GENERALIZED CHARACTERISTICS METHOD FOR ELASTIC WAVE PROPAGATION PROBLEMS

M. ZIV
Department of Aeronautical Engineering
Technion - Israel Institute of Technology
Haifa, Israel.

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ABSTRACT

Characterstic equations are derived in generalized curvilinear coordinates. Linear elastic, isotropic, and homogeneous constitutive equations have been used in the derivation. The generalized characteristic equations readily lend themselves to any requirements of space and dimension and geometry. A simple boundary value problem is solved to indicate the applicability of these equations.

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NOTATION

g^{mk}	contravariant components of the metric tensor
G_L	longitudinal wave speed
G_S	shear wave speed
n_k	unit vector normal to the wave front
t	time
V_k	particle velocity vector
x^k	generalized curvilinear coordinates
$\Gamma^m_{\ell k}$	Christofell symbol of the second kind
δ^k_ℓ	Kronecker delta
$\varepsilon_{\ell pd}$	covariant permutation tensor
λ, μ	Lamé's constants
ρ	density of the material
σ^k_ℓ	stress tensor
$\cdot$	above a variable - time derivative
;	covariant differentiation

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I. INTRODUCTION

The method of characteristics, when applied in the solution of wave propagation problems, involves first the determination of the characteristic equations along the generated waves. The derivation process of these equations is rather lengthy and tedious. In the past, the process for a given boundary value problem was specialized according to the relevant space dimension and the prescribed geometry (Ziv, 1969).

In this study it is proposed to present characteristic equations in a generalized curvilinear tensor form. Once these generalized characteristic equations are given, they readily lend themselves to any specific boundary value problem regardless of space dimension and geometry.

The derivation intended is a generalization of the method proposed by Ziv (1969); namely, it is based on kinematical relations across a surface of discontinuity.

The constitutive equations used are of the linear elastic, isotropic, and homogeneous type.

A simple boundary value problem is solved in order to demonstrate the applicability of the generalized characteristic equations to a given problem. This example consists of a uniform oblique load suddenly applied on an infinite flat plate.

II. GENERALIZED CHARACTERISTIC EQUATIONS

Characteristic equations are derived with respect to generalized curvilinear coordinates. The analysis is based on the method given by Ziv (1969). Dynamical field equations are presented first.

The linear equations of motion are

$$\sigma^{k}{}_{\ell;k} = \rho \frac{\partial v_\ell}{\partial t} \quad . \tag{1}$$

The constitutive equations for linear, isotropic, and homogeneous elastic material are

$$\frac{\partial \sigma^{k}{}_{\ell}}{\partial t} = \mu(v^{k}{}_{;\ell} + g^{mk} v_{\ell;m}) + \lambda v^{m}{}_{;m} \delta^{k}{}_{\ell} \quad . \tag{2}$$

The above equations represent first-order partial differential equations of the hyperbolic type which govern the dynamical deformation in the medium.

To induce a characteristic surface for the dependent variables, their total differentials are introduced as follows:

$$\dot{v}_\ell = v_{\ell;k} \dot{x}^k + \frac{\partial v_\ell}{\partial t} \tag{3}$$

$$\dot{\sigma}^{k}{}_{\ell} = \sigma^{k}{}_{\ell;m} \dot{x}^m + \frac{\partial \sigma^{k}{}_{\ell}}{\partial t} \quad . \tag{4}$$

Equations (1), (2), (3) and (4) are combined and become

$$\square \sigma^{k}{}_{\ell;k} \square + \rho \dot{x}^{k} \square v_{\ell;k} \square = \rho \dot{v}_\ell \tag{5}$$

$$\dot{x}^m \square \sigma^k{}_{\ell;m} \square + \mu(\square v^k{}_{;\ell} \square + g^{mk} \square v_{\ell;m} \square) + \lambda \square v^m{}_{;m} \square \delta^k{}_\ell = \dot{\sigma}^k{}_\ell \tag{6}$$

where the symbol $\square\ \square$ is introduced to imply that the first partial derivatives are intederminate if Eqs. (5) and (6) are transformed along the wave surfaces. It follows that to obtain the characteristic equations, Eqs. (5) and (6) must be solved for the indeterminate quantities included in $\square\ \square$. Physically, these derivatives can be considered the derivatives on the rear of the wave surface and are created as the wave surface is being formed due to an excitation.

