N72-30032

AN ANALYSIS OF GENERAL CHAIN SYSTEMS

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CONTENTS

ABSTRACT

A general analysis of dynamical systems consisting of connected rigid bodies is presented. The number of bodies and their manner of connection is arbitrary so long as no closed loops are formed. In essence, the analysis represents a new dynamical finite-element method, which is computer oriented and designed so that non-working, interval constraint forces are automatically eliminated. The method is based upon Lagranges form of d'Alembert's principle. Shifter matrix transformations are used with the geometrical aspects of the analysis. The method is illustrated with a space manipulator.

SUMMARY

The analysis presented represents a new kind of finite-element analysis applicable with a broad class of chain-like dynamical systems. It is computer oriented and designed so that non-working constraint forces are automatically eliminated.

The method is applicable with any dynamical system which can be modelled by a series of connected rigid bodies provided only that no closed loops are formed by the bodies. Manipulator systems and teleoperators are thus prime candidates for analysis by this method. The method is also directly applicable with human body models and cable problems. Furthermore, by introducing spring and torsion forces at the joints the analysis becomes a nonlinear finite-element elastic analysis.

Finally, the analysis is developed in a way that allows for either forces or displacements to be specified with the unknown resulting displacements or forces then determined.

The research for this work was supported by NASA Grant NGL 36-004-014 to the University of Cincinnati, Institute of Space Sciences under the University Sustaining Grant program.

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INTRODUCTION

Significant changes have occurred during the past decade in methods of analysis of complex structural systems. New matrix methods and finite-element techniques have been and are continuing to replace traditional approximation methods. Furthermore, new advances in matrix and numerical techniques are continuing to refine and expand the accuracy and capability of these methods. To a large extent, this rapid success and development of these new methods can be directly traced to the advent and rapid development of the digital computation machines.

The study contained herein represents a new kind of finite-element method applicable in the analysis of chain-like continuous dynamical systems. This method, which is also designed for digital computation, uses the approach of considering a dynamical system as a general, nonlinear, chain system. The kinematics of the links of the chain are then developed. Expressions for the forces and moments exerted on and between the links are then obtained. Lagrange's form of d'Alembert's principle (1) is then used to obtain the governing differential equations, which are developed and solved numerically.

Although the notion of using such finite elements with dynamical systems is not new, the particular approach and techniques developed in this study are, for the most part, new and original.

Analysis of chain systems occurred as early as 1732 when Dan Bernoulli developed and solved equations of motion for a hanging chain (2). However, he modelled the chain as a continuous string or cable, and since that time chains have almost always been approximated as flexible, but inextensible cables or strings. The works of Coughey (3), Huang and Dareing (4), and Woodward (5) are modern examples of this approach. Moreover, these analyses consider the chain, cable, or string as being linear, that is, continuous without branches or "tree-like" features. It is interesting to note that in 1967, Elnan and Evert (6) reversed the procedure by modelling a cable by a chain composed of a system of pin connected rods. Their analysis, like the analysis of the work herein was a finite-element approximation of a continuous dynamical system. Indeed the methods of (6) are similar to those herein, although (6) was restricted to linear systems. The study herein considers chains nonlinear, that is, composed of finite connected, rigid links which may contain branches or "tree-like" features.

Most finite element methods are used in the analysis of static structural systems following the approach of Turner, et.al. (7) and that exposited by Zienkiewicz (8) and Odin (9). In the work herein, however, the primary concern is with dynamical systems of rigid bodies such as chains, booms, skeletal systems, or human body models. Hence, this finite element approach is tailored to analyze such systems.

There have been some other recent attemps to obtain general analyses of dynamical systems of rigid bodies, particularly in the area of human biomechanics. In 1969, Roberts and Robbins (10) examined mathematical human body models in crash simulation. In 1970, Kane and Scher (11) studied human self-reorientation in free fall. Also, in 1970, Passerello and Huston (12) produced an analysis of human attitude control in free fall. This latter work was generalized (13) in 1971 to consider human body models in arbitrary force fields. None of these analyses however, provide a general dynamical theory. Indeed, most are directed toward specific applications and most employ the traditional methods of classical mechanics.

In 1970, Young (14) provided a more extensive analysis of a human body model subjected to impulsive forces. Young's analysis uses Lagrange's equations, but the algebraic computation becomes extremely involved. In 1971, Chace and Bayazitoglu (15) presented a general theory of dynamical systems also employing Lagrange's equations. They avoid some of the algebraic tedium by considering the kinetic energy function in several parts. In 1969, Hooker (16) and later in 1972, Fleischer (17) introduced and used the concept of "barycenters" to alleviate the algebraic problems of Lagrange's equations. However, this method while being quite ingenious, also involves considerable algebraic computation.

The analysis of general chain systems presented herein avoids most of the difficulties of the above analyses. That is, by using Lagrange's form of d'Alembert's principle as developed and exploited by Kane (18,1), the governing dynamical equations are obtained without needing to resort to extensive algebraic computations. Basically, Lagrange's form of d'Alembert's principle employs vector quantities whose derivatives may be obtained by vector multiplication and hence on a digital computer. Also, the principle provides for the automatic elimination of non-working constraint forces which are generally of no interest. These two advantages make possible the systematic development of a general theory of chain systems subject to arbitrary prescribed external or internal forces or motions.

The study is divided into five chapters with the first chapter providing the preliminary considerations and background needed in the analysis. The general chain dynamics theory is then developed in Chapters II and III. Effects of impulsive forces are considered in Chapter IV. Application to manipulators and teleoperators is then given in Chapter V.

