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Department of Electrical Engineering

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This report is based on the unaltered thesis of Raymond Hon-Sing Kwong submitted in partial fulfillment of the requirements for the degree of Master of Science at the Massachusetts Institute of Technology in May, 1972. This research was conducted at the Decision and Control Sciences Group of the M.I.T. Electronic Systems Laboratory, with support extended by NASA/AMES under grant NASA-NGL-22-009-124 and by the Air Force under grants AFOSR-70-1941 and AFOSR-72-2273.

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ABSTRACT

The stochastic control of two weakly coupled linear systems with different controllers is considered. Each controller only makes measurements about his own system; no information about the other system is assumed to be available. Based on the noisy measurements, the controllers are to generate independently suitable control policies which minimize a quadratic cost functional.

To account for the effects of weak coupling directly, an approximate model, which involves replacing the influence of one system on the other by a white noise process is proposed. Simple suboptimal control problem for calculating the covariances of these noises is solved using the matrix minimum principle. The overall system performance based on this scheme is analysed as a function of the degree of intersystem coupling. The results are compared to those obtained using complete centralization and the Separation Theorem. Tradeoffs between the various approaches are discussed.

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CHAPTER 1 - INTRODUCTION

The development of modern control theory has been characterized by its numerous ramifications into various areas of interest. On the theoretical side, researchers have drawn on such diverse mathematical techniques as abstract algebra, calculus of variations, partial differential equations, functional analysis, etc. Control theory has been cast into such a sophisticated mathematical framework that it not only interests the engineers, but has attracted the attention of scientists and mathematicians as well. On the practical side, control theory has found applications in the guidance and navigation of space vehicles, industrial chemical process control, air traffic control, etc. In these various areas where the study of control theory is relevant, there is one particular area which is of great theoretical as well as practical interest, namely, the study of large scale systems. What is the best way of decision making for a firm and its branches? How should an automated ground transportation system be set up? How should the government coordinate the various sectors of the economy? These problems had not been investigated to any appreciable degree in the past owing to their great complexity. With the advent of modern computer technology, the powerful computational facilities made available render the systematic study of large scale systems possible. Various approaches to this problem have been suggested [1], [2], although few concrete results have been obtained.

A large system can be viewed as a single huge object and its overall behavior analyzed as a unit. Very often, however, it is divided into subsystems, each with its own controller. In this situation, the amount of interaction between the subsystems plays a crucial role in determining the system performance. If the subsystems are strongly coupled, the division into subsystems will not help us much in the analysis. We still have to take everything fully into account. However, if the coupling between the subsystems is weak, intuitively at least, it seems that some simplification may be possible. It is to the latter situation that this thesis will address itself.

There are many examples of weakly coupled systems with different controllers. Consider the various branches of a firm. Normally, each of these branches will operate almost in an autonomous manner. The manager of each branch will make his own decision without too much knowledge or concern about how the others are operating. Thus, even though they belong to the same large system, their interaction is, in general, very weak. Other typical examples can be found in process control applications, transportation systems, and economic systems.

We can therefore pose the following general problem for weakly coupled systems with different controllers: given a set of dynamical systems, each of which interacts only slightly with the rest, what is the best set of controls that the controllers should choose in order to minimize some cost criterion?

Both in the deterministic and stochastic case, if all the controllers are organized under a central agency (the so-called completely centralized situation) who collects all measurements and determines all controls, we will have a mathematically optimal design. However, as has been pointed out by Athans et al. for the deterministic case [3], this will require a large and expensive communication system, which for technical and/or economic reasons may not be desirable. Furthermore, it may be very inefficient for the central agency to make all the decisions and then transmit them to the various controllers. These considerations prompted Athans et al. [3] to suggest a suboptimal control scheme which resulted in some form of decentralized control.

In the stochastic case, the centralized optimal control for the linear-quadratic-Gaussian problem is given by the Separation Theorem[4]. When a central agency is absent and the information patterns of the controllers are different, Chong [5] has shown that, in general, the Separation Theorem does not hold. At the present time, there is no general theory that one can appeal to when the policy of complete centralization is not desired. In this situation, a variety of approaches have been used to tackle our general problem. For example, using the theory of nonzero sum differential games [6], [7], or that of team theory [8]. However, many of these approaches suffer from the fact that although a certain degree of decentralization is obtained, the effects of weak coupling are not brought into focus and utilized in the system design. Intuitively, a very reasonable approach to the problem is to try to account for the weak coupling directly.

In this thesis, the stochastic control of two weakly coupled stochastic linear systems with quadratic cost is considered. Each system has its own controller with different available information. A model for calculating the controls in which the weak coupling is approximated by "fake" white noise processes, is proposed. This model is essentially due to Chong [5]. The intensities of the white noise processes are viewed as pseudo-control variables (the reasons for adopting this terminology will be given in Chapter 2) whose determination generates the real physical controls. A matrix optimal control problem is then solved and the complete suboptimal control scheme specified.

The structure of the remainder of the thesis is as follows: In Chapter 2, the precise problem under consideration is stated. We define the class of admissible physical controls and, after a discussion of the problem, propose a model for calculating the physical controls. A justification for the model is given. We then define pseudo-controls to be the covariances of the "fake" white noises introduced in the proposed model and the considerable simplification obtained is discussed. For fixed but arbitrary pseudo-controls, the optimal physical controls are obtained. The problem now becomes one in the determination of the pseudo-controls.

In Chapter 3 a reformulation of the problem using the model proposed in Chapter 2 is stated. A deterministic matrix optimal control problem with the pseudo-control variables defined in Chapter 2 viewed as controls is stated. The interpretation of this problem in relation to the original problem is given. The unconventional constraints on the pseudo-controls are discussed. Auxiliary variables are then defined to remove these constraints.

The solution of the matrix optimal control problem is given in Chapter 4. The optimal pseudo-controls are shown to be a function of the degree of intersystem coupling. The complete scheme for generating the physical controls for the original problem is discussed. Computational aspects of the solution are considered. The asymptotic case in which the coupling goes to zero is analyzed. The optimal pseudo-controls are then shown to be zero, which is what we should expect for originally uncoupled systems.

In Chapter 5,the performance of the system with controls calculated using the proposed scheme is investigated. The results obtained are compared to the mathematically optimal policy of complete centralization and an intuitive interpretation is given. The advantagesin using the proposed scheme over that of complete centralization are stated. The trade-offs involved between the choice of policies are discussed.

Chapter 6 summarizes the results given in the preceding chapter. Topics for future research are also suggested.

The main contribution of this thesis lies in the consideration of an approximate model which considerably simplifies the analysis of the stochastic control of weakly coupled systems. The results show that this simple suboptimal scheme will be very important when trade-offs between various control policies are studied. The philosophy of using white noise intensities as pseudo-control variables, and the general method of solution to the problem also indicate an approach to many stochastic problems in which white noise processes are used as approximations. This should find applications to various filtering and control problems.

CHAPTER 2 - THE STOCHASTIC CONTROL PROBLEM

In this chapter we first give a precise statement of the stochastic optimal control problem for two weakly coupled linear systems with different controllers. We then discuss the implications of weak coupling and show how an approximate model can be used to considerably simplify the system structure. We also give some intuitive arguments which justify the use of the approximate model. We define the covariance matrices of the "fake" white noises introduced in the proposed model to be the pseudo-control variables and show how the actual physical controls can be generated using the pseudo-controls. How the pseudo-controls are to be determined will be the subject of the following chapters.

2.1 Statement of the Stochastic Optimal Control Problem

Consider the following weakly coupled linear systems (see Fig. 1):

$$\dot{\underline{x}}_1(t) = \underline{A}_{11}(t)\underline{x}_1(t)+\underline{B}_{11}(t)\underline{u}_1(t)+\varepsilon\underline{A}_{12}(t)\underline{x}_2(t)+\underline{\lambda}_1(t)$$

$$\underline{x}_1(t_0) = \underline{x}_{10} \qquad (2.1.1)$$

$$\dot{\underline{x}}_2(t) = \underline{A}_{22}(t)\underline{x}_2(t)+\underline{B}_{22}(t)\underline{u}_2(t)+\varepsilon\underline{A}_{21}(t)\underline{x}_1(t)+\underline{\lambda}_2(t)$$

$$\underline{x}_2(t_0) = \underline{x}_{20} \qquad (2.1.2)$$

where $\underline{x}_1(t) \in R^{n_1}$ is the state of system 1,

$\underline{x}_2(t) \in R^{n_2}$ is the state of system 2,

$\underline{u}_1(t) \in R^{p_1}$ is the control for system 1,

$\underline{u}_2(t) \in R^{p_2}$ is the control for system 2,

$\underline{A}_{11}(t)$ is an $n_1 x n_1$ matrix

$\underline{A}_{12}(t)$ is an $n_1 x n_2$ matrix

$\underline{A}_{22}(t)$ is an $n_2 x n_2$ matrix

$\underline{A}_{21}(t)$ is an $n_2 x n_1$ matrix

$\underline{B}_{11}(t)$ is an $n_1 x p_1$ matrix

$\underline{B}_{22}(t)$ is an $n_2 x p_2$ matrix

and ε, which is assumed to be much smaller in magnitude than any other quantity associated with the systems, is a scalar coupling parameter.

We assume that $\underline{\lambda}_1(t)$ and $\underline{\lambda}_2(t)$ are zero mean, mutually independent Gaussian white noise processes driving the systems, with

$$\text{cov}\ \{\underline{\lambda}_1(t);\ \underline{\lambda}_1(\tau)\} = \underline{\Lambda}_1(t)\delta(t-\tau)$$
$$\text{cov}\ \{\underline{\lambda}_2(t);\ \underline{\lambda}_2(\tau)\} = \underline{\Lambda}_2(t)\delta(t-\tau)$$

The initial states $\underline{x}_{10}$ and $\underline{x}_{20}$ are assumed to be mutually independent Gaussian random vectors with statistics:

$$E\{\underline{x}_{10}\} = \bar{\underline{x}}_{10}$$

$$\text{cov}\{\underline{x}_{10};\underline{x}_{10}\} = \underline{\Sigma}_{011}$$

$$E\{\underline{x}_{20}\} = \bar{\underline{x}}_{20}$$

$$\text{cov}\{\underline{x}_{20};\underline{x}_{20}\} = \underline{\Sigma}_{022}$$

$\underline{x}_{10}$ and $\underline{x}_{20}$ are also assumed to be independent of $\underline{\lambda}_1(t)$ and $\underline{\lambda}_2(t)$, for all t.

As has been noted in Chapter 1, it is often (technically or economically) not feasible to allow each controller to measure the states of both systems.

We consider, therefore, the case in which each controller only makes measurements about his own system output and has no access to the measurements carried out by the other system. In general, the measurements are corrupted by noise and so, we assume that the observation equations can be written as

$$\underline{z}_1(t) = \underline{C}_{11}(t)\underline{x}_1(t)+\underline{\theta}_1(t) \tag{2.1.3}$$

and

$$\underline{z}_2(t) = \underline{C}_{22}(t)\underline{x}_2(t)+\underline{\theta}_2(t) \tag{2.1.4}$$

where $\underline{z}_1(t) \in R^{r_1}$ is the output of system 1,
$\underline{z}_2(t) \in R^{r_2}$ is the output of system 2,
$\underline{C}_{11}(t)$ is a $r_1 x n_1$ matrix,
$\underline{C}_{22}(t)$ is a $r_2 x n_2$ matrix.

The stochastic processes $\underline{\theta}_1(t)$ and $\underline{\theta}_2(t)$ are assumed to be zero mean, mutually independent Gaussian white noise processes with

$$\text{cov}\{\underline{\theta}_1(t);\ \underline{\theta}_1(\tau)\} = \underline{\Theta}_1(t)\delta(t-\tau)$$

$$\text{cov}\{\underline{\theta}_2(t);\ \theta_2(\tau)\} = \underline{\Theta}_2(t)\delta(t-\tau)$$

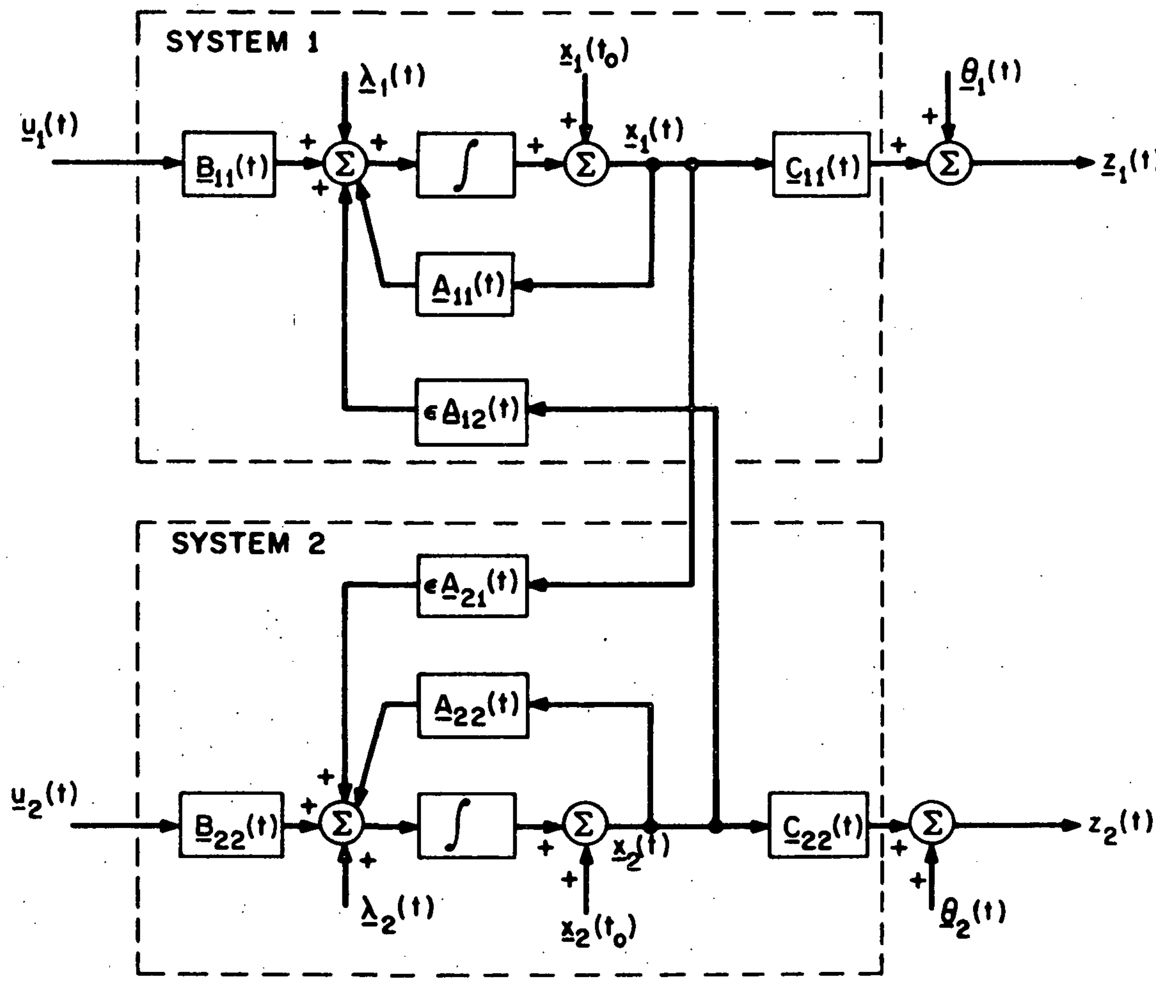

Fig. 1 The Structure of Two Weakly Coupled Linear Systems

$\theta_1(t)$ and $\underline{\theta}_2(t)$ are also assumed to be independent of $\underline{\lambda}_1(\tau)$, $\underline{\lambda}_2(\tau)$, $\underline{x}_{10}$, and $\underline{x}_{20}$ for all t and τ.

Our control objective is to regulate the states of the two systems without expending too much control energy. Since each of our systems is operating almost in an autonomous fashion, we take as the overall performance measure to be the sum of the performance measures of the two individual systems. Hence, we take as our system performance measure the cost functional

$$J(\underline{u}_1,\underline{u}_2) = J_1(\underline{u}_1,\underline{u}_2) + J_2(\underline{u}_1,\underline{u}_2)$$

where

$$J_1(\underline{u}_1,\underline{u}_2) = E\{\underline{x}_1'(T)\underline{F}_1\underline{x}_1(T)+ \int_{t_0}^{T}(\underline{x}_1'(t)\underline{Q}_1(t)\underline{x}_1(t) +\underline{u}_1'(t)\underline{R}_1(t)\underline{u}_1(t))dt\} \tag{2.1.5}$$

$$J_2(\underline{u}_1,\underline{u}_2) = E\{\underline{x}_2'(T)\underline{F}_2\underline{x}_2(T)+ \int_{t_0}^{T}(\underline{x}_2'(t)\underline{Q}_2(t)\underline{x}_2(t) \underline{u}_2'(t)\underline{R}_2(t)\underline{u}_2(t))dt\} \tag{2.1.6}$$

$$\underline{F}_1 = F_1' \geq \underline{0},\quad \underline{Q}_1 = \underline{Q}_1' \geq \underline{0},\quad \underline{R}_1 = \underline{R}_1' > \underline{0}$$

$$\underline{F}_2 = \underline{F}_2' \geq \underline{0},\quad \underline{Q}_2 = \underline{Q}_2' \geq \underline{0},\quad \underline{R}_2 = \underline{R}_2' > \underline{0}$$

are weighting matrices of appropriate dimensions. Note that in general, J_1 depends on both $\underline{u}_1(t)$ and $\underline{u}_2(t)$ as the choice of $\underline{u}_2(t)$ affects system 1 through the weak coupling, and vice-versa.

Since each controller only makes measurements about his own system, we require $\underline{u}_1(t)$ to be a function only of the measurements made on system 1, together with the a priori information about the two systems, which is shared by both controllers. Similarly, we require $\underline{u}_2(t)$ to be a function only of the measurements made on system 2, together with the a priori information.

To be more precise, we define the information sets

$$Y_1(t) = \{\underline{z}_1(\tau);\ t_0 \leq \tau \leq t\} \cup \{\text{a priori information}\}$$

$$Y_2(t) = \{\underline{z}_2(\tau);\ t_0 \leq \tau \leq t\} \cup \{\text{a priori information}\}$$

Then $\underline{u}_i(t)$ is an admissable control if, and only if, it is of the form

$$\underline{u}_i(t) = \underline{\phi}_i(Y_i(t),\ t) \qquad i=1,2 \tag{2.1.7}$$

Our control objective consists of finding the best $\underline{u}_1(t)$ and $\underline{u}_2(t)$ such that the cost functional $J(\underline{u}_1, \underline{u}_2)$ is minimized.

In other words, the stochastic optimal control problem being considered is solved if we can find controls $\underline{u}_1^*(t)$ and $\underline{u}_2^*(t)$ satisfying the properties that they are admissible and that for any other set of admissible controls $\underline{u}_1(t)$ and $\underline{u}_2(t)$,

$$J(\underline{u}_1^*,\underline{u}_2^*) \leq J(\underline{u}_1,\underline{u}_2) \tag{2.1.8}$$

$\underline{u}_1^*(t)$ and $\underline{u}_2^*(t)$ are called the optimal controls for our problem.

The way in which the control strategies depend on the information structure of the system is crucial for our problem. The celebrated Separation Theorem gives the solution to the linear-quadratic-Gaussian problem for the case where all controls are generated by a central agency who makes all measurements and has perfect memory [4]. Since, for reasons mentioned earlier, the centralized scheme is not allowed, the Separation Theorem cannot be applied directly to our problem [5]. Considerations in stochastic differential game theory suggest that the true optimal controls may be very complicated. We shall, therefore, propose an approximate model for our original system, which will simplify the analysis considerably.

2.2 An Approximate Model for Calculations of the Controls

We observe first of all that the influence of one system on the other is quite small, as the intersystem coupling is weak (by virtue of the fact that ε is small). Secondly, since each controller does not have access to the measurements about the other system, he can view the state of the other system essentially as a completely random quantity. Thus, the perturbational affects of the weak coupling on each system are similar to that of a noise. For controller 1, therefore, the term $\varepsilon \underline{A}_{12}(t)\underline{x}_2(t)$ looks like an additional driving noise process. Since $\underline{x}_2(t)$ is the state of a stochastic linear system driven by white Gaussian noise, it is a Gaussian but generally colored process. However, controller 1 has almost no knowledge about $\underline{x}_2(t)$ and so the perturbational effects of $\underline{x}_2(t)$ on system 1 are more or less entirely unpredictable. For this reason, we may try to approximate the influence of $\underline{x}_2(t)$ upon system 1 by a white noise process. Furthermore, since our aim is to maintain the states near zero, we can model $\varepsilon \underline{A}_{12}(t)\underline{x}_2(t)$, which is much smaller than $\underline{x}_2(t)$, as having zero mean. Similarly, controller 2 also models the term $\varepsilon \underline{A}_{21}(t)\underline{x}_1(t)$ as a zero mean, Gaussian white noise process.

