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CONTROL OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FUNCTION SPACE BOUNDARY CONDITIONS[†]

by

H. T. Banks^{*}

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

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1. Introduction.

It has been known for a number of years that functional differential equations play an important role in the modeling of many mechanical and electrical systems. Interest in equations of this type has continued to grow as it has become apparent that they are also of importance in areas of biomedical modeling (physiological and hormonal control systems). Many authors have contributed to the growing literature on the mathematical theory of control of functional differential equations: Halanay (Rumania), Lee and coworkers (U. Minnesota), Delfour and Mitter (MIT), Weiss and coworkers (U. Maryland) are just a few in the long list of contributors. Mathematicians from the USSR have made numerous advances in this area: Kharatishvili, Kirillova, Gabasov, Curakova, Krasovskii and a number of investigators at Patrice Lumumba University (Moscow) [see Trudy Seminara po Teorii Differencial'nye Uravnenija s Otklonyayushchimsya Argumentom, Moscow, Vol. 1-7, 1962-1970] should be included in this group. Much of the work of these and other authors has dealt with problems having terminal or target sets in R^n . Several review papers and articles with extensive bibliographies on these results have appeared ([1], [5], [15], [29] and the volumes of Trudy Seminara cited above.)

We shall motivate below problems involving functional differential equations with terminal conditions in function space. Study of problems of this type has been much less extensive and we shall attempt a brief survey in which we report those results known to us at the present. Many of these results are very recent while some of the older investigations (before 1970) due to Soviet mathematicians appear to be unknown to some researchers in this country.

We shall, for ease in exposition, restrict our discussions mainly to the simplest linear neutral (retarded if $A_1 \equiv 0$) n -vector system with lags

$$(1.1) \quad \dot{x}(t) = A_1(t)\dot{x}(t-h) + A_2(t)x(t) + A_3(t)x(t-h) + B(t)u(t),$$

although, as we shall point out whenever appropriate below, a great deal of the work reported here has been carried out for much more general (including nonlinear) systems of retarded and/or neutral type. The subject of our subsequent paragraphs involves the control of system (1.1) from $x_{t_0} = \varphi$ to $x_{t_1} = \zeta$, where x_t denotes the segment $x(t+s)$, $-h \leq s \leq 0$, of the trajectory x (i.e. $x_t(\theta) = x(t+\theta)$, $\theta \in [-h, 0]$), and φ, ζ are given functions in some properly chosen space of functions on $[-h, 0]$ into R^n .

As a first example where problems of this type occur we suppose that (1.1) represents a system which we would like to drive to $x = 0$ and have it remain there if we shut off the controller and no other disturbances are present (i.e., the so-called "regulator" or "settling" problem). Then it is quite obvious that the desired terminal condition is $x_{t_1} = 0$, not $x(t_1) = 0$.

A second example involves boundary control of the wave equation (for a more detailed discussion see [6], [23]). Suppose $w(x, t)$ is a solution to

$$(1.2) \quad w_{tt} - c^2 w_{xx} = 0 \quad (x, t) \in [0, 1] \times [0, t_1]$$

with boundary conditions

$$\begin{aligned}
 (1.3) \quad & a_0(t)w_t(0,t) + b_0(t)w_x(0,t) = f_0(t,w(0,t)) \\
 & a_1(t)w_t(1,t) + b_1(t)w_x(1,t) = f_1(t,w(1,t))
 \end{aligned}
 \quad t \in [0, t_1]$$

and initial-terminal conditions

$$\begin{aligned}
 (1.4) \quad & w(x,0) = \alpha_0(x) \quad w(x,t_1) = \alpha_1(x) \\
 & w_t(x,0) = \beta_0(x) \quad w_t(x,t_1) = \beta_1(x)
 \end{aligned}
 \quad x \in [0,1].$$