I. PRELIMINARY CONSIDERATIONS

Consider a set of N rigid bodies joined to each other in a chain or link system such as is shown in Figure 1. The bodies may be joined together in an arbitrary fashion provided only that (1) any two adjacent, connecting bodies have one common point and (2) that no closed loops are formed by the system nor by any of its chain or link segments. The physical dimensions and the masses of the bodies are arbitrary. Thus the system may be used to model a variety of actual physical systems including the human body.

Consider next that this chain system of rigid bodies is subjected to a general force field such that each body has an arbitrary system of forces exerted upon it. Also, let there be, in general, moments exerted by adjacent, connecting bodies on each other.

The primary objective of this analysis is to obtain governing dynamical equations of motion for a general chain system such as this, and subjected to a general arbitrary force field. Furthermore, it is an objective of this analysis to develop these governing equations so that: (1) if the force field is specified, the configuration and motion of the system is determined; (2) if the configuration and motion of the system is specified, the force field is determined; and (3) if a combination of a portion of the force field and a portion of the configuration and motion is specified, the remaining (unknown) portions are determined, by the equations. Finally, it is an objective of this analysis to apply these equations with a specific physical system: the action of a manipulator or teleoperator.

To begin this analysis it is helpful to first consider some preliminary ideas regarding the development of equations of motion and some ideas pertaining to the geometrical and kinematical relations between two adjoining rigid bodies.

Development of Equations of Motion

Consider writing equations of motion for an N-body system such as is shown in Figure 1. Conceptually, the simplest approach is to use Newton's laws and write equations of motion for each individual body of the system. However, this approach has the disadvantage of introducing excessive, unnecessary computation and analysis. For example, this approach would lead to 6N equations involving as unknowns non-working constraint forces between the bodies of the system. These non-working constraint forces are usually of no interest and thus they would need to be systematically eliminated from the 6N equations. In general there are 3(N-1) of these forces. (Three between each adjoining body.) Hence, this procedure would ultimately lead to $6N - 3(N-1) = 3N + 3$ equations of motion to be solved.

It is possible to avoid the computation associated with the elimination of these non-working constraint forces. Two methods are available: The best known and most widely used is the method of Lagrange's equations. In this method the N-body system is treated as a unit, the non-working constraint forces are automatically eliminated, and the 3N + 3 equations of motion are obtained directly. These equations may be written in the form

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{x}_r} - \left(\frac{\partial K}{\partial x_r}\right) = F_r \qquad (r = 1,\ldots,3N + 3) \qquad (1.1)$$

where K is the kinetic energy of the system, x_r $(r = 1,\ldots, 3N + 3)$ are the generalized coordinates of the system (one for each degree of freedom) and F_r $(r = 1,\ldots,3N + 3)$ are the generalized active forces acting on the system. If the externally applied forces acting on the system of bodies are replaced by an equivalent set of forces consisting of N forces and N couples acting on the N respective bodies of the system, then the generalized forces F_r may be written

$$F_r = \sum_{j=1}^{N} \left(\frac{\partial \overline{V}^{Gj}}{\partial \dot{x}_r} \cdot \overline{F}_j + \frac{\partial \overline{\omega}^{Bj}}{\partial \dot{x}_r} \cdot \overline{M}_j\right) \qquad (r = 1,\ldots,3N + 3) \qquad (1.2)$$

where $\overline{F}_j$ and $\overline{M}_j$ $(j = 1,\ldots,15)$ represents the equivalent forces and couple torques acting on the respective bodies B_j and where $\overline{F}_j$ has its line of action passing through point G_j of B. $\overline{V}^{Gj}$ and $\overline{\omega}^{Bj}$ represent the velocity and angular velocity of G_j and B_j in an inertial reference frame. (In some cases, it may be of interest to consider internal (working) moments between adjoining bodies. In these cases these moments may be included in $\overline{M}_j$.) The quantities $\partial\overline{V}^{Gj}/\partial\dot{x}_r$ and $\partial\overline{\omega}^{Bj}/\partial\dot{x}_r$ $(r = 1,\ldots,15)$

are called "partial rates of change of position" and "orientation" respectively.

While providing a number of advantages such as those listed above, Lagrange's equations also lead to serious disadvantages--particularly with complex systems such as in Figure 1. The principle disadvantage is that the computation of the derivatives in Eqs. (1.1) is extremely tedious and is actually unwieldy with systems containing many bodies (14,18).