We shall replace the coupling terms by the white noise processes $\underline{\xi}_1(t)$ and $\underline{\xi}_2(t)$ so that the system equations (2.1.1) and (2.1.2) beoomes (see Fig. 2)

$$\dot{\underline{x}}_1(t) = \underline{A}_{11}(t)\underline{x}_1(t)+\underline{B}_{11}(t)\underline{u}_1(t)+\underline{\lambda}_1(t)+\underline{\xi}_1(t) \tag{2.2.1}$$

$$\dot{\underline{x}}_2(t) = \underline{A}_{22}(t)\underline{x}_2(t)+\underline{B}_{22}(t)\underline{u}_2(t)+\underline{\lambda}_2(t)+\underline{\xi}_2(t) \tag{2.2.2}$$

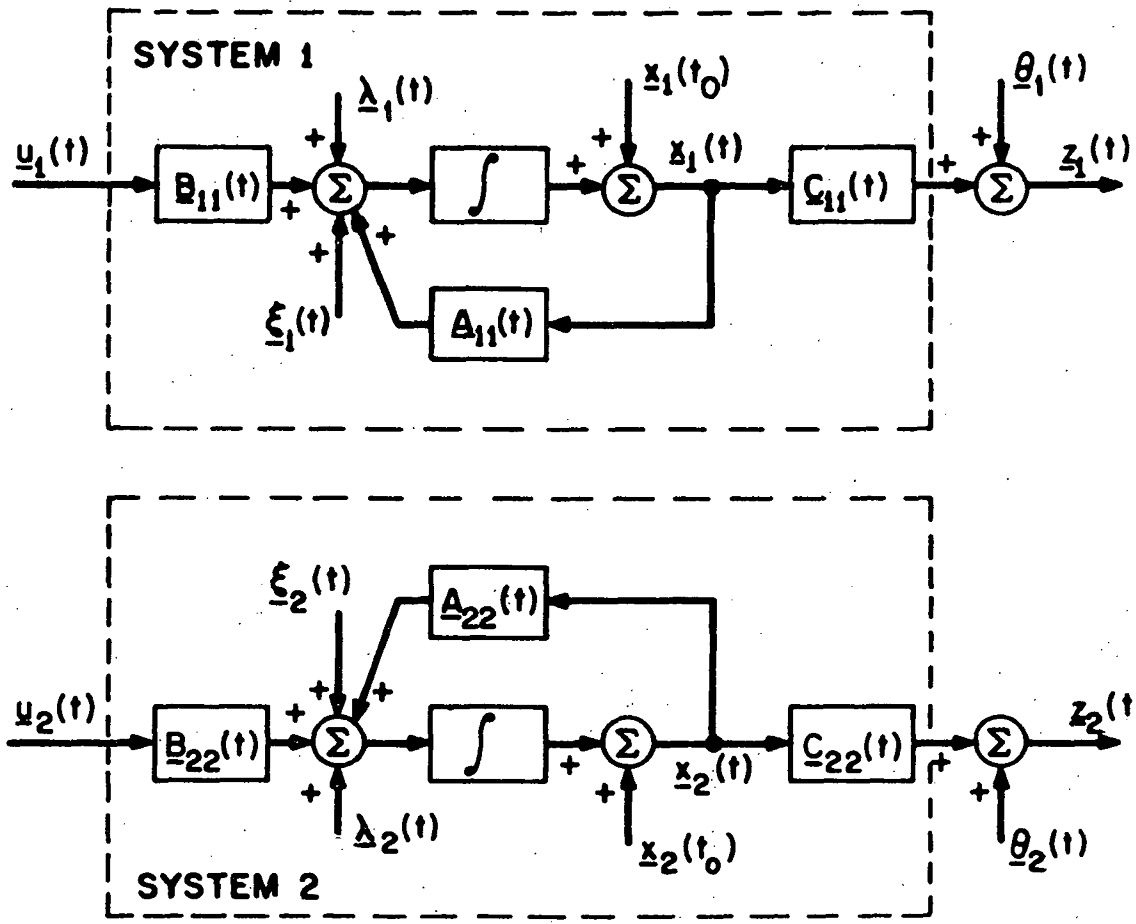

Fig. 2 Assumed Model for Control

where we model $\underline{\xi}_1(t)$ and $\underline{\xi}_2(t)$ as zero mean, independent Gaussian "fake" white noises with

$$\text{cov}\{\underline{\xi}_1(t); \underline{\xi}_1(\tau)\} = \underline{\Xi}_1(t)\delta(t-\tau)$$

$$\text{cov}\{\underline{\xi}_2(t); \underline{\xi}_2(\tau)\} = \underline{\Xi}_2(t)\delta(t-\tau)$$

Considering the nature of the quantities which $\underline{\xi}_1(t)$ and $\underline{\xi}_2(t)$ model, they are assumed to be also independent of the other noise processes. To specify these fake white noises completely, we must determine the covariance or intensity matrices $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. Intuitively, there should be some sort of "optimal" choices, for $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ so that we can get the best results possible with the kind of approximations that we are making. However, being covariance matrices, they are constrained to belong to the class of symmetric and positive semidefinite matrices.

With the above model, the structure of the entire system is greatly simplified. System 1 is completely decoupled from system 2. They are driven by the mutually independent equivalent white noises

$$\underline{w}_1(t) = \underline{\lambda}_1(t) + \underline{\xi}_1(t)$$

and

$$\underline{w}_2(t) = \underline{\lambda}_2(t) + \underline{\xi}_2(t)$$

respectively. Our original control problem is also modified to one which completely decomposes into two stochastic optimal control problems, one for system 1 and the other for system 2. If we fix $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ so that the statistics of $\underline{w}_1(t)$ and $\underline{w}_2(t)$ are completely specified, both of these problems are linear-quadratic-Gaussian problems with centralized information structure. The Separation Theorem can thus be applied to these two problems individually. We see that $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ now play the role of control variables in the sense that their specification generates the solutions to our modified problem. We shall therefore call $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ the <u>pseudo-controls</u> for our modified problem. In contrast, $\underline{u}_1(t)$ and $\underline{u}_2(t)$ will be called the <u>physical controls</u>. To be more specific, for fixed but arbitrary $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$, we can use the Separation Theorem to obtain the following set of optimal physical controls for our modified problem

$$\underline{u}_1(t) = -\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)\hat{\underline{x}}_{11}(t) \tag{2.2.3}$$

$$\underline{u}_2(t) = -\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)\hat{\underline{x}}_{22}(t) \tag{2.2.4}$$

where $\underline{K}_1(t)$ and $\underline{K}_2(t)$ are the solutions of the following matrix Riccati differential equations

$$\dot{\underline{K}}_1(t)=-\underline{K}_1(t)\underline{A}_{11}(t)-\underline{A}_{11}'(t)\underline{K}_1(t)$$
$$+\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)-\underline{Q}_1(t); \quad \underline{K}_1(T)=\underline{F}_1 \tag{2.2.5}$$

$$\dot{\underline{K}}_2(t)=-\underline{K}_2(t)\underline{A}_{22}(t)-\underline{A}'_{22}(t)\underline{K}_2(t)+\underline{K}_2(t)\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}'_{22}(t)\underline{K}_2(t)$$

$$-\underline{Q}_2(t)\underline{K}_2(T)=\underline{F}_2 \quad (2.2.6)$$

and $\hat{\underline{x}}_{11}(t)$ and $\hat{\underline{x}}_{22}(t)$ are estimates of the states $\underline{x}_1(t)$ and $\underline{x}_2(t)$ generated by the "uncoupled" Kalman-Bucy filters

$$\dot{\hat{\underline{x}}}_{11}(t)=\underline{A}_{11}(t)\hat{\underline{x}}_{11}(t)+\underline{B}_{11}(t)\underline{u}_1(t)+\underline{G}_1(t)[\underline{z}_1(t)-\underline{C}_{11}(t)\hat{\underline{x}}_{11}(t)]$$

$$\hat{\underline{x}}_{11}(t_0)=\bar{\underline{x}}_{10} \quad (2.2.7)$$

$$\dot{\hat{\underline{x}}}_{22}(t)=\underline{A}_{22}(t)\hat{\underline{x}}_{22}(t)+\underline{B}_{22}(t)\underline{u}_2(t)+\underline{G}_2(t)[\underline{z}_2(t)-\underline{C}_{22}(t)\hat{\underline{x}}_{22}(t)]$$

$$\hat{\underline{x}}_{22}(t_0)=\bar{\underline{x}}_{20} \quad (2.2.8)$$

with $$\underline{G}_1(t) = \underline{\Sigma}_1(t)\underline{C}'_{11}(t)\underline{\Theta}_1^{-1}(t) \quad (2.2.9)$$

$$\underline{G}_2(t) = \underline{\Sigma}_2(t)\underline{C}'_{22}(t)\underline{\Theta}_2^{-1}(t) \quad (2.2.10)$$

The matrices $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ satisfy another set of matrix Riccati differential equations

$$\dot{\underline{\Sigma}}_1(t)=\underline{A}_{11}(t)\underline{\Sigma}_1(t)+\underline{\Sigma}_1(t)\underline{A}'_{11}(t)-\underline{\Sigma}_1(t)\underline{C}'_{11}(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)$$

$$+\underline{\Lambda}_1(t)+\underline{\Xi}_1(t);\ \underline{\Sigma}_1(t_0)=\underline{\Sigma}_{011} \quad (2.2.11)$$

$$\dot{\underline{\Sigma}}_2(t)=\underline{A}_{22}(t)\underline{\Sigma}_2(t)+\underline{\Sigma}_2(t)\underline{A}'_{22}(t)-\underline{\Sigma}_2(t)\underline{C}'_{22}(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)\underline{\Sigma}_2(t)$$

$$+\underline{\Lambda}_2(t)+\underline{\Xi}_2(t);\ \underline{\Sigma}_2(t_0)=\underline{\Sigma}_{022} \quad (2.2.12)$$

We note that the optimal physical controls given by (2.2.3) and (2.2.4) for our modified problem are admissible controls for our original problem. The adoption of the approximate model not only decouples the two systems, but it also helps to generate suboptimal physical controls for our original problem. The work now remains to be done is to choose the pseudo-controls, i.e., the intensity matrices $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$, through an optimization procedure.

2.3 Discussion

It is important to get a precise interpretation of the results of the previous section. We have proposed a model which is very reasonable in view of the nature of the problem. It also brings us a great deal of simplification to the original problem we considered. We have not yet proved mathematically, however, that the model is acceptable in the sense that the physical controls generated by using the model, as given in equations (2.2.3) and (2.2.4), are nearly optimal. Nor have we made any comparison between the true optimal cost and the cost resulting from the adoption of the model. These matters will be given due attention in Chapter 5, where we shall prove that the model is indeed acceptable.

We remark that $\hat{\underline{x}}_{11}(t)$ and $\hat{\underline{x}}_{22}(t)$ are not the optimal estimates associated with our original problem, nor can we interpret $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ as the error covariances of the estimates. $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ are simply part of the weighting matrices which generate the state estimates $\hat{\underline{x}}_{11}(t)$ and $\hat{\underline{x}}_{22}(t)$, which in turn give us the physical controls. Since $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ depend on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ respectively, it is clear from the above discussion that the physical controls $\underline{u}_1(t)$ and $\underline{u}_2(t)$ depend on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. Hence, the cost associated with the problem also depends on the choice of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. If we view equations (2.2.11) and (2.2.12) as state equations with the pseudo-controls $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ as controls, we should be able to formulate an optimal control problem in $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. Exactly how this is accomplished will be the concern of the next chapter.

CHAPTER 3 - THE ASSOCIATED DETERMINISTIC CONTROL PROBLEM

We have seen in Chapter 2 that by adopting a very reasonable approximate model, the physical controls will be generated as long as we specify the covariance matrices of the "fake" white noisesintroduced. We also showed that the cost associated with the modified problem depends on these intensity matrices, and it is, therefore, appropriate to view them as pseudo-control variables. In this chapter, we take the physical controls to be generated according to the model proposed in Chapter 2 and reformulate our stochastic optimal control problem into a deterministic matrix optimal control problem with the pseudo-controls as the control variables. The interpretation of the deterministic problem in relation to our original stochastic problem is given. This, together with certain "technical" aspects of the optimization problem, prompt us to introduce additional terms into the original cost functional. (Further discussion on this problem will be given.) The complete deterministic optimal control problem is then stated.

3.1 Formulation of the Associated Deterministic Optimal Control Problem

Taking our physical controls to be generated by equations (2.2.3) and (2.2.4), we first of all substitute equations (2.2.3) and (2.2.4) into (2.1.1) and (2.1.2).

$$\dot{\underline{x}}_1(t) = \underline{A}_{11}(t)\underline{x}_1(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)\hat{\underline{x}}_{11}(t)$$

$$+ \varepsilon\underline{A}_{12}(t)\underline{x}_2(t)+\underline{\lambda}_1(t) \qquad (3.1.1)$$

$$\dot{\underline{x}}_2(t) = \underline{A}_{22}(t)\underline{x}_2(t)-\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)\hat{\underline{x}}_{22}(t)$$

$$+ \varepsilon\underline{A}_{21}(t)\underline{x}_1(t)+\underline{\lambda}_2(t) \qquad (3.1.2)$$

Subtract equation (2.2.7) from (3.1.1) and (2.2.8) from (3.1.2) to get

$$\dot{\underline{x}}_1(t)-\dot{\hat{\underline{x}}}_{11}(t)=[\underline{A}_{11}(t)-\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)][\underline{x}_1(t)-\hat{\underline{x}}_{11}(t)]$$

$$+ \varepsilon\underline{A}_{12}(t)\underline{x}_2(t)+\underline{\lambda}_1(t)-\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{\theta}_1(t) \qquad (3.1.3)$$

and

$$\underline{x}_2(t)-\hat{\underline{x}}_{22}(t)=[\underline{A}_{22}(t)-\underline{\Sigma}_2(t)\underline{C}_{22}'(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)][\underline{x}_2(t)-\hat{\underline{x}}_{22}(t)]$$

$$+ \varepsilon\underline{A}_{21}(t)\underline{x}_1(t)+\underline{\lambda}_2(t)-\underline{\Sigma}_2(t)\underline{C}_{22}'(t)\underline{\Theta}_2^{-1}(t)\underline{\theta}_2(t) \qquad (3.1.4)$$

Since $\hat{\underline{x}}_{11}(t)$ and $\hat{\underline{x}}_{22}(t)$ are the estimates of $\underline{x}_1(t)$ and $\underline{x}_2(t)$ respectively, the above two equations can be interpreted as the equations for the estimation errors.

If we now define

$$\underline{m}'(t) \triangleq [\underline{x}_1'(t) \;\; \underline{x}_2'(t) \;\; (\underline{x}_1'(t)-\hat{\underline{x}}_{11}'(t))\,(\underline{x}_2'(t)-\hat{\underline{x}}_{22}'(t))] \qquad (3.1.5)$$

then straightforward computation shows

$$\dot{\underline{m}}(t) = \hat{\underline{A}}(t)\underline{m}(t)+\hat{\underline{B}}(t)\hat{\underline{\theta}}(t) \tag{3.1.6}$$

where (dependence on t has been omitted for notational simplicity)

$$\underline{A}(t) \triangleq \begin{bmatrix} \underline{A}_{11}-\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \varepsilon\underline{A}_{12} & \underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \underline{0} \\ \varepsilon\underline{A}_{21} & \underline{A}_{22}-\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 & \underline{0} & \underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 \\ \underline{0} & \varepsilon\underline{A}_{12} & \underline{A}_{11}-\underline{\Sigma}_1\underline{C}_{11}'\underline{\Theta}_1^{-1}\underline{C}_{11} & \underline{0} \\ \varepsilon\underline{A}_{21} & \underline{0} & \underline{0} & \underline{A}_{22}-\underline{\Sigma}_2\underline{C}_{22}'\underline{\Theta}_2^{-1}\underline{C}_{22} \end{bmatrix} \tag{3.1.7}$$

$$\hat{\underline{B}}(t) \triangleq \begin{bmatrix} \underline{I} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{I} & \underline{0} & \underline{0} \\ \underline{I} & \underline{0} & -\underline{\Sigma}_1\underline{C}_{11}'\underline{\Theta}_1^{-1} & \underline{0} \\ \underline{0} & \underline{I} & \underline{0} & -\underline{\Sigma}_2\underline{C}_{22}'\underline{\Theta}_2^{-1} \end{bmatrix} \tag{3.1.8}$$

$$\hat{\underline{\theta}}(t) \triangleq \begin{bmatrix} \underline{\lambda}_1(t) \\ \underline{\lambda}_2(t) \\ \underline{\theta}_1(t) \\ \underline{\theta}_2(t) \end{bmatrix} \tag{3.1.9}$$

Equation (3.1.6) is still a vector stochastic differential equation. To obtain deterministic equations, we define the symmetric second moment matrix $\underline{M}(t)$ to be

$$\underline{M}(t) \triangleq \begin{bmatrix} \underline{M}_{11} & \underline{M}_{12} & \underline{M}_{13} & \underline{M}_{14} \\ \underline{M}_{21} & \underline{M}_{22} & \underline{M}_{23} & \underline{M}_{24} \\ \underline{M}_{31} & \underline{M}_{32} & \underline{M}_{33} & \underline{M}_{34} \\ \underline{M}_{41} & \underline{M}_{42} & \underline{M}_{43} & \underline{M}_{44} \end{bmatrix} = E\{\underline{m}(t)\underline{m}'(t)\} \qquad (3.1.10)$$

Then $\underline{M}(t)$ satisfies

$$\dot{\underline{M}}(t) = \hat{\underline{A}}(t)\underline{M}(t)+\underline{M}(t)\hat{\underline{A}}'(t)+\hat{\underline{B}}(t)\hat{\underline{\Theta}}(t)\hat{\underline{B}}'(t);$$

$$\underline{M}(t_0)= \begin{bmatrix} \underline{\Sigma}_{011}+\bar{\underline{x}}_{10}\bar{\underline{x}}_{10}' & \bar{\underline{x}}_{10}\bar{\underline{x}}_{20}' & \underline{\Sigma}_{011} & \underline{0} \\ \bar{\underline{x}}_{20}\bar{\underline{x}}_{10}' & \underline{\Sigma}_{022}+\bar{\underline{x}}_{20}\bar{\underline{x}}_{20}' & \underline{0} & \underline{\Sigma}_{022} \\ \underline{\Sigma}_{011} & \underline{0} & \underline{\Sigma}_{011} & \underline{0} \\ \underline{0} & \underline{\Sigma}_{022} & \underline{0} & \underline{\Sigma}_{022} \end{bmatrix} \qquad (3.1.11)$$

where

$$\hat{\underline{\Theta}}(t) = \begin{bmatrix} \underline{\Lambda}_1 & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{\Lambda}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{\Theta}_1 & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{\Theta}_2 \end{bmatrix} \qquad (3.1.12)$$

Equation (3.1.11) is now a deterministic matrix differential equation. Our plan is to use this as one of the state equations in a deterministic matrix optimal control problem. To this end, we substitute equations (2.2.3) and (2.2.4) into equations (2.1.5) and (2.1.6). We can then express the cost functional in terms of the components of $\underline{M}(t)$ as

$$J=tr[\underline{F}_1\underline{M}_{11}(T)+\underline{F}_2\underline{M}_{22}(T)] + \int_{t_0}^{T} tr[\underline{Q}_1\underline{M}_{11}+\underline{Q}_2\underline{M}_{22}$$

$$+\underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1(\underline{M}_{11}-\underline{M}_{13}-\underline{M}_{31}+\underline{M}_{33})$$

$$+\underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2(\underline{M}_{22}-\underline{M}_{24}-\underline{M}_{42}+\underline{M}_{44})]dt \qquad (3.1.13)$$

This can be written in a more compact form by defining

$$\hat{\underline{F}} \triangleq \begin{bmatrix} \underline{F}_1 & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{F}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix} \qquad (3.1.14)$$

$$\underline{\hat{Q}}(t) \triangleq \begin{bmatrix} \underline{Q}_1+\underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \underline{0} & -\underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \underline{0} \\ \underline{0} & \underline{Q}_2+\underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 & \underline{0} & -\underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 \\ -\underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \underline{0} & \underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \underline{0} \\ \underline{0} & -\underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 & \underline{0} & \underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 \end{bmatrix} \tag{3.1.15}$$

Then

$$J = \mathrm{tr}[\underline{\hat{F}}\underline{M}(T)] + \int_{t_0}^{T} \mathrm{tr}[\underline{\hat{Q}}(t)\underline{M}(t)]\, dt \tag{3.1.16}$$

Since $\underline{\Sigma}_1(t)$ depends on $\underline{\Xi}_1(t)$ and $\underline{\Sigma}_2(t)$ depends on $\underline{\Xi}_2(t)$, while $\underline{M}(t)$ in turn depends on $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ through $\underline{\hat{A}}(t)$, we see that in general, the cost J will depend on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. It seems possible then to take $\underline{M}(t)$, $\underline{\Sigma}_1(t)$, and $\underline{\Sigma}_2(t)$ as the states and the pseudo-controls $\underline{\Xi}_1(t)$, $\underline{\Xi}_2(t)$ as the control variables which affect $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ by equations (2.2.11) and (2.2.12).

