y \\ \Omega(A-C) & 0 & -\Omega^2 m\bar{\Gamma}_z & -\Omega^2 m\bar{\Gamma}_x \\ 0 & 0 & 2\Omega m\bar{\Gamma}_x & 0 \end{vmatrix}$$

$$+ C \begin{vmatrix} \Omega\bar{\Gamma}_z & 0 & \omega_x^2 & 0 \\ \Omega\bar{\Gamma}_x & \Omega\bar{\Gamma}_y & 0 & \omega_z^2 \\ 0 & \Omega(C-B) & 0 & \Omega^2 m\bar{\Gamma}_y \\ \Omega(A-C) & 0 & -\Omega^2 m\bar{\Gamma}_z & -\Omega^2 m\bar{\Gamma}_x \end{vmatrix} > 0$$

Therefore,

$$(2\Omega\bar{\Gamma}_x)(2\Omega m\bar{\Gamma}_x) \begin{vmatrix} \Omega\bar{\Gamma}_x & \Omega\bar{\Gamma}_y & \omega_z^2 \\ 0 & \Omega(C-B) & \Omega^2 m\bar{\Gamma}_y \\ \Omega(A-C) & 0 & -\Omega^2 m\bar{\Gamma}_x \end{vmatrix}$$

$$+ C\Omega\bar{\Gamma}_z \begin{vmatrix} \Omega\bar{\Gamma}_y & 0 & \omega_z^2 \\ \Omega(C-B) & 0 & \Omega^2 m\bar{\Gamma}_y \\ 0 & -\Omega^2 m\bar{\Gamma}_z & -\Omega^2 m\bar{\Gamma}_x \end{vmatrix}$$

$$+ C\omega_x^2 \begin{vmatrix} \Omega\bar{\Gamma}_x & \Omega\bar{\Gamma}_y & \omega_z^2 \\ 0 & \Omega(C-B) & \Omega^2 m\bar{\Gamma}_y \\ \Omega(A-C) & 0 & -\Omega^2 m\bar{\Gamma}_x \end{vmatrix} > 0$$

Upon further expansion one finds:

$$\left(C\omega_x^2 + 4m\Omega^2\bar{\Gamma}_x^2\right)\Bigg[\Omega\bar{\Gamma}_x\left[-m\Omega^3(C-B)\bar{\Gamma}_x\right]$$

$$+\Omega(A-C)\left[m\Omega^3\bar{\Gamma}_y^2 - \omega_z^2\Omega(C-B)\right]\Bigg]$$

$$+C\Omega\bar{\Gamma}_z\left[\Omega\bar{\Gamma}_y\left(m^2\Omega^4\bar{\Gamma}_y\bar{\Gamma}_x\right) + \omega_z^2\left(-m\Omega^3\bar{\Gamma}_z(C-B)\right)\right] > 0$$

which leads to the necessary condition

$$\left(C\omega_x^2 + 4m\Omega^2\bar{\Gamma}_x^2\right)\Big[\omega_z^2\Omega^2(C-A)(C-B)$$

$$-m\Omega^4\bar{\Gamma}_y^2(C-A) - m\Omega^4\bar{\Gamma}_x^2(C-B)\Big]$$

$$-\omega_z^2 m\Omega^4\bar{\Gamma}_z^2 C(C-B) + m^2\Omega^6\bar{\Gamma}_y^2\bar{\Gamma}_z^2 C > 0 \qquad (43)$$

First consider the case where $\bar{\Gamma}_x = \bar{\Gamma}_z = 0$. Then

$$C\omega_x^2\left[\omega_z^2\Omega^2(C-A)(C-B) - m\Omega^4\bar{\Gamma}_y^2(C-A)\right] > 0$$

$$\Rightarrow C\omega_x^2\Omega^2(C-A)\left[\omega_z^2(C-B) - m\Omega^2\bar{\Gamma}_y^2\right] > 0$$

$$\Rightarrow \left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_y^2}{C-B}$$

which is the same criterion derived earlier, for a single particle on the $\hat{y}$-axis. As before, a similar condition emerges for the case where $\bar{\Gamma}_y = \bar{\Gamma}_z = 0$, i.e.,

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_x^2}{(C-A)}$$

However, a condition we have not seen before arises for the case where $\bar{\Gamma}_x = \bar{\Gamma}_y = 0$:

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_z^2}{(C-A)} \quad .$$

Although not analyzed one surmises (by similarity) that for $k_x = \infty$, $k_y \neq \infty$, and $\bar{\Gamma}_x = \bar{\Gamma}_y = 0$ the following condition would result

$$\left(\frac{\omega_y}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_z^2}{(C-B)}$$

Equation (43) could also be used to generate stability criteria for the case where only $\bar{\Gamma}_z = 0$, a simplification leading to criteria observed a number of times in this dissertation, i. e.,

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_y^2(C-A) + m\bar{\Gamma}_x^2(C-B)}{(C-A)(C-B)}$$

A more interesting case occurs if one allows only $\bar{\Gamma}_x = 0$, namely

$$C\omega_x^2\left[\omega_z^2\Omega^2(C-A)(C-B) - m\Omega^4\bar{\Gamma}_y^2(C-A)\right]$$

$$-\omega_z^2 m\Omega^4\bar{\Gamma}_z^2\, C(C-B) + m^2\Omega^6\bar{\Gamma}_y^2\bar{\Gamma}_z^2\, C > 0$$

$$\Rightarrow \omega_z^2\left[\omega_x^2\, C(C-A)(C-B) - m\Omega^2\bar{\Gamma}_z^2\, C(C-B)\right]$$

$$-C\omega_x^2 m\Omega^2\bar{\Gamma}_y^2(C-A) + m^2\Omega^4\bar{\Gamma}_y^2\bar{\Gamma}_z^2\; C > 0$$

$$\Rightarrow \omega_z^2\, C(C-B)\left[\omega_x^2(C-A) - m\Omega^2\bar{\Gamma}_z^2\right]$$

$$-m\Omega^2\bar{\Gamma}_y^2\, C\left[\omega_x^2(C-A) - m\Omega^2\bar{\Gamma}_z^2\right] > 0$$

$$\Rightarrow \left[\omega_x^2(C-A) - m\Omega^2\,\bar{\Gamma}_z^2\right]\left[\omega_z^2(C-B) - m\Omega^2\bar{\Gamma}_y^2\right] > 0$$

Disregarding the possibility that both brackets may be negative the above leads to:

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_z^2}{(C-A)}$$

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_y^2}{(C-B)}$$

Clearly $\bar{\Gamma}_z \neq 0$ requires the satisfaction of additional stability criteria. For the case where only $\bar{\Gamma}_y = 0$ the results are even more interesting, to wit

$$\left(C\omega_x^2 + 4m\Omega^2\bar{\Gamma}_x^2\right)\left[\omega_z^2\Omega^2(C-A)(C-B) - m\Omega^4\bar{\Gamma}_x^2(C-B)\right]$$

$$-\omega_z^2 m\Omega^4\bar{\Gamma}_z^2 C(C-B) > 0$$

$$\Rightarrow \omega_z^2 > \frac{m\Omega^2\bar{\Gamma}_x^2\left(C\omega_x^2 + 4m\Omega^2\bar{\Gamma}_x^2\right)}{(C-A)\left(C\omega_x^2 + 4m\Omega^2\bar{\Gamma}_x^2\right) - m\Omega^2\bar{\Gamma}_z^2 C}$$

For the first time we observe in the stability criteria a coupling between the frequencies. With the assumption that the denominator is greater than zero, i.e.,

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_z^2}{(C-A)} - \frac{4m\bar{\Gamma}_x^2}{C}$$

then one concludes that the following is required for stability.