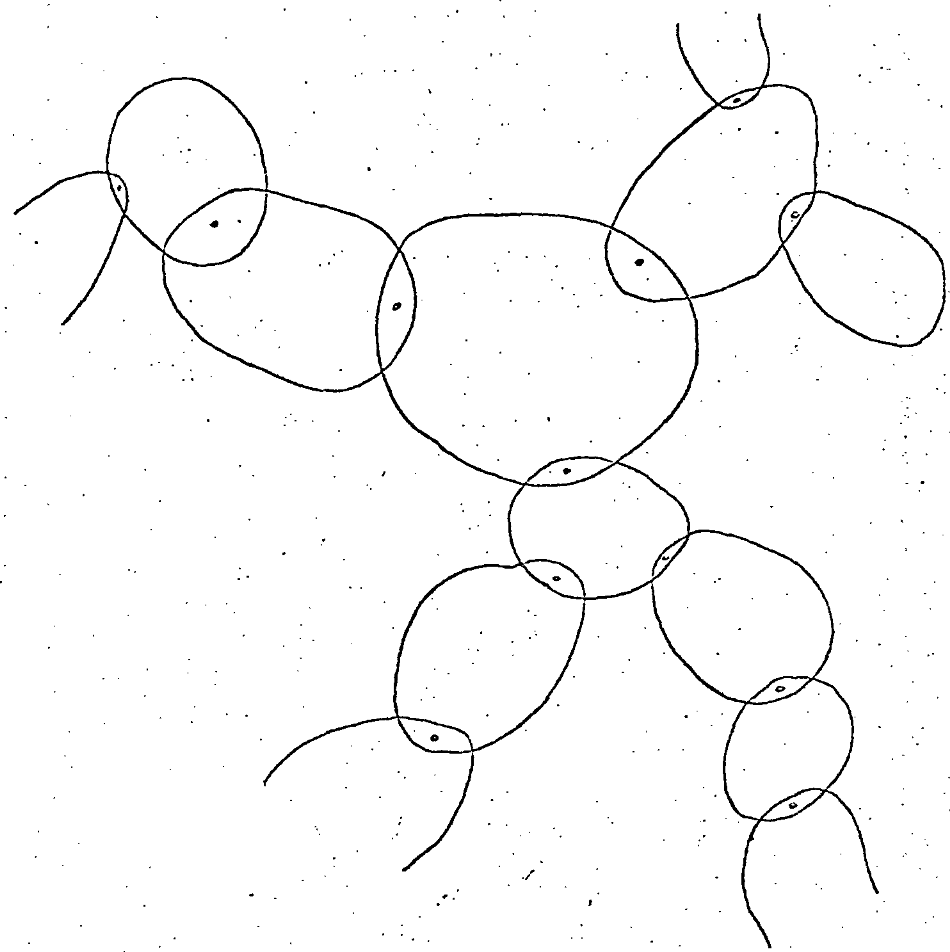

Figure 1.

The other method referred to above, retains the advantages of Lagrange's equations but it avoids the differentiation problems. This method was developed by Kane (1) in 1961, and it is based upon the notion of generalized inertia forces. Specifically, it involves replacing the left side of Eqs. (1.1) by the expression

$$-F_r^* = -\sum_{j=1}^{N} (\bar{F}_j^* \cdot \frac{\partial \bar{V}^{Gj}}{\partial \dot{x}_r} + \bar{T}_j^* \cdot \frac{\partial \bar{\omega}^{Bj}}{\partial \dot{x}_r}) \qquad (1.3)$$

where $\bar{F}_j^*$ and $\bar{T}_j^*$ are the inertia force and torque respectively and are given by

$$\bar{F}_j^* = -m_j \, \bar{a}^{Gj} \qquad (1.4)$$

and

$$\bar{T}_j^* = -\bar{\bar{I}}_j \cdot \bar{\alpha}^{Bj} - \bar{\omega}^{Bj} \times (\bar{\bar{I}}_j \cdot \bar{\omega}^{Bj}) \qquad (1.5)$$

where G_j is now the mass center of B_j, m_j is the mass of B_j, $\bar{\bar{I}}_j$ is the inertia dyadic of B_j relative to G_j, and $\bar{\alpha}^{Bj}$ is the angular acceleration of B_j in the inertial reference frame, F_r^* $(r = 1,\ldots,3N+3)$ is called a "generalized inertial force." Hence

$$F_r^* = -\frac{d}{dt}(\frac{\partial K}{\partial \dot{x}_r}) + \frac{\partial K}{\partial x_r} \qquad (1.6)$$

and the equations of motion are

$$F_r^* + F_r = 0 \qquad (r = 1,\ldots,3N + 3) \qquad (1.7)$$

This method provides the same advantages as the method of Lagrange's equations (principally, the elimination of the non-working constraint forces) together with the advantage of avoiding the differentiation required with Lagrange's equations. However, the replacement function F_r^* (see eq. 1.6) also introduces derivatives. But, in this case, the derivatives are fundamental vector quantities and these derivatives may be calculated by vector multiplication. Given the necessary algorithms, a computer may be used to perform these calculations. Therefore, in view of these advantages, this method of Kane's is used in the sequel to develop the equations of motion for the N-body system of Figure 1. Some of the necessary algorithms for computer application are developed in the following section.

Geometrical and Kinematical Relations between Adjoining Bodies

Consider two typical adjoining bodies of the system such as shown in Figure 2, where B_k and B_ℓ are the names of the bodies, and $\bar{n}_{ki}$ and $\bar{n}_{\ell i}$ $(i = 1,2,3)$ represent sets of mutually perpendicular unit vectors in B_k and B_ℓ respectively. It is the objective of this section to develop convenient relations describing the relative orientation and the relative rate of change of orientation, that is, the angular velocity, of two adjoining bodies such as shown in Figure 2.