In view of this description, Hadamard's (1949) kinematical relations across a surface of discontinuity are modified and interpreted as follows:

$$\square v_{\ell;k} \square = [\frac{\partial v_\ell}{\partial n}] n_k + v_{\ell;k} \tag{7}$$

$$\square \sigma^k{}_{\ell;m} \square = [\frac{\partial \sigma^k{}_\ell}{\partial n}] n_m + \sigma^k{}_{\ell;m} \tag{8}$$

where the square brackets represent the discontinuity of the first partial derivatives of continuous variables. The quantity included in these square brackets is defined to be equal to its value on the rear of the wave surface minus its value on the front of this surface. The derivatives on the front of the wave, which are the last terms of the right-hand side of Eqs. (7) and (8), are known from prescribed conditions. The n is the distance measured along the normal to the wave surface. The n_k is the unit vector normal to the wave surface.

The discontinuity relations (7) and (8) are substituted in Eqs. (5) and (6) to yield the following:

$$n_k\left[\frac{\partial\sigma^k{}_\ell}{\partial n}\right] + \rho G\left[\frac{\partial V_\ell}{\partial n}\right] = R_\ell \qquad (9)$$

$$G\left[\frac{\partial\sigma^k{}_\ell}{\partial n}\right] + \mu\left(n_\ell\left[\frac{\partial V^k}{\partial n}\right] + n^k\left[\frac{\partial V_\ell}{\partial n}\right]\right) + $$
$$+ \lambda\delta^k{}_\ell n_m\left[\frac{\partial V^m}{\partial n}\right] = R^k{}_\ell \qquad (10)$$

$$G = \dot{x}^\ell n^\ell \qquad \text{or} \qquad \dot{x}^\ell = Gn^\ell \qquad (11)$$

$$R_\ell = \rho(\dot{V}_\ell - Gn^k V_{\ell;k}) - \sigma^k{}_{\ell;k} \qquad (12)$$

$$R^k{}_\ell = \dot{\sigma}^k{}_\ell - Gn^m\sigma^k{}_{\ell;m} - \mu(V^k{}_{;\ell} + g^{mk}V_{\ell;m}) - \lambda V^m{}_{;m}\delta^k_\ell . \qquad (13)$$

Equations (9) and (10) are denoted as the dynamical conditions, since it is from these relations that the wave surfaces and the characteristic equations will be derived.

To obtain the characteristic equations, Eqs. (9) and (10) are solved simultaneously for the discontinuous first partial derivatives in the following way:

Equation (9) is multiplied by Gn^ℓ, i.e.,

$$Gn_k n^\ell\left[\frac{\partial\sigma^k{}_\ell}{\partial n}\right] + \rho G^2 n^\ell\left[\frac{\partial V_\ell}{\partial n}\right] = Gn^\ell R_\ell \qquad (14)$$

Equation (10) is multiplied by $n^\ell n_k$

$$Gn_k n^\ell [\frac{\partial \sigma^k_\ell}{\partial n}] + \mu (n_k [\frac{\partial V^k}{\partial n}] + n^\ell [\frac{\partial V_\ell}{\partial n}]) + \lambda \delta^k_\ell n_m n^\ell n_k [\frac{\partial V^m}{\partial n}] = n^\ell n_k R^k_\ell .$$

This Equation can be written as follows:

$$Gn_k n^\ell [\frac{\partial \sigma^k_\ell}{\partial n}] + (\lambda + 2\mu) n^\ell [\frac{\partial V_\ell}{\partial n}] = R^k_\ell n^\ell n_k \tag{15}$$

Simultaneous solution of Eqs. (14) and (15) gives

$$(\lambda + 2\mu - \rho G^2) [\frac{\partial V_\ell}{\partial n}] n^\ell = (R^k_\ell n_k - GR_\ell) n^\ell$$

or

$$[\frac{\partial V_\ell}{\partial n}] n^\ell = \frac{(R^k_\ell n_k - GR_\ell) n^\ell}{\lambda + 2\mu - \rho G^2} . \tag{16}$$

Equation (9) is multiplied by $G\varepsilon_{\ell pd} n^p$ as follows:

$$G\varepsilon_{\ell pd} n^p n_k [\frac{\partial \sigma^k_\ell}{\partial n}] + \rho G^2 \varepsilon_{\ell pd} n^p [\frac{\partial V_\ell}{\partial n}] = GR_\ell \varepsilon_{\ell pd} n^p \tag{17}$$

and Eq. (10) is multiplied by $\varepsilon_{\ell pd} n^p n_k$ as follows:

$$G\varepsilon_{\ell pd} n^p n_k [\frac{\partial \sigma^k_\ell}{\partial n}] + \mu \varepsilon_{\ell pd} n^p n_k n^k [\frac{\partial V_\ell}{\partial n}]$$

$$+ \mu \varepsilon_{\ell pd} n_\ell n^p n_k [\frac{\partial V^k}{\partial n}] + \lambda \varepsilon_{\ell pd} n^p n_m \delta^k_\ell n_k [\frac{\partial V^m}{\partial n}] =$$

$$= R^k_\ell \varepsilon_{\ell pd} n^p n_k$$

The last equation becomes:

$$G\varepsilon_{\ell pd} n^p n_k [\frac{\partial \sigma^k_\ell}{\partial n}] + \mu \varepsilon_{\ell pd} n^p [\frac{\partial V_\ell}{\partial n}] = R^k_\ell \varepsilon_{\ell pd} n^p n_k \tag{18}$$