We note, however, that we now have a different problem from the one we started with. Originally we were concerned with calculating optimal physical controls using the above quadratic performance index. The physical controls we had obtained would be optimal if equations (2.2.1) and (2.2.2) represented the true system dynamics. Since they are an approximation, this should be reflected in the cost functional for the deterministic problem. In other words, we should add to the original cost functional terms which measure the effects of our approximation and subsequent employment of the Separation Theorem. Additional motivation to do so is provided by the fact that the pseudo-controls $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ appear only linearly in the state equations for $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$. If we use the original cost functional which does not contain any terms explicitly in the pseudo-controls, the Hamiltonian associated with such an optimal control problem will be linear in $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. We will then be faced with a complex singular matrix optimal control problem.

The simplest way to achieve the above objective is to add to the cost functional the term

$$\int_{t_0}^{T} \mathrm{tr}[\underline{\Xi}_1(t)\underline{\Xi}'_1(t)+\underline{\Xi}_2(t)\underline{\Xi}'_2(t)]dt$$

Essentially this is equivalent to penalizing the magnitude of the noise intensities for becoming too large. The rationale behind this is that our physical controls used come from a model in which "fake" white plant noises are used to approximate the coupling between the systems. This procedure is justifiable only when the coupling is weak. Clearly, if the model is to work at all, the noise intensities must not be large compared to the quantities they approximate. Also, by keeping the noise intensities as low as possible, the state estimates obtained would be a more faithful representation of the true states. Furthermore, the quadratic nature of the additional term is intuitively very appealing. Since this is not something which arises automatically out of the original problem, we should expect that other choices for the additional term are also justifiable. A discussion pertaining to this observation will be given in Chapter 5.

Since $\Xi_1(t)$ and $\Xi_2(t)$ are covariance matrices of white noise processes, they must be symmetric and at least positive semidefinite. Hence the optimal pseudo-control are constrained to lie in this class of matrices.

Putting together the above development we arrive at the following deterministic matrix optimal control problem, whose solution gives the physical controls for the original stochastic optimal control problem:

Problem: Given the system described by the matrix differential equations:

$$\dot{\underline{M}}(t) = \hat{\underline{A}}(t)\underline{M}(t)+\underline{M}(t)\hat{\underline{A}}'(t)+\hat{\underline{B}}(t)\hat{\underline{\Theta}}(t)\hat{\underline{B}}'(t) \tag{3.1.17}$$

$$\dot{\underline{\Sigma}}_1(t) = \underline{A}_{11}(t)\underline{\Sigma}_1(t)+\underline{\Sigma}_1(t)\underline{A}'_{11}(t)-\underline{\Sigma}_1(t)\underline{C}'_{11}(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t) +\underline{\Lambda}_1(t)+\underline{\Xi}_1(t) \tag{3.1.18}$$

$$\dot{\underline{\Sigma}}_2(t) = \underline{A}_{22}(t)\underline{\Sigma}_2(t)+\underline{\Sigma}_2(t)\underline{A}'_{22}(t)-\underline{\Sigma}_2(t)\underline{C}'_{22}(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)\underline{\Sigma}_2(t) +\underline{\Lambda}_2(t)+\underline{\Xi}_2(t) \tag{3.1.19}$$

with initial conditions

$$\underline{M}(t_0)= \begin{bmatrix} \underline{\Sigma}_{011}+\bar{\underline{x}}_{10}\bar{\underline{x}}'_{10} & \bar{\underline{x}}_{10}\bar{\underline{x}}'_{20} & \underline{\Sigma}_{011} & \underline{0} \\ \bar{\underline{x}}_{20}\bar{\underline{x}}'_{10} & \underline{\Sigma}_{022}+\bar{\underline{x}}_{20}\bar{\underline{x}}'_{20} & \underline{0} & \underline{\Sigma}_{022} \\ \underline{\Sigma}_{011} & \underline{0} & \underline{\Sigma}_{011} & \underline{0} \\ \underline{0} & \underline{\Sigma}_{022} & \underline{0} & \underline{\Sigma}_{022} \end{bmatrix} \tag{3.1.20}$$

$$\underline{\Sigma}_1(t_0) = \underline{\Sigma}_{011} \tag{3.1.21}$$

$$\underline{\Sigma}_2(t_0) = \underline{\Sigma}_{022} \tag{3.1.22}$$

and the cost functional

$$\hat{J}(\underline{\Xi}_1,\underline{\Xi}_2)=\text{tr}[\underline{\hat{F}M}(T)] + \int_{t_0}^{T} \text{tr}[\underline{\hat{Q}}(t)\underline{M}(t)+\underline{\Xi}_1(t)\underline{\Xi}_1'(t) + \underline{\Xi}_2(t)\underline{\Xi}_2'(t)]dt \qquad (3.1.23)$$

Find symmetric and at least positive semidefinite matrices $\underline{\Xi}_1{}^*(t)$ and $\underline{\Xi}_2{}^*(t)$ such that the cost functional $\hat{J}$ is minimized.

3.2 Discussion

A careful interpretation must be given to the equations derived in the above section. Although the submatrix $\underline{M}_{33}(t)$ satisfies an equation of the form satisfied by $\underline{\Sigma}_1(t)$, they are, in general, altogether different objects. $\underline{M}_{33}(t)$ is the error covariance associated with the estimate $\underline{\hat{x}}_{11}(t)$ while $\underline{\Sigma}_1(t)$ is simply part of the gain matrix used in the Kalman filter for generating $\underline{\hat{x}}_{11}(t)$. This difference is more vividly illustrated by looking at the cost functional $\hat{J}(\underline{\Xi}_1,\underline{\Xi}_2)$. Only the components of $\underline{M}(t)$ appear in $\hat{J}(\underline{\Xi}_1,\underline{\Xi}_2)$; there are no terms involving $\underline{\Sigma}_1(t)$ or $\underline{\Sigma}_2(t)$ explicitly. As will be shown in Chapter 5, $\underline{M}_{33}(t)$ and $\underline{\Sigma}_1(t)$ will both represent the estimation error covariance only when $\varepsilon=0$, i.e., when the two systems are uncoupled to start with. In this particular case, we should expect $\underline{\Xi}_1{}^*(t)$ and $\underline{\Xi}_2{}^*(t)$ to be both equal to zero. For if we do not have any coupling, there is clearly no need to add fictitious noise to the system to approximate nothing. This asymptotic case will be treated in Chapter 4. In general, we should also expect $\underline{\Xi}_1{}^*(t)$ and $\underline{\Xi}_2{}^*(t)$ to be a function of the degree of intersystem coupling because this is precisely what $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ are approximating.

Requiring the control variables to lie in the class of symmetric and positive semidefinite matrices is an unconventional constraint. If we try to solve the constrained problem directly, we shall find it difficult to ensure that the resulting controls will meet the constraints. However, we can circumvent this difficulty by defining the auxiliary variables $\underline{N}_1(t)$ and $\underline{N}_2(t)$ where

$$\underline{\Xi}_1(t) = \underline{N}'_1(t)\underline{N}_1(t) \tag{3.2.1}$$

$$\underline{\Xi}_2(t) = \underline{N}'_2(t)\underline{N}_2(t) \tag{3.2.2}$$

By writing $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ in the above form, the constraints are automatically satisfied. Therefore, instead of seeking the optimal $\underline{\Xi}_1{}^*(t)$ and $\underline{\Xi}_2{}^*(t)$ directly, we shall optimize the choice of $\underline{N}_1(t)$ and $\underline{N}_2(t)$. The optimal pseudo-controls then follow immediately. We note that $\underline{N}_1(t)$ and $\underline{N}_2(t)$ are not necessarily square matrices. Nor would the factorization of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ into the above forms in general be unique. If $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ are positive definite, however, $\underline{N}_1(t)$ and $\underline{N}_2(t)$ can be taken to be nonsingular and unique.

In the next chapter, we shall see how such an unconventional control problem can be solved using standard optimal control techniques.

CHAPTER 4: SOLUTION OF THE DETERMINISTIC PROBLEM AND THE COMPLETE STOCHASTIC CONTROL STRATEGY

In Chapter 3, we posed a deterministic optimal control problem using the pseudo-control variables. The unconventional constraints were removed by defining auxiliary variables. We shall now proceed to solve this deterministic problem. Our main tool is the matrix minimum principle [9], and so the results derived here will be the necessary conditions satisfied by the optimal pseudo-controls. After obtaining the optimal pseudo-controls, we display the complete procedure for generating the physical controls to our original stochastic problem. Some computational aspects are then considered and the asymptotic case in which the intersystem coupling goes to zero is treated. The results are shown to be in keeping with the intuitive discussion given in section 3.2.

4.1 Solution of the Deterministic Optimal Control Problem

In order to simplify the notation, we shall define

$$\underline{\Gamma}_1(t) \triangleq \underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t) \tag{4.1.1}$$

$$\underline{\Gamma}_2(t) \triangleq \underline{C}_{22}'(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t) \tag{4.1.2}$$

Let $\underline{P}(t)$, $\underline{S}_1(t)$, and $\underline{S}_2(t)$ be the costate matrices associated with $\underline{M}(t)$, $\underline{\Sigma}_1(t)$, and $\underline{\Sigma}_2(t)$ respectively. Also define

$$\underline{P}(t)=\begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} & \underline{P}_{13} & \underline{P}_{14} \\ \underline{P}_{21} & \underline{P}_{22} & \underline{P}_{23} & \underline{P}_{24} \\ \underline{P}_{31} & \underline{P}_{32} & \underline{P}_{33} & \underline{P}_{34} \\ \underline{P}_{41} & \underline{P}_{42} & \underline{P}_{43} & \underline{P}_{44} \end{bmatrix} \qquad (4.1.3)$$

For any real symmetric nxn matrix $\underline{W}$, we use $\sigma(\underline{W})$ to denote its spectrum, i.e., the set of eigenvalues $\{\lambda_i\}$ i=1,2....n of $\underline{W}$. Diag $\sigma(\underline{W})$ will denote the nxn matrix with the only nonzero entries the numbers $\lambda_1, \lambda_2, \ldots \lambda_n$ along the diagonal:

$$\text{diag } \sigma(\underline{W}) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdot & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & & & \lambda_n \end{bmatrix} \qquad (4.1.4)$$

Diag $|\sigma(\underline{W})|$ is similarly defined with $|\lambda_i|$ replacing λ_i in equation (4.1.4). The eigenvalues λ_i need not be distinct. However, for any real symmetric nxn matrix $\underline{W}$, we can always find an orthogonal matrix which diagonalizes $\underline{W}$ into the form diag $\sigma(\underline{W})$.

Using the matrix minimum principle with the above notation, we arrive at the following theorem:

Theorem:

The optimal pseudo-controls are given by (* denotes evaluation along the optimal trajectory)

$$\underline{\Xi}_1^*(t) = \frac{1}{4}\,\underline{T}_1(t)\,\mathrm{diag}|\sigma(\underline{S}_1^*(t))|\,\underline{T}_1'(t) - \frac{1}{4}\underline{S}_1^*(t) \qquad (4.1.5)$$

$$\underline{\Xi}_2^*(t) = \frac{1}{4}\,\underline{T}_2(t)\,\mathrm{diag}|\sigma(\underline{S}_2^*(t))|\,\underline{T}_2'(t) - \frac{1}{4}\underline{S}_2^*(t) \qquad (4.1.6)$$

where $\underline{T}_1(t)$ and $\underline{T}_2(t)$ are the orthogonal matrices which diagonalize $\underline{S}_1^*(t)$ and $\underline{S}_2^*(t)$ respectively for every $t\,\varepsilon[t_o,T]$.

$\underline{M}^*(t)$, $\underline{P}^*(t)$, $\underline{\Sigma}_1^*(t)$, $\underline{S}_1^*(t)$, $\underline{\Sigma}_2^*(t)$, and $\underline{S}_2^*(t)$ satisfy the following set of differential equations:

$$\dot{\underline{M}}^*(t)=\hat{\underline{A}}^*(t)\underline{M}^*(t)+\underline{M}^*(t)\hat{\underline{A}}^{*\prime}(t)+\hat{\underline{B}}^*(t)\hat{\underline{\Theta}}(t)\hat{\underline{B}}^{*\prime}(t)$$
$$\underline{M}^*(t_0) = \underline{M}(t_0) \qquad (4.1.7)$$

$$\dot{\underline{P}}^*(t) = -\hat{\underline{A}}^{*\prime}(t)\underline{P}^*(t)-\underline{P}^*(t)\hat{\underline{A}}^*(t)-\hat{\underline{Q}}(t)$$
$$\underline{P}^*(T) = \hat{\underline{F}} \qquad (4.1.8)$$

$$\dot{\underline{\Sigma}}_1{}^*(t)=\underline{A}_{11}(t)\underline{\Sigma}_1{}^*(t)$$

$$+\underline{\Sigma}_1{}^*(t)\underline{A}'_{11}(t)-\underline{\Sigma}_1{}^*(t)\underline{C}'_{11}(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1{}^*(t)$$

$$+\underline{\Lambda}_1(t)+\underline{\Xi}_1^*(t)$$

$$\underline{\Sigma}_1{}^*(t_0)=\underline{\Sigma}_{011} \qquad (4.1.9)$$

$$\dot{\underline{S}}_1{}^*(t)=-\underline{A}'_{11}(t)\underline{S}_1{}^*(t)-\underline{S}_1{}^*(t)\underline{A}_{11}(t)+S_1{}^*(t)\underline{\Sigma}_1{}^*(t)\underline{\Gamma}_1(t)$$

$$+\underline{\Gamma}_1(t)\underline{\Sigma}_1{}^*(t)\underline{S}_1{}^*(t)$$

$$-\underline{P}^*_{33}(t)\underline{\Sigma}^*_1(t)\underline{\Gamma}_1(t)-\underline{\Gamma}_1(t)\underline{\Sigma}^*_1(t)\underline{P}^*_{33}(t)+\underline{P}^*_{31}(t)\underline{M}^{*\prime}_{31}(t)\underline{\Gamma}_1(t)$$

$$+\underline{P}^*_{32}(t)\underline{M}^{*\prime}_{32}(t)\underline{\Gamma}_1(t)$$

$$+\underline{P}^*_{33}(t)\underline{M}^{*\prime}_{33}(t)\underline{\Gamma}_1(t)+\underline{P}^*_{34}(t)\underline{M}^{*\prime}_{34}(t)\underline{\Gamma}_1(t)+\underline{\Gamma}_1(t)\underline{M}^{*\prime}_{13}(t)\underline{P}_{13}(t)$$

$$+\underline{\Gamma}_1(t)\underline{M}^{*\prime}_{23}(t)\underline{P}^*_{23}(t)$$

$$+\underline{\Gamma}_1(t)\underline{M}^{*\prime}_{33}(t)\underline{P}^*_{33}(t)+\underline{\Gamma}_1(t)\underline{M}^{*\prime}_{43}(t)\underline{P}^*_{43}(t);$$

$$\underline{S}^*_1(T) = \underline{0} \qquad (4.1.10)$$

$$\dot{\underline{\Sigma}}^*_2(t)=\underline{A}_{22}(t)\underline{\Sigma}^*_2(t)$$

$$+\underline{\Sigma}^*_2(t)\underline{A}'_{22}(t)-\underline{\Sigma}^*_2(t)\underline{C}'_{22}(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)\underline{\Sigma}^*_2(t)$$

$$+\underline{\Lambda}_2(t)+\underline{\Xi}^*_2(t)$$

$$\underline{\Sigma}^*_2(t_0)=\underline{\Sigma}_{022} \qquad (4.1.11)$$

$$\dot{\underline{S}}_2^*(t) = -\underline{A}_{22}'(t)\underline{S}_2^*(t)-\underline{S}_2^*(t)\underline{A}_{22}(t)+\underline{S}_2^*(t)\underline{\Sigma}_2^*(t)\underline{\Gamma}_2(t) +\underline{\Gamma}_2(t)\underline{\Sigma}_2^*(t)\underline{S}_2^*(t)$$

$$- \underline{P}_{44}^*(t)\underline{\Sigma}_2^*(t)\underline{\Gamma}_2(t)-\underline{\Gamma}_2(t)\underline{\Sigma}_2^*(t)\underline{P}_{44}^*(t)+\underline{P}_{41}^*(t)\underline{M}_{41}^{*\prime}(t)\underline{\Gamma}_2(t) +\underline{P}_{42}^*(t)\underline{M}_{42}^{*\prime}(t)\underline{\Gamma}_2(t)$$

$$+ \underline{P}_{43}^*(t)\underline{M}_{43}^{*\prime}(t)\underline{\Gamma}_2(t)+\underline{P}_{44}^*(t)\underline{M}_{44}^{*\prime}(t)\underline{\Gamma}_2(t) +\underline{\Gamma}_2(t)\underline{M}_{14}^{*\prime}(t)\underline{P}_{14}^*(t)$$

$$+\underline{\Gamma}_2(t)\underline{M}_{24}^{*\prime}(t)\underline{P}_{24}^*(t)+\underline{\Gamma}_2(t)\underline{M}_{34}^{*\prime}(t)\underline{P}_{34}^*(t) +\underline{\Gamma}_2(t)\underline{M}_{44}^{*\prime}(t)\underline{P}_{44}^*(t)$$

$$\underline{S}_2^*(T) = \underline{0} \qquad (4.1.12)$$

The optimal cost is

$$\hat{J}^* = tr[\hat{\underline{F}}\underline{M}^*(T)]+ \int_{t_0}^{T} tr[\hat{Q}\_(t)\underline{M}^*(t)+\underline{\Xi}_1^*(t)\underline{\Xi}_1^{*\prime}(t) +\underline{\Xi}_2^*(t)\underline{\Xi}_2^{*\prime}(t)]dt \qquad (4.1.13)$$

Proof: See Appendix A.

To obtain the optimal pseudo-controls, we have to solve three two-point boundary value problems represented by equations (4.1.7) and (4.1.12). This can be done, for example, using the gradient method. We emphasize that all the calculations required to solve these equations are off-line and so the optimal $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$ are precomputable. Although it may be objected that solving three nonlinear, coupled two-point boundary value problems is no easy matter, we remark that in Chapter 5, we will show that in many cases of interest, $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$ can be taken to be zero. In these situations, there is even no need to go through computationally any of the steps of this deterministic problem. This does not imply, however, that solving the deterministic problem is a futile and meaningless exercise. It gives us the limitations as well as the advantages of using this approach. These matters will all be dealt with in the next chapter.

We note also the definiteness property of $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$ is governed by the definiteness of $\underline{S}_1^*(t)$ and $\underline{S}_2^*(t)$ respectively. When $\underline{S}_1^*(t) \geq \underline{0}$, $\underline{\Xi}_1^*(t) = \underline{0}$. This represents the minimum that $\underline{\Xi}_1^*(t)$ can take on. If $S_1^*(t) < \underline{0}$, $\underline{\Xi}_1^*(t) = -\frac{1}{2}\underline{S}_1^*(t)$ and is therefore positive definite. This is the maximum of $\underline{\Xi}_1^*(t)$. If $\underline{S}_1^*(t)$ is negative semidefinite or indefinite, then $\underline{\Xi}_1^*(t)$ is only positive semidefinite. Completely analogous remarks hold for $\underline{\Xi}_2^*(t)$. Hence by analyzing the behavior of $\underline{S}_1^*(t)$ and $\underline{S}_2^*(t)$, we will obtain all the properties of $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$. We shall follow this line of approach in the sequel.