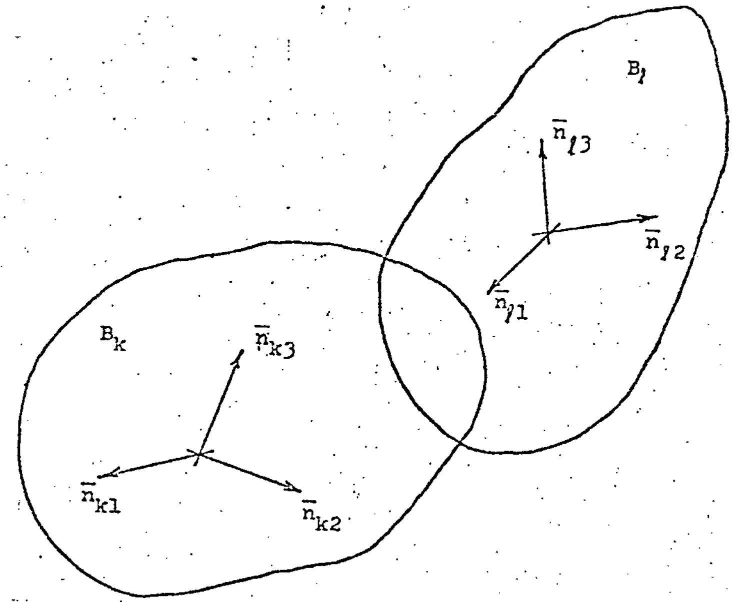

Figure 2.

Since the unit vector sets are fixed in the respective bodies, the relative orientation of the bodies is determined by the relative orientation of the unit vector sets. Hence, consider the matrix defined as:

$$SKL_{ij} = \bar{n}_{ki} \cdot \bar{n}_{\ell j} \qquad (i,j = 1,2,3) \tag{1.8}$$

This 3 x 3 square matrix defines the relative orientation of the unit vector sets since it provides the scalar components of $\bar{n}_{ki}$ along $\bar{n}_{\ell j}$. This matrix also provides the familiar transformation relation between the components of a vector referred to each set. That is, suppose a vector $\bar{V}$ is expressed as

$$\bar{V} = v_i^{(k)} \; \bar{n}_{ki} = v_j^{(\ell)} \; \bar{n}_{\ell j} \tag{1.9}$$

where following the summation convention, there is a sum from

1 to 3 over the repeated subscripts. Then, $V_i^{(k)}$ and $V_j^{(\ell)}$ are related by the expressions

$$
\begin{aligned}
V_i^{(k)} &= SKL_{ij}\ V_j^{(\ell)} \\
V_i^{(\ell)} &= SKL_{ji}\ V_j^{(k)}
\end{aligned}
\tag{1.10}
$$

where again there is a sum over the repeated subscripts. The expressions of Eqs. (1.10) are obtained immediately by taking the dot products of Eq. (1.9) with $\overline{n}_{ki}$ and $\overline{n}_{\ell j}$ respectively. The matrix SKL_{ij} is thus seen to be the familiar transformation matrix encountered in elementary tensor analysis. It is frequently called the "shifter" matrix (12) because of its shifting properties as displayed in Eqs. (1.10).

The shifter matrix also has the property of being an "orthogonal" matrix. That is,

$$
\begin{aligned}
SKL_{ij}\ SKL_{kj} &= \delta_{ik} \\
SKL_{ji}\ SKL_{jk} &= \delta_{ik}
\end{aligned}
\tag{1.11}
$$

where δ_{ik} is Kronecker's delta function defined as 1 for $i = k$ and 0 for $i \neq k$. In matrix notation, these relations may be expressed as

$$(SKL)(SKL)^T = (SKL)(SLK) = I \tag{1.12}$$

where the superscript T denotes the transpose and I is the identity matrix (elements δ_{ij}). Equations (1.11) and (1.12) follow immediately from the definition of SKL_{ij} of Eq. (1.8). Finally, one other property which also follows immediately from Eq. (1.8) is the "chain rule". That is, K, L, and M refer to three sets of unit vectors, then

$$SKM = (SKL)(SLM) \tag{1.13}$$

The chain rule of Eq. (1.13) together with the shifting property of Eq. (1.10) provides for the transformations of the components of a vector referred to unit vectors of any body into components (of the same vector) referred to unit vectors of any other body. For example, there are advantages in expressing a vector in terms of unit vectors fixed in an inertial reference frame because such unit vectors maintain constant orientation. Hence, if SOK_{ij} represents the shifter matrix between the unit vectors of body B_k and the unit vectors of the inertial reference frame, then the components $V_i^{(o)}$ of a vector referred to the inertial reference frame may be expressed in terms of the components $V_j^{(k)}$ referrred to the unit vectors of B_k as (See eq. (1.10))

$$V_i^{(o)} = SOK_{ij}\; V_j^{(k)} \tag{1.14}$$

The shifter matrix SOK_{ij} may be obtained by repeated application of the chain rule Eq. (1.13). However, to use the chain rule it is necessary to know the shifter matrices between the respective adjoining bodies. In this regard, Huston and Passerello (12,13) have developed a systematic scheme for obtaining these individual shifters. This scheme is briefly outlined in the following paragraphs.

Consider again the two adjoining bodies of Figure 2. Introduce coordinate axes X_{ki} and $X_{\ell i}$ $(i = 1,2,3)$ in bodies B_k and B_ℓ respectively and let these axes be respectively parallel to the unit vector sets. Next, imagine the bodies B_k and B_ℓ to be oriented relative to each other such that these axes are respectively parallel as shown in Figure 3. This orientation, when the respective axes are parallel, is called the "reference configuration" between two bodies. Next, imagine three successive dextral rotations of B_ℓ relative to B_k about the axes $X_{\ell 1}$, $X_{\ell 2}$, and $X_{\ell 3}$, through angles $\alpha_{k\ell}$, $\beta_{k\ell}$, $\gamma_{k\ell}$ respectively. (The subscripts on the anges refer to the bodies B_k and B_ℓ.) Then, these three rotations bring B_ℓ into "general configuration" with respect to B_k as shown in Figure 2.