Equations (17) and (18) will yield the following

$$[\frac{\partial v_\ell}{\partial n}]\varepsilon_{\ell pm}n^p = \frac{(R^k_\ell n_k - GR_\ell)\varepsilon_{\ell pm}n^p}{\mu - \rho G^2} \tag{19}$$

Eq. (10) is multiplied by $n_k n^\ell$

$$Gn_k n^\ell[\frac{\partial\sigma^k_\ell}{\partial n}] + (\lambda + 2\mu)n_\ell[\frac{\partial v_\ell}{\partial n}] = R^k_\ell n_k n^\ell \tag{20}$$

and the same Eq. (10) is now multiplied by δ^ℓ_k to give

$$G\delta^\ell_k[\frac{\partial\sigma^k_\ell}{\partial n}] + (\lambda\delta^m_m + 2\mu)n_\ell\ [\frac{\partial v_\ell}{\partial n}] = R^k_\ell\delta^\ell_k \tag{21}$$

Simultaneous solution of Eqs. (20) and (21) gives

$$[\frac{\partial\sigma^k_\ell}{\partial n}](\delta^\ell_k - \frac{(\lambda\delta^m_m + 2\mu)}{\lambda + 2\mu}n_k n^\ell) = \frac{(\delta^\ell_k - \frac{(\lambda\delta^m_m + 2\mu)}{\lambda + 2\mu}n_k n^\ell)R^k_\ell}{G} \tag{22}$$

Finally, Eq. (9) is multiplied by $n_k n^\ell$

$$[\frac{\partial\sigma_{k\ell}}{\partial n}]n^\ell + \rho G[\frac{\partial v_\ell}{\partial n}]n_k n^\ell = R_\ell n_k n^\ell \tag{23}$$

and the same Eq. (9) is now multiplied by δ^ℓ_k to give

$$[\frac{\partial\sigma_{k\ell}}{\partial n}]n^\ell + \rho G[\frac{\partial v_\ell}{\partial n}]\delta^\ell_k = R_\ell\delta^\ell_k \tag{24}$$

Simultaneous solution of Eqs. (23) and (24) gives

$$[\frac{\partial v_\ell}{\partial n}](n_k n^\ell - \delta^\ell_k) = \frac{(n_k n^\ell - \delta^\ell_k)R_\ell}{\rho G} \tag{25}$$

Since it is necessary that the derivatives be discontinuous, the denominators of Eqs. (16), (19), (22) and (25) are made to disappear. This implies, in view of Eqs. (16), (19), (22) and (25) that $G^2 = \frac{\lambda + 2\mu}{\rho}$, $G^2 = \frac{\mu}{\rho}$ and $G = 0$, respectively. Furthermore, a necessary condition for finite solutions to exist is that the numerators of Eqs. (16), (19), (22) and (25) must vanish. Thus, the dot product form of the left-hand side of Eq. (16) which is equal to $\frac{0}{0}$ implies that the vector $\vec{V}$ is parallel to vector $\vec{n}$ or that the material particle at the wave front moves parallel to the propagating wave. Therefore $G^2 = \frac{\lambda + 2\mu}{\rho}$ represents the squared celerity of longitudinal waves and will be denoted as $G_L^2 = \frac{\lambda + 2\mu}{\rho}$. Similarly, the cross product of the left-hand side of Eq. (19) implies that $G^2 = \frac{\mu}{\rho}$ represents the speed of shear waves and, therefore, is denoted as $G_S^2 = \frac{\mu}{\rho}$. $G = 0$ represents an equilibrium condition. The results of these operations are the desired generalized characteristic equations which are presented below:

$$n^\ell (n^k \dot{\sigma}_{k\ell} - \rho G_L \dot{V}_\ell) = \lambda (V^m_{;m} - n^k n_\ell V^\ell_{;k}) - G_L n_\ell (\sigma^{\ell k}_{;k} - n^m n_k \sigma^{\ell k}_{;m}) \tag{26}$$

$$\varepsilon_{\ell pd} n^p \{ n^k \dot{\sigma}_{k\ell} - \rho G_S \dot{V}_\ell - n_k \mu V^k_{;\ell} + G_S (\sigma^k_{\ell;k} - n^m n_k \sigma^k_{\ell;m}) \} = 0 \tag{27}$$