4.2 The Complete Stochastic Control Strategy

Collecting all the results we have developed so far, we are now in a position to state the complete stochastic control strategy for two weakly coupled linear systems using quadratic criteria:

Step 1 - Solve the two matrix Riccati differential equations (2.2.5) and (2.2.6) to yield $\underline{K}_1(t)$ and $\underline{K}_2(t)$. These are off-line calculations and can be precomputed. The control gain matrices $\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)$ and $\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)$ are then completely specified.

Step 2 - Formulate and solve the deterministic optimal control problem as given by equations (4.1.7) to (4.1.12). We emphasize that these are again off-line calculations and can again be precomputed. We will determine $\underline{\Sigma}_1^*(t)$ and $\underline{\Sigma}_2^*(t)$ in the process, and hence the filter gain matrices $\underline{\Sigma}_1^*(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)$ and $\underline{\Sigma}_2^*(t)\underline{C}_{22}'(t)\underline{\Theta}_2^{-1}(t)$ are completely specified.

Step 3 - Use the filter gains determined from Step 2 in the Kalman filter equations of (2.2.7) and (2.2.8) to generate the state estimates $\hat{\underline{x}}_{11}(t)$ and $\hat{\underline{x}}_{22}(t)$.

Step 4 - Implement the controls $\underline{u}_1(t)$ and $\underline{u}_2(t)$ by putting together the control gain matrices and the state estimates (see Fig. 3).

We remark that since all the complex calculations can be carried out off-line, and that the controls are simply linear transformations of the state estimates, there will be no difficulties with on-line implementation.

4.3 Qualitative Properties of the Optimal Pseudo-Controls

Throughout our proposed model, we have assumed that $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ approximate small coupling elements. The philosophy of introducing additional terms in the cost functional when we were formulating the deterministic problem is indeed to make $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ small themselves. But if $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ are small, it should be possible to view them as perturbations and to use perturbational analysis, with ε as the small parameter to investigate the qualitative effects $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ have on the system.

To be precise, we expand the variables in a power series in ε and equate terms of the same power in ε. In other words, we write (all the analysis in the sequel will be done along the optimal trajectory; * has been omitted for simplicity)

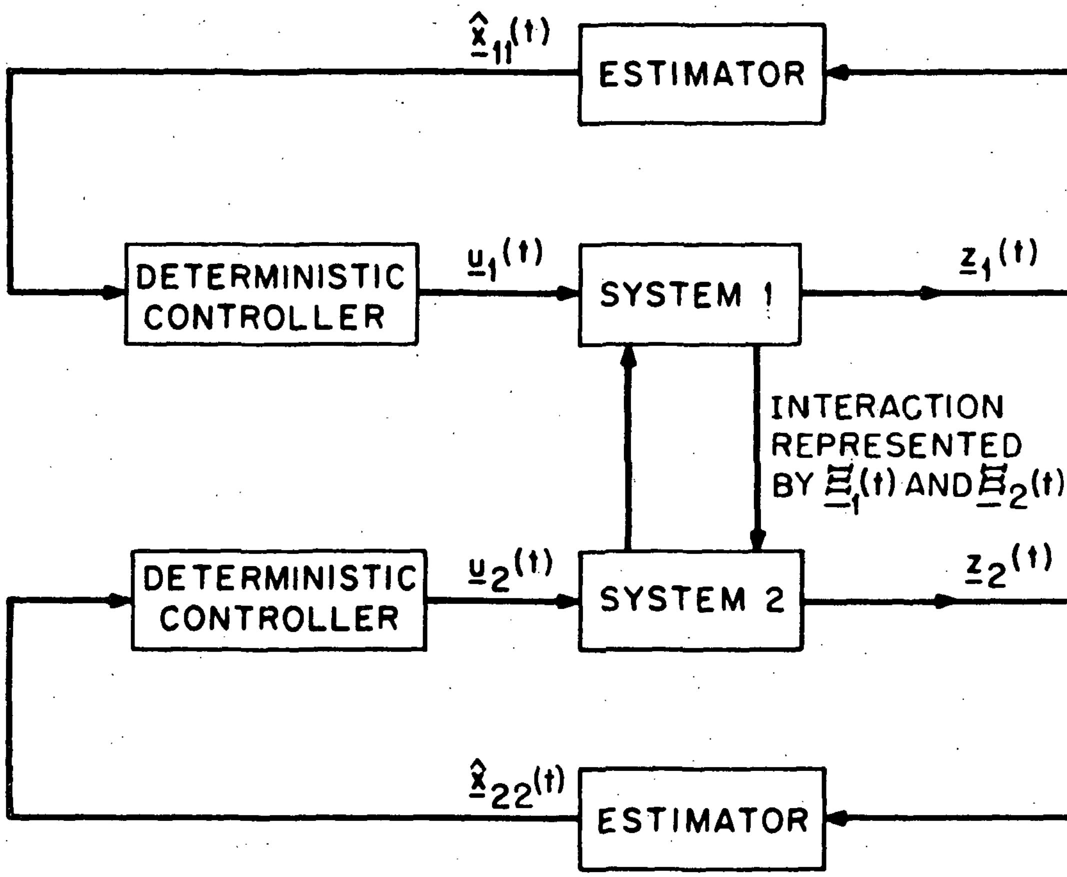

Fig. 3 Structure of the Proposed Control Scheme

$$\underline{M}_{ij}(t) = \underline{M}_{ij}^{(0)}(t)+\varepsilon\underline{M}_{ij}^{(1)}(t)+\varepsilon^2\underline{M}_{ij}^{(2)}(t)+\underline{0}(\varepsilon^2,t);\quad i=1,2,3,4;\ j=1,2,3,4$$

$$\underline{\Sigma}_i(t) = \underline{\Sigma}_i^{(0)}(t)+\varepsilon\underline{\Sigma}_i^{(1)}(t)+\varepsilon^2\underline{\Sigma}_i^{(2)}(t)+\underline{0}(\varepsilon^2,t);\quad i=1,2$$

$$\underline{P}_{ij}(t) = \underline{P}_{ij}^{(0)}(t)+\varepsilon\underline{P}_{ij}^{(1)}(t)+\varepsilon^2\underline{P}_{ij}^{(2)}(t)+\underline{0}(\varepsilon^2,t);\quad i=1,2,3,4$$

$$\underline{S}_i(t) = \underline{S}_i^{(0)}(t)+\varepsilon\underline{S}_i^{(1)}(t)+\varepsilon^2\underline{S}_i^{(2)}(t)+\underline{0}(\varepsilon^2,t);\quad i=1,2$$

$$\underline{\Xi}_i(t) = \underline{\Xi}_i^{(0)}(t)+\varepsilon\underline{\Xi}_i^{(1)}(t)+\varepsilon^2\underline{\Xi}_i^{(2)}(t)+\underline{0}(\varepsilon^2,t);\quad i=1,2$$

where $\lim_{\varepsilon\to 0} \frac{0(\varepsilon^2,t)}{\varepsilon^2} = \underline{0}$ uniformly in t

The various terms in the power series are obtained by differentiating the original quantities with respect to ε an appropriate number of times and then setting ε equal to zero in the final result. For example

$$\underline{M}_{ij}^{(k)}(t) = \left.\frac{\partial^{(k)}M_{ij}(t)}{\partial\varepsilon^k}\right|_{\varepsilon=0} \qquad k=0,1\ldots \tag{4.3.1}$$

Consider first the zeroth order terms. This is obtained by simply setting $\varepsilon=0$. This corresponds precisely to the case in which the two systems are in fact uncoupled. It can be shown that $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$ are then identically zero (see Appendix B). The interpretation is, of course, that $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ were introduced originally to approximate the coupling terms and hence decouple the systems.

If the systems were uncoupled to start with, such a trick would obviously be unnecessary. Furthermore, since we had added terms in $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ to the cost functional, the cost will increase for no purpose at all if $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ were nonzero. This result is certainly what we would expect and require of the optimal pseudo-controls.

We can actually say quite a bit more about the optimal $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$. Intuitively, since $\underline{\xi}_1(t)$ and $\underline{\xi}_2(t)$ approximate the terms $\varepsilon\underline{A}_{12}(t)\underline{x}_2(t)$ and $\varepsilon\underline{A}_{21}(t)\underline{x}_1(t)$ respectively, $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ should both be of second order in ε. A direct calculation shows that this is indeed the case and that, in general, second and higher order terms are nonzero (see Appendix C). This has an important interpretation:

If we adopt the approximate model, and if either the second order effects are small enough so that they can be neglected, or the second order corrections are not important in the design, then we can simply set the coupling to zero and work with the decoupled systems.

The above remark is, of course, obvious if we look at the original stochastic system equations. Nevertheless, it is worthwhile to check that our optimal solutions have the "correct" properties.

4.4 Discussion

Two more things now remain to be done. First, all the preceding analysis assumes that the proposed model is acceptable to start with. We still have not proved mathematically that this is so. Secondly, we have not yet undertaken an analysis of the system performance, a critical issue which concerns the designer. Both of these are, of course, part of the same question: how does the performance of the system using the proposed control strategy compare to the truly optimal one? What are the trade-offs between choosing these various schemes? These will be the subject matter of the next chapter.

CHAPTER 5: PERFORMANCE OF THE SYSTEM

In Chapter 4, we have obtained the complete scheme for generating the physical controls by way of the solution of the associated deterministic problem. Since these results are derived using an approximate model, the physical controls are clearly suboptimal. Although the scheme is appealing in view of its simplicity, it would not be acceptable if it is "too" suboptimal. In order to justify the proposed design, we must show that the system performance is close to the truly optimal one.

For a general linear system, the mathematically optimal design is to adopt complete centralization and apply the Separation Theorem. In our case, we can treat the two weakly coupled systems as a big unit and assume that the controllers are now administered by a central agency. We can then compare the cost obtained by using the complete centralization scheme to that obtained by using the proposed design.

We will show that these two costs are the same up to linear terms in ε. Since ε is small, we may therefore conclude that our proposed scheme is approximately optimal. We will also show that using $\underline{\Xi}_1^*(t) = \underline{\Xi}_2^*(t) = \underline{0}$ gives us the same performance up to ε^3 terms as that obtained by using the optimal $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$. We then turn to a discussion of the advantages of the proposed design and the trade-offs between the various choices of control strategy. The practical implications of our proposed strategy are also explored.

5.1 Analysis of the System Performance

To investigate the performance of the system using our control scheme, we look at the cost incurred. For any fixed $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$, it can be shown that the cost is given by (see Appendix D)

$$\hat{J} = tr[\underline{K}_1(t_o)(\underline{\Sigma}_{011}+\underline{\bar{x}}_{10}\underline{\bar{x}}_{10}')+\underline{K}_2(t_o)(\underline{\Sigma}_{022}+\underline{\bar{x}}_{20}\underline{\bar{x}}_{20}')] + \int_{t_o}^{T} tr(\underline{K}_1\underline{\Lambda}_1+\underline{K}_2\underline{\Lambda}_2)dt$$

$$+ \int_{t_o}^{T} tr[\underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1\underline{M}_{33}+\underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2\underline{M}_{44} + 2\varepsilon(\underline{K}_1\underline{A}_{12}\underline{M}_{21}+\underline{K}_2\underline{A}_{21}\underline{M}_{12})$$

$$+ \underline{\Xi}_1\underline{\Xi}_1' + \underline{\Xi}_2\underline{\Xi}_2']dt \qquad (5.1.1)$$

Following the technique used in Chapter 4, we can expand $\hat{J}$ in a power series in ε.

$$\hat{J} = \hat{J}^{(0)} + \varepsilon\hat{J}^{(1)} + \varepsilon^2\hat{J}^{(2)} + 0(\varepsilon^2) \qquad (5.1.2)$$

The calculations in Appendices B and C show that in addition to (all the analysis below will be done along the optimal trajectory; * has been omitted)

$$\underline{\Xi}_1^{(0)}(t) = \underline{\Xi}_2^{(0)}(t) = \underline{\Xi}_1^{(1)}(t) = \underline{\Xi}_2^{(1)}(t) = \underline{0} \qquad (5.1.3)$$

we also have

$$\underline{M}_{33}^{(1)}(t) = \underline{\Sigma}_1^{(1)}(t) = \underline{0} \qquad (5.1.4)$$

$$\underline{M}_{44}^{(1)}(t) = \underline{\Sigma}_2^{(1)}(t) = \underline{0} \qquad (5.1.5)$$

Thus, the optimal cost up to linear terms in ε is given by

$$\begin{aligned}\hat{J}^{(0)}+\varepsilon\hat{J}^{(1)} = \mathrm{tr}[&\underline{K}_1(t_o)(\underline{\Sigma}_{011}+\underline{\bar{x}}_{10}\underline{\bar{x}}'_{10})+\underline{K}_2(t_o)(\underline{\Sigma}_{022}+\underline{\bar{x}}_{20}\underline{\bar{x}}'_{20})] \\ &+ \int_{t_o}^{T} \mathrm{tr}[\underline{K}_1\underline{\Lambda}_1+\underline{K}_2\underline{\Lambda}_2)dt \\ &+ \int_{t_o}^{T}\mathrm{tr}[\underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}'_{11}\underline{K}_1\underline{M}_{33}^{(0)}+\underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}'_{22}\underline{K}_2\underline{M}_{44}^{(0)} \\ &+2\varepsilon(\underline{K}_1\underline{A}_{12}\underline{M}_{21}^{(0)}+\underline{K}_2\underline{A}_{21}\underline{M}_{12}^{(0)})]dt \end{aligned} \qquad (5.1.6)$$

As a preliminary remark, we observe that since the zeroth order terms do not depend on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$, $\hat{J}^{(0)}+\varepsilon\hat{J}^{(1)}$ is independent of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. Further discussion on the dependence of $\hat{J}$ on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ will be given later in the chapter.

Let us now compare the optimal cost up to linear terms in ε in our scheme to that given by complete centralization. To this end, we treat our coupled systems as a big unit and define

$$\underline{x}'(t) = [\underline{x}_1'(t) \ \ \underline{x}_2'(t)]$$

$$\underline{A}(t) = \begin{bmatrix} \underline{A}_{11}(t) & \varepsilon\underline{A}_{12}(t) \\ \varepsilon\underline{A}_{21}(t) & \underline{A}_{22}(t) \end{bmatrix}$$

$$\underline{B}(t) = \begin{bmatrix} \underline{B}_{11}(t) & \underline{0} \\ \underline{0} & \underline{B}_{22}(t) \end{bmatrix}$$

$$\underline{C}(t) = \begin{bmatrix} \underline{C}_{11}(t) & \underline{0} \\ \underline{0} & \underline{C}_{22}(t) \end{bmatrix}$$

$$\underline{u}(t) = \begin{bmatrix} \underline{u}_1(t) \\ \underline{u}_2(t) \end{bmatrix}$$

$$\underline{z}(t) = \begin{bmatrix} \underline{z}_1(t) \\ \underline{z}_2(t) \end{bmatrix}$$

$$\underline{\lambda}(t) = \begin{bmatrix} \underline{\lambda}_1(t) \\ \underline{\lambda}_2(t) \end{bmatrix}$$

$$\underline{\theta}(t) = \begin{bmatrix} \underline{\theta}_1(t) \\ \underline{\theta}_2(t) \end{bmatrix}$$

The composite system can be represented by the following set of equations:

$$\underline{\dot{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) + \underline{\lambda}(t)$$

$$\underline{x}(t_o) = \underline{x}_o = \begin{bmatrix} \underline{x}_{10} \\ \underline{x}_{20} \end{bmatrix} \qquad (5.1.7)$$

$$\underline{z}(t) = \underline{C}(t)\underline{x}(t) + \underline{\theta}(t) \qquad (5.1.8)$$

The white noise processes $\underline{\lambda}(t)$ and $\underline{\theta}(t)$ are zero mean with covariances

$$\underline{\Lambda}(t) = \begin{bmatrix} \underline{\Lambda}_1(t) & \underline{0} \\ \underline{0} & \underline{\Lambda}_2(t) \end{bmatrix}$$

$$\underline{\Theta}(t) = \begin{bmatrix} \underline{\Theta}_1(t) & \underline{0} \\ \underline{0} & \underline{\Theta}_2(t) \end{bmatrix}$$

$\underline{x}(t_o)$ is assumed to be a Gaussian random vector and has statistics

$$E\{\underline{x}(t_o)\} = \underline{\bar{x}}_o = \begin{bmatrix} \underline{\bar{x}}_{10} \\ \underline{\bar{x}}_{20} \end{bmatrix}$$

$$\text{cov}\{\underline{x}(t_o);\underline{x}(t_o)\} = \underline{\Sigma}_o = \begin{bmatrix} \underline{\Sigma}_{011} & \underline{\Sigma}_{012} \\ \underline{\Sigma}_{021} & \underline{\Sigma}_{022} \end{bmatrix}$$

Define $\underline{Q}(t) = \begin{bmatrix} \underline{Q}_1(t) & \underline{0} \\ \underline{0} & \underline{Q}_2(t) \end{bmatrix}$

$$\underline{R}(t) = \begin{bmatrix} \underline{R}_1(t) & \underline{0} \\ \underline{0} & \underline{R}_2(t) \end{bmatrix}$$

$$\underline{F} = \begin{bmatrix} \underline{F}_1 & \underline{0} \\ \underline{0} & \underline{F}_2 \end{bmatrix}$$

The cost functional can then be expressed as

$$J = E\{\underline{x}'(T)\underline{F}\underline{x}(T) + \int_{t_o}^{T} (\underline{x}'(t)\underline{Q}(t)\underline{x}(t)+\underline{u}'(t)\underline{R}(t)\underline{u}(t))dt\}$$

Define $\underline{K}(t) = \begin{bmatrix} \underline{K}_{11}(t) & \underline{K}_{12}(t) \\ \underline{K}_{21}(t) & \underline{K}_{22}(t) \end{bmatrix}$

$$\underline{\Sigma}(t) = \begin{bmatrix} \underline{\Sigma}_{11}(t) & \underline{\Sigma}_{12}(t) \\ \underline{\Sigma}_{21}(t) & \underline{\Sigma}_{22}(t) \end{bmatrix}$$

$\underline{K}(t)$ and $\underline{\Sigma}(t)$ satisfy the usual Riccati differential equations for the control and filtering problems respectively.

It can be shown that by using the Separation Theorem, the optimal cost is given by [10], [11]

$$J = \mathrm{tr}[\underline{K}(t_o)(\underline{\Sigma}_o+\underline{\bar{x}}_o\underline{\bar{x}}_o')] + \int_{t_o}^{T} \mathrm{tr}[\underline{K}(t)\underline{\Lambda}(t)]dt$$

$$+ \int_{t_o}^{T}\mathrm{tr}[\underline{K}(t)\underline{B}(t)\underline{R}^{-1}(t)\underline{B}'(t)\underline{K}(t)\underline{\Sigma}(t)]dt \tag{5.1.9}$$

We can again expand this optimal cost incurred by using complete centralization in a power series in ε. Up to linear terms in ε, the result is given by [5]

$$J^{(0)}+\varepsilon J^{(1)}=\mathrm{tr}[\underline{K}_{11}^{(0)}(t_o)(\underline{\Sigma}_{011}+\underline{\bar{x}}_{10}\underline{\bar{x}}_{10}')+\underline{K}_{22}^{(0)}(t_o)(\underline{\Sigma}_{022}$$

$$+\underline{\bar{x}}_{20}\underline{\bar{x}}_{20}') + \int_{t_o}^{T}\mathrm{tr}[\underline{K}_{11}^{(0)}(t)\underline{\Lambda}_1(t)+\underline{K}_{22}^{(0)}(t)\underline{\Lambda}_2(t)]dt$$

$$+ \int_{t_o}^{T} \mathrm{tr}[\underline{K}_{11}^{(0)}\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_{11}^{(0)}\underline{\Sigma}_{11}^{(0)}$$

$$+\underline{K}_{22}^{(0)}\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_{22}^{(0)}\underline{\Sigma}_{22}^{(0)}]dt$$

$$+ \varepsilon \mathrm{tr}[\underline{K}_{12}^{(1)}(t_o)\underline{\bar{x}}_{20}\underline{\bar{x}}_{10}'+\underline{K}_{21}^{(1)}(t_o)\underline{\bar{x}}_{10}\underline{\bar{x}}_{20}'] \tag{5.1.10}$$

It can be shown that the cost (see Appendix D)

$$\hat{J}^{(0)}+\varepsilon\hat{J}^{(1)}= J^{(0)}+\varepsilon J^{(1)} \tag{5.1.11}$$

What this says is that even if we allow the mathematically optimal design of complete centralization, the improvement over our scheme is of the order of ε^2. Of course, since we are constraining our physical controls to be a function only of the individual system measurements respectively, the controls generated by complete centralization are not admissible. However, no matter what the optimal physical controls for our problem are, they can certainly not give a lower cost than the one obtained by using complete centralization. In other words, our system will perform optimally at least up to linear terms in ε. Since the coupling is weak (ε small), terms in quadratic and higher orders of ε do not contribute much to the overall system behavior. We may therefore conclude that our approach gives an approximately optimal design. This vindicates our claim that our complete control strategy is acceptable as far as the mathematical cost criterion is concerned.