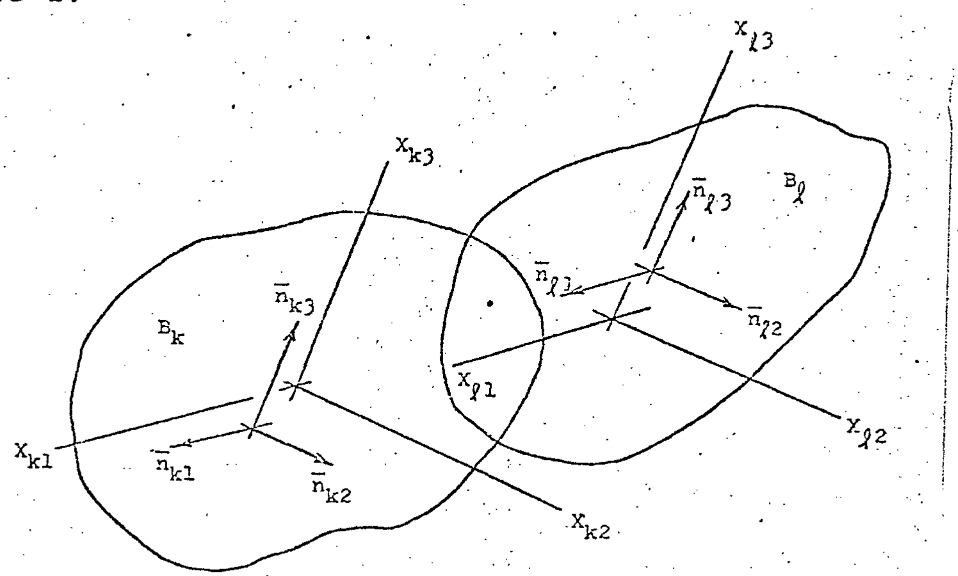

Figure 3.

This process is schematically described by the configguration chart of Figure 4. Configuration charts (12) provide a tabular representation of the relationship between sets of unit vectors. Each dot in the chart represents a unit vector indexed in the far left column and identified in the bottom row. The respective reference frames are listed in the top row. The two intermediate reference frames and their unit vectors are not named. These are the reference frames of the intermediate positions of X_k in the successive rotation process described above. The horizontal lines in the chart connect dots associated with common axes and the angle written beneath is the respective rotation angle about these axes. The inclined lines are used to develop the relations between the unit vectors in adjacent columns: If two dots are connected by an inclined line, the corresponding unit vectors are related by a positive sine term. If two dots in adjacent columns do not lie on any line, the corresponding unit vectors are related by a negative sine term. Unit vectors corresponding to dots in a common row in adjacent columns are related with a positive cosine term, unless they are equal (as when the dots are connected by a horizontal line). The argument of these trigonometric functions is the angle between the respective columns.

As an illustration of this, suppose the unit vectors represented by the second column of dots in Figure 4 are named $\bar{N}_{ki}$ (i = 1,2,3). Then following the instructions outlined above, the following relations between $\bar{n}_{ji}$ and $\bar{N}_{ji}$ are obtained:

$$
\begin{aligned}
\bar{n}_{k1} &= \bar{N}_{k1} \\
\bar{n}_{k2} &= c\alpha_{k\ell}\bar{N}_{k2} - s\alpha_{k\ell}\bar{N}_{k3} \\
\bar{n}_{k3} &= s\alpha_{k\ell}\bar{N}_{k2} + c\alpha_{k\ell}\bar{N}_{k3} \\
\bar{N}_{k2} &= c\alpha_{k\ell}\bar{n}_{k2} + s\alpha_{k\ell}\bar{n}_{k3} \\
\bar{N}_{k3} &= -s\alpha_{k\ell}\bar{n}_{k2} + c\alpha_{k\ell}\bar{n}_{k3}
\end{aligned}
\tag{1.15}
$$

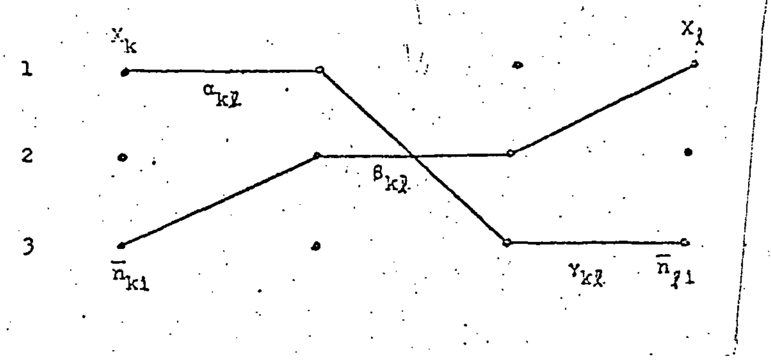

Figure 4.

where $s\alpha_{k\ell}$ and $c\alpha_{k\ell}$ are abbreviations for sin $\alpha_{k\ell}$ and cos $\alpha_{k\ell}$ respectively. It is easy to verify these relations with a simple sketch of the two unit vector sets as shown in Figure 5. Note also that when $\alpha_{k\ell} = \beta_{k\ell} = \gamma_{k\ell}$ the respective unit vectors are equal and the bodies are in reference configuration.