$$(n_k n^\ell - \delta^\ell_k)(\rho \dot{V}_\ell - \sigma^m_{\ell;m}) = 0 \tag{28}$$

$$(\delta^\ell_k - \frac{\lambda \delta^p_p + 2\mu}{\lambda + 2\mu} n_k n^\ell)\{\dot{\sigma}^k_\ell - \mu (V^k_{;\ell} + g^{mk} V_{\ell;m}) - \lambda V^m_{;m} \delta^k_\ell\} = 0 \tag{29}$$

where

$$G_L^2 = \frac{\lambda + 2\mu}{\rho}$$

$$G_S^2 = \frac{\mu}{\rho}$$

and from Eq. (11) it follows that the bicharacteristic curves, i.e., the wave fronts on which the characteristic equations hold, are

$\dot{x}^\ell = G_L n^\ell, \dot{x}^\ell = G_S n^\ell$ and $\dot{x}^\ell = 0$ for Eqs. (26), (27), (28) and (29) respectively. A pictorial representation of these bicharacteristics for orthogonal coordinate systems is presented in Fig. 1.

Next, for demonstration purposes, a simple boundary problem is solved as an indication of the applicability of the characteristic equations.

III. UNIFORM PLANE OBLIQUE LOAD SUDDENLY APPLIED ON AN INFINITE FLAT PLATE

A detailed solution to a problem where impulsive loads exist, is solved here. A bounded half-space is impacted by a step normal (V) and tangential (W) velocities, as described in Figs. 2 and 3 and Eq. (30). This combined step load is uniformly distributed over the surface z = 0.

$$\bar{U},\bar{W} = \begin{cases} 1 \text{ when } \tau \geq 0 \\ 0 \text{ when } \tau < 0 \end{cases} \tag{30}$$

where $\bar{U},\bar{W}$, and τ are dimensionless quantities in the following way:

$$\bar{U} = \frac{U}{G_L}, \quad \bar{W} = \frac{W}{G_L}, \quad \tau = \frac{G_L t}{\text{Length}}$$

also

$$\bar{x} = \frac{x}{\text{Length}}, \quad \bar{z} = \frac{z}{\text{Length}}, \quad \bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\lambda + 2\mu} \tag{31}$$

$$\bar{\sigma}_{zx} = \frac{\sigma_{xz}}{\lambda + 2\mu}, \quad \bar{\sigma}_{zz} = \frac{\sigma_{zz}}{\lambda + 2\mu}, \quad \bar{\sigma}_{yy} = \frac{\sigma_{yy}}{\lambda + 2\mu} .$$

This problem may simulate, for example; an impact landing of an infinite flat plate on a rigid target at rest (Fig. 4a). For the convenience of having zero initial conditions, an oblique velocity $\vec{V}$ (such that $\vec{V} = \vec{W} + \vec{U}$) is added to the whole system (Fig. 4b). This implies that the motion is relative to the initial steady motion of the plate.

It follows that at $\tau = 0$ the rigid target is suddenly impacting on the plate and at this instant ($\tau = 0$) the plate and the target are at rest. Since the given actual velocity of the plate is constant, the added constant velocity will not affect the emanating stress waves.

The following analysis is done by using the Lagrangian view point: an observer is assigned to the origin of the coordinate system at the impact face of plate (z = 0). The observer fixes his attention on a specific particle in the plate at time equal to zero, when the plate is still at rest and therefore unstrained. This implies that the stresses will be measured from the initial unstrained condition.

The initial conditions are then

$$\left.\begin{aligned} \bar{V} &= 0 \\ \bar{\sigma}_{ij} &= 0 \end{aligned}\right\} \quad \text{at} \quad \tau = 0 \quad \text{and} \quad 0 \leqq \bar{z} \leqq 1$$

and the boundary conditions are

$$\left.\begin{aligned} \bar{V} &= \sqrt{2} \text{ at } \bar{z} = 0 \\ \bar{\sigma}_{ij} &= 0 \text{ at } \bar{z} = 1 \end{aligned}\right\} \quad \text{for} \quad \tau > 0.$$