A moment's reflection tells us that the results we have obtained are very reasonable. We have seen in Chapter 4 that $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ are both of the order of ε^2. Hence, we should expect that their effects on the system performance will also be of the order of ε^2, and so the optimal cost up to linear terms in ε will be independent of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$.

Another way to interpret the result of equation (5.1.11) is to go back to our original stochastic problem. Recall that in the formulation of the stochastic problem in Chapter 2, we have used quadratic terms in $\underline{x}_1(t)$ and $\underline{x}_2(t)$ in the cost functional. In the proposed model, we have replaced the coupling terms $\varepsilon\underline{A}_{12}(t)\underline{x}_2(t)$ and $\varepsilon\underline{A}_{21}(t)\underline{x}_1(t)$ by the zero mean, Gaussian white noise processes $\underline{\xi}_1(t)$ and $\underline{\xi}_2(t)$. What we have implicitly done is that we have modeled the effects of coupling terms on the systems to be zero on the average. Now since the coupling terms $\varepsilon\underline{A}_{12}(t)\underline{x}_2(t)$ and $\varepsilon\underline{A}_{21}(t)\underline{x}_1(t)$ influence $\underline{x}_1(t)$ and $\underline{x}_2(t)$ in a linear fashion, their effects on quadratic cost functions will be of the order of ε^2. Thus the overall cost will not be affected up to linear terms in ε on the average if we replace the coupling terms by things with zero mean. Suppose we now adopt the complete centralization scheme and apply the Separation Theorem directly to our model represented by equations (2.2.1) and (2.2.2). As we have argued, the resulting cost will be the same up to linear terms in ε as the one obtained if we have not adopted the model, and used the complete centralization policy to begin with. Notice, however, that applying centralized control policy <u>to our model</u> results in exactly the <u>same</u> physical controls as those given in our proposed strategy because the two systems are now decoupled. These arguments indicate, therefore, that equation (5.1.11) should hold. Our intuition indeed agrees well with the analytical results.

C.2

5.2 The Dependence of $\hat{J}$ on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$

We remarked earlier in this chapter that $\hat{J}^{(0)}+\epsilon\hat{J}^{(1)}$ does not depend on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. It can be shown that if $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ are of the order of ϵ^2, i.e., $\underline{\Xi}_1^{(0)}(t) = \underline{\Xi}_1^{(1)}(t) = \underline{\Xi}_2^{(0)}(t) = \underline{\Xi}_2^{(1)}(t)=\underline{0}$, then only $\hat{J}^{(4)}$ and the higher order terms of the cost will depend on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ (see Appendix E). In other words, any $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ which go to zero as ϵ^2 would up to ϵ^3 terms be the same as the optimal cost obtained when $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$ are used. Thus, for most purposes we may assume that the two systems are uncoupled. Only when we want to improve the performance of the system beyond the fourth terms in ϵ that we have to solve the two-point boundary value problem to find $\underline{\Xi}_1^*(t)$ and $\underline{\Xi}_2^*(t)$. This does not mean, of course, that the formulation and solution of the deterministic problem is a waste of time, because the calculations in Appendix E are based on the optimal solution. Only by investigating the properties of the optimal solution can we make such a statement concerning the choice of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$.

This result has important implications in studying the tradeoffs between the choice of control schemes and will be further discussed in the next section. It also leads up to the remark given in Chapter 3 concerning the choice of the additional term to be introduced in the cost functional. If instead of adding the term $\int_{t_o}^{T} tr[\underline{\Xi}_1(t)\underline{\Xi}_1'(t)+\underline{\Xi}_2(t)\underline{\Xi}_2'(t)]dt$ to the cost functional, we add a term of the form $\int_{t_o}^{T} tr\{[\underline{\Xi}_1(t)-\underline{D}_1(t)][\underline{\Xi}_1(t)-\underline{D}_1(t)]'$ $+ [\underline{\Xi}_2(t)-\underline{D}_2(t)][\underline{\Xi}_2-\underline{D}_2(t)]'\}dt$ where $\underline{D}_1(t)$ and $\underline{D}_2(t)$ are symmetric matrices formed from linear combinations of the components of $\underline{M}(t)$, which are zero up to first order in ε and go to zero as ε^2. An entirely analogous calculation to those given in Appendix A shows that the optimal pseudo-controls are then of the form

$$\underline{\Xi}_1^*(t) = \frac{1}{4}\,\underline{T}_3(t)\text{diag}|\sigma\ (\underline{S}_1^*(t)-2\underline{D}_1^*(t))|\underline{T}_3'(t) - \frac{1}{4}\,\underline{S}_1^*(t) + \frac{1}{2}\,\underline{D}_1^*(t) \tag{5.2.1}$$

$$\underline{\Xi}_2^*(t) = \frac{1}{4}\,\underline{T}_4(t)\text{diag}|\sigma\ (\underline{S}_2^*(t)-2\underline{D}_2^*(t))|\underline{T}_4'(t) - \frac{1}{4}\underline{S}_2^*(t) + \frac{1}{2}\,\underline{D}_2^*(t) \tag{5.2.2}$$

where $\underline{T}_3(t)$ and $\underline{T}_4(t)$ are orthogonal matrices which diagonalize $\underline{S}_1^*(t)-2\underline{D}_1^*(t)$ and $\underline{S}_2^*(t)-2\underline{D}_2^*(t)$ respectively. Since these pseudo-controls are of the order of ε^2, we see that they also have all the "correct" properties noted in our development, and they yield a cost which is the same as the one we have obtained in the main development up to ε^3 terms. The philosophy behind this choice of the additional term is that we want to track matrices, which may have been omitted in our approximate model, by the pseudo-controls. For example, we may take $\underline{D}_1(t)$ to be $\varepsilon^2\underline{A}_{12}(t)\underline{M}_{22}(t)\underline{A}'_{12}(t)$ because we have replaced $\varepsilon\underline{A}_{12}(t)\underline{x}_2(t)$ by $\underline{\xi}_1(t)$. Clearly, such a choice is justifiable. It is neither superior nor inferior to the one we used in the main development. Whether one adds a term to the cost functional to minimize the noise intensities or to track some system states will be a decision left to the discretion of the designer.

5.3 Optimal vs. Suboptimal

In this section, we turn to the question of the application of our proposed control strategy. What are the significant advantages obtained by using the proposed design as opposed to using complete centralization? The first thought that comes to one's mind is, of course, the tremendous reduction in the communication between the two systems. Indeed, this has been our original motivation in seeking other control schemes than complete centralization. Enough has been said in the previous pages on the technical and economic reasons for desiring this reduction, and this point will not be further belaboured here.

The second point that we wish to stress here is that in the completely centralized scheme, the state estimates are generated <u>on line</u> by a (n_1+n_2) dimensional Kalman filter, while in our proposed scheme, the state estimates are generated on line by two Kalman filters, one of dimension n_1 and the other n_2. When n_1 and n_2 are large, it is much easier to integrate two vector differential equations with dimension n_1 and n_2 respectively, than to integrate a single vector differential equation, but with dimension (n_1+n_2). It is thus computationally advantageous to use the proposed design in generating the state estimates and hence the physical controls.

Of course, one can think of many more advantages of working with the uncoupled systems, though the above gives only two important ones. These considerations lead to the question of tradeoffs between various choices of control strategies. There is no universal answer to such a question. The choice depends crucially on the problem at hand and the objectives of the designer. If computation time and communication facilities are not a problem, as in systems of very low dimension, then absolute optimality may be the primary goal and we may want to adopt complete centralization. However, if the dimension of the systems is large, and there are constraints on the available facilities, one may want to adopt the control scheme proposed here. The tradeoff lies precisely in whether we want to get optimal performance beyond linear terms in ε, regardless of the increased expenditure and technical difficulties involved, or we want a simple and efficient system, and are willing to accept approximate optimal performance as satisfactory. Such a decision is ultimately a test of the designer's experience and engineering judgement which no mathematical analysis can replace.

CHAPTER 6 - CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In the preceding chapters, we have considered the stochastic control of two weakly coupled linear systems using quadratic performance indices. Each system is equipped with its own controller who has access only to the noisy measurements made on the output of his system. The physical control input to each system is constrained to be a function only of the measurements carried out on that system, in order to reduce the required amount of communication between the systems. Instead of trying to solve for the optimal physical controls directly, we model the weak coupling as additional "fake" white plant noises. This completely decouples the two systems and the Separation Theorem is applied to each system individually to obtain simple, though somewhat suboptimal physical controls.

The need to completely specify the "fake" white plant noise processes prompts us to seek a systematic method for finding the covariances of these white noises. A deterministic matrix optimal control problem is formulated using the white noise covariances as control variables. Necessary conditions satisfied by the optimal covariances are derived using the matrix minimum principle. These involve the solution of three two-point boundary value problems. However, if the coupling between the systems is small enough, the white noise covariances can be chosen to be simply zero. In this situation, we need only to analyze the uncoupled systems.

We then compare the proposed control scheme to the mathematically optimal one of complete centralization. We show that the proposed strategy is approximately optimal. The practical advantages of using the design over that of complete centralization are given and the tradeoffs between the various choices of control scheme discussed.

In the course of the research reported here, a number of related problems have appeared. They represent interesting areas for future research.

1. We have shown in this thesis that if the coupling between the systems is weak enough, i.e., if quadratic and higher order terms in ε can be neglected, our proposed scheme is both mathematically optimal as well as computationally efficient. If ε is not small enough so that second order effects are still important, we must investigate how these are compared to the truly optimal design, before we can study the tradeoffs between the various control schemes. This important question will be complicated to examine because of the messy equations involved.

2. In [5], Chong has suggested another scheme of controlling weakly coupled linear stochastic systems by using partial centralization and cooperation between two controllers with different information sets. This scheme

is also a suboptimal one. It would be interesting to relate his method to our design and compare the system performance in both cases. A study of the tradeoffs between these two schemes are just as important from a practical viewpoint as the study of tradeoffs given in this thesis.

3. The approach of using white noises to approximate colored noise processes has also been used in filtering problems [10]. However, in most of these cases, the covariances of the white noises are obtained by trial-and-error. It seems that in some cases, it may be possible to formulate an optimization problem in the choice of the covariances in a manner suggested in this thesis. What is a suitable choice for the cost functional and how much loss in accuracy is incurred in the approximation represent interesting theoretical as well as practical questions for future research.

4. The power and practical significance of perturbation analysis does not seem to be very much appreciated in optimal control literature, though it has been widely used in stability theory. An important theoretical research topic would be to study the control of weakly nonlinear systems using the perturbational approach. In the deterministic case, application of the approach to the Hamilton-Jacobi equation seems to be fruitful. In the general stochastic setting, it becomes a much more complex but vastly challenging question.

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APPENDIX A - PROOF OF THEOREM IN SECTION 4.1

Using the definitions and notations in section 4.1, the scalar Hamiltonian function H for the deterministic problem posed in section 3.1 is given by

$$H = tr[\hat{\underline{Q}}\ \underline{M} + (\hat{\underline{A}}\ \underline{M} + \underline{M}\ \hat{\underline{A}}' + \hat{\underline{B}}\ \hat{\underline{\Theta}}\ \hat{\underline{B}}')\underline{P}' + (\underline{A}_{11}\underline{\Sigma}_1 + \underline{\Sigma}_1\underline{A}_{11}' - \underline{\Sigma}_1\underline{C}_{11}'\underline{\Theta}_1^{-1}\underline{C}_{11}\underline{\Sigma}_1$$

$$+ \underline{\Lambda}_1 + \underline{\Xi}_1)\underline{S}_1' + (\underline{A}_{22}\underline{\Sigma}_2 + \underline{\Sigma}_2\underline{A}_{22}' - \underline{\Sigma}_2\underline{C}_{22}'\underline{\Theta}_2^{-1}\underline{C}_{22}\underline{\Sigma}_2 + \underline{\Lambda}_2 + \underline{\Xi}_2)\underline{S}_2'$$

$$+ \underline{\Xi}_1\underline{\Xi}_1' + \underline{\Xi}_2\underline{\Xi}_2'] \tag{A.1}$$

Let $\underline{\Xi}_1(t) = \underline{N}_1'(t)\underline{N}_1(t)$, $\underline{\Xi}_2(t) = \underline{N}_2'(t)\underline{N}_2(t)$

Using the matrix minimum principle, the necessary conditions for optimality are, after a lot of manipulations

$$(1)\quad \dot{\underline{M}}^*(t) = \left.\frac{\partial H}{\partial \underline{P}}\right|_* = \hat{\underline{A}}^*(t)\underline{M}^*(t) + \underline{M}^*(t)\hat{\underline{A}}^{*\prime}(t) + \hat{\underline{B}}^*(t)\hat{\underline{\Theta}}(t)\hat{\underline{B}}^{*\prime}(t);$$

$$\underline{M}^*(t_0) = M(t_0) \tag{A.2}$$

$$(2)\quad \dot{\underline{P}}^*(t) = -\left.\frac{\partial H}{\partial \underline{M}}\right|_* = -\hat{\underline{A}}^{*\prime}P^*(t) - \underline{P}^*(t)\hat{\underline{A}}^*(t) - \hat{\underline{Q}}(t);$$

$$\underline{P}^*(T) = \hat{\underline{F}} \tag{A.3}$$

$$(3)\quad \dot{\underline{\Sigma}}_1^*(t) = \left.\frac{\partial H}{\partial \underline{S}_1}\right|_* = \underline{A}_{11}(t)\underline{\Sigma}_1^*(t)$$

$$+ \underline{\Sigma}_1^*(t)\underline{A}_{11}'(t) - \underline{\Sigma}_1^*(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^*(t)$$

$$+ \underline{\Lambda}_1(t) + \underline{\Xi}_1^*(t);\ \underline{\Sigma}_1^*(t_0) = \underline{\Sigma}_{011} \tag{A.4}$$

$$(4)\quad \dot{\underline{S}}_1{}^*(t) = -\frac{\partial H}{\partial \underline{\Sigma}_1}\Big|_* = -\underline{A}_{11}'(t)\underline{S}_1^*(t)-\underline{S}_1^*(t)\underline{A}_{11}(t)$$

$$+\underline{S}_1^*(t)\underline{\Sigma}_1^*(t)\underline{\Gamma}_1(t)$$

$$+\underline{\Gamma}_1(t)\underline{\Sigma}_1^*(t)\underline{S}_1^*(t)-\underline{P}_{33}^*(t)\underline{\Sigma}_1^*(t)\underline{\Gamma}_1(t)$$

$$-\underline{\Gamma}_1(t)\underline{\Sigma}_1^*(t)\underline{P}_{33}^*(t)$$

$$+\underline{P}_{31}^*(t)\underline{M}_{31}^{*\prime}(t)\underline{\Gamma}_1(t)+\underline{P}_{32}^*(t)\underline{M}_{32}^{*\prime}(t)\underline{\Gamma}_1(t)$$

$$+\underline{P}_{33}^*(t)\underline{M}_{33}^{*\prime}(t)\underline{\Gamma}_1(t)$$

$$+\underline{P}_{34}^*(t)\underline{M}_{34}^{*\prime}(t)\underline{\Gamma}_1(t)+\underline{\Gamma}_1(t)\underline{M}_{13}^{*\prime}(t)\underline{P}_{13}^*(t)$$

$$+\underline{\Gamma}_1(t)\underline{M}_{23}^{*\prime}(t)\underline{P}_{23}^*(t)$$

$$+\underline{\Gamma}_1(t)\underline{M}_{33}^{*\prime}(t)\underline{P}_{33}^*(t)+\underline{\Gamma}_1(t)\underline{M}_{43}^{*\prime}(t)\underline{P}_{43}^*(t);$$

$$\underline{S}_1^*(T) = \underline{0} \qquad \text{(A.5)}$$

where $\underline{\Gamma}_1(t) = \underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)$

$$(5)\quad \dot{\underline{\Sigma}}_2^*(t) = \frac{\partial H}{\partial \underline{S}_1}\Big|_* = \underline{A}_{22}(t)\underline{\Sigma}_2^*+\underline{\Sigma}_2^*(t)\underline{A}_{22}'(t)$$

$$-\underline{\Sigma}_2^*(t)\underline{C}_{22}'(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)\underline{\Sigma}_2^*(t)$$

$$+\underline{\Lambda}_2(t)+\underline{\Xi}_2^*;$$

$$\underline{\Sigma}_2^*(t_0) = \underline{\Sigma}_{022} \qquad \text{(A.6)}$$

$$(6)\quad \dot{\underline{S}}_2^*(t) = -\left.\frac{\partial H}{\partial \underline{\Sigma}_2}\right|_* = -\underline{A}_{22}'(t)\underline{S}_2^*(t) - \underline{S}_2^*(t)\underline{A}_{22}(t)$$
$$+\underline{S}_2^*(t)\underline{\Sigma}_2^*(t)\underline{\Gamma}_2(t) + \underline{\Gamma}_2(t)\underline{\Sigma}_2^*(t)\underline{S}_2^*(t)$$
$$-\underline{P}_{44}^*(t)\underline{\Sigma}_2^*(t)\underline{\Gamma}_2(t) + \underline{\Gamma}_2(t)\underline{\Sigma}_2^*(t)\underline{P}_{44}^*(t)$$
$$+\underline{P}_{41}^*(t)\underline{M}_{41}^{*\prime}(t)\underline{\Gamma}_2(t)$$
$$+\underline{P}_{42}^*(t)\underline{M}_{42}^{*\prime}(t)\underline{\Gamma}_2(t) + \underline{P}_{43}^*(t)\underline{M}_{43}^{*\prime}(t)\underline{\Gamma}_2(t)$$
$$+\underline{P}_{44}^*(t)\underline{M}_{44}^{*\prime}(t)\underline{\Gamma}_2(t)$$
$$+\underline{\Gamma}_2(t)\underline{M}_{14}^{*\prime}(t)\underline{P}_{14}^*(t) + \underline{\Gamma}_2(t)\underline{M}_{24}^{*\prime}(t)\underline{P}_{24}^*(t)$$
$$+\underline{\Gamma}_2(t)\underline{M}_{34}^{*\prime}(t)\underline{P}_{34}^*(t) + \underline{\Gamma}_2(t)\underline{M}_{44}^{*\prime}(t)\underline{P}_{44}^*(t)$$
$$\underline{S}_2^*(T) = \underline{0}$$

(A.7)

where $\underline{\Gamma}_2(t) = \underline{C}_{22}'(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)$

$$(7)\quad \underline{0} = \left.\frac{\partial H}{\partial \underline{N}_1}\right|_* = \underline{N}_1^*(t)[\underline{S}_1^*(t) + \underline{S}_1^{*\prime}(t)] + 4\underline{N}_1^*(t)\underline{N}_1^{*\prime}(t)\underline{N}_1^*(t)$$

(A.8)

On multiplying throughout by $\underline{N}_1^{*\prime}(t)$, we get

$$\underline{0} = \underline{\Xi}_1^*(t)[\underline{S}_1^*(t) + \underline{S}_1^{*\prime}(t) + 4\underline{\Xi}_1^*(t)] \qquad (A.9)$$

$$(8)\quad \underline{0} = \left.\frac{\partial H}{\partial \underline{N}_2}\right|_* = \underline{N}_2^*(t)[\underline{S}_2^*(t) + \underline{S}_2^{*\prime}(t)]$$
$$+4\underline{N}_2^*(t)\underline{N}_2^{*\prime}(t)\underline{N}_2^*(t) \qquad (A.10)$$

On multiplying throughout by $\underline{N}_2^{*\prime}(t)$, we get

$$\underline{0} = \underline{\Xi}_2^*(t)[\underline{S}_2^*(t) + \underline{S}_2^{*\prime}(t) + 4\underline{\Xi}_2^*(t)] \qquad (A.11)$$

Taking the transpose of (A.9), we get

$$4\underline{\Xi}_1^{*\prime}(t)\underline{\Xi}_1^{*\prime}(t)+\underline{S}_1^*(t)\underline{\Xi}_1^{*\prime}(t)+\underline{S}_1^{*\prime}(t)\underline{\Xi}_1^{*\prime}(t)=\underline{0} \tag{A.12}$$

Since both $\underline{\Xi}_1^*(t)$ and $\underline{S}_1^*(t)$ are symmetric, we get, on comparing with equation (A.9),

$$\underline{\Xi}_1^*(t)\underline{S}_1^*(t) = \underline{S}_1^*(t)\underline{\Xi}_1^*(t) \tag{A.13}$$

Thus, we can rewrite equation (A.9) as

$$4\underline{\Xi}_1^{*2}(t)+\underline{\Xi}_1^*(t)\underline{S}_1^*(t)+\underline{S}_1^*(t)\underline{\Xi}_1^*(t) = \underline{0} \tag{A.14}$$

On completing the square, we have

$$[\underline{\Xi}_1^*(t)+ \frac{1}{4}\,\underline{S}_1^*(t)]^2 = [\frac{1}{4}\,\underline{S}_1^*(t)]^2 \tag{A.15}$$

On taking the square root, we have

$$\underline{\Xi}_1^*(t) = \frac{1}{4}(\underline{S}_1^{*2}(t))^{\frac{1}{2}} - \frac{1}{4}\,\underline{S}_1^*(t) \tag{A.16}$$

Since we cannot make any <u>a priori</u> statements about the definitions of $\underline{S}_1^*(t)$, in general (A.16) has many admissible solutions. We will now demonstrate that one of these minimizes the Hamiltonian and is therefore the optimal control we seek.