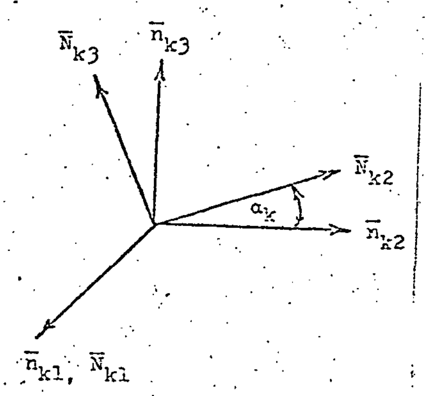

Figure 5.

The configuration chart may now be conveniently used to determine the shifter matrix SKL: Let αKL be the matrix defined as:

$$\alpha KL = \begin{bmatrix} 1 & 1 & 0 \\ 0 & c\alpha_{k\ell} & -s\alpha_{k\ell} \\ 0 & s\alpha_{k\ell} & c\alpha_{k\ell} \end{bmatrix} \tag{1.16}$$

Then it is easily seen by Eqs. (1.15) that

$$\alpha KL_{ij} = \bar{n}_{ki} \cdot \bar{N}_{kj} \tag{1.17}$$

Hence, αKL is a shifter matrix between $\bar{n}_{ki}$ and $\bar{N}_{ki}$. Then by the chain rule of Eq. (1.13)

$$SKL = (\alpha KL)(\beta KL)(\gamma KL) \tag{1.18}$$

where βKL and γKL are the matrices

$$\beta KL = \begin{bmatrix} c\beta_{k\ell} & 0 & s\beta_{k\ell} \\ 0 & 1 & 0 \\ -s\beta_{k\ell} & 0 & c\gamma_{k\ell} \end{bmatrix} \tag{1.19}$$

$$\gamma KL = \begin{bmatrix} c\gamma_{k\ell} & -s\gamma_{k\ell} & 0 \\ s\gamma_{k\ell} & c\gamma_{k\ell} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.20}$$

where the sine and cosine are again abbreviated. The shifter matrices αKL, βKL, and γKL may be determined from the configuration chart by inspection by noting the following: Each matrix has the integer 1 and cosines on the diagonal. The integer occurs in the same row as the horizontal line of the configuration chart. The remaining elements in the row and column of the integer are zero. The other elements are + and - sine's. The + occurs in the lower row if the slope of the inclined line is positive.

The configuration chart of Figure 4 is valid for any two typical adjoining bodies of the N-body system of Figure 1. This means that all the configuration charts of adjoining bodies have the same form and hence, all the shifter matrices between adjoining bodies have the same form. Therefore, by using Eqs. (1.16), (1.18), (1.19) and (1.20), a computer subprogram may be written to compute all the shifter matrices between adjoining bodies. The chain rule of Eq. (1.13) may the be used to compute any shifter matrix, such as SOK.

The shifter matrices determine the relative orientation of the bodies of the system in terms of the various rotation-orientation angles α, β, and γ. As mentioned above, it is also of interest to obtain expressions for the relative rate of change of orientation between the bodies, that is, the relative angular velocities. To this end, consider again the two typical adjoining bodies B_k and B_ℓ (Figure 2). Recall that B_ℓ is brought into general configuration relative to B_k by three successive dextral rotations about the axes $X_{\ell 1}$, $X_{\ell 2}$, and $X_{\ell 3}$ through the angles $\alpha_{k\ell}$, $\beta_{k\ell}$, $\gamma_{k\ell}$ respectively. Hence, the addition formula for angular velocity (see for example, Kane (1), and Eq. (1.24) below leads to the following expression for the angular velocity of β_ℓ relative to B_k:

$$ {}^{k}\bar{\omega}^{\ell} = \dot{\alpha}_{k\ell}\, \bar{n}_{k1} + \dot{\beta}_{k\ell}\, \bar{N}_{k2} + \dot{\gamma}_{k\ell}\, \bar{n}_{\ell 3} \tag{1.21} $$

where the dots denote time differentiation. The vector $\bar{N}_{k2}$ is parallel to $X_{\ell 2}$ after the first rotation (through $\alpha_{k\ell}$).

Although Eq. (1.21) follows immediately from the addition formula and the manner of bringing B_ℓ into general configuration relative to B_k, the equation may also be determined by inspection from the configuration chart (Figure 4) by making the following observations: The relative angular velocity of the unit vectors associated with adjacent columns of dots (Figure 4) is simple angular velocity (1) directed along the common unit vector (horizontal line) with magnitude proportional to the derivative of the corresponding rotation angle (written under the horizontal line). Hence, Eq. (1.21) may be obtained by the sum of products of the orientation angle derivatives and the associated unit vectors corresponding to the horizontal lines of the configuration charts. Furthermore, if ${}^{k}\bar{\omega}^{\ell}$ is expressed in terms of $\bar{n}_{ki}$ as

$$ {}^{k}\bar{\omega}^{\ell} = {}^{k}\omega^{\ell}\, \bar{n}_{ki} \tag{1.22} $$

the components ${}^{k}\omega^{\ell}_{i}$ $(i = 1,2,3)$ are given by

$$ {}^{k}\omega^{\ell}_{i} = \dot{\alpha}_{k\ell}\, \delta_{i1} + \dot{\beta}_{k\ell}\, \alpha KL_{i2} + \dot{\gamma}_{k\ell}\, \alpha KL_{ij}\, \beta KL_{j3} \tag{1.23} $$

The angular velociety of B_ℓ in an inertial reference frame R may now be determined by repeated use of the addition formula for angular velocity, that is,

$$^{o}\bar{\omega}^{\ell} = {}^{k}\bar{\omega}^{\ell} + {}^{o}\bar{\omega}^{k} \tag{1.24}$$

and the components may all be referred to unit vectors $\bar{n}_{oi}$ ($i = 1,2,3$) fixed in R by multiplication of the appropriate shifter matrices.