Although this is a one-spatial dimensional problem, it may be considered as a step towards a strictly two-spatial dimensional problem in the sense of having two kinds of waves and five unknown dependent variables. It is seen that the five unknown dependent variables are σ_{xx}, σ_{zz}, σ_{xz}, U, and W which are functions of z only. Also, it can be readily seen that a shear wave and a longitudinal wave will be generated here. In view of these observations the characteristic Eqs. (26), (27), (28) and (29) will be immediately reduced to the following:

$$d\bar{\sigma}_{zz} - d\bar{W} = 0 \text{ along } \frac{d\bar{z}}{d\tau} = 1$$

$$d\bar{\sigma}_{zz} + d\bar{W} = 0 \text{ along } \frac{d\bar{z}}{d\tau} = -1$$

$$d\bar{\sigma}_{zx} - \frac{G_S}{G_L} d\bar{U} = 0 \text{ along } \frac{d\bar{z}}{d\tau} = \frac{G_S}{G_L}$$

$$d\bar{\sigma}_{zx} + \frac{G_S}{G} d\bar{U} = 0 \text{ along } \frac{d\bar{z}}{d\tau} = -\frac{G_S}{G_L}$$

$$\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = \frac{\nu}{1 - \nu} \bar{\sigma}_{zz}$$

ν is Poisson's ratio which was chosen for this example to be $\nu = 0.25$. Hence, the above equations become, after bars have been dropped,

$$d\sigma_{zz} - dW = 0 \qquad \text{along } \frac{dz}{d\tau} = 1 \tag{32}$$

$$d\sigma_{zz} + dW = 0 \qquad \text{along } \frac{dz}{d\tau} = -1 \tag{33}$$

$$d\sigma_{zx} - \frac{1}{\sqrt{3}} dU = 0 \text{ along } \frac{dz}{d\tau} = \frac{1}{\sqrt{3}} \tag{34}$$

$$d\sigma_{zx} + \frac{1}{\sqrt{3}} dU = 0 \text{ along } \frac{dz}{d\tau} = -\frac{1}{\sqrt{3}} \tag{35}$$

$$\sigma_{xx} = \sigma_{yy} = \frac{1}{3}\sigma_{zz}$$

$$\sigma_{xy} = \sigma_{zy} = 0$$

To evaluate numerically the dependent variables in the solution domain, it is first necessary to compute these variables along the leading wave front. Because of the abrupt load, discontinuities may occur in the dependent variables along the leading wave front. Therefore, the characteristic equation (32), which holds along the first bicharacteristic curve $z = \tau$, has to be reformulated to accommodate the discontinuities in the variables themselves. Thus, in view of Hadamard's (1949) definition of discontinuity, Eq. (32), and Fig. 5, the following relation exists for point P on the leading wave front

$$d[\sigma_{zz}] - d[W] = 0 \tag{36}$$

Equation (36) is now integrated as the bicharacteristic curve ($z - \tau$ = constant) approaches the leading front $z - \tau = 0$, i.e.,

$$[\sigma_{zz}] - [W] = K \tag{37}$$

where K is the constant of integration.

Since the material is at rest in front of the leading wave

$$(\sigma_{zz_p})_{front} = (W_p)_{front} = 0$$

and Eq. (35) is rewritten as follows:

$$(\sigma_{zz_p})_{rear} - (W_p)_{rear} = K \tag{38}$$

On the other hand, the characteristic Eq. (33) which holds along the bicharacteristic curve $z = -\tau$ + constant may be used as it stands for point P. This can be done since there are no abrupt changes across $z = -\tau$ + constant. Thus, the following relation exists:

$$d\sigma_{zz} + dW = 0 \quad \text{along} \quad z = -\tau + \text{constant}$$

or by the finite difference method

$$(\sigma_{zz_p})_{rear} - (\sigma_{zz_p})_{front} + (W_p)_{rear} - (W_p)_{front} = 0 \tag{39}$$

Again, since $(\sigma_{zz_p})_{front} = (W_p)_{front} = 0$, Eq. (22) is written as follows:

$$(\sigma_{zz_p})_{rear} = -(W_p)_{rear} \qquad (40)$$

It is worthwhile to point out that this last result is simply the Rankine-Hugoniot relation.

In view of the Eqs. (38) and (40) it is concluded that $(\sigma_{zz_p})_{rear}$ and $(W_p)_{rear}$ remain constant along the leading wave front. Now, since the applied load at the origin $(z = 0, \tau > 0)$ is $W = 1$, it follows from Eq. (40) that $\sigma_{zz} = -1$ at the origin. It then follows immediately that σ_{zz} and W are equal to -1 and 1 respectively along the leading wave front $(\sigma_{zz_p} = -1$ and $W_p = 1)$ until the wave front reaches the boundary $z = 1$ (Fig. 6). The above analysis is then similarly repeated for the reflected wave. Equations (34) and (35) are treated similarly. Figure 6 represents the stress velocity distribution obtained in the xz-plane.

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REFERENCES

1. Hadamard, J., 1949, Leçons sur la Propagation des Ondes et les Equations de l'Hydrodynamique, Chelsea Publishing Co.
2. Ziv, M., 1969, Two-Spatial Dimensional Elastic Wave Propagation by the Theory of Characteristics, Int. J. Solids Struct., 5, 1135.