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_x^2}{(C-A) - \left[\dfrac{m\bar{\Gamma}_z^2 C}{C\left(\dfrac{\omega_x}{\Omega}\right)^2 + 4m\bar{\Gamma}_x^2}\right]}$$

Finally, consider the case where $\bar{\Gamma}_x \neq 0$, $\bar{\Gamma}_y \neq 0$, and $\bar{\Gamma}_z \neq 0$.

$$\Rightarrow \left(C\omega_x^2 + 4m\Omega^2\bar{\Gamma}_x^2\right)\Big[\omega_z^2\Omega^2(C-A)(C-B) - m\Omega^4\bar{\Gamma}_y^2(C-A) - m\Omega^4\bar{\Gamma}_x^2(C-B)\Big] - \omega_z^2 m\Omega^4\bar{\Gamma}_z^2 C(C-B) + m^2\Omega^6\bar{\Gamma}_y^2\bar{\Gamma}_z^2 C > 0$$

$$\Rightarrow \left[\omega_z^2(C-A)(C-B) - m\Omega^2\left[\bar{\Gamma}_y^2(C-A) + \bar{\Gamma}_x^2(C-B)\right]\right] - \frac{\Omega^2 C m\bar{\Gamma}_z^2}{C\omega_x^2 + 4m\Omega^2\bar{\Gamma}_x^2}\left[\omega_z^2(C-B) - m\Omega^2\bar{\Gamma}_y^2\right] > 0$$

$$\Rightarrow \left(\frac{\omega_z}{\Omega}\right)^2(C-A)(C-B) - m\left[\bar{\Gamma}_y^2(C-A) + \bar{\Gamma}_x^2(C-B)\right] - \frac{Cm\bar{\Gamma}_z^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\bar{\Gamma}_x^2}\left[\left(\frac{\omega_z}{\Omega}\right)^2(C-B) - m\bar{\Gamma}_y^2\right] > 0$$

$$\Rightarrow \left(\frac{\omega_z}{\Omega}\right)^2\left[(C-A)(C-B) - \frac{Cm\bar{\Gamma}_z^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\bar{\Gamma}_x^2}(C-B)\right]$$

$$> m\left[\bar{\Gamma}_y^2(C-A) + \bar{\Gamma}_x^2(C-B)\right] - \frac{Cm^2\bar{\Gamma}_y^2\bar{\Gamma}_z^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\bar{\Gamma}_x^2}$$

As the first bracket above can be written as

$$(C-B)\left[(C-A) - \frac{C\,m\,\bar{\Gamma}_z^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\,\bar{\Gamma}_x^2}\right]$$

we disregard the possibility that it can be less than zero. The reasoning is as follows: If the expression above is less than zero, then for

$$\bar{\Gamma}_z = 0 \Rightarrow (C-A) < 0$$

$$\bar{\Gamma}_x = 0 \Rightarrow \left(\frac{\omega_x}{\Omega}\right) < \frac{m\,\bar{\Gamma}_z^2}{(C-A)} \quad .$$

Both of which are in violation of previously established criteria. Hence, the sought after stability criterion can be written as:

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\left[\bar{\Gamma}_y^2(C-A) + \bar{\Gamma}_x^2(C-B)\right] - \dfrac{C\,m^2\,\bar{\Gamma}_z^2\,\bar{\Gamma}_y^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\,\bar{\Gamma}_x^2}}{(C-B)\left[(C-A) - \dfrac{C\,m\,\bar{\Gamma}_z^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\,\bar{\Gamma}_x^2}\right]}$$

and

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_z^2}{(C-A)} - \frac{4m\,\bar{\Gamma}_x^2}{C}$$

Exploring only superficially into the problem where $\Gamma_z \neq 0$ leads to two general conclusions:

1) The problem is greatly compounded.
2) The stability boundary is degraded.

Control of Passively Unstable Spacecraft

Finally, we consider techniques for enhancing stability of a spinning flexible body through active control. Clearly this subject could constitute a dissertation in itself. As such we explore the subject on a limited scale. Nevertheless the results show conclusively that with relatively simple control techniques a passively unstable configuration could be stabilized. Two approaches are considered: (1) A rigid spinning rotor with its momentum directed along the vehicle spin axis, and (2) An idealized linear controller which in the first approximation is representative of control moment gyros.

Consider the expressions representative of the wobble motion with the appendage equations truncated to one mode and with the vehicle containing a constant speed axisymmetric rigid rotor with angular velocity relative to the core in the $+\hat{\underline{z}}$ direction:

$$\ddot{\eta}_1 + 2\zeta\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = \delta_{x1}(\dot{w}_y - \Omega w_x) - \delta_{y1}(\dot{w}_x + \Omega w_y)$$

$$A\dot{w}_x - \Omega w_y(B - C) + w_y h + \Omega^2\delta_{y1}\eta_1 + \delta_{y1}\ddot{\eta}_1 = 0$$

$$B\dot{w}_y + \Omega w_x(A - C) - w_x h - \Omega^2\delta_{x1}\eta_1 - \delta_{x1}\ddot{\eta}_1 = 0$$

where $h\hat{\underline{z}}$ is the relative angular momentum of the rotor, $h > 0$. Identify a modified inertia C^* as

$$C^* = C + \frac{h}{\Omega} \quad .$$

The wobble equations then become:

$$\ddot{\eta}_1 + 2\zeta\omega\dot{\eta}_1 + \omega_1^2\eta_1 = \delta_{x1}(\dot{w}_y - \Omega w_x) - \delta_{y1}(\dot{w}_x + \Omega w_y)$$

$$A\dot{w}_x - \Omega w_y(B - C^*) + \Omega^2\delta_{y1}\eta_1 + \delta_{y1}\ddot{\eta}_1 = 0$$

$$B\dot{w}_y + \Omega w_x(A - C^*) - \Omega^2\delta_{x1}\eta_1 - \delta_{x1}\ddot{\eta}_1 = 0$$

which are identical in form to the case without the rigid rotor. Thus, conditions both necessary and sufficient for asymptotic stability of the wobble equations are:

$$C^* > A \quad \text{and} \quad C^* > B$$

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^2 (C^* - B) + \delta_{y1}^2 (C^* - A)}{(C^* - A)(C^* - B)}$$

Expanding the latter one finds:

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^2 \left(C + \frac{h}{\Omega} - B\right) + \delta_{y1}^2 \left(C + \frac{h}{\Omega} - A\right)}{\left(C + \frac{h}{\Omega} - A\right) \left(C + \frac{h}{\Omega} - B\right)}$$

That the rotor can enhance stability is made even more apparent by rewriting the above in a slightly different form, i. e.,

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^2}{\left(C + \frac{h}{\Omega} - A\right)} + \frac{\delta_{y1}^2}{\left(C + \frac{h}{\Omega} - B\right)}$$

Clearly the right hand side decreases as the relative momentum h increases.