It is assumed that controls are contained in the terms f_i . (For example, we might have $f_i(t,w) = d_i w + e_i u_i(t)$, $i = 0,1$.) Assuming a solution in terms of D'Alembert wave functions

$$(1.5) \quad w(x,t) = \Phi\left(t + \frac{x}{c}\right) + \Psi\left(t - \frac{x}{c}\right),$$

we substitute into (1.3) and differentiate once. A few algebraic manipulations yield the neutral equations for the differentiated wave functions

$$(\Phi', \Psi') = (y, z)$$

$$\begin{aligned}
 (1.6) \quad & \dot{y}(t) + r(t)\dot{z}\left(t - \frac{2}{c}\right) = F_0(t, y(\cdot), z(\cdot)) \\
 & \dot{z}(t) + s(t)\dot{y}\left(t - \frac{2}{c}\right) = F_1(t, y(\cdot), z(\cdot))
 \end{aligned}$$

for $t \in [\frac{1}{c}, t_1]$, where the F_0, F_1 terms contain the controls and depend on $y(s), z(s)$, $s \leq t$, in a complicated but precise way. Under appropriate

assumptions [6] the data in (1.4) can be transformed into data for y on $[0, \frac{1}{c}]$ and at t_1 while the values of z are given on $[-\frac{1}{c}, \frac{1}{c}]$ and $[t_1 - \frac{2}{c}, t_1]$. This initial data for (y, z) is sufficient to solve (1.6) for $(y, z) = (\Phi', \Psi')$ in $W_2^{(1)}$ (absolutely continuous functions possessing L_2 derivatives). It can be argued that (1.5) then yields a weak solution for (1.2) subject to (1.3), (1.4) in the sense that w is C^1 with w_t, w_x in $W_2^{(1)}$ and the equation (1.2) is satisfied a.e.

That a relationship between hyperbolic partial differential equations and functional differential equations of neutral type exists has been known for some time and this idea has been explored by a number of authors [6, 8, 9, 12, 23, 35]. We remark that the boundary conditions (1.3) include as special cases those usually associated with transverse vibrations of a string or longitudinal vibrations of a rod, the ends of the string or rod being elastically supported.

In the following sections we shall report results of investigations on three important questions for problems of the type formulated above: controllability, existence of optimal controls, and necessary and sufficient conditions for optimality.

2. Controllability.

Kirillova and her colleagues [14, 15, 25, 26] were among the first to study controllability of delayed systems, their main emphasis being on retarded systems with constant coefficients

$$(2.1) \quad \dot{x}(t) = A_2 x(t) + A_3 x(t-h) + Bu(t),$$

$x \in R^n$, $u \in R^r$, for which they sought computable (testable) criteria for controllability. They carefully formulated two types of controllability of interest:

Definition 2.1. System (2.1) is relatively (null) controllable on $[0, t_1]$ if, given any $\varphi \in \Phi$, there exists an admissible control u such that $x(t_1; \varphi, u) = 0$.

Definition 2.2. System (2.1) is (null) controllable on $[0, t_1]$ if, given any $\varphi \in \Phi$, there exists an admissible control u such that $x_{t_1}(\varphi, u) = 0$.

Here and elsewhere throughout this note $x(\cdot, \varphi, u)$ will denote the solution to the system being discussed (system (2.1) in the above definitions) corresponding to initial data $x_0 = \varphi$ ($t_0 = 0$) and control u . In their investigations Kirillova, Curakova, and Gabasov use the class of piecewise continuous controllers as the admissible class while $\Phi = \{\varphi: [-h, 0] \rightarrow R^n \mid \varphi \text{ piecewise continuous}\}$. We shall refer to the controllability defined in Definition 2.1 as Euclidean space (null) controllability while that defined in Definition 2.2 will be called function space (null) controllability. Furthermore, we shall say (see [25]) that (2.1) is relatively controllable if given

$\varphi \in \Phi$, there exist admissible u and t_1 (depending possibly on φ) such that $x(t_1; \varphi, u) = 0$. A similar definition will be taken for controllability of system (2.1).

In [25] the authors give necessary conditions and sufficient conditions for relative controllable in terms of rank conditions on certain matrices P and Q respectively, P and Q being formed from the coefficient matrices in the system (2.1). They show that for $B = b \in R^n$ ($r = 1$) and $n \leq 3$, $\text{rank } P = \text{rank } Q$ and the conditions are necessary and sufficient. They offer a number of additional results:

R1: For pure delay systems ($A_2 \equiv 0$ in (2.1)) with $B = b \in R^n$ ($r = 1$), controllability \Leftrightarrow relative controllability.