Let $\underline{T}_1(t)$ be the orthogonal matrix which diagonalizes the symmetric matrix $\underline{S}_1^*(t)$ [12]. Then $\underline{T}_1(t)$ also diagonalizes $\underline{S}_1^{*2}(t)$. So,

$$[\text{diag}\ \sigma(\underline{S}_1^*(t))]^2 = \underline{T}_1'(t)\underline{S}_1^{*2}(t)\underline{T}_1(t) \tag{A.17}$$

On taking the square root and using the properties of functions of a matrix, we get

$$\{[\text{diag}\ \sigma(\underline{S}_1^*(t))]^2\}^{\frac{1}{2}} = \underline{T}_1'(t)[\underline{S}_1^{*2}(t)]^{\frac{1}{2}}\underline{T}_1(t) \tag{A.18}$$

(A.16) can now be written as

$$\underline{\Xi}_1^*(t) = \tfrac{1}{4}\underline{T}_1(t)\{[\text{diag}\ \sigma(\underline{S}_1^*(t))]^2\}^{\frac{1}{2}}\underline{T}_1'(t) - \tfrac{1}{4}\underline{S}_1^*(t) \tag{A.19}$$

(A.19) still has many admissible solutions, but we now claim that the solution

$$\underline{\Xi}_1^*(t) = \tfrac{1}{4}\underline{T}_1(t)\ \text{diag}|\sigma(\underline{S}_1^*(t))|\underline{T}_1'(t) - \tfrac{1}{4}\underline{S}_1^*(t) \tag{A.20}$$

minimizes the Hamiltonian. To see this, we first note that we need only to consider the behavior of the terms

$$\text{tr}[\underline{\Xi}_1^*(t)\underline{S}_1^{*\prime}(t)+\underline{\Xi}_1^*(t)\underline{\Xi}_1^{*\prime}(t)] = \text{tr}[\underline{\Xi}_1^*(t)\underline{S}_1^*(t)+\underline{\Xi}^{*2}(t)]$$

Since $\underline{T}_1(t)$ simultaneously diagonalizes $\underline{\Xi}_1^*(t)$ and $\underline{S}_1^*(t)$, these two terms are, by (A.19), just

$$\mathrm{tr}[\underline{\Xi}_1^*(t)\underline{S}_1^*(t)+\underline{\Xi}_1^{*2}(t)]$$

$$= \mathrm{tr}\ \{\tfrac{1}{4}[(\mathrm{diag}^2\sigma(\underline{S}_1^*(t)))^{\frac{1}{2}}-\mathrm{diag}\ \sigma(\underline{S}_1^*(t))]\ \mathrm{diag}\ \sigma(\underline{S}_1^*(t))\}$$

$$+ \mathrm{tr}\ \{\tfrac{1}{16}[(\mathrm{diag}^2\ \sigma(\underline{S}_1^*(t)))^{\frac{1}{2}}-\mathrm{diag}\ \sigma(\underline{S}_1^*(t))]^2\} \qquad (A.21)$$

There are three cases to consider. If $\underline{S}_1^*(t)$ is positive semidefinite, then the only positive semidefinite solution for $\underline{\Xi}_1^*(t)$ is, from (A.19),

$$\underline{\Xi}_1^*(t) = \underline{0} \qquad (A.22)$$

If $\underline{S}_1^*(t)$ is negative semidefinite, then taking the negative square roots in (A.19),

$$\underline{\Xi}_1^*(t) = -\tfrac{1}{4}\underline{T}_1(t)\ \mathrm{diag}\ \sigma(\underline{S}_1^*(t))\underline{T}_1'(t)-\tfrac{1}{4}\underline{S}_1^*(t) \qquad (A.23)$$

will minimize the terms given in (A.21). If $\underline{S}_1^*(t)$ is indefinite, taking the square roots in (A.19) such that the resulting diagonal matrix is positive semidefinite, i.e.,

$$\underline{\Xi}_1^*(t) = \tfrac{1}{4}\underline{T}_1(t)\mathrm{diag}|\sigma(\underline{S}_1^*(t))|\underline{T}_1'(t)-\tfrac{1}{4}\underline{S}_1^*(t) \qquad (A.24)$$

will again minimize the terms in (A.21). But equations (A.22) to (A.24) together say that the solution of $\underline{\Xi}_1^*(t)$ given in (A.20) is the admissible solution which minimizes the Hamiltonian H for any $\underline{S}_1^*(t)$. Hence it is the desired optimal control.

An entirely analogous development shows that

$$\underline{\Xi}_2^*(t) = \frac{1}{4}\underline{T}_2(t)\ \mathrm{diag}|\sigma(\underline{S}_2^*(t))|\underline{T}_2'(t) - \frac{1}{4}\underline{S}_2^*(t) \qquad (A.25)$$

where $\underline{T}_2(t)$ is the orthogonal matrix which diagonalizes $\underline{S}_2^*(t)$, is the desired optimal control.

Equations (A.2) to (A.7) and (A.20), (A.25) are just the results which comprise the theorem.

APPENDIX B - ANALYSIS OF THE UNCOUPLED CASE

We have seen in section 4.1 that $\underline{\Xi}_1^*(t)$ lies in the "range" of $\underline{0}$ to $-\frac{1}{2}\underline{S}_1^*(t)$ depending on whether $\underline{S}_1^*(t)$ is positive semidefinite, indefinite, or negative semidefinite. Since the solution of $\underline{\Xi}_1^*(t) = \underline{0}$ completely decouples the equations, it is the easiest to analyze. However, since we do not know the definitions of $\underline{S}_1^*(t)$ before we have completely solved the two-point boundary value problems, we do not know that by analyzing only the case of $\underline{\Xi}_1^*(t) = \underline{0}$, which corresponds to $\underline{S}_1^*(t) \geq \underline{0}$, will give us all the properties of the optimal solution. If we also analyze the case of $\underline{\Xi}_1^*(t) = -\frac{1}{2}\underline{S}_1^*(t)$, which corresponds to $\underline{S}_1^*(t) \leq \underline{0}$, we would have obtained the solution properties both for the maximum and the minimum $\underline{\Xi}_1^*(t)$. Intuitively, we feel that we would then be in a position to state all the qualitative properties of the optimal solutions. It will turn out that by analyzing the maximum solution $\underline{\Xi}_1^*(t) = -\frac{1}{2}\underline{S}_1^*(t)$, we can already make a lot of statements about the system behavior. Thus, we will always examine the case of $\underline{\Xi}_1^*(t) = -\frac{1}{2}\underline{S}_1^*(t)$ first. If it proves necessary to analyze the other cases, we will then proceed to do so.

We now turn to the business at hand, i.e., to examine the situation in which the systems were uncoupled to start with.

If $\varepsilon = 0$, then

$$\underline{\hat{A}}(t) = \begin{bmatrix} \underline{A}_{11}-\underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \underline{0} & \underline{B}_{11}\underline{R}_1^{-1}\underline{B}_{11}'\underline{K}_1 & \underline{0} \\ \underline{0} & \underline{A}_{22}-\underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 & \underline{0} & \underline{B}_{22}\underline{R}_2^{-1}\underline{B}_{22}'\underline{K}_2 \\ \underline{0} & \underline{0} & \underline{A}_{11}-\underline{\Sigma}_1\underline{C}_1'\underline{\Theta}_1^{-1}\underline{C}_{11} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{A}_{22}-\underline{\Sigma}_2\underline{C}_{22}'\underline{\Theta}_2^{-1}\underline{C}_{22} \end{bmatrix} \quad (B.1)$$

We have the following equations:

$$\begin{aligned} \dot{\underline{M}}_{33}(t) = {} & \underline{A}_{11}(t)\underline{M}_{33}(t)+\underline{M}_{33}(t)\underline{A}_{11}'(t)-\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{M}_{33}(t) \\ & - \underline{M}_{33}(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t) \\ & \qquad +\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t) \\ & + \underline{\Lambda}_1(t) \\ & \underline{M}_{33}(t_o) = \underline{\Sigma}_{011} \end{aligned} \quad (B.2)$$

$$\begin{aligned} \dot{\underline{\Sigma}}_1(t) = {} & \underline{A}_{11}(t)\underline{\Sigma}_1(t)+\underline{\Sigma}_1(t)\underline{A}_{11}'(t)-\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t) \\ & \qquad +\underline{\Lambda}_1(t)+\underline{\Xi}_1(t) \\ & \underline{\Sigma}_1(t_o) = \underline{\Sigma}_{011} \end{aligned} \quad (B.3)$$

$$\underline{\Xi}_1(t) = -\tfrac{1}{2}\underline{S}_1(t) \quad (B.4)$$

$$\underline{S}_1(t) = -\underline{A}_{11}'(t)\underline{S}_1(t)-\underline{S}_1(t)\underline{A}_{11}(t)+\underline{S}_1(t)\underline{\Sigma}_1(t)\underline{\Gamma}_1(t)$$
$$+\underline{\Gamma}_1(t)\underline{\Sigma}_1(t)\underline{S}_1(t)$$
$$-\underline{P}_{33}(t)\underline{\Sigma}_1(t)\underline{\Gamma}_1(t)-\underline{\Gamma}_1(t)\underline{\Sigma}_1(t)\underline{P}_{33}(t)$$
$$+\underline{P}_{31}(t)\underline{M}_{31}'(t)\underline{\Gamma}_1(t)$$
$$+\underline{P}_{32}(t)\underline{M}_{32}'(t)\underline{\Gamma}_1(t)+\underline{P}_{33}(t)\underline{M}_{33}'(t)\underline{\Gamma}_1(t)$$
$$+\underline{P}_{34}(t)\underline{M}_{34}'(t)\underline{\Gamma}_1(t)$$
$$+\underline{\Gamma}_1(t)\underline{M}_{13}'(t)\underline{P}_{13}(t)+\underline{\Gamma}_1(t)\underline{M}_{23}'(t)\underline{P}_{23}(t)$$
$$+\underline{\Gamma}_1(t)\underline{M}_{33}'(t)\underline{P}_{33}(t)$$
$$+\underline{\Gamma}_1(t)\underline{M}_{43}'(t)\underline{P}_{43}(t)$$
$$\underline{S}_1(T) = \underline{0} \tag{B.5}$$

All quantities are evaluated along the optimal trajectory. * has been omitted for simplicity.

On writing out the equations for the components of the costate $\underline{P}(t)$, we have

$$\dot{\underline{P}}_{31}(t) = -\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{P}_{11}(t)-[\underline{A}_{11}'(t)$$
$$-\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)]\underline{P}_{31}(t)$$
$$-\underline{P}_{31}(t)[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]$$
$$+\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}(t)\underline{K}_1(t)$$
$$\underline{P}_{31}(T) = \underline{0} \tag{B.6}$$

and $\dot{\underline{P}}_{11}(t)=-[\underline{A}_{11}'(t)-\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)]\underline{P}_{11}(t)$

$$-\underline{P}_{11}(t)[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]$$

$$-[Q_1(t)+\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)B_{11}'(t)\underline{K}_1(t)]$$

$$\underline{P}_{11}(T) = \underline{F}_1 \qquad (B.7)$$

Recalling that

$$\dot{\underline{K}}_1(t)=-\underline{A}_{11}'(t)\underline{K}_1(t)-\underline{K}_1(t)\underline{A}_{11}(t)$$

$$+\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)-\underline{Q}_1(t)$$

$$\underline{K}_1(T) = \underline{F}_1$$

we see that $\underline{P}_{11}(t) = \underline{K}_1(t)$ is a solution to equation (B.7). Since the equation is linear, it is the unique solution.

Substituting $\underline{P}_{11}(t) = \underline{K}_1(t)$ into (B.6), we get

$$\dot{\underline{P}}_{31}(t)=-[\underline{A}_{11}'(t)-\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)]\underline{P}_{31}(t)$$

$$-\underline{P}_{31}(t)[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]$$

$$\underline{P}_{31}(T) = \underline{0} \qquad (B.8)$$

We see that $\underline{P}_{31}(t)=\underline{0}$. Similarly we can prove that $\underline{P}_{32}(t)=\underline{P}_{34}(t)=\underline{P}_{13}(t)=\underline{P}_{23}(t)=\underline{P}_{43}(t)=\underline{0}$. Equation (B.5) is reduced to

$$\dot{\underline{S}}_1(t) = -\underline{A}_{11}'(t)\underline{S}_1(t) - \underline{S}_1(t)\underline{A}_{11}(t) + \underline{S}_1(t)\underline{\Sigma}_1(t)\underline{\Gamma}_1(t)$$
$$+\underline{\Gamma}_1(t)\underline{\Sigma}_1(t)\underline{S}_1(t)$$
$$-\underline{P}_{33}(t)\underline{\Sigma}_1(t)\underline{\Gamma}_1(t) - \underline{\Gamma}_1(t)\underline{\Sigma}_1(t)\underline{P}_{33}(t) + \underline{P}_{33}(t)\underline{M}_{33}'(t)\underline{\Gamma}_1(t)$$
$$+\underline{\Gamma}_1(t)\underline{M}_{33}'(t)\underline{P}_{33}(t)$$
$$\underline{S}_1(T) = \underline{0} \qquad \text{(B.9)}$$

Furthermore, $\underline{M}_{33}(t)$ is symmetric. Using equations (B.2) to (B.4), and (B.9), we obtain

$$\dot{\underline{M}}_{33}(t) - \underline{\Sigma}_1(t) = \underline{A}_{11}(t)[\underline{M}_{33}(t) - \underline{\Sigma}_1(t)]$$
$$+[\underline{M}_{33}(t) - \underline{\Sigma}_1(t)]\underline{A}_{11}'(t)$$
$$-\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)[\underline{M}_{33}(t) - \underline{\Sigma}_1(t)]$$
$$-[\underline{M}_{33}(t) - \Sigma_1(t)]\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)$$
$$+ \frac{1}{2}\underline{S}_1(t) \qquad \text{(B.10)}$$

Let $\underline{M}_{33}(t) - \underline{\Sigma}_1(t) = \tilde{\underline{M}}_\Sigma(t)$

Then

$$\dot{\tilde{\underline{M}}}_\Sigma(t) = [\underline{A}_{11}(t) - \underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]\tilde{\underline{M}}_\Sigma(t)$$
$$+\tilde{\underline{M}}_\Sigma(t)[\underline{A}_{11}'(t)$$
$$-\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)] + \frac{1}{2}\underline{S}_1(t)$$
$$\tilde{\underline{M}}_\Sigma(t_o) = \underline{0} \qquad \text{(B.11)}$$

$$\dot{\underline{S}}_1(t) = -[\underline{A}_{11}'(t) - \underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)]\underline{S}_1(t)$$
$$-\underline{S}_1(t)[\underline{A}_{11}(t) - \underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]$$
$$+\underline{P}_{33}(t)\tilde{\underline{M}}_\Sigma(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)$$
$$+\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\tilde{\underline{M}}_\Sigma(t)\underline{P}_{33}(t)$$
$$\underline{S}_1(T) = \underline{0} \quad \text{(B.12)}$$

$\tilde{\underline{M}}_\Sigma(t) = \underline{0}$ and $\underline{S}_1(t) = \underline{0}$ is a solution. Again, since the equations are linear, it is the unique solution. Thus we see that

$$\underline{\Xi}_1(t) = \underline{0} \quad \text{(B.13)}$$

$$\underline{M}_{33}(t) = \underline{\Sigma}_1(t) \quad \text{(B.14)}$$

Similarly, we can prove

$$\underline{\Xi}_2(t) = \underline{0} \quad \text{(B.15)}$$

$$\underline{M}_{44}(t) = \underline{\Sigma}_2(t) \quad \text{(B.16)}$$

Since $\underline{S}_1(t)$ and $\underline{S}_2(t)$ are themselves equal to zero, equations (B.13) and (B.15) represent the only solution for $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. In this case, we have no ambiguities.

APPENDIX C - FIRST AND SECOND ORDER PROPERTIES OF THE OPTIMAL SOLUTION

The equations of interest are:

$$\dot{\underline{M}}_{33}(t)=[\underline{A}_{11}(t)-\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]\underline{M}_{33}(t)$$
$$+\underline{M}_{33}(t)[\underline{A}_{11}'(t)-\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)]$$
$$+\varepsilon[\underline{A}_{12}(t)\underline{M}_{23}(t)\underline{+M}_{32}(t)\underline{A}_{12}'(t)]$$
$$+\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)+\underline{\Lambda}_1(t)$$
$$\underline{M}_{33}(t_0)=\underline{\Sigma}_{011} \qquad \text{(C.1)}$$

$$\dot{\underline{\Sigma}}_1(t)=\underline{A}_{11}(t)\underline{\Sigma}_1(t)+\underline{\Sigma}_1(t)\underline{A}_{11}'(t)-\underline{\Sigma}_1(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1(t)$$
$$+\underline{\Lambda}_1(t)-\frac{1}{2}\underline{S}_1(t)$$
$$\underline{\Sigma}_1(t_0)=\underline{\Sigma}_{011} \qquad \text{(C.2)}$$

$$\dot{\underline{S}}_1(t)=-\underline{A}_{11}'(t)\underline{S}_1(t)-\underline{S}_1(t)\underline{A}_{11}(t)+\underline{S}_1(t)\underline{\Sigma}_1(t)\underline{\Gamma}_1(t)$$
$$+\underline{\Gamma}_1(t)\underline{\Sigma}_1(t)S_1(t)-\underline{P}_{33}(t)\underline{\Sigma}_1(t)\underline{\Gamma}_1(t)$$
$$-\underline{\Gamma}_1(t)\underline{\Sigma}_1(t)\underline{P}_{33}(t)+\underline{P}_{31}(t)\underline{M}_{31}'(t)\underline{\Gamma}_1(t)$$
$$+\underline{P}_{32}(t)\underline{M}_{32}'(t)\underline{\Gamma}_1(t)+\underline{P}_{33}(t)\underline{M}_{33}'(t)\underline{\Gamma}_1(t)$$
$$+\underline{P}_{34}(t)\underline{M}_{34}'(t)\underline{\Gamma}_1(t)+\underline{\Gamma}_1(t)\underline{M}_{13}'(t)\underline{P}_{13}(t)$$
$$+\underline{\Gamma}_1(t)\underline{M}_{23}'(t)\underline{P}_{23}(t)+\underline{\Gamma}_1(t)\underline{M}_{33}'(t)\underline{P}_{33}(t)$$
$$+\underline{\Gamma}_1(t)\underline{M}_{43}'(t)\underline{P}_{43}(t)$$
$$\underline{S}_1(T)=\underline{0} \qquad \text{(C.3)}$$