If instead of a rigid rotor we add control torques proportional to the transverse inertial angular velocity components, the equations of motion take the form

$$\ddot{\eta}_1 + 2\zeta\omega_1\dot{\eta}_1 + \omega_1^2 \eta_1 = \delta_{x1}(\dot{w}_y - \Omega w_x) - \delta_{y1}(\dot{w}_x + \Omega w_y)$$

$$A\dot{w}_x - \Omega w_y (B - C) + \Omega^2 \delta_{y1} \eta_1 + \delta_{y1} \ddot{\eta}_1 = - k w_y$$

$$B\dot{w}_y + \Omega w_x (A - C) - \Omega^2 \delta_{x1} \eta_1 - \delta_{x1} \ddot{\eta}_1 = k w_x$$

The linear control policy could represent to the first approximation an idealized control moment gyro system. Clearly other control policies could be implemented. However, the one chosen, as with a rigid rotor, can be analyzed by similarity. That is, by identifying

$$\overline{C} = C + \frac{k}{\Omega}$$

the equations again reduce to the previously analyzed uncontrolled case with $\overline{C}$ replacing C. Thus the stability criteria can again be written by inspection, to wit

$$\overline{C} > A \quad \text{and} \quad \overline{C} > B$$

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^{\,2}(\overline{C} - B) + \delta_{y1}^{\,2}(\overline{C} - A)}{(\overline{C} - A)(\overline{C} - B)}$$

The latter expands to:

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^{\,2}\left(C + \frac{k}{\Omega} - B\right) + \delta_{y1}^{\,2}\left(C + \frac{k}{\Omega} - A\right)}{\left(C + \frac{k}{\Omega} - A\right)\left(C + \frac{k}{\Omega} - B\right)}$$

Combining both a rigid rotor and a linear controller yields the following stability criteria:

$$\left(C + \frac{k}{\Omega} + \frac{h}{\Omega}\right) > A \quad \text{and} \quad \left(C + \frac{k}{\Omega} + \frac{h}{\Omega}\right) > B$$

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^{\,2}\left(C + \frac{k}{\Omega} + \frac{h}{\Omega} - B\right) + \delta_{y1}^{\,2}\left(C + \frac{k}{\Omega} + \frac{h}{\Omega} - A\right)}{\left(C + \frac{k}{\Omega} + \frac{h}{\Omega} - B\right)\left(C + \frac{k}{\Omega} + \frac{h}{\Omega} - A\right)}$$

A catalogue of necessary conditions for stability, terminating with this the final example, can be found in Appendix III.

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APPENDIX I

PARTICLE EQUATIONS OF MOTION LINEARIZED ABOUT THE LOADED STRUCTURE CONFIGURATION

The appendage equation for a spinning flexible vehicle wherein the appendage is assumed to consist of a collection of particles, may be derived from Equation (80) of Reference 11 which represents the most general formulation of the appendage equation. This follows by identifying C as the identity matrix and permitting e to be zero. The first identification assures that the appendage is not gimballed with respect to the core, and the latter restricts center of mass shifts solely to those due to appendage deformations. These restrictions permit Equation (80) of Reference 11 to be written as

$$F^i = m_i \Big[\frac{F_e}{\mathcal{M}} + \ddot{u}^i - \sum \frac{m_j}{\mathcal{M}} \ddot{u}^j + 2\tilde{\omega}\Big(\dot{u}^i - \sum \frac{m_j}{\mathcal{M}} \dot{u}^j\Big)$$

$$- \Big(\tilde{R} + \tilde{r}^i\Big)\dot{\omega} + \tilde{\dot{\omega}}\Big(u^i - \sum \frac{m_j}{\mathcal{M}} u^j\Big)$$

$$+ \tilde{\omega}\tilde{\omega}\Big(R + r^i + u^i - \sum \frac{m_j u^j}{\mathcal{M}}\Big)\Big] \qquad \text{(I-1)}$$

Where F_e is representative of external forces and the remaining variables are as defined in Figure 5 with the sum $\underline{R}' + \underline{C}'$ identified as $\underline{R}$ and $\underline{r}^{i'} + \underline{\Delta}^i$ identified as $\underline{r}^i$. Letting

$$\underline{\Gamma}^i = \underline{R} + \underline{r}^i$$

$$= \underline{R}' + \underline{r}^{i'} + \underline{\Delta}^i - \frac{1}{\mathcal{M}} \sum_j m_j \underline{\Delta}^j$$

$$F_e = 0$$

$$\omega = w + \Omega$$

and linearizing about u, w (and their derivatives) permits Equation (I-1) to be written as

$$F^i = m_i \Big[\ddot{u}^i - \frac{1}{\mathcal{M}} \sum m_j \ddot{u}^j + \dot{\tilde{w}} \Gamma^i + 2\tilde{\Omega} \left(\dot{u}^i - \frac{1}{\mathcal{M}} \sum m_j \dot{u}^j \right)$$

$$+ (\tilde{\Omega}\tilde{w} + \tilde{w}\tilde{\Omega}) \Gamma^i + \tilde{\Omega}\tilde{\Omega} \left(\Gamma^i + u^i - \frac{1}{\mathcal{M}} \sum m_j u^j \right) \Big] \qquad \text{(I-2)}$$

The force on the i^{th} particle is given by

$$F^i = F^i_{ss} + F^i_d + F^i_k$$

Where F^i_d is representative of damping forces given solely as a function of deformation derivatives; F^i_k is representative of deformation and spin dependent stiffness forces; and F^i_{ss} is the steady state force on the i^{th} mass due to spin, independent of deformations. The latter persists whether or not the structure is flexible and cancels that portion of the centripetal acceleration independent of deformations, i. e.,

$$F^i_{ss} = \tilde{\Omega}\tilde{\Omega} \Gamma^i m_i$$

Equation (I-2) can then be written as

$$m_i \left(\ddot{u}^i - \frac{1}{\mathcal{M}} \sum m_j \ddot{u}^j \right) + 2m_i \tilde{\Omega} \left(\dot{u}^i - \frac{1}{\mathcal{M}} \sum m_j \dot{u}^j \right) - F^i_d$$

$$+ m_i \tilde{\Omega}\tilde{\Omega} \left(u^i - \frac{1}{\mathcal{M}} \sum m_j u^j \right) - F^i_k = m_i \tilde{\Gamma}^i \dot{w} - m_i (\tilde{\Omega}\tilde{w} + \tilde{w}\tilde{\Omega}) \Gamma^i$$

$$\text{(I-3)}$$

With the structure spinning about the body $\hat{z}$ axis, term by term expansion allows Equation (I-3) to take the form

$$m_i \left[\ddot{u}^i_x - \frac{1}{\mathcal{M}} \left(\sum m_j \ddot{u}^j \right)_x \right] - 2m_i \Omega \left[\dot{u}^i_y - \frac{1}{\mathcal{M}} \left(\sum m_j \dot{u}^j \right)_x \right] - F^i_{Dx}$$

$$- m_i \Omega^2 \left[u^i_x - \frac{1}{\mathcal{M}} \left(\sum m_j u^j \right)_x \right] - F^i_{kx} = m_i \left(\dot{w}_z \Gamma^i_y - \dot{w}_y \Gamma^i_z \right)$$

$$- m_i \left(-2\Omega w_z \Gamma^i_x + \Omega w_x \Gamma^i_z \right) \qquad \text{(I-4a)}$$

$$m_i\left[\ddot{u}_y^i - \frac{1}{\mathcal{M}}\left(\sum m_j \ddot{u}^j\right)_y\right] + 2m_i\Omega\left[\dot{u}_x^i - \frac{1}{\mathcal{M}}\left(\sum m_j \dot{u}^j\right)_y\right] - F_{Dy}^i$$

$$- m_i\Omega^2\left[u_y^i - \frac{1}{\mathcal{M}}\left(\sum m_j u^j\right)_y\right] - F_{ky}^i = m_i\left(\dot{w}_x\Gamma_z^i - \dot{w}_z\Gamma_x^i\right)$$

$$- m_i\left(-2\Omega w_z \Gamma_y^i + \Omega w_y \Gamma_z^i\right) \qquad \text{(I-4b)}$$

$$m_i\left[\ddot{u}_z^i - \frac{1}{\mathcal{M}}\left(\sum m_j \ddot{u}^j\right)_z\right] - F_{Dz}^i - F_{kz}^i$$

$$= m_i\left(\dot{w}_y\Gamma_x^i - \dot{w}_x\Gamma_y^i\right) - m_i\left(\Omega w_y \Gamma_y^i + \Omega w_x \Gamma_x^i\right) \qquad \text{(I-4c)}$$

where the subscript denotes the corresponding components.