R2: If B is nonsingular ($\Rightarrow B$ square, $r = n$), then (2.1) is controllable. {Actually, it is easily seen that if B has rank n ($r \geq n$), then BB^* is invertible and (2.1) is controllable. One can choose $u(t) = -B^*(BB^*)^{-1}A_3x(t-h)$ for $t \in [t_1-h, t_1]$ after having chosen u on $[0, t_1-h]$ so that $x(t_1-h) = 0$.}

R3: If $A_3 = \begin{pmatrix} 0 \\ \tilde{A}_3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix}$ where \tilde{A}_3 is $r \times n$, \tilde{B} is $r \times r$ and if \tilde{B} has rank r , $r < n$, then relative controllability \Rightarrow controllability. This can be used to show that the n^{th} order scalar retarded equations

$$x^{(n)}(t) + \sum_{i=1}^n a_i x^{(n-i)}(t) + \sum_{i=1}^n b_i x^{(n-i)}(t-h) = bu$$

are controllable if $b \neq 0$. {Using arguments similar to those in a

in a closure lemma of Banks and Jacobs [7; Lemma 3.1 and Remark 3.4] one can actually establish that n^{th} order scalar neutral equations

$$x^{(n)}(t) + \sum_{i=1}^n a_i x^{(n-i)}(t) + \sum_{i=0}^n b_i x^{(n-i)}(t-h) = bu$$

are controllable (for $b \neq 0$) from an arbitrary φ such that $\varphi^{(n-1)}$ is absolutely continuous with L_2 derivative (i.e., $\varphi \in W_2^{(n)}([-h, 0], \mathbb{R}^1)$) to an arbitrary ζ in $W_2^{(n)}([-h, 0], \mathbb{R}^1)$. The functions φ, ζ may be taken in $W_1^{(n)}$ if L_1 controls are used in place of L_2 controls (see [7]).

In [26] the emphasis is on resolving the discrepancy between necessary conditions and sufficient conditions for relative controllability (although extensions of results to systems with time-varying coefficients are also made). These ideas are developed further in [14, 15]. Since [15] contains all the results of [26] and [14] (some in improved form), we shall restrict our comments to results in [15].

The equation

$$Q_k(s) = A_2 Q_{k-1}(s) + A_3 Q_{k-1}(s-h), \quad s \geq 0, \quad k = 1, 2, \dots$$

$$Q_0(0) = B, \quad Q_0(s) = 0 \quad \text{for } s \neq 0,$$

is called the defining equation for (2.1). One then defines for every $\alpha > 0$

$$\Pi_\alpha = \{Q_k(s) \mid k = 0, 1, \dots, n-1; s \in [0, \alpha h]\}$$

and says that the defining equation is α non-degenerate if $\text{rank } \Pi_\alpha = n$. The authors' main result is (an analogous result is derived for neutral systems (1.1) with constant coefficients):

Theorem 2.1. System (2.1) is relatively controllable on $[0, t_1]$ if and only if the defining equation for (2.1) is α non-degenerate for $\alpha = [t_1/h]$.

The authors also sharpen the results mentioned above from [25]. For the system discussed in R3, one actually has relative controllability \Leftrightarrow controllability. Also, for systems with pure delay ($A_2 \equiv 0$) with rank $A_3 = n$ and rank $B = r$, they argue that relative controllability \Leftrightarrow controllability. As was pointed out in R1 above, if $B = b \in R^n$ ($r = 1$), the condition "rank $A_3 = n$ " can be omitted.

In the results described above the authors have observed that for certain types of systems relative controllability is equivalent to controllability in which case the algebraic criteria of Theorem 2.1 are applicable. Examples of systems which are relatively controllable but not controllable are given in [15] and demonstrate that in general these concepts are quite different.