From the above discussion, and from the earlier remarks, it is seen to be convenient computationally, to express all vectors in terms of $\bar{n}_{oi}$, the unit vectors of R, the inertial reference frame. Also, this may be done conveniently through shifter matrices such as SOK (see Eq. (1.14)). However, it is sometimes necessary to differentiate the vector components which are referred to $\bar{n}_{oi}$. This means that shifters such as SOK will need to be differentiated. As also mentioned above, however, these shifter derviatives may be obtained by a multiplication algorithm. To obtain this algorithm, consider SOK to be given by

$$SOK_{ij} = \bar{n}_{oi} \cdot \bar{n}_{kj} \tag{1.25}$$

Then since $\bar{n}_{oi}$ is fixed in R (and are therefore constant),

$$\frac{d}{dt}(SOK_{ij}) = \bar{n}_{oi} \cdot \frac{{}^{R}d\bar{n}_{kj}}{dt} \tag{1.26}$$

But since $\bar{n}_{kj}$ are fixed in B_k

$$\frac{{}^{R}d\bar{n}_{kj}}{dt} = {}^{o}\bar{\omega}^{k} \times \bar{n}_{kj} \tag{1.27}$$

Hence $d(SOK_{ij})/dt$ becomes

$$\frac{d}{dt}(SOK_{ij}) = \bar{n}_{oi} \cdot {}^{o}\bar{\omega}^{k} X \bar{n}_{kj}$$

$$= -{}^{o}\bar{\omega}^{k} X \bar{n}_{oi} \cdot \bar{n}_{kj}$$

$$= -e_{imn} \, {}^{o}\omega_{n}^{k} \, \bar{n}_{om} \cdot \bar{n}_{kj}$$

or

$$\frac{d}{dt}(SOK_{ij}) = WOK_{im} \; SOK_{mj} \tag{1.28}$$

where WOK_{im} is a matrix defined as

$$WOK_{im} = -e_{imn} \; {}^{o}\omega_{n}^{k} \tag{1.29}$$

where ${}^{o}\omega_{n}^{k}$ are the components of ${}^{o}\bar{\omega}^{k}$ referred to $\bar{n}_{on}$ and e_{imn} is the permutation symbol (19) defined as

$$e_{imn} = \begin{cases} 1 & i,m,n \quad \text{distinct and cyclic} \\ -1 & i,m,n \quad \text{distinct and anticyclic} \\ 0 & i,m,n \quad \text{not distinct} \end{cases} \tag{1.30}$$

WOK_{im} is simply the matrix whose dual vector (19) is ${}^{o}\bar{\omega}^{k}$. Eq. (1.28) then provides the desired multiplication algorithm.

<u>Summary</u>

To summarize then, the underlying principle of the analysis is to formulate the equations so they may be adapted to programming on a digital computer. This is done by using Kane's dynamical equations, Eqs. (1.7) and by expressing vector quantities in terms of unit vectors in an inertial reference frame. The corresponding component transformation such as Eq. (1.14), is obtained through the shifter matrices which are in turn obtained from configuration charts (Figure 4). The vector quantities are then easy to differentiate since the unit vectors in the inertial reference frame are constant. Furthermore, the necessary shifter derivatives may be obtained through multiplication (Eq. (1.28)) and may thus be performed by the computer.

These perliminary notions provide a basis for the analysis of the following chapters.

II. KINEMATICS OF GENERAL CHAIN SYSTEMS

Consider introducing the following notation as shown in Figure 6. Number the bodies in the system such that the numbers increase along chains originating from B_1. Let O_1 be an arbitrary point of B_1. Let O_j be a point common to B_j and the adjacent lower numbered body. Denote the center of mass B_j by G_j, the vector which locates O_j relative to O_k by $\overline{\xi}_j$ where B_k is the adjacent lower numbered body to B_j. Note here that $\overline{r}_j$ is fixed in B_j and $\overline{\xi}_j$ is fixed in the adjacent lower numbered body to B_j. Finally, let $\overline{q}_i$ be the vector from O_1 to G_i.