The rotational equation may be obtained from Equation (128) of Reference 11 which represents the most general vector rotational equation. Thus, after dropping terms representative of rigid rotors, dampers, and elements of the appendage consisting of rigid bodies (as opposed to particles), and premultiplying by $\{\underline{b}\}$, equation (128) of Reference 11 becomes

$$T = I^*\dot{\omega} + \tilde{\omega} I^*\omega + \sum m_i\left[2\left(R + r^i\right)^T u^i E\right.$$

$$\left. - \left(R + r^i\right)u^{iT} - u^i\left(R + r^i\right)^T\right]\dot{\omega} + \tilde{\omega}\sum m_i\left[2\left(R + r^i\right)^T u^i E\right.$$

$$\left. - \left(R + r^i\right)u^{iT} - u^i\left(R + r^i\right)^T\right]\omega + \sum m_i\left[2\left(R + r^i\right)^T \dot{u}^i E\right.$$

$$\left. - \left(R + r^i\right)\dot{u}^{iT} - \dot{u}^i\left(R + r^i\right)^T\right]\omega$$

$$+ \widetilde{\omega R}\sum m_i \dot{u}^i + \tilde{R}\sum m_i \ddot{u}^i + \tilde{R}\tilde{\omega}\sum m_i \dot{u}^i$$

$$+ \widetilde{\omega \sum r^i m_i \dot{u}^i} + \sum \tilde{r}^i\left(m_i \ddot{u}^i + \tilde{\omega} m_i \dot{u}^i\right) \qquad \text{(I-5)}$$

Where I^* is the inertia matrix of the spinning structure distorted under steady spin but otherwise undeformed.

Prior to linearization the above equation may be simplified by combining certain terms. In particular, consider the last six terms

$$\widetilde{\widetilde{\omega} R} \sum m_i \dot{u}^i + \widetilde{R}\widetilde{\omega} \sum m_i \dot{u}^i + \widetilde{\omega}\widetilde{\sum r^i m_i \dot{u}^i}$$

$$+ \sum \widetilde{r}^i \widetilde{\omega} m_i \dot{u}^i + \widetilde{R} \sum m_i \ddot{u}^i + \sum \widetilde{r}^i m_i \ddot{u}^i$$

$$= \sum \left(\widetilde{R} + \widetilde{r}^i \right) \widetilde{\omega} m_i \dot{u}^i + \widetilde{\omega} \widetilde{\sum \left(R + r^i\right) m_i \dot{u}_i} + \sum \left(\widetilde{R} + \widetilde{r}^i\right) m_i \ddot{u}^i$$

of which the first two terms combine to

$$\widetilde{\omega} \sum \left(\widetilde{R} + \widetilde{r}^i \right) m_i \dot{u}^i$$

which results from the vector identity

$$B \times (A \times C) + (A \times B) \times C = A \times (B \times C)$$

Hence, the last five terms in (I-5) reduce to

$$\widetilde{\omega} \sum \left(\widetilde{R} + \widetilde{r}^i \right) m_i \dot{u}^i + \sum \left(\widetilde{R} + \widetilde{r}^i \right) m_i \ddot{u}^i$$

The result of which allows us to rewrite (I-5) as

$$T = I^* \dot{\omega} + \widetilde{\omega} I^* \omega + \sum m_i \Big[2 \left(R + r^i\right)^T u^i E$$

$$- \left(R + r^i\right) u^{iT} - u^i \left(R + r^i\right)^T \Big] \dot{\omega}$$

$$+ \widetilde{\omega} \sum m_i \Big[2 \left(R + r^i\right)^T u^i E - \left(R + r^i\right) u^{iT} - u^i \left(R + r^i\right)^T \Big] \omega$$

$$+ \sum m_i \Big[2 \left(R + r^i\right)^T \dot{u}^i E - \left(R + r^i\right) \dot{u}^{iT} - \dot{u}^i \left(R + r^i\right)^T \Big] \omega$$

$$+ \widetilde{\omega} \sum \left(\widetilde{R} + \widetilde{r}^i \right) m_i \dot{u}^i + \sum \left(\widetilde{R} + \widetilde{r}^i \right) m_i \ddot{u}^i$$

With the identity $\Gamma^i = R + r^i$ and the substitution of $w + \Omega$ for ω, the above becomes

$$T = I^* \dot{w} + \tilde{w} I^* \Omega + \tilde{\Omega} I^* w + \tilde{\Omega} I^* \Omega$$
$$+ \Omega \left\{ \sum m_i \left[\left(2 \Gamma^{iT} u^i \right) E - u^i \Gamma^{iT} - \Gamma^i u^{iT} \right] \right\} \Omega$$
$$+ \left\{ \sum m_i \left[\left(2 \Gamma^{iT} \dot{u}^i \right) E - \dot{u}^i \Gamma^{iT} - \Gamma^i \dot{u}^{iT} \right] \right\} \Omega$$
$$+ \sum m_i \left[\tilde{\Gamma}^i \ddot{u}^i + \Omega \tilde{\Gamma}^i \dot{u}^i \right] \qquad \text{(I-6)}$$

Assuming that the structure is unforced and that the vector basis in which the equations are written corresponds to a principal axis (identified as A, B, and C) the above equation expands to

$$A \dot{w}_x - \Omega w_y (B - C) + \Omega^2 \sum m_i \left(\Gamma_y^i u_z^i + \Gamma_z^i u_y^i \right)$$
$$+ \sum m_i \left(\Gamma_y^i \ddot{u}_z^i - \Gamma_z^i \ddot{u}_y^i \right) - 2\Omega \sum m_i \Gamma_z^i \dot{u}_x^i = 0 \qquad \text{(I-7a)}$$

$$B \dot{w}_y + \Omega w_x (A - C) - \Omega^2 \sum m_i \left(\Gamma_x^i u_z^i + \Gamma_z^i u_x^i \right)$$
$$+ \sum m_i \left(\Gamma_z^i \ddot{u}_x^i - \Gamma_x^i \ddot{u}_z^i \right) - 2\Omega \sum m_i \Gamma_z^i \dot{u}_y^i = 0 \qquad \text{(I-7b)}$$

$$C \dot{w}_z + \sum m_i \left(\Gamma_x^i \ddot{u}_y^i - \Gamma_y^i \ddot{u}_x^i \right) + 2\Omega \sum m_i \left(\Gamma_x^i \dot{u}_x^i + \Gamma_y^i \dot{u}_y^i \right) = 0 \qquad \text{(I-7c)}$$