We use the perturbation approach and expand the equations in powers of ε. The relations for the zeroth order terms are the same as those obtained in Appendix B. Hence

$$\underline{M}_{33}^{(0)}(t) = \underline{\Sigma}_1^{(0)}(t) \tag{C.4}$$

$$\underline{S}_1^{(0)}(t) = \underline{0} \tag{C.5}$$

The first order terms satisfy the following equations

$$\begin{aligned}\dot{\underline{M}}_{33}^{(1)}(t) &= [\underline{A}_{11}(t)\underline{\Sigma}_1^{(0)}(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]\underline{M}_{33}^{(1)}(t)\\ &+\underline{M}_{33}^{(1)}(t)[\underline{A}_{11}'(t)-\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^{(0)}(t)]\\ \underline{M}_{33}^{(1)}(t_0) &= \underline{0}\end{aligned} \tag{C.6}$$

$$\begin{aligned}\dot{\underline{\Sigma}}_1^{(1)}(t) &= [\underline{A}_{11}(t)-\underline{\Sigma}_1^{(0)}(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]\underline{\Sigma}_1^{(1)}(t)\\ &+\underline{\Sigma}_1^{(1)}(t)[\underline{A}_{11}'(t)-\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^{(0)}(t)]\\ &-\tfrac{1}{2}\underline{S}_1^{(1)}(t)\\ \underline{\Sigma}_1^{(1)}(t_0) &= \underline{0}\end{aligned} \tag{C.7}$$

$$\begin{aligned}\dot{\underline{S}}_1^{(1)}(t) &= -\underline{A}_{11}'(t)\underline{S}_1^{(1)}(t)-\underline{S}_1^{(1)}(t)\underline{A}_{11}(t)+\underline{S}_1^{(1)}(t)\underline{\Sigma}_1^{(0)}(t)\underline{\Gamma}_1(t)\\ &+\underline{\Gamma}_1(t)\underline{\Sigma}_1^{(0)}(t)\underline{S}_1^{(0)}(t)-\underline{P}_{33}^{(0)}(t)\underline{\Sigma}_1^{(1)}(t)\underline{\Gamma}_1(t)\\ &-\underline{\Gamma}_1(t)\underline{\Sigma}_1^{(1)}(t)\underline{P}_{33}^{(0)}(t)+\underline{P}_{33}^{(0)}(t)\underline{M}_{33}^{(1)}(t)\underline{\Gamma}_1(t)\\ &+\underline{\Gamma}_1(t)\underline{M}_{33}^{(1)}(t)\underline{P}_{33}^{(0)}(t)+\underline{P}_{31}^{(1)}(t)\underline{M}_{31}^{(0)\prime}(t)\underline{\Gamma}_1(t)\\ &+\underline{\Gamma}_1(t)\underline{M}_{13}^{(0)\prime}(t)\underline{P}_{13}^{(1)}(t)+\underline{P}_{32}^{(1)}(t)\underline{M}_{32}^{(0)\prime}(t)\underline{\Gamma}_1(t)\end{aligned}$$

$$+\Gamma_1(t)M_{23}^{(0)\prime}(t)P_{23}^{(1)}(t)+P_{34}^{(1)}(t)M_{34}^{(0)\prime}(t)\Gamma_1(t)$$

$$+\Gamma_1(t)M_{43}^{(0)\prime}(t)P_{43}(t)$$

$$S_1^{(1)}(T) = 0 \tag{C.8}$$

where we have used the results of Appendix B and Eq. (C.4) and (C.5). Since

$$\dot{M}_{34}^{(0)}(t)=[A_{11}(t)-\Sigma_1^{(0)}(t)C_{11}'(t)\Theta_1^{-1}(t)C_{11}(t)]M_{34}^{(0)}(t)$$

$$+M_{34}^{(0)}(t)[A_{22}(t)-C_{22}'(t)\Theta_2^{-1}(t)C_{22}(t)\Sigma_2^{(0)}(t)]$$

$$M_{34}^{(0)}(t_0)=0 \tag{C.9}$$

we have $M_{34}^{(0)}(t) = 0$ (C.10)

Similarly, we can show that

$$M_{14}^{(0)}(t)=M_{41}^{(0)}(t)=M_{23}^{(0)}(t)=M_{32}^{(0)}(t)=M_{43}^{(0)}(t)=0 \tag{C.11}$$

Now $\dot{P}_{13}^{(1)}(t)=-[A_{11}'(t)-K_1(t)B_{11}(t)R_1^{-1}(t)B_{11}'(t)]P_{13}^{(1)}(t)$

$$-P_{13}^{(1)}(t)[A_{11}(t)-\Sigma_1^{(0)}(t)C_{11}'(t)\Theta_1^{-1}(t)C_{11}(t)]$$

$$-P_{11}^{(1)}(t)B_{11}(t)R_1^{-1}(t)B_{11}'(t)K_1(t)$$

$$P_{13}^{(1)}(T) = 0 \tag{C.12}$$

and

$$\dot{P}_{11}^{(1)}(t)=-[A_{11}'(t)-K_1(t)B_{11}(t)R_1^{-1}(t)B_{11}'(t)]P_{11}^{(1)}(t)$$

$$-P_{11}^{(1)}(t)[A_{11}(t)-B_{11}(t)R_1^{-1}(t)B_{11}'(t)K_1(t)]$$

$$P_{11}^{(1)}(T) = 0 \tag{C.13}$$

Hence $\underline{P}_{11}^{(1)}(t) = \underline{0}$ (C.14)

This in turn implies that

$$\underline{P}_{13}^{(1)}(t) = \underline{0} \tag{C.15}$$

Thus Eq. (C.8) is reduced to

$$\begin{aligned}\dot{\underline{S}}_1^{(1)}(t)=&-[\underline{A}_{11}'(t)-\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^{(0)}(t)]\underline{S}_1^{(1)}(t)\\&-\underline{S}_1^{(1)}(t)[\underline{A}_{11}(t)-\underline{\Sigma}_1^{(0)}(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]\\&+\underline{P}_{33}^{(0)}(t)[\underline{M}_{33}^{(1)}(t)-\underline{\Sigma}_1^{(1)}(t)]\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\\&+\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)[\underline{M}_{33}^{(1)}(t)-\underline{\Sigma}_1^{(1)}(t)]\underline{P}_{33}^{(0)}(t)\\&\underline{S}_1^{(1)}(T) = \underline{0}\end{aligned} \tag{C.16}$$

Simultaneous solutions for Eqs. (C.6), (C.7), and (C.16) give

$$\underline{M}_{33}^{(1)}(t) = \underline{\Sigma}_1^{(1)}(t) = \underline{0} \tag{C.17}$$

$$\underline{S}_1^{(1)}(t) = \underline{0} \tag{C.18}$$

We can similarly prove that

$$\underline{M}_{44}^{(1)}(t)=\underline{\Sigma}_2^{(1)}(t) = \underline{0} \tag{C.19}$$

$$\underline{S}_2^{(1)}(t) = \underline{0} \tag{C.20}$$

Hence $\quad \underline{\Xi}_1^{(1)}(t) = \underline{0}$ (C.21)

$$\underline{\Xi}_2^{(1)}(t) = \underline{0} \tag{C.22}$$

Since $\underline{S}_1^{(0)}(t)=\underline{S}_1^{(1)}(t)=\underline{S}_2^{(0)}(t)=\underline{S}_2^{(1)}(t)=\underline{0}$, we have, as in Appendix B,

$$\underline{\Xi}_1^{(0)}(t) = \underline{\Xi}_1^{(1)}(t) = \underline{\Xi}_2^{(0)}(t) = \underline{\Xi}_2^{(1)}(t) = \underline{0}$$

as the unique optimal solutions for the zeroth and first order terms. The second order terms satisfy the following relations

$$\begin{aligned}\dot{\underline{M}}_{33}^{(2)}(t)=&[\underline{A}_{11}(t)-\underline{\Sigma}_1^{(0)}(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]\underline{M}_{33}^{(2)}(t)\\ &+\underline{M}_{33}^{(2)}(t)[\underline{A}_{11}'(t)=\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^{(0)}(t)]\\ &+\underline{A}_{12}(t)\underline{M}_{23}^{(1)}(t)+\underline{M}_{32}^{(1)}(t)\underline{A}_{12}'(t)\\ &\underline{M}_{33}^{(2)}(t_0) = \underline{0}\end{aligned} \tag{C.23}$$

$$\begin{aligned}\dot{\underline{\Sigma}}_1^{(2)}(t)=&[\underline{A}_{11}(t)-\underline{\Sigma}_1^{(0)}(t)\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)]\underline{\Sigma}_1^{(2)}(t)\\ &+\underline{\Sigma}_1^{(2)}(t)[\underline{A}_{11}'(t)-\underline{C}_{11}'(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^{(0)}(t)]\\ &-\tfrac{1}{2}\underline{S}_1^{(2)}(t)\\ &\underline{\Sigma}_1^{(2)}(t_0) = \underline{0}\end{aligned} \tag{C.24}$$

$$\dot{\underline{S}}_1^{(2)}(t)=-[\underline{A}_{11}'(t)-\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^{(0)}(t)]\underline{S}_1^{(2)}(t)$$
$$-\underline{S}_1^{(2)}(t)[\underline{A}_{11}(t)-\underline{\Sigma}_1^{(0)}(t)\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)]$$
$$+\underline{P}_{33}^{(0)}(t)[\underline{M}_{33}^{(2)}(t)-\underline{\Sigma}_1^{(2)}(t)]\underline{C}_{11}(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)$$
$$+\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)[\underline{M}_{33}^{(2)}(t)-\underline{\Sigma}_1^{(2)}(t)]\underline{P}_{33}^{(0)}(t)$$
$$+\underline{P}_{31}^{(2)}(t)\underline{M}_{31}^{(0)'}(t)\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)$$
$$+\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{M}_{13}^{(0)'}(t)\underline{P}_{13}^{(2)}(t)$$
$$+\underline{P}_{32}^{(1)}(t)\underline{M}_{32}^{(1)'}(t)\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)$$
$$+\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{M}_{23}^{(1)'}(t)\underline{P}_{23}^{(1)}(t)$$
$$+\underline{P}_{34}^{(1)}(t)\underline{M}_{34}^{(1)'}(t)\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)$$
$$+\underline{C}_{11}'(t)\underline{\theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{M}_{43}^{(1)'}(t)\underline{P}_{43}^{(1)}(t)$$
$$\underline{S}_1^{(2)}(T) = \underline{0} \qquad \text{(C.25)}$$

If we analyze the forcing terms in the equation (C.25), we see that a lot of them are nonzero. For example, $\underline{P}_{31}^{(2)}(t)$ is nonzero, and depends, in a very complicated way, on the various other components of the matrix $\underline{P}(t)$. Therefore, we see that $\underline{S}_1^{(2)}(t)$ is nonzero, and so $\underline{\Xi}_1^{(2)}(t)$ is nonzero. Furthermore, $\underline{M}_{33}^{(2)}(t)$ is not equal to $\underline{\Sigma}_1^{(2)}(t)$. Similarly, $\underline{S}_2^{(2)}(t)$, $\underline{\Xi}_2^{(2)}(t)$ are nonzero and $\underline{M}_{44}^{(2)}(t)$ is not equal to $\underline{\Sigma}_2^{(2)}(t)$. We conclude that the optimal pseudo-controls will affect the <u>states</u> in quadratic and higher orders in ε.

APPENDIX D - PROOF OF THE APPROXIMATE OPTIMALITY OF THE PROPOSED DESIGN

Consider

$$- \frac{d}{dt}[\underline{x}_1'(t)\underline{K}_1(t)\underline{x}_1(t)]+\dot{\underline{x}}_1'(t)\underline{K}_1(t)\underline{x}_1(t)+\underline{x}_1'(t)\dot{\underline{K}}_1(t)\underline{x}_1(t) + \underline{x}_1'(t)\underline{K}_1(t)\dot{\underline{x}}_1(t)=\underline{0} \quad (D.1)$$

Using equations (2.1.1), (2.2.1), and (2.2.3) we obtain

$$\begin{aligned} &- \frac{d}{dt}[\underline{x}_1'(t)\underline{K}_1(t)\underline{x}_1(t)]+\underline{x}_1'(t)\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)\underline{x}_1(t) \\ &-\hat{\underline{x}}_{11}'(t)\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)\underline{x}_1(t) \\ &-\underline{x}_1'(t)\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\hat{\underline{x}}_{11}(t) \\ &+\varepsilon\underline{x}_2'(t)\underline{A}_{12}'(t)\underline{K}_1(t)\underline{x}_1(t)+\varepsilon\underline{x}_1'(t)\underline{K}_1(t)\underline{A}_{12}(t)\underline{x}_2(t) \\ &+\underline{\lambda}_1'(t)\underline{K}_1(t)\underline{x}_1(t)+\underline{x}_1'(t)\underline{K}_1(t)\underline{\lambda}_1(t) \\ &= \underline{0} \end{aligned} \quad (D.2)$$

Adding equation (D.2) to (2.1.5) we get

$$\begin{aligned} J_1 = E\{&\underline{x}_1'(T)\underline{F}_1\underline{x}_1(T)+ \int_{t_o}^{T} - \frac{d}{dt}[\underline{x}_1'(t)\underline{K}_1(t)\underline{x}_1(t)] \\ &+\varepsilon\underline{x}_2'(t)\underline{A}_{12}'(t)\underline{K}_1(t)\underline{x}_1(t)+\varepsilon\underline{x}_1'(t)\underline{K}_1(t)\underline{A}_{12}(t)\underline{x}_2(t) \\ &+[\underline{x}_1(t)-\hat{\underline{x}}_{11}(t)]'\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)[\underline{x}_1(t) \\ &-\hat{\underline{x}}_{11}(t)]+\underline{\lambda}_1'(t)\underline{K}_1(t)\underline{x}_1(t)+\underline{x}_1'(t)\underline{K}_1(t)\underline{\lambda}_1(t)dt\} \end{aligned}$$

$$= E\{\underline{x}_1'(t_o)\underline{K}_1(t_o)\underline{x}_1(t_o) + \int_{t_o}^{T}[\underline{x}_1(t)$$

$$- \hat{\underline{x}}_{11}(t)]'\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)[\underline{x}_1(t)-\hat{\underline{x}}_{11}(t)]$$

$$+ \varepsilon\underline{x}_1'(t)\underline{K}_1(t)\underline{A}_{12}(t)\underline{x}_2(t)+\varepsilon\underline{x}_2'(t)\underline{A}_{12}'(t)\underline{K}_1(t)\underline{x}_1(t)dt\}$$

$$+ \int_{t_o}^{T} tr[\underline{K}_1(t)\underline{\Lambda}_1(t)]dt \qquad (D.3)$$

Using the definition of the $\underline{M}_{ij}(t)$'s, we obtain

$$J_1 = tr[\underline{K}_1(t_o)(\underline{\Sigma}_{011}+\bar{\underline{x}}_{10}\bar{\underline{x}}_{10}')] + \int_{t_o}^{T} tr[\underline{K}_1(t)\underline{\Lambda}_1(t)]dt$$

$$+ \int_{t_o}^{T} tr[\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)\underline{M}_{33}(t)$$

$$+ 2\varepsilon\underline{K}_1(t)\underline{A}_{12}(t)\underline{M}_{21}(t)]dt \qquad (D.4)$$

Similarly J_2 can be shown to be

$$J_2 = tr[\underline{K}_2(t_o)(\underline{\Sigma}_{022}+\bar{\underline{x}}_{20}\bar{\underline{x}}_{20}')] + \int_{t_o}^{T} tr[\underline{K}_2(t)\underline{\Lambda}_2(t)]dt$$

$$+ \int_{t_o}^{T} tr[\underline{K}_2(t)\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)\underline{M}_{44}(t)$$

$$+ 2\varepsilon\underline{K}_2(t)\underline{A}_{21}(t)\underline{M}_{12}(t)]dt \qquad (D.5)$$

Since $\hat{J}=J_1+J_2+\int_{t_o}^{T}tr[\underline{\Xi}_1(t)\underline{\Xi}_1'(t)+\underline{\Xi}_2(t)\underline{\Xi}_2'(t)]dt$ we obtain equation (5.1.1).

Now, using (5.1.9) and expanding

$$\underline{\Sigma}_{ij}(t)=\underline{\Sigma}_{ij}^{(0)}(t)+\varepsilon\underline{\Sigma}_{ij}^{(1)}(t)+0(\varepsilon,t) \quad i=1,2;\ j=1,2.$$

$$\underline{K}_{ij}(t)=\underline{K}_{ij}^{(0)}(t)+\varepsilon\underline{K}_{ij}^{(1)}(t)+0(\varepsilon,t)$$

where $\lim_{\varepsilon\to 0}\frac{0(,t)}{\varepsilon}=\underline{0}$ uniformly in t, it can be shown that the optimal cost up to linear terms in ε is given by [5]

$$\begin{aligned}J^{*(0)}+\varepsilon J^{*(1)}=&tr[\underline{K}_{11}^{(0)}(t_o)(\underline{\Sigma}_{011}+\underline{\bar{x}}_{10}'\underline{\bar{x}}_{10}')+\underline{K}_{22}^{(0)}(t_o)(\underline{\Sigma}_{022}\\&+\underline{\bar{x}}_{20}\underline{\bar{x}}_{20}')]\\&+\int_{t_o}^{T}tr[\underline{K}_{11}^{(0)}(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_{11}^{(0)}(t)\underline{\Sigma}_{11}^{(0)}(t)\\&+\underline{K}_{22}^{(0)}(t)\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_{22}^{(0)}(t)\underline{\Sigma}_{22}^{(0)}(t)]dt\\&+\int_{t_o}^{T}tr[\underline{K}_{11}^{(0)}(t)\underline{\Lambda}_1(t)\\&+\underline{K}_{22}^{(0)}(t)\underline{\Lambda}_2(t)]dt+\varepsilon tr[\underline{K}_{12}^{(1)}(t_o)\underline{\bar{x}}_{20}\underline{\bar{x}}_{10}'+\underline{K}_{21}^{(1)}(t_o)\underline{\bar{x}}_{10}\underline{\bar{x}}_{20}']\end{aligned} \tag{D.6}$$

where $\underline{K}_{11}^{(0)}$, $\underline{K}_{22}^{(0)}(t)$, $\underline{K}_{12}^{(1)}(t)$, $\underline{K}_{21}^{(1)}(t)$, $\underline{\Sigma}_{11}^{(0)}(t)$, and $\underline{\Sigma}_{22}^{(0)}(t)$ satisfy the following equations

$$\underline{\dot{K}}_{11}^{(0)}(t) = -\underline{A}_{11}'(t)\underline{K}_{11}^{(0)}(t)-\underline{K}_{11}^{(0)}(t)\underline{A}_{11}(t)$$

$$+\underline{K}_{11}^{(0)}(t)\underline{B}_{11}(t)\underline{R}_{1}^{-1}(t)\underline{B}_{11}'(t)\underline{K}_{11}^{(0)}(t)-\underline{Q}_{1}(t)$$

$$\underline{K}_{11}^{(0)}(T) = \underline{F}_{1} \qquad (D.7)$$

$$\underline{\dot{K}}_{22}^{(0)}(t) = -\underline{A}_{22}'(t)\underline{K}_{22}^{(0)}(t)-\underline{K}_{22}^{(0)}(t)\underline{A}_{22}(t)$$

$$+\underline{K}_{22}^{(0)}\underline{B}_{22}(t)\underline{R}_{2}^{-1}(t)\underline{B}_{22}'(t)\underline{K}_{22}^{(0)}(t)-\underline{Q}_{2}(t)$$

$$\underline{K}_{22}^{(0)}(T) = \underline{F}_{2} \qquad (D.8)$$

$$\underline{\dot{K}}_{12}^{(1)} =[-\underline{A}_{11}'(t)+\underline{K}_{11}^{(0)}(t)\underline{B}_{11}(t)\underline{R}_{1}^{-1}(t)\underline{B}_{11}'(t)]\underline{K}_{12}^{(1)}(t)$$

$$+\underline{K}_{12}^{(1)}(t)[-\underline{A}_{22}(t)+\underline{B}_{22}(t)\underline{R}_{2}^{-1}(t)\underline{B}_{22}'(t)\underline{K}_{22}^{(0)}(t)]$$

$$-\underline{A}_{21}'(t)\underline{K}_{22}^{(0)}(t)-\underline{K}_{11}^{(0)}(t)\underline{A}_{12}(t)$$

$$\underline{K}_{12}^{(1)}(T) = \underline{0} \qquad (D.9)$$

$$\underline{\dot{K}}_{21}^{(1)}(t) = [-\underline{A}_{22}'(t)+\underline{K}_{22}^{(0)}(t)\underline{B}_{22}(t)\underline{R}_{2}^{-1}(t)\underline{B}_{22}'(t)]\underline{K}_{21}^{(1)}(t)$$

$$+\underline{K}_{21}^{(1)}(t)[-\underline{A}_{11}(t)+\underline{B}_{11}(t)\underline{R}_{1}^{-1}(t)\underline{B}_{11}'(t)\underline{K}_{11}^{(0)}(t)]$$

$$-\underline{A}_{12}'(t)\underline{K}_{11}^{(0)}(t)-\underline{K}_{22}^{(0)}(t)\underline{A}_{21}(t)$$

$$\underline{K}_{21}^{(1)}(T) = \underline{0} \qquad (D.10)$$

$$\underline{\dot{\Sigma}}_{11}^{(0)}(t) = \underline{A}_{11}(t)\underline{\Sigma}_{11}^{(0)}(t)+\underline{\Sigma}_{11}^{(0)}(t)\underline{A}_{11}'(t)$$

$$-\underline{\Sigma}_{11}^{(0)}(t)\underline{C}_{11}'(t)\underline{\Theta}_{1}^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_{11}^{(0)}(t)+\underline{\Lambda}(t)$$

$$\underline{\Sigma}_{11}^{(0)}(t_o) = \underline{\Sigma}_{011} \qquad (D.11)$$

$$\dot{\underline{\Sigma}}_{22}^{(0)}(t) = \underline{A}_{22}(t)\underline{\Sigma}_{22}^{(0)}(t)+\underline{\Sigma}_{22}^{(0)}(t)\underline{A}'_{22}(t)$$
$$-\underline{\Sigma}_{22}^{(0)}(t)\underline{C}'_{22}(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)\underline{\Sigma}_{22}^{(0)}(t)+\underline{\Lambda}_2(t)$$
$$\underline{\Sigma}_{22}^{(0)}(t_o)=\underline{\Sigma}_{022} \tag{D.12}$$