The authors discuss a general scheme (applied to several examples) for investigating controllability of certain systems of type (2.1). Finally, they offer conditions that are sufficient for controllability of (2.1) whenever the delay h is sufficiently small.

Weiss in [36] considers essentially (null) controllability as defined in Definition 2.2 above for n -vector retarded systems

$$(2.2) \quad \dot{x}(t) = A_2(t)x(t) + A_3(t)x(t-h) + B(t)u(t)$$

with bounded measurable controls u . His main result is:

Theorem 2.2. System (2.2) is controllable on $[t_0, t_1]$ if

- (i) $\text{rank } G(t_0, t_1-h) = n$
- (ii) for every $\varphi \in C([-h, 0], \mathbb{R}^n)$ and for some bounded measurable u on $[t_0, t_1-h]$ such that $x(t_1-h; \varphi, u) = 0$ (x is the solution to (2.2) on $[t_0, t_1-h]$ with control u and initial data $x_{t_0} = \varphi$), the equation $B(t)v(t) = -A_3(t)x(t-h; \varphi, u)$ has a bounded measurable solution v on $[t_1-h, t_1]$.

Here $G(t_0, t_1-h) \equiv \int_{t_0}^{t_1-h} X(t_1-h, s)B(s)B^*(s)X^*(t_1-h, s)ds$ where $X(t, s)$ is the usual "fundamental" matrix solution appearing in the variation of parameters formula and satisfying (as a function of s) the well-known adjoint to the uncontrolled form of (2.2). As the author points out, (ii) will always hold if A_3 has the form $A_3(t) = B(t)D(t)$ for some bounded measurable $n \times r$ matrix function $t \rightarrow D(t)$.

In [36] Weiss also discusses extensions of certain standard controllability arguments for nonlinear ordinary differential equation systems to establish controllability results for nonlinear retarded systems.

A somewhat different approach to the controllability question has been taken by Halanay [17] and Popov [34]. Both authors use transfer function techniques to consider the n -vector system with scalar controls

$$(2.3) \quad \dot{x}(t) = A_2 x(t) + A_3 x(t-h) + bu(t)$$

where $b \in R^n$. Halanay, motivated by the n^{th} order scalar retarded equations already mentioned above, makes the further assumption that $A_3 = ba_3^*$ where a_3 is an n -column vector. Thus to obtain (null) controllability on $[0, t_1]$ (as in Definition 2.2), it suffices to effect $x(t_1-h; \varphi, u) = 0$ by choice of a control u on $[0, t_1-h]$ and then take $u(t) = -a_3^* x(t-h)$ for $t \in [t_1-h, t_1]$. Halanay shows that this system is controllable if and only if $\dot{x}(t) = A_2 x(t) + bu(t)$ is controllable in the usual sense and therefore reduces the study of (2.3) to a study of ordinary differential equation controllability. He uses piecewise continuous controls as his class of admissible controls. We note that in fact the results of Kirillova, et.al. [14, 15, 25, 26] discussed above are directly applicable to these systems since in this case relative controllability on $[0, t_1-h]$ implies controllability on $[0, t_1]$.

Popov [34] introduces the concept of complete reachability: System (2.3) is completely reachable if there exists a positive integer p such that for every ϵ , $0 < \epsilon < h$, and every $w \in C^p([-h+\epsilon, 0], R^n)$, there exist $t_1 \geq h$ and continuous control u on $[0, t_1]$ such that $x(t; 0, u) = w(t-t_1)$ for $t \in [t_1-h+\epsilon, t_1]$. Defining the function $g(s, z) = (sI - A_2 - zA_3)^{-1}b$ {recall that the transfer function is $g(s, e^{-sh}) = (sI - A_2 - e^{-sh}A_3)^{-1}b$ }, the author notes that g can be written

$$g(s, z) = \frac{P(s)v(z)}{\det(sI - A_2 - zA_3)}$$

where $v(z)$ is an n -column vector with coordinates $1, z, z^2, \dots, z^{n-1}$ and $P(s)$ is a polynomial of degree $n-1$ with $n \times n$ matrix coefficients. Popov then proves:

Theorem 2.3. The system (2.3) is completely reachable if and only if there exists s_0 such that $P(s_0)$ is nonsingular.

