Abstract

This note gives a short and elementary proof of MacLane's theorem on the embedding of graphs in a 2-sphere.

(NASA-CR-127463) A SHORT PROOF OF MACLANE'S PLANARITY THEOREM P.V. ONeil (College of William and Mary) [1970] 4 p CSCL 12A

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The purpose of this note is to give a short and elementary proof of a theorem by Saunders MacLane on the embedding of graphs in the 2-sphere [4]. Existing proofs are the original ones of MacLane and an algebraic topology proof by Lefschetz [3]. Our proof is by Kuratowski's theorem [2].

Terminology follows [4] and [1], with the exception that we shall call MacLane's 2-fold complete set of circuits a P-base.

Let G be a nonseparable graph.

Theorem. If G has a P-base, then G is planar.

Proof. Let $C_1,\ldots,C_n$ form a P-base for G, and suppose that G is nonplanar. Then $n > 1$ and, by Kuratowski's theorem, G has a subgraph H homeomorphic to K_5 or to $K_{3,3}$. We claim that H also has a P-base. This is immediate by induction if it is first shown that G − e has a P-base for each arc e of G. But, if e is in exactly one C_i, say C_1, then $C_2,\ldots,C_n$ form a P-base for G − e, and if e is in two C_i's, say C_1 and C_2, then $C_3,\ldots,C_n$, $C_{n+1} = \sum_{i=1}^{n} C_i$ form a P-base for G − e.

Thus H, hence also K_5 or $K_{3,3}$, has a P-base. We now show that this is impossible.

If $C_1,\ldots,C_6$ form a P-base for K_5, then each of the ten branches of K_5 is in exactly two of the circuits $C_1,\ldots,C_6$, $C_7 = \sum_{i=1}^{6} C_i$. But each circuit has at least three branches, so

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$$\sum_{i=1}^{7} (\text{number of branches in } C_i) = 20 \not\geq 21.$$

Similarly, if $C_1,\ldots,C_4$ form a P-base for $K_{3,3}$, then set $C_5 = \sum_{i=1}^{4} C_i$. Since each circuit in $K_{3,3}$ has at least four arcs, then

$$\sum_{i=1}^{5} (\text{number of branches in } C_i) = 18 \not\geq 20,$$

completing the proof.

The converse of the theorem is of course also true, but the proof of this is trivial.

References

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