From Appendix B, we see that the zeroth order terms in $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ satisfy the equations

$$\dot{\underline{\Sigma}}_1^{(0)}(t)=\underline{A}_{11}(t)\underline{\Sigma}_1^{(0)}(t)+\underline{\Sigma}_1^{(0)}(t)\underline{A}'_{11}(t)$$
$$-\underline{\Sigma}_1^{(0)}(t)\underline{C}'_{11}(t)\underline{\Theta}_1^{-1}(t)\underline{C}_{11}(t)\underline{\Sigma}_1^{(0)}(t)+\underline{\Lambda}_1(t)$$
$$\underline{\Sigma}_1^{(0)}(t_o) = \underline{\Sigma}_{011} \tag{D.13}$$

$$\dot{\underline{\Sigma}}_2^{(0)}(t)=\underline{A}_{22}(t)\underline{\Sigma}_2^{(0)}(t)+\underline{\Sigma}_2^{(0)}(t)\underline{A}'_{22}(t)$$
$$-\underline{\Sigma}_2^{(0)}(t)\underline{C}'_{22}(t)\underline{\Theta}_2^{-1}(t)\underline{C}_{22}(t)\underline{\Sigma}_2^{(0)}(t)+\underline{\Lambda}_2(t)$$
$$\underline{\Sigma}_2^{(0)}(t_o) = \underline{\Sigma}_{022} \tag{D.14}$$

On comparing equations (D.7), (D.8), (D.11) and (D.12) to (2.2.5), (2.2.6), (D.13) and (D.14), we see that

$$\underline{K}_{11}^{(0)}(t) = \underline{K}_1(t) \tag{D.15}$$

$$\underline{K}_{22}^{(0)}(t) = \underline{K}_2(t) \tag{D.16}$$

$$\underline{\Sigma}_{11}^{(0)}(t) = \underline{\Sigma}_1^{(0)}(t) \tag{D.17}$$

$$\underline{\Sigma}_{22}^{(0)}(t) = \underline{\Sigma}_2^{(0)}(t) \tag{D.18}$$

Recalling that

$$\hat{J}^{*(0)}+\epsilon\hat{J}^{*(1)}=tr[\underline{K}_1(t_o)(\underline{\Sigma}_{011}+\underline{\bar{x}}_{10}\underline{\bar{x}}'_{10})+\underline{K}_2(t_o)(\underline{\Sigma}_{022}+\underline{\bar{x}}_{20}\underline{\bar{x}}'_{20})]+\int_{t_o}^{T} tr[\underline{K}_1(t)\underline{\Lambda}_1(t)+\underline{K}_2(t)\underline{\Lambda}_2(t)]dt$$
$$+\int_{t_o}^{T} tr[\underline{K}_1\underline{B}_{11}\underline{R}_1^{-1}\underline{B}'_{11}\underline{K}_1\underline{M}_{33}^{(0)}+\underline{K}_2\underline{B}_{22}\underline{R}_2^{-1}\underline{B}'_{22}\underline{K}_2\underline{M}_{44}^{(0)}+2\epsilon(\underline{K}_1\underline{A}_{12}\underline{M}_{21}^{(0)}+\underline{K}_2\underline{A}_{21}\underline{M}_{12}^{(0)})]dt \qquad (D.19)$$

and that

$$\underline{M}_{33}^{(0)}(t) = \underline{\Sigma}_1^{(0)}(t) \qquad (D.20)$$

$$\underline{M}_{44}^{(0)}(t) = \underline{\Sigma}_2^{(0)}(t) \qquad (D.21)$$

we see, on comparing equations (D.6) and (D.19), that $J^{*(0)}+\epsilon J^{*(1)}$ looks different from $J^{*(0)}+\epsilon J^{*(1)}$ only in so far as the former contains the term

$$\epsilon tr[\underline{K}_{12}^{(1)}(t_o)\underline{\bar{x}}_{20}\underline{\bar{x}}'_{10}+\underline{K}_{21}^{(1)}(t_o)\underline{\bar{x}}_{10}\underline{\bar{x}}'_{20}]$$

while the latter contains instead the terms

$$\int_{t_o}^{T} tr[2\epsilon(\underline{K}_1\underline{A}_{12}\underline{M}_{21}^{(0)}+\underline{K}_2\underline{A}_{21}\underline{M}_{12}^{(0)})]dt$$

In the following, we shall show that these two terms are in fact equal. To this end, we consider the equations for $\underline{M}_{12}(t)$ and $\underline{M}_{21}(t)$.

$$\begin{aligned}\dot{\underline{M}}_{12}(t)=&[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]\underline{M}_{12}(t)\\ &+\underline{M}_{12}(t)[\underline{A}_{22}(t)-\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)]\\ &+\varepsilon\,\underline{A}_{12}(t)\underline{M}_{22}(t)+\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)\underline{M}_{32}(t)\\ &+\varepsilon\underline{M}_{11}(t)\underline{A}_{21}'(t)+\underline{M}_{14}(t)\underline{K}_2(t)\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\\ \underline{M}_{12}(t_o) =&\ \bar{\underline{x}}_{10}\bar{\underline{x}}_{20}'\end{aligned} \tag{D.22}$$

$$\begin{aligned}\dot{\underline{M}}_{21}(t)=&[\underline{A}_{22}(t)-\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)]\underline{M}_{21}(t)\\ &+\underline{M}_{21}(t)[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]'\\ &+\varepsilon\underline{A}_{21}(t)\underline{M}_{11}(t)+\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)\underline{M}_{41}(t)\\ &+\varepsilon\underline{M}_{22}(t)\underline{A}_{12}'(t)+\underline{M}_{23}(t)\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\\ \underline{M}_{21}(t_o) =&\ \bar{\underline{x}}_{20}\bar{\underline{x}}_{10}'\end{aligned} \tag{D.23}$$

Since $\underline{M}_{14}^{(0)}(t)=\underline{M}_{23}^{(0)}(t)=\underline{M}_{32}^{(0)}(t)=\underline{M}_{41}^{(0)}(t)=\underline{0}$, the zeroth order terms satisfy

$$\begin{aligned}\dot{\underline{M}}_{12}^{(0)}(t)=&[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]\underline{M}_{12}^{(0)}(t)\\ &+\underline{M}_{12}^{(0)}(t)[\underline{A}_{22}(t)-\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)]\\ \underline{M}_{12}^{(0)}(t_o) =&\ \bar{\underline{x}}_{10}\bar{\underline{x}}_{20}'\end{aligned} \tag{D.24}$$

$$\dot{\underline{M}}_{21}^{(0)}(t)=[\underline{A}_{22}(t)-\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)]\underline{M}_{21}^{(0)}(t)$$
$$+\underline{M}_{21}^{(0)}(t)[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]'$$
$$\underline{M}_{21}^{(0)}(t_o)=\bar{\underline{x}}_{20}\bar{\underline{x}}_{10}' \qquad (D.25)$$

Let the transition matrices $\underline{\Psi}_1(t,t_o)$, $\underline{\Psi}_2(t,t_o)$, $\underline{\Phi}_1(t,t_o)$ and $\underline{\Phi}_2(t,t_o)$ be defined as follows:

$$\dot{\underline{\Psi}}_1(t,t_o)=[\underline{A}_{11}(t)-\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)]\underline{\Psi}_1(t,t_o);$$
$$\underline{\Psi}_1(t_o,t_o) = \underline{I} \qquad (D.26)$$

$$\dot{\underline{\Psi}}_2(t,t_o)=[\underline{A}_{22}(t)-\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)]\underline{\Psi}_2(t,t_o);$$
$$\underline{\Psi}_2(t_o,t_o) = \underline{I} \qquad (D.27)$$

$$\dot{\underline{\Phi}}_1(t,t_o)=[-\underline{A}_{22}'(t)+\underline{K}_{22}^{(0)}\underline{B}_{22}(t)\underline{R}_1^{-1}(t)\underline{B}_{22}'(t)]\underline{\Phi}_1(t,t_o);$$
$$\underline{\Phi}_1(t_o,t_o) = \underline{I} \qquad (D.28)$$

$$\dot{\underline{\Phi}}_2(t,t_o)=[-\underline{A}_{11}'(t)+\underline{K}_{11}^{(0)}(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)]\underline{\Phi}_2(t,t_o);$$
$$\underline{\Phi}_2(t_o,t_o) = \underline{I} \qquad (D.29)$$

With above definition and using the matrix variation of constants formula,

$$\underline{M}_{12}^{(0)}(t)=\underline{\Psi}_1(t,t_o)\bar{\underline{x}}_{10}\bar{\underline{x}}_{20}'\underline{\Psi}_2'(t,t_o) \qquad (D.30)$$

$$\underline{M}_{21}^{(0)}(t) = \underline{\Psi}_2(t,t_o)\bar{\underline{x}}_{20}\bar{\underline{x}}_{10}'\underline{\Psi}_1'(t,t_o) \tag{D.31}$$

Then $\int_{t_o}^{T}\text{tr}[2\varepsilon(\underline{K}_1(t)\underline{A}_{12}(t)\underline{M}_{21}^{(0)}(t)+\underline{K}_2(t)\underline{A}_{21}(t)\underline{M}_{12}^{(0)}(t))]dt$

$$= 2\varepsilon \int_{t_o}^{T} \text{tr}[\underline{K}_1(t)\underline{A}_{12}(t)\underline{\Psi}_2(t,t_o)\bar{\underline{x}}_{20}\bar{\underline{x}}_{10}'\underline{\Psi}_1'(t,t_o)$$

$$+\underline{K}_2(t)\underline{A}_{21}(t)\underline{\Psi}_1(t,t_o)\bar{\underline{x}}_{10}\bar{\underline{x}}_{20}'\underline{\Psi}_2'(t,t_o)]dt \tag{D.32}$$

Furthermore, applying the matrix variation of constants formula to equations (D.9) and (D.10), we get

$$\underline{K}_{21}^{(1)}(t) = \int_{t}^{T}\underline{\Phi}_1(t,\sigma)[\underline{A}_{12}'(\sigma)\underline{K}_{11}^{(0)}(\sigma)$$

$$-\underline{K}_{22}^{(0)}(\sigma)\underline{A}_{21}(\sigma)]\underline{\Phi}_2'(t,\sigma)d\sigma \tag{D.33}$$

$$\underline{K}_{12}^{(1)}(t) = \int_{t}^{T}\underline{\Phi}_2(t,\sigma)[\underline{A}_{21}'(\sigma)\underline{K}_{22}^{(0)}(\sigma)$$

$$+\underline{K}_{11}^{(0)}(\sigma)\underline{A}_{12}(\sigma)]\underline{\Phi}_1'(t,\sigma)d\sigma \tag{D.34}$$

Thus $\varepsilon\text{tr}[\underline{K}_{12}^{(1)}(t_0)\bar{\underline{x}}_{20}\bar{\underline{x}}_{10}'+\underline{K}_{21}^{(1)}(t_0)\bar{\underline{x}}_{10}\bar{\underline{x}}_{20}']$

$$= \varepsilon\text{tr}\ \{\int_{t_o}^{T}\underline{\Phi}_2(t_0,\sigma)[\underline{A}_{21}'(\sigma)\underline{K}_{22}^{(0)}(\sigma)$$

$$+\underline{K}_{11}^{(0)}(\sigma)\underline{A}_{12}(\sigma)]\underline{\Phi}_1'(t_0,\sigma)d\sigma\bar{\underline{x}}_{20}\bar{\underline{x}}_{10}'$$

$$+ \int_{t_0}^{T} \underline{\Phi}_1(t_0,\sigma)[\underline{A}_{12}'(\sigma)\underline{K}_{11}^{(0)}(\sigma)$$

$$+\underline{K}_{22}^{(0)}(\sigma)\underline{A}_{21}(\sigma)]\Phi_2'(t_0,\sigma)d\sigma\bar{\underline{x}}_{10}\bar{\underline{x}}_{20}'\} \tag{D.35}$$

Using the definitions given in equations (D.26) to (D.29) and the fact that

$$\underline{K}_1(t) = \underline{K}_{11}^{(0)}(t)$$

$$\underline{K}_2(t) = \underline{K}_{22}^{(0)}(t)$$

we see that
$$\underline{\Psi}_1(t,t_0) = \underline{\Phi}_2'(t_0,t) \tag{D.36}$$

$$\underline{\Psi}_2(t,t_0) = \underline{\Phi}_1'(t_0,t) \tag{D.37}$$

Rewriting equation (D.35) in terms of $K_1(t)$, $\underline{K}_2(t)$, $\underline{\Psi}_1(t,t_o)$, and $\underline{\Psi}_2(t,t_0)$ we get

$$\begin{aligned}
&\varepsilon tr[\underline{K}_{12}^{(1)}(t_0)\underline{\bar{x}}_{20}\underline{\bar{x}}_{10}'+\underline{K}_{21}^{(1)}(t_0)\underline{\bar{x}}_{10}\underline{\bar{x}}_{20}'] \\
&=\varepsilon\int_{t_0}^{T} tr\{\underline{\Psi}_1'(t,t_0)\underline{A}_{21}'(t)\underline{K}_2(t)\underline{\Psi}_2(t,t_0)\underline{\bar{x}}_{20}\underline{\bar{x}}_{10}' \\
&+ \underline{\Psi}_1'(t,t_0)\underline{K}_1(t)\underline{A}_{12}(t)\underline{\Psi}_2(t,t_0)\underline{\bar{x}}_{20}\underline{\bar{x}}_{10}' \\
&+\underline{\Psi}_2'(t,t_0)\underline{A}_{12}'(t)\underline{K}_1(t)\underline{\Psi}_1(t,t_0)\underline{\bar{x}}_{10}\underline{\bar{x}}_{20}' \\
&+\underline{\Psi}_2'(t,t_0)\underline{K}_2(t)\underline{A}_{21}(t)\underline{\Psi}_1(t,t_0)\underline{\bar{x}}_{10}\underline{\bar{x}}_{20}'\}dt
\end{aligned} \tag{D.38}$$

Using the fact that

$$tr(\underline{A}) = tr(\underline{A}')$$

$$tr(\underline{AB}) = tr(\underline{BA})$$

we can write equation (D.38) in the form

$$\begin{aligned}
&\varepsilon tr[\underline{K}_{12}^{(1)}(t_0)\underline{\bar{x}}_{20}\underline{\bar{x}}_{10}'+\underline{K}_{21}^{(1)}(t_0)\underline{\bar{x}}_{10}\underline{\bar{x}}_{20}' \\
&= 2\varepsilon\int_{t_0}^{T} tr[\underline{K}_1(t)\underline{A}_{12}(t)\underline{\Psi}_2(t,t_0)\underline{\bar{x}}_{20}\underline{\bar{x}}_{10}'\underline{\Psi}_1'(t,t_0) \\
&+\underline{K}_2(t)\underline{A}_{21}(t)\underline{\Psi}_1(t,t_0)\underline{\bar{x}}_{10}\underline{\bar{x}}_{20}'\underline{\Psi}_2'(t,t_0)]dt
\end{aligned} \tag{D.39}$$

On comparing equations (D.32 and (D.39) we see that

$$\int_{t_0}^{T} tr[2\varepsilon(\underline{K}_1(t)\underline{A}_{12}(t)\underline{M}_{21}^{(0)}(t)+\underline{K}_2(t)\underline{A}_{21}(t)\underline{M}_{12}^{(0)}(t))]dt$$

$$=\varepsilon tr[\underline{K}_{12}^{(1)}(t_0)\bar{\underline{x}}_{20}\bar{\underline{x}}_{10}'+\underline{K}_{21}^{(1)}(t_0)\bar{\underline{x}}_{10}\bar{\underline{x}}_{20}']$$

and hence

$$\hat{J}^{*(0)}+\varepsilon\hat{J}^{*(1)}=J^{*(0)}+\varepsilon J^{*(1)}$$

APPENDIX E - CALCULATION ON THE DEPENDENCE OF $\hat{J}$ ON $\underline{\Xi}_1(t)$ AND $\underline{\Xi}_2(t)$

Recall equation (5.1.1) gives the cost

$$\hat{J}=tr[\underline{K}_1(t_0)(\underline{\Sigma}_{011}+\underline{\bar{x}}_{10}\underline{\bar{x}}_{10}')+\underline{K}_2(t_0)(\underline{\Sigma}_{022}+\underline{\bar{x}}_{20}\underline{\bar{x}}_{20}')]$$
$$+\int_{t_0}^{T} tr[\underline{K}_1(t)\underline{\Lambda}_1(t)+\underline{K}_2(t)\underline{\Lambda}_2(t)]dt$$
$$+\int_{t_0}^{T} tr[\underline{K}_1(t)\underline{B}_{11}(t)\underline{R}_1^{-1}(t)\underline{B}_{11}'(t)\underline{K}_1(t)\underline{M}_{33}(t)$$
$$+\underline{K}_2(t)\underline{B}_{22}(t)\underline{R}_2^{-1}(t)\underline{B}_{22}'(t)\underline{K}_2(t)\underline{M}_{44}(t)$$
$$+2\varepsilon(\underline{K}_1(t)\underline{A}_{12}(t)\underline{M}_{21}(t)+\underline{K}_2(t)\underline{A}_{21}(t)\underline{M}_{12}(t))]dt$$
$$+\int_{t_0}^{T} tr[\underline{\Xi}_1(t)\underline{\Xi}_1'(t)+\underline{\Xi}_2(t)\underline{\Xi}_2'(t)]dt \qquad (E.1)$$

The term due to the initial conditions and the integral involving $\underline{\Lambda}_1(t)$ and $\underline{\Lambda}_2(t)$ are independent of the choice of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. Hence we need only consider the effect of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ on the last two integrals.

Suppose $\underline{\Xi}_1^{(0)}(t) = \underline{\Xi}_1^{(1)}(t) = \underline{\Xi}_2^{(0)}(t) = \underline{\Xi}_2^{(1)}(t) = \underline{0}$; the choice of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$ will affect $\underline{\Sigma}_1(t)$ and $\underline{\Sigma}_2(t)$ only in quadratic and higher order terms in the power series expansion in ε.

Consider the equation (C.23) for $\underline{M}_{33}^{(2)}(t)$. It depends only on $\underline{\Sigma}_1^{(0)}(t)$ and so $\underline{M}_{33}^{(2)}(t)$ is independent of $\underline{\Xi}_1(t)$. Pursuing this argument further by analyzing higher order terms, we see that $\underline{M}_{33}^{(4)}(t)$ will depend on $\underline{\Sigma}_1^{(2)}(t)$ and hence on $\underline{\Xi}_1(t)$. Similarly we can show that $\underline{M}_{44}^{(4)}(t)$, $\underline{M}_{21}^{(3)}(t)$, and $\underline{M}_{12}^{(3)}(t)$, but not the lower order ones, are dependent on the choice of $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$. From equation (E.1) we see immediately that $\hat{J}^{(4)}$ and higher order terms but notthe lower order ones, will depend on $\underline{\Xi}_1(t)$ and $\underline{\Xi}_2(t)$, provided $\underline{\Xi}_1^{(0)}(t) = \underline{\Xi}_1^{(1)}(t) = \underline{\Xi}_2^{(0)}(t) = \underline{\Xi}_2^{(1)}(t) = \underline{0}$.

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