We shall now utilize the set of Equations (I-4) and (I-7) to derive equations of motion of the simple particle model shown in Figure 1. Expanding (I-4) for the two particle model one finds:

Particle # 1 Equations

$$\ddot{u}_x^1 - \frac{m}{\mathcal{M}}\left(\ddot{u}_x^1 + \ddot{u}_x^2\right) - 2\Omega\left[\dot{u}_y^1 - \frac{m}{\mathcal{M}}\left(\dot{u}_x^1 + \dot{u}_x^2\right)\right] + 2\zeta\sigma_x\dot{u}_x^1$$

$$-\Omega^2\left[u_x^1 - \frac{m}{\mathcal{M}}\left(u_x^1 + u_x^2\right)\right] + \sigma_x^2 u_x^1 = \dot{w}_z\overline{\Gamma}_y$$

$$\ddot{u}_y^1 - \frac{m}{\mathcal{M}}\left(\ddot{u}_y^1 + \ddot{u}_y^2\right) + 2\Omega\left[\dot{u}_x^1 - \frac{m}{\mathcal{M}}\left(\dot{u}_x^1 + \dot{u}_x^2\right)\right] + 2\zeta\sigma_y\dot{u}_y^1$$

$$-\Omega^2\left[u_y^1 - \frac{m}{\mathcal{M}}\left(u_y^1 + u_y^2\right)\right] + \sigma_y^2 u_y^1 = 2\Omega\, w_z\overline{\Gamma}_y$$

$$\ddot{u}_z^1 - \frac{m}{\mathcal{M}}\left(\ddot{u}_z^1 + \ddot{u}_z^2\right) + 2\zeta\sigma_z\dot{u}_z^1 + \sigma_z^2 u_z^1 = \dot{w}_x\overline{\Gamma}_y - \Omega\, w_y\overline{\Gamma}_y$$

Particle # 2 Equations

$$\ddot{u}_x^2 - \frac{m}{\mathcal{M}}\left(\ddot{u}_x^1 + \ddot{u}_x^2\right) - 2\Omega\left[\dot{u}_y^2 - \frac{m}{\mathcal{M}}\left(\dot{u}_x^1 + \dot{u}_x^2\right)\right] + 2\zeta\sigma_x\dot{u}_x^2$$

$$-\Omega^2\left[u_x^2 - \frac{m}{\mathcal{M}}\left(u_x^1 + u_x^2\right)\right] + \sigma_x^2 u_x^2 = -\dot{w}_z\overline{\Gamma}_y$$

$$\ddot{u}_y^2 - \frac{m}{\mathcal{M}}\left(\ddot{u}_y^1 + \ddot{u}_y^2\right) + 2\Omega\left[\dot{u}_x^2 - \frac{m}{\mathcal{M}}\left(\dot{u}_x^1 + \dot{u}_x^2\right)\right] + 2\zeta\sigma_y\dot{u}_y^2$$

$$-\Omega^2\left[u_y^2 - \frac{m}{\mathcal{M}}\left(u_y^1 + u_y^2\right)\right] + \sigma_y^2 u_y^2 = -2\Omega\, w_z\overline{\Gamma}_y$$

$$\ddot{u}_z^2 - \frac{m}{\mathcal{M}}\left(\ddot{u}_z^1 + \ddot{u}_z^2\right) + 2\zeta\sigma_z\dot{u}_z^2 + \sigma_z^2 u_z^2 = \dot{w}_x\overline{\Gamma}_y + \Omega\, w_y\overline{\Gamma}_y$$

Rotational Equations

$$A\dot{w}_x - \Omega\, w_y(B - C) + m\Omega^2\overline{\Gamma}_y\left(u_z^1 - u_z^2\right) + m\overline{\Gamma}_y\left(\ddot{u}_z^1 - \ddot{u}_z^2\right) = 0$$

$$B\dot{w}_y + \Omega\, w_x(A - C) = 0$$

$$C\dot{w}_z - m\overline{\Gamma}_y\left(\ddot{u}_x^1 - \ddot{u}_x^2\right) + 2\Omega m\overline{\Gamma}_y\left(\dot{u}_y^1 - \dot{u}_y^2\right) = 0$$

In the above the superscript on the deformation variable denotes the particle under consideration (e.g., u_x^2 corresponds to the second particle deformation in the $\hat{x}$ direction), whereas the superscript on the variable descriptive of the natural frequency is an exponent (e.g., σ_x^2 implies $\sigma_x \cdot \sigma_x$).

It is observed that equations descriptive of w_x and w_y and deformations in the $\hat{z}$ direction separate from the remaining set of equations. This set, descriptive of the "wobble" motion, is repeated below.

$$\ddot{u}_z^1 - \frac{m}{\mathcal{M}}\left(\ddot{u}_z^1 + \ddot{u}_z^2\right) + 2\zeta\sigma_z\dot{u}_z^1 + \sigma_z^2 u_z^1 = -\dot{w}_x\overline{\Gamma}_y - \Omega w_y\overline{\Gamma}_y$$

$$\ddot{u}_z^2 - \frac{m}{\mathcal{M}}\left(\ddot{u}_z^1 + \ddot{u}_z^2\right) + 2\zeta\sigma_z\dot{u}_z^2 + \sigma_z^2 u_z^2 = \dot{w}_x\overline{\Gamma}_y + \Omega w_y\overline{\Gamma}_y$$

$$A\dot{w}_x - \Omega w_y(B-C) + m\Omega^2\overline{\Gamma}_y\left(u_z^1 - u_z^2\right) + m\overline{\Gamma}_y\left(\ddot{u}_z^1 - \ddot{u}_z^2\right) = 0$$

$$B\dot{w}_y + \Omega w_x(A-C) = 0$$

The first two of the above equations can be replaced by their sum and difference;

$$\ddot{\eta} - \frac{2m}{\mathcal{M}}\ddot{\eta} + 2\zeta\sigma_z\dot{\eta} + \sigma_z^2\eta = 0$$

$$\ddot{\mu} + 2\zeta\sigma_z\dot{\mu} + \sigma_z^2\mu = -2\dot{w}_x\overline{\Gamma}_y - 2\Omega w_y\overline{\Gamma}_y$$

where the sum of the first and second deformations is identified as η and the difference as μ, i.e.,

$$\eta = u_z^1 + u_z^2$$

$$\mu = u_z^1 - u_z^2$$

With these definitions we can describe η as the symmetrical mode and μ as the asymmetrical mode.