Furthermore, he shows that complete reachability implies controllability, in the sense that: For every $\epsilon > 0$ and every φ, ζ in $C^p([-h+\epsilon, 0], \mathbb{R}^n)$, there exists a continuous control u on $[0, t_1]$ such that some solution x to (2.3) corresponding to u satisfies $x(t) = \varphi(t)$, $t \in [-h+\epsilon, 0]$, $x(t) = \zeta(t-t_1)$, $t \in [t_1-h+\epsilon, t_1]$.

It is obvious that while this is a type of function space controllability, it is not equivalent to the concept introduced in Definition 2.2 above (i.e., where one has terminal boundary conditions on x_t , the "state" of the system). It is not yet clear how Popov's results might be of use in studying these more natural concepts of controllability.

Another type of function space controllability that differs from that under consideration here was studied by Zmood [37]: System (2.1) is completely function space controllable at time $t_1 > h$ if for every $\epsilon > 0$, every $\varphi \in C([-h, 0], \mathbb{R}^n)$, and every $\zeta \in L_2([-h, 0], \mathbb{R}^n)$, there exists $u \in L_2([0, t_1], \mathbb{R}^r)$ such that $\|\zeta - x_{t_1}(\varphi, u)\|_2 < \epsilon$, where $\|\cdot\|_2$ denotes the usual norm in $L_2([-h, 0], \mathbb{R}^n)$. Like Popov's results, Zmood's work yields little more than a density result for the problems under discussion in this note. Results of this type are not adequate to yield the desired information for the "regulator" or "settling" problems ($x_{t_1} = 0$).

A number of other authors [2, 13] have investigated controllability of functional differential equations to terminal functions and have derived results in terms of certain abstract mappings related to these systems. These theorems have not, as yet, led to any computable or testable criteria for controllability.

It is evident that much remains to be done in the area of function space controllability as the concept is formulated in the introduction of this note.

3. Existence of optimal controls.

The question of existence of optimal controls for problems with function space terminal conditions has been answered much more completely than has the question of controllability posed in the preceding section. A number of authors have given satisfactory treatments to the existence question, showing in most cases that, with proper modifications, the hypotheses and arguments used in existence theorems for ordinary differential equation control systems can be employed to obtain theorems for control problems with functional differential equation systems.

One of the first to consider existence of optimal controls for problems of the type being discussed here was Angell [2, 3]. Using the methods and closure ideas of Cesari [10, 11] and an extension of Filippov's lemma due to McShane and Warfield [30], Angell obtains existence of optimal controls for problems involving general nonlinear retarded systems and function space boundary conditions. In [4] he extends these arguments to systems governed by neutral equations.

In his thesis [23] Kent shows that quite general existence results such as those of Jacobs [21] can be extended to establish results for problems involving certain nonlinear neutral functional differential equations. Existence theorems for a general class of linear-in-the-state, nonlinear-in-the-controls neutral systems were derived by Banks and Kent in [6] through use of attainable sets arguments in $C([-h, 0], \mathbb{R}^n)$. Unlike the situation for ordinary and functional differential systems with terminal targets in \mathbb{R}^n [5, 21, 31], it is observed in [6] that the usual convexity assumptions

for the control term in the system is crucial. (The well-known Liapunov theorem arguments (see [5]) employed in the proof for finite dimensional terminal sets do not extend to infinite dimensional cases.) Furthermore, the authors in [6] show by example that the regularity (existence in smoother classes of controls) and bang-bang results usually associated with linear ordinary differential equation control systems [18, 19, 20] do not obtain for problems with terminal function conditions.

Finally, Jacobs and Kao [22] (see also [6]) show that the usual weak compactness arguments in L_2 can be used to obtain existence theorems for problems of the type being considered with linear retarded system equations and Lagrange payoffs with quadratic integrands.