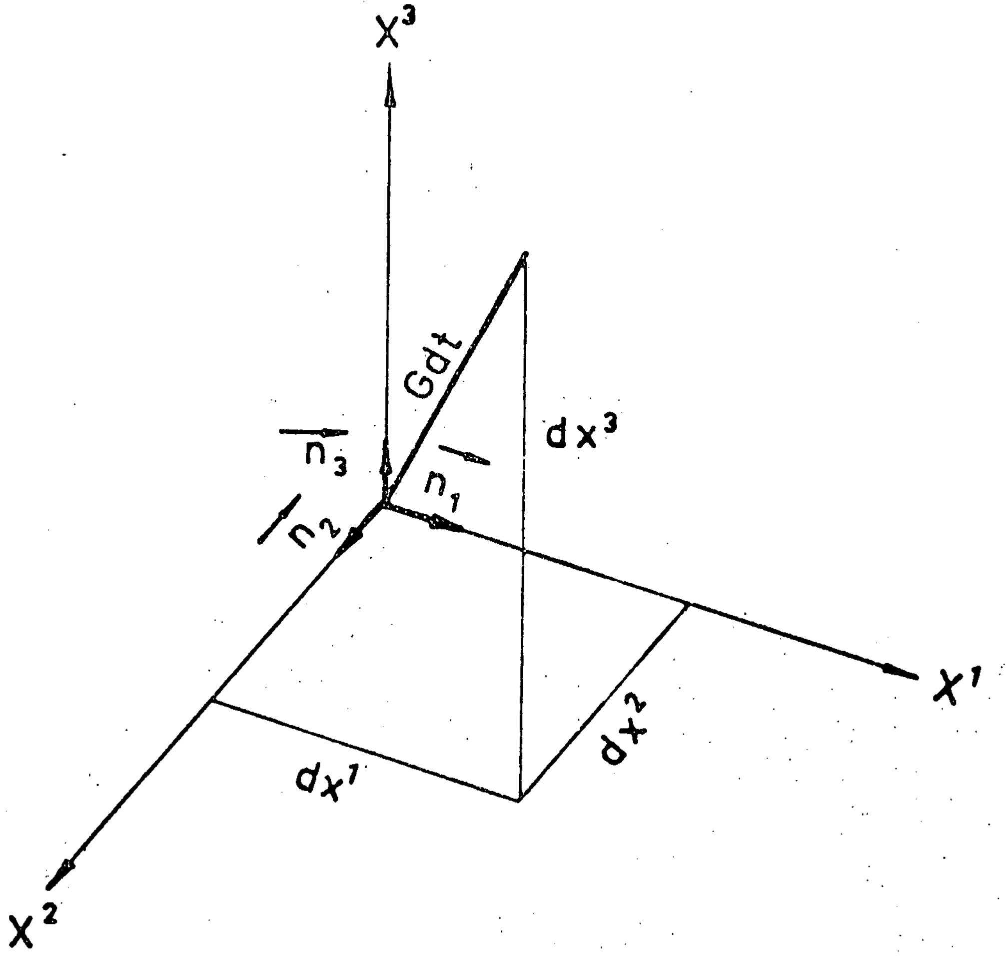

Fig. 1. Representation of orthogonal bicharacteristic curves.

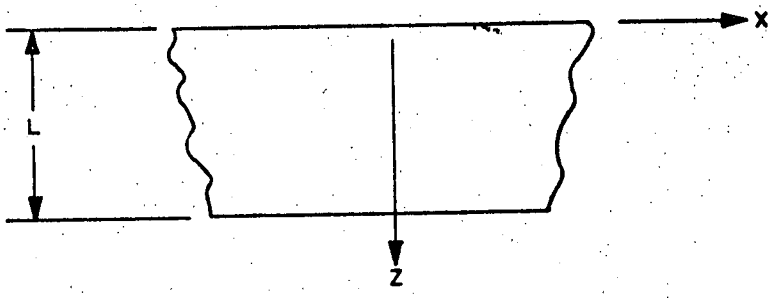

Figure 2. A bounded half-space.

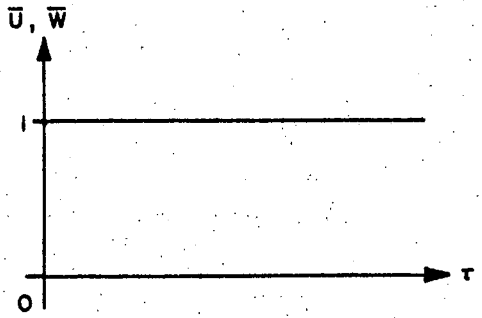

Figure 3. Step normal and tangential velocities.