It is clear that only the asymmetrical mode is of importance since the symmetrical mode is uncoupled with the rotational equations. So that the equations descriptive of wobble motion can be written as follows

$$A \dot{w}_x - \Omega w_y (B - C) + m \bar{\Gamma}_y (\ddot{\mu} + \Omega^2 \mu) = 0 \tag{I-8a}$$

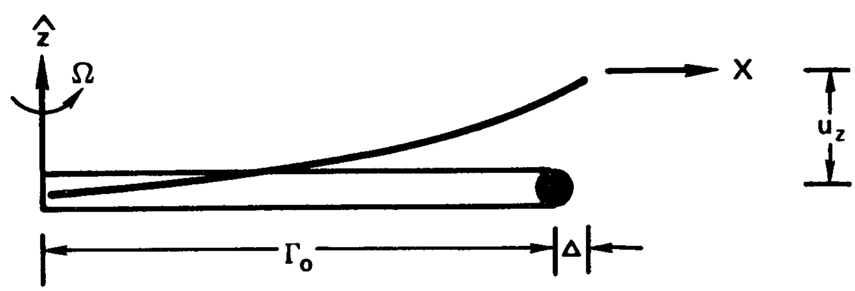
$$B \dot{w}_y + \Omega w_x (A - C) = 0 \tag{I-8b}$$

$$\ddot{\mu} + 2 \zeta \sigma_z \dot{\mu} + \sigma_z^2 \mu + 2 \bar{\Gamma}_y \dot{w}_x + 2 \Omega \bar{\Gamma}_y w_y = 0 \tag{I-8c}$$

APPENDIX II

LOADED FREQUENCY OF A MASSLESS CANTILEVERED BEAM WITH A TIP MASS

Consider a massless cantilever beam of length Γ_o having a tip mass m



The bending stiffness when loaded in tension is*

$$k_z = \frac{\hat{k}_z (\eta \Gamma_o)^2}{3\left[1 - \frac{\tanh(\eta \Gamma_o)}{\eta \Gamma_o}\right]}$$

where $\hat{k}_z$ is the unloaded stiffness prescribed as a function of the beams flexural rigidity EI and Γ_o, i.e.,

$$\hat{k}_z = \frac{3EI}{\Gamma_o^3}$$

and η is given as a function of the applied tensile force P due solely to spin, to wit

$$\eta^2 = \frac{P}{EI}$$

$$P = m(\Gamma_o + \Delta)\Omega^2 = m\bar{\Gamma}_y \Omega^2$$

The combination of the above permits the following:

---

*Taken from a set of unpublished JPL notes written by E. Weiner.

$$(\eta \Gamma_o)^2 = \frac{m \bar{\Gamma}_y \Omega^2 \Gamma_o^2}{EI}$$

$$= m \bar{\Gamma}_y \Omega^2 \frac{3}{\hat{k}_z \Gamma_o}$$

Defining $\hat{\sigma}_z$ as $\hat{k}_x / m$ allows

$$(\eta \Gamma_o)^2 = \frac{3 \bar{\Gamma}_y \Omega^2}{\hat{\sigma}_z^2 \Gamma_o}$$

Substition into the expression for k_z results in the following

$$k_z = \frac{\hat{k}_z \left(\frac{3 \bar{\Gamma}_y \Omega^2}{\hat{\sigma}_z^2 \Gamma_o} \right)}{3 \left[1 - \frac{\tanh\left(\frac{\Omega}{\hat{\sigma}_z} \sqrt{\frac{3 \bar{\Gamma}_y}{\Gamma_o}} \right)}{\frac{\Omega}{\hat{\sigma}_z} \sqrt{\frac{3 \bar{\Gamma}_y}{\Gamma_o}}} \right]}$$

Therefore,

$$\sigma_z^2 \triangleq \frac{k_z}{m} = \frac{\frac{\hat{k}_z}{m} \left(\frac{\bar{\Gamma}_y \Omega^2}{\hat{\sigma}_z^2 \Gamma_o} \right)}{\left[1 - \frac{\tanh\left(\frac{\Omega}{\hat{\sigma}_z} \sqrt{\frac{3 \bar{\Gamma}_y}{\Gamma_o}} \right)}{\frac{\Omega}{\hat{\sigma}_z} \sqrt{\frac{3 \bar{\Gamma}_y}{\Gamma_o}}} \right]}$$

so that

$$\left(\frac{\sigma_z}{\Omega}\right) = \frac{\left(\frac{\bar{\Gamma}_y}{\Gamma_o}\right)^{1/2}}{\left[1 - \frac{\tanh\left(\frac{\Omega}{\hat{\sigma}_z}\sqrt{\frac{3\bar{\Gamma}_y}{\Gamma_o}}\right)}{\frac{\Omega}{\hat{\sigma}_z}\sqrt{\frac{3\bar{\Gamma}_y}{\Gamma_o}}}\right]^{1/2}}$$

For small arguments the hyperbolic tangent function simplifies to:

$$\tanh x \approx x - \frac{x^3}{3} + \frac{2x^5}{15}$$

$$\frac{\tanh x}{x} \approx 1 - \frac{x^2}{3} + \frac{2x^4}{15}$$

$$1 - \frac{\tanh x}{x} \approx x^2 \left(\frac{1}{3} - \frac{2x^2}{15}\right)$$

$$\frac{1}{1 - \frac{\tanh x}{x}} \approx \frac{\sqrt{3}}{x} \left(1 + \frac{x^2}{5}\right)$$

Hence,

$$\left(\frac{\sigma_z}{\Omega}\right) \approx \frac{\hat{\sigma}_z}{\Omega} \left[1 + \frac{3}{5} \frac{\bar{\Gamma}_y}{\Gamma_o} \left(\frac{\Omega}{\hat{\sigma}_z}\right)^2\right]$$

Squaring both sides provides the following

$$\left(\frac{\sigma_z}{\Omega}\right)^2 \approx \left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 \left[1 + \frac{6}{5} \frac{\bar{\Gamma}_y}{\Gamma_o} \left(\frac{\Omega}{\hat{\sigma}_z}\right)^2 + ---\right]$$

For an extremely stiff rod $\bar{\Gamma}_y \approx \Gamma_o$ allowing the loaded frequency approximation

$$\sigma_z^2 \approx \hat{\sigma}_z^2 + 1.2\,\Omega^2$$

The approximated expression for the loaded frequency is compared with the true expression in the table below. Certainly for $\hat{\sigma}_z/\Omega > 1$ the simplified expression is an excellent approximation.

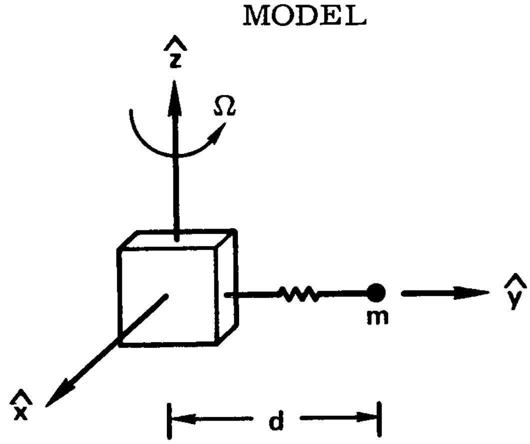
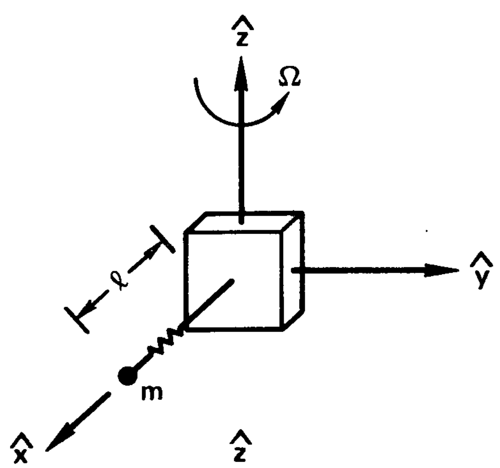
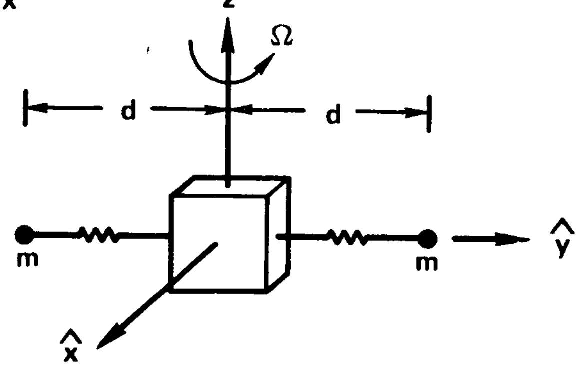
	True Value $\left(\frac{\sigma_z}{\Omega}\right)^2 = \frac{1}{\left[1 - \frac{\tanh\left(\sqrt{3}\,\frac{\Omega}{\hat{\sigma}_z}\right)}{\left(\sqrt{3}\,\frac{\Omega}{\hat{\sigma}_z}\right)}\right]}$		Approximate Value $\left(\frac{\sigma_z}{\Omega}\right)^2 \approx \left(\frac{\hat{\sigma}_z}{\Omega}\right)^2 + 1.2$	
$\frac{\hat{\sigma}_z}{\Omega}$	$\left(\frac{\sigma_z}{\Omega}\right)^2$	$\left(\frac{\sigma_z}{\Omega}\right)$	$\left(\frac{\sigma_z}{\Omega}\right)^2$	$\left(\frac{\sigma_z}{\Omega}\right)$
0	1.0	1.0	1.2	1.095
.1	1.061	1.030	1.21	1.1
.5	1.405	1.185	1.45	1.204
1	2.184	1.478	2.2	1.483
5	26.199	5.119	26.2	5.119
10	101.20	10.06	101.20	10.06