4. Necessary and sufficient conditions for optimality.

Necessary conditions for optimality for the problems under discussion in this note have been obtained recently in [6, 7, 22, 23, 24]. We shall present one statement of these results here and then discuss variations of this theorem as found in the references cited. Consider then the problem: Minimize $J = \int_{t_0}^{t_1} f^0(x(t), u(t), t) dt$ subject to (1.1) on $[t_0, t_1]$ and $x_{t_0} = \varphi$, $x_{t_1} = \zeta$, where φ, ζ are given functions in $W_2^{(1)}([-h, 0], \mathbb{R}^n)$, and u is to be chosen from $\mathcal{U} \equiv \{u: [t_0, t_1] \rightarrow \mathbb{R}^r \mid u \text{ is bounded measurable, } u(t) \in U \text{ for } t \in [t_0, t_1]\}$ with U a given nonempty subset of \mathbb{R}^r .

Theorem 4.1. Suppose (x^*, u^*) is a solution to the above problem. Then there exist $\alpha^0 \leq 0$, $\hat{\psi}: [t_0, \infty) \rightarrow \mathbb{R}^{n+1}$, $\mu: \mathbb{R}^1 \rightarrow \mathbb{R}^n$ with $\hat{\psi}, \mu$ of bounded variation and left continuous, μ constant outside $[-h, 0]$, such that

(i) $\hat{\psi} = (\psi^0, \psi) = (\psi^0, \psi^1, \dots, \psi^n)$ satisfies

$$\psi^0 = \alpha^0 \leq 0$$

$$\psi(s) = 0 \quad s > t_1$$

$$\begin{aligned} \psi(s) = & -\mu(s-t_1) + \psi(s+h)A_1(s+h) + \int_{s+h}^{t_1} \psi(\theta)A_3(\theta)d\theta \\ & + \int_s^{t_1} \{\psi(\theta)A_2(\theta) + \psi^0 \frac{\partial f^0}{\partial x}(x^*(\theta), u^*(\theta), \theta)\} d\theta \quad \text{for } s \in [t_0, t_1] \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \int_{t_0}^{t_1} \{\psi^0 f^0(x^*(\theta), u(\theta), \theta) + \psi(\theta)B(\theta)u(\theta)\} d\theta \\ & \leq \int_{t_0}^{t_1} \{\psi^0 f^0(x^*(\theta), u^*(\theta), \theta) + \psi(\theta)B(\theta)u^*(\theta)\} d\theta \end{aligned}$$

for every $u \in \mathcal{U}$.

The above theorem is a special case of results proven in [6].

Before discussing this further, we point out that the adjoint variables (co-states, multipliers) ψ are, for the neutral case, in general only of

bounded variation on $[t_0, t_1]$. If $A_1 \equiv 0$ (retarded systems), then ψ is absolutely continuous on $[t_0, t_1-h]$ where it satisfies the differentiated form of the "adjoint" equation in (i) above. Furthermore, in this case $\{\psi(s) + \mu(s-t_1)\}$ is easily seen to be absolutely continuous on $[t_1-h, t_1]$ where it satisfies

$$\frac{d}{ds} \{ \psi + \mu \} = -\psi^0 \frac{\partial f^{0*}}{\partial x} - \psi A_2.$$

In [6] Banks and Kent derive the above theorem (with, of course, the appropriate changes in the statement) for general neutral systems of the form

$$\frac{d}{dt} \{ x(t) - \int_{t_0-h}^t d_s v(t, s) x(s) \} = f(x(\cdot), u(t), t)$$

where $v(t, \cdot)$ is a measure depending also on the parameter t and satisfying certain technical but not very restrictive hypotheses. Here

$f(x(\cdot), u(t), t)$ denotes dependence of $f(\cdot, u(t), t)$ on any or all of the

values $x(s)$, $t_0-h \leq s \leq t$. The approach taken in the proofs evolved from

ideas involving bounded state variable techniques and utilizes the abstract multiplier rule for extremals as developed by Neustadt [32, 33] and

Gamkrelidze [16]. Comments on the similarities between the behavior (jumps, etc.) of μ and that usually associated with multipliers for bounded state

state variable problems can be found in [6]. In case $f^0 = g^0(x, t) + k^0(u, t)$ with $x \rightarrow g^0(x, t)$ convex and the system is linear, the authors show that if x^* is a trajectory corresponding to $u^* \in \mathcal{U}$ and the conditions of the above theorem can be satisfied for some α^0, ψ, μ with $\alpha^0 < 0$ (normality), then (x^*, u^*) is a solution to the problem. That is, for linear systems, convexity of g^0 and normality imply the necessary conditions are also sufficient.