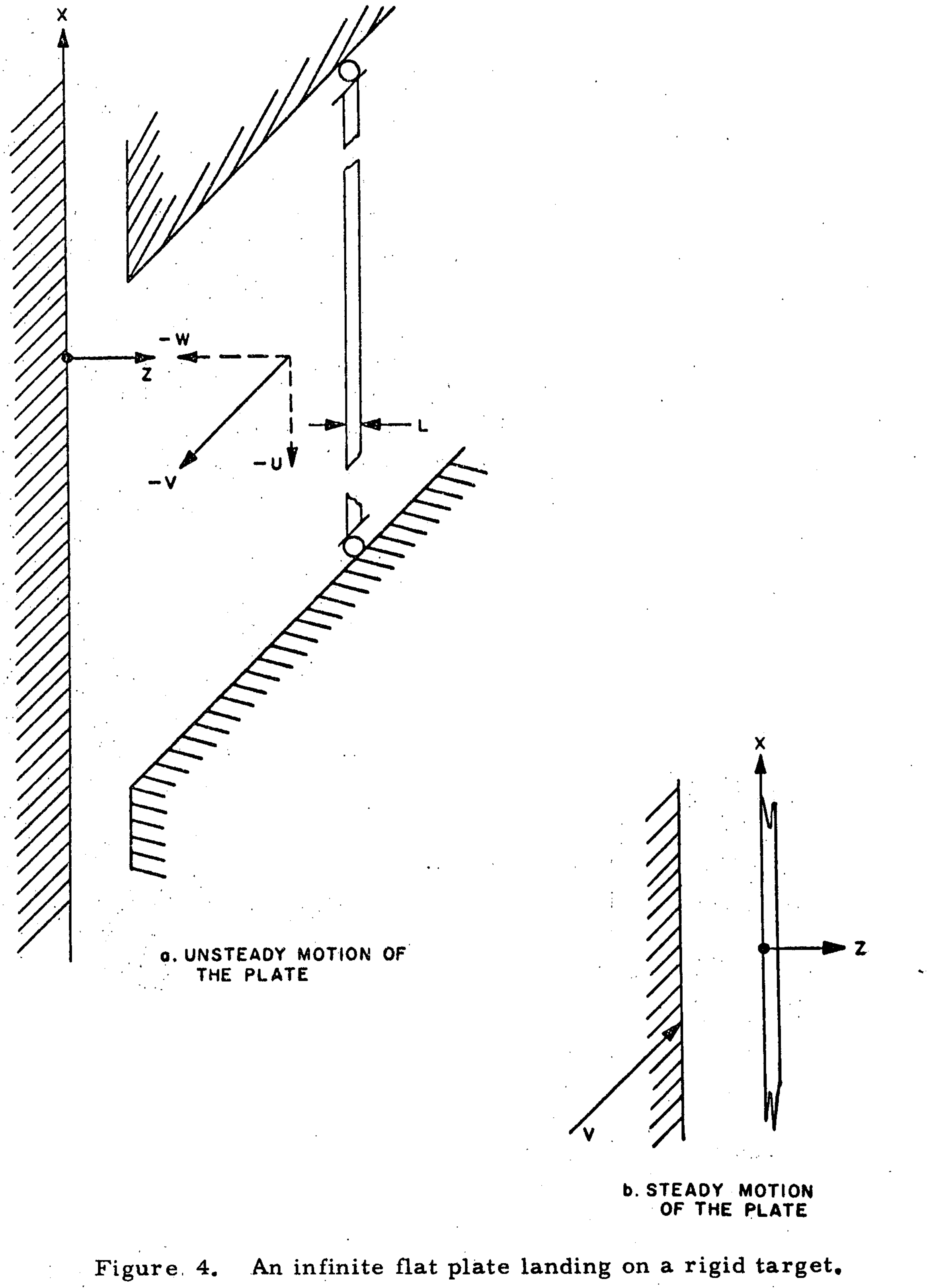

Figure. 4. An infinite flat plate landing on a rigid target.