APPENDIX III

SUMMARY OF STABILITY CRITERIA

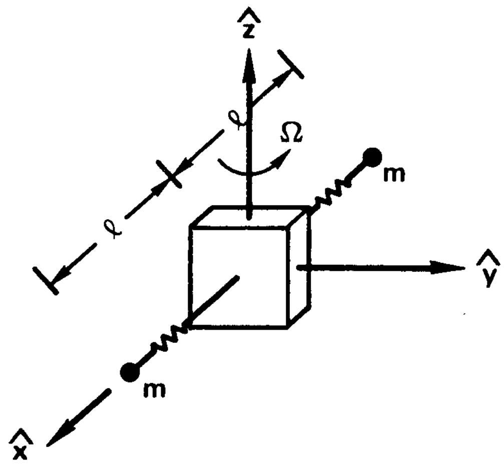
The following criteria have been formally established as necessary conditions* for asymptotic stability of spin.

C > A AND C > B IN ALL MODELS

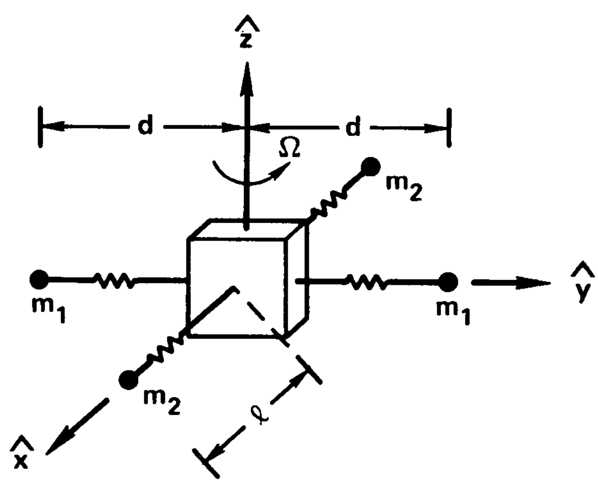
MODEL	CRITERION
	$\left(\frac{\omega}{\Omega}\right)^2 > \frac{m\,d^2}{(C-B)}$
	$\left(\frac{\omega}{\Omega}\right)^2 > \frac{m\,\ell^2}{(C-A)}$
	$\left(\frac{\omega}{\Omega}\right)^2 > \frac{2\,m\,d^2}{(C-B)}$

*In some cases sufficiency has also been proven formally.

MODEL | CRITERION



$$\left(\frac{\omega}{\Omega}\right)^2 > \frac{2 m \ell^2}{(C-A)}$$

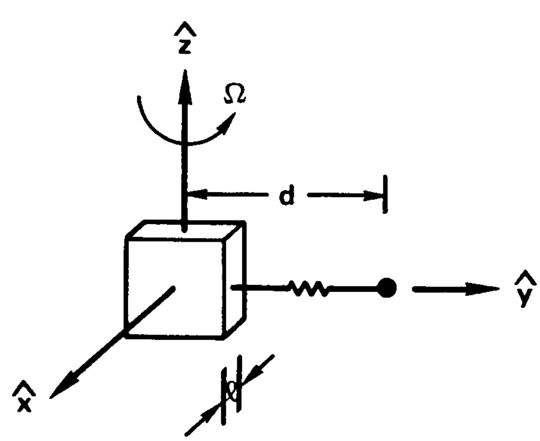


$$\left(\frac{\omega}{\Omega}\right)^2 > \frac{2 m_1 d^2}{2 m_1 d^2 + C' - B'}$$

$$\left(\frac{\sigma}{\Omega}\right)^2 > \frac{2 m_2 \ell^2}{2 m_2 \ell^2 + C' - A'}$$

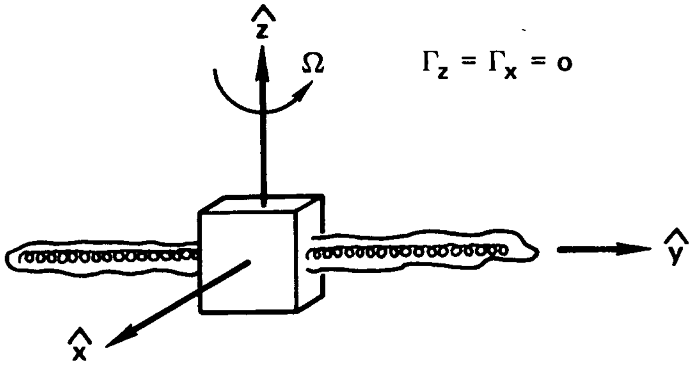
ω ~ loaded frequency associated with m_1

σ ~ loaded frequency associated with m_2



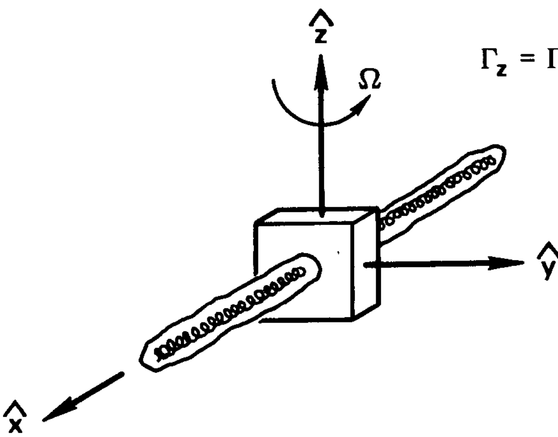
$$\left(\frac{\omega}{\Omega}\right)^2 > \frac{(C-A) m d^2 + (C-B) m \ell^2}{(C-A)(C-B)}$$

MODEL CRITERION



$\Gamma_z = \Gamma_x = o$

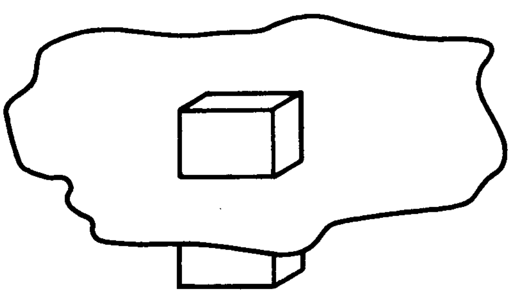
$$\left(\frac{1}{\Omega}\right)^2 > \frac{\delta_y^T\left[\omega^2\right]^{-1}\delta_y}{(C-B)}$$