The theorems obtained in the above cited paper suffer two notable deficiencies: (a) the authors offer no general class of problems for which normality can be established; (b) indeed, no proof that $(\alpha^0, \mu) \neq 0$ for some class of problems (non-triviality of the necessary conditions) is presented. However, the authors do discuss a number of solved examples which demonstrates that the class of problems for which the conditions are necessary and sufficient, and for which they offer nontrivial conditions, is non-vacuous.

A number of other examples, along with necessary conditions for somewhat more general problems involving variable endpoints in function space can be found in Kent's thesis [23; see also 24].

Jacobs and Kao [22] also consider the Lagrange problem above but for retarded systems with delays in the controls $\dot{x}(t) = f(x(t), x(t-h), u(t), u(t-\tau), t)$. Using the Lagrange Multiplier Rule [28] in $W_2^{(1)}$ for an unconstrained class of controls $\mathcal{U} = L_2([t_0 - \tau, t_1], R^r)$, they derive necessary conditions in a form analogous to $\frac{\partial H}{\partial u} = 0$, where H is the properly defined Hamiltonian function for these systems:

$$H(\psi, x, u, t) \equiv f^0(x(t), u(t), t) + \psi(t)f(x(t), x(t-h), u(t), u(t-\tau), t) \\ + \psi(t+\tau)f(x(t+\tau), x(t+\tau-h), u(t+\tau), u(t), t+\tau).$$

As can be seen with relatively straightforward arguments, the conditions obtained in [6] and [22] are essentially equivalent (with one exception) whenever the problem is such that the results of both papers are applicable. The one exception concerns normality ($\alpha^0 \neq 0$), which Jacobs and Kao obtain automatically via use of a Lagrange multiplier rule that requires a very restrictive assumption. For linear retarded ($A_1 \equiv 0$) systems (1.1) this assumption takes the form: $\text{rank } B(t) = n$ for a.e. t in $[t_1-h, t_1]$. We note that a version of Jacobs' and Kao's results could have been obtained in the absence of this restrictive assumption by use of a corollary to the multiplier rule cited above [28; p. 244]. However then normality is not obtained and the results suffer the same deficiencies as those of Banks and Kent mentioned above.

Jacobs and Kao also discuss sufficiency of their conditions under hypotheses similar to those already detailed in the preceding paragraphs.

Banks and Jacobs [7] use a geometric approach to the problem stated above. Using a class of unconstrained controls $\mathcal{U} = L_2([t_0, t_1], R^r)$ they employ attainable sets arguments (see [27]) in $W_2^{(1)}([t_1-h, t_1], R^n)$ to derive essentially the conditions in Theorem 4.1 with normality (and hence sufficiency) assured. The assumptions under which normality is guaranteed (controllability type assumptions on the system and the relatively weak hypothesis on B : $\rho(t) \equiv \text{rank } B(t)$ is constant on $[t_1-h, t_1]$) are much less restrictive than those in [22]. As is pointed out in [7], these

assumptions are weak enough to include the general n^{th} order scalar neutral equation with scalar control in the class of systems to which the results are applicable.

One difference between the form of Theorem 4.1 derived in [7] and that given above is that in [7] the multiplier μ is only L_2 , not necessarily of bounded variation, so that for the neutral systems (1.1) the multiplier ψ may also only be in $L_2([t_0, t_1], R^n)$. For retarded systems ($A_1 \equiv 0$ in (1.1)) one again does find that $\{\psi(s) + \mu(s-t_1)\}$ and $\psi(s)$ are absolutely continuous on $[t_1-h, t_1]$ and $[t_0, t_1-h]$ respectively.

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