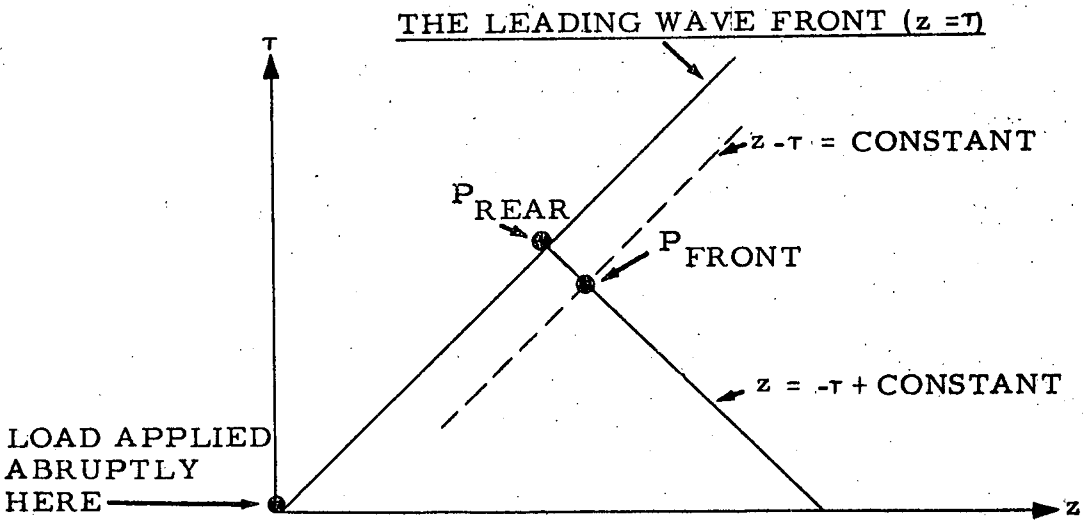

Figure 5. Discontinuity in the dependent variables across the leading wave front

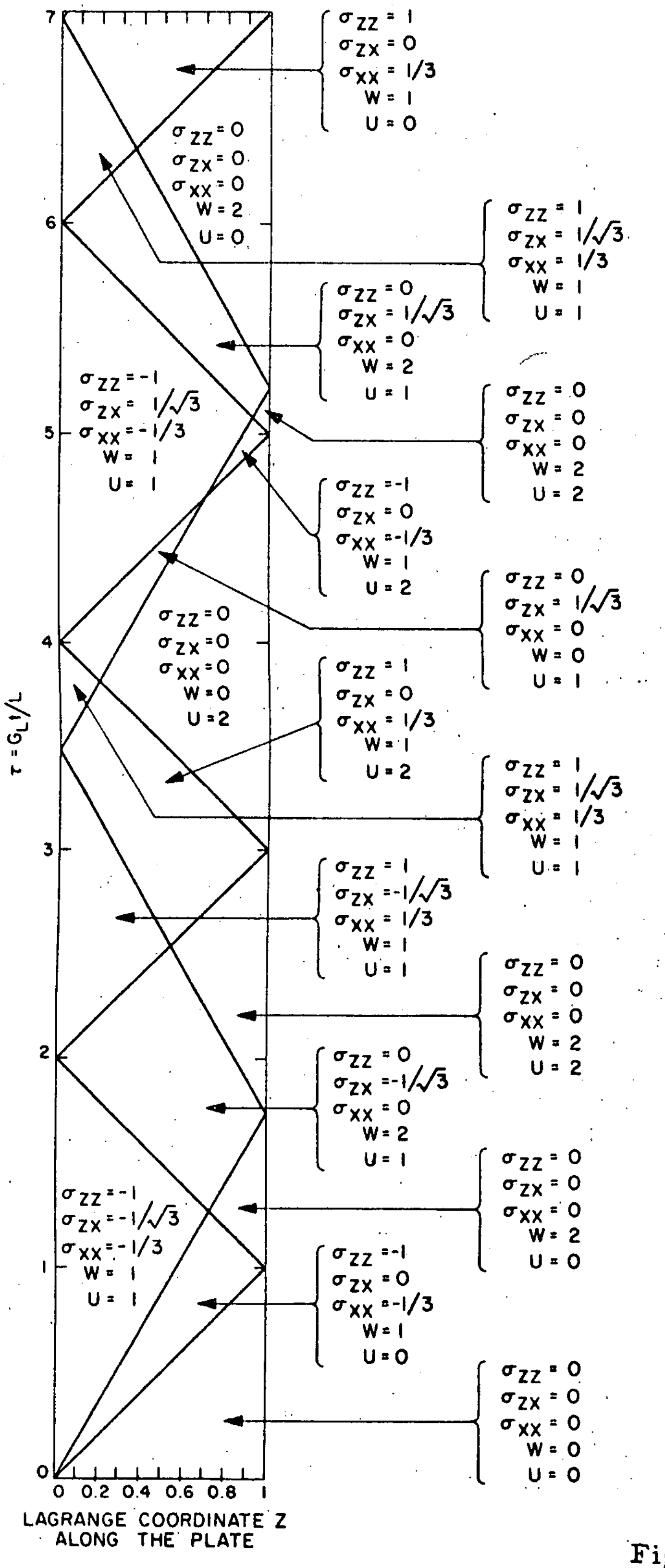

Figure 6. Bicharacteristic grid