$\Gamma_z = \Gamma_y = o$

$$\left(\frac{1}{\Omega}\right)^2 > \frac{\delta_x^T\left[\omega^2\right]^{-1}\delta_x}{(C-A)}$$

$\Gamma_z = o$



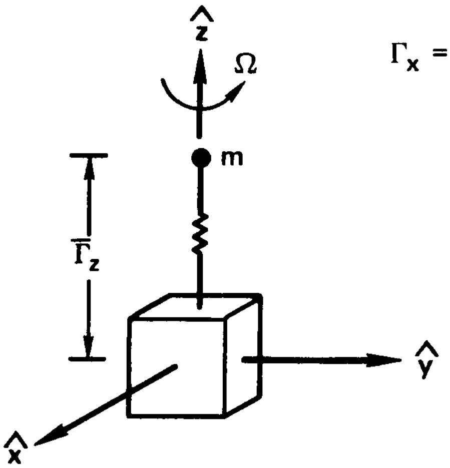
1st Mode

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{(C-A)\,\delta_{y1}^2 + (C-B)\,\delta_{x1}^2}{(C-A)(C-B)}$$

Nth Mode $(\delta_x\,\delta_y^T - \delta_y\,\delta_x^T) = 0$

$$\left(\frac{1}{\Omega}\right) > \frac{(C-A)\,\delta_y^T\left[\omega^2\right]^{-1}\delta_y + (C-B)\,\delta_x^T\left[\omega^2\right]^{-1}\delta_x}{(C-A)(C-B)}$$

MODEL CRITERION



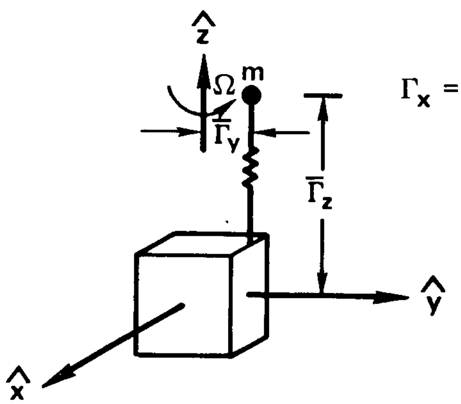
$\Gamma_x = \Gamma_y = o$

For $k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\overline{\Gamma}_z^{\,2}}{(C-A)}$$

For $k_x = \infty$

$$\left(\frac{\omega_y}{\Omega}\right)^2 > \frac{m\,\overline{\Gamma}_z^{\,2}}{(C-B)}$$

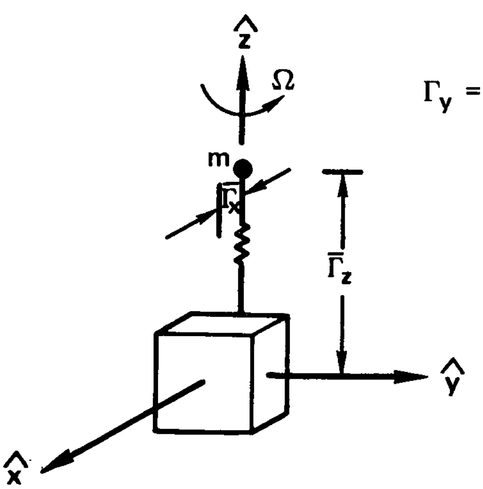


$\Gamma_x = o$

$k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\overline{\Gamma}_z^{\,2}}{(C-A)}$$

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\,\overline{\Gamma}_y^{\,2}}{(C-B)}$$

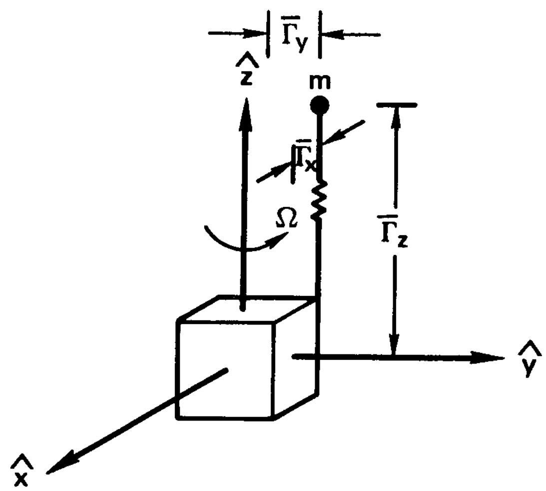


$\Gamma_y = o$

$k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\overline{\Gamma}_z^{\,2}}{(C-A)} - \frac{4\,m\,\overline{\Gamma}_x^{\,2}}{C}$$

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\,\overline{\Gamma}_x^{\,2}}{(C-A) - \dfrac{m\,\overline{\Gamma}_z^{\,2}\,C}{\left[C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\,\overline{\Gamma}_x^{\,2}\right]}}$$



$k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_z^{\,2}}{(C-A)} - \frac{4m\,\bar{\Gamma}_x^{\,2}}{C}$$

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_y^{\,2}(C-A) + \bar{\Gamma}_x^{\,2}(C-B)}{D} - \frac{\left[\dfrac{C\,m^2\bar{\Gamma}_y^{\,2}\bar{\Gamma}_z^{\,2}}{C\left(\dfrac{\omega_x}{\Omega}\right)^2 + 4m\,\bar{\Gamma}_x^{\,2}}\right]}{D}$$

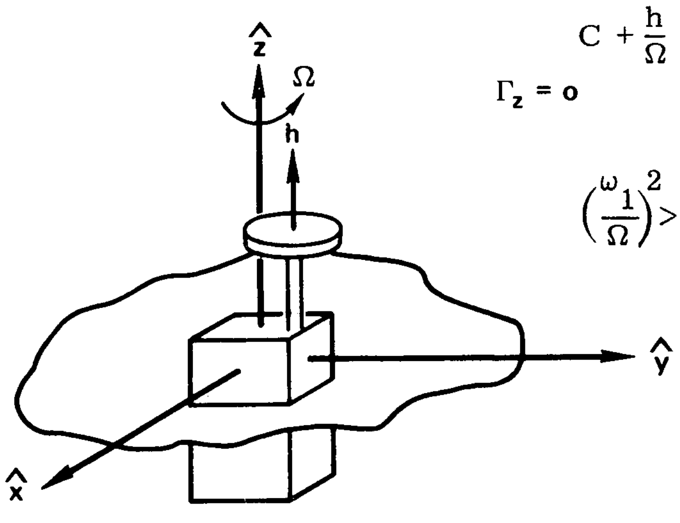
where

$$D = (C-B)\left[(C-A) - \frac{C m\,\bar{\Gamma}_z^{\,2}}{C\left(\dfrac{\omega_x}{\Omega}\right)^2 + 4m\,\bar{\Gamma}_x^{\,2}}\right]$$

First Mode

$$C + \frac{h}{\Omega} > A \text{ and } C + \frac{h}{\Omega} > B$$

$\Gamma_z = o$



$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^{\,2}}{\left(C + \frac{h}{\Omega} - A\right)} + \frac{\delta_{y1}^{\,2}}{\left(C + \frac{h}{\Omega} - B\right)}$$