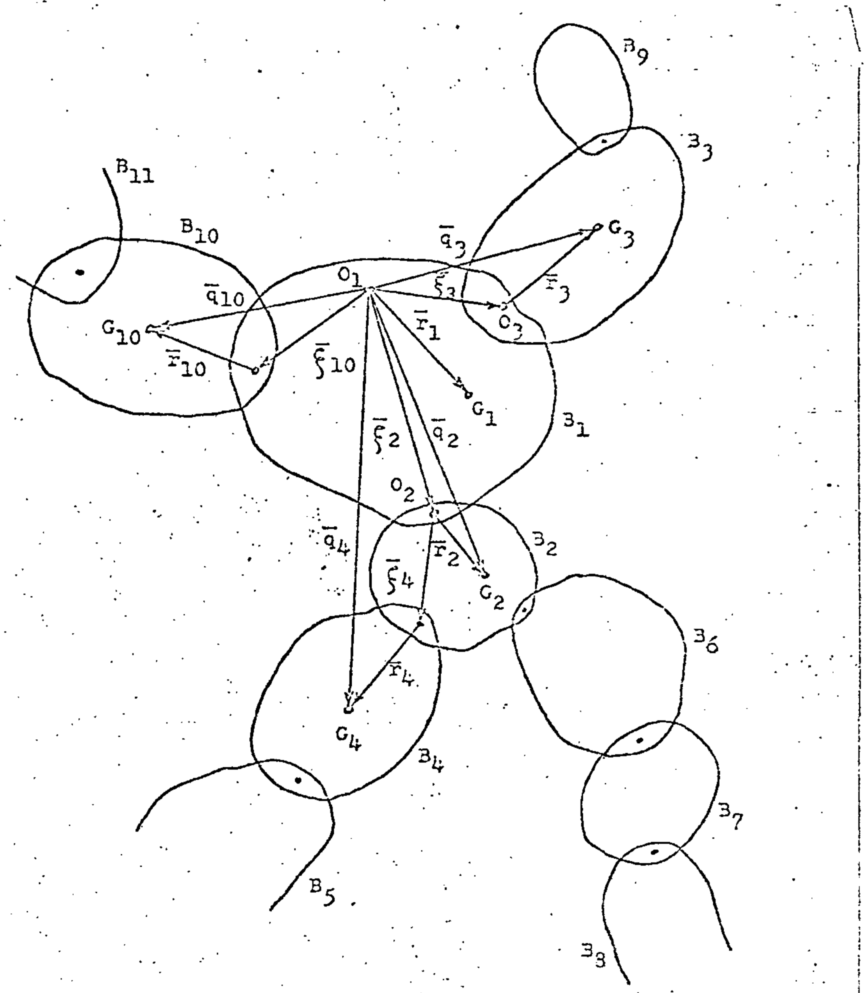

Figure 6.

Now consider naming the generalized coordinates needed to specify the position of the system as follows. Let x_1, x_2, x_3 be the components of a vector referred to basis $\bar{n}_{oi}$ locating O_1 relative to a point fixed in R_o. Let x_4, x_5, x_6 be the angles which specify the orientation of B_1 in R_o. For bodies B_i $(i = 2,N)$ let x_{3i+1}, x_{3i+2}, x_{3i+3} be the angles specifying the orientation of B_i in the adjacent lower numbered body.

It is now possible to derive the kinematical quantities needed in the equations of motion. These are $\bar{\omega}^i$ the angular velocity in B_i in R_o, $\bar{\omega}^i_{\dot{x}_r}$ the partial rate of change of orientation of B_i, $\bar{\alpha}^i$ the angular acceleration of B_i in R_o, $\bar{V}^{Gi}$ the velocity of G_i in R_o, $\bar{V}^{G_i}_{\dot{x}_r}$ the partial rate of change of position of G_i and $\bar{a}^{Gi}$ the acceleration of G_i in R_o.

Consider $\bar{\omega}^i$ first: From Eq. (1.24)

$$\bar{\omega}^i = \bar{\omega}^1 + {}^1\bar{\omega}^m + \ldots + {}^k\bar{\omega}^j + {}^j\bar{\omega}^i \tag{2.1}$$

$$i = 2,3,\ldots,N$$

where from Eq. (1.23)

$${}^k\bar{\omega}^j = SOK_{\ell i}(\dot{x}_{3j+1}\,\delta_{i1} + \dot{x}_{3j+2}\,\alpha KJ_{i2} + \dot{x}_{3j+3}\,\alpha KJ_{i\ell}\beta KJ_{\ell 3})\bar{n}_{o\ell} \tag{2.2}$$

Hence, in general

$$\bar{\omega}^i = \omega_{ijk}\dot{x}_j\bar{n}_{ok} \qquad i = 1,\ldots,N \tag{2.3}$$

where $\omega_{ijk} = 0$ for $i = 1,15$; $j = 1,3$; $k = 1,3$ (x_1, x_2, x_3 are position coordinates of O_1). Comparing equations (2.1) and (2.3), the non-zero ω_{ijk} take one of three forms

$$\omega_{ijk} = \begin{cases} SOK_{k1} \\ SOK_{km}\ \alpha KI_{m2} \\ SOK_{km}\ \alpha KI_{m\ell}\beta KI_{\ell 3} \end{cases} \tag{2.4}$$

depending on whether j is the first, second or third dextral angle in defining the orientation of B_i with respect to its adjacent lower numbered body.

The angular acceleration of B_i in R_o is obtained by differentiating $\bar{\omega}^i$. Hence

$$\bar{\alpha}^i = (\omega_{ijk}\ \ddot{x}_j + \dot{\omega}_{ijk}\ \dot{x}_j)\ \bar{n}_{ok} \tag{2.5}$$

Note here that non-zero $\dot{\omega}_{ijk}$ takes one of three forms (see Eq. (2.4))

$$\dot{\omega}_{ijk} = \begin{cases} S\dot{O}K_{k1} \\ S\dot{O}K_{km}\alpha KI_{m2} + SOK_{km}\alpha\dot{K}I_{m2} \\ S\dot{O}K_{km}\alpha KI_{m\ell}\beta KI_{\ell 3} + SOK_{km}\alpha\dot{K}I_{m}\beta SKI_{\ell 3} + SOK_{km}\alpha KI_{m\ell}\beta\dot{K}I_{\ell 3} \end{cases} \tag{2.6}$$

where $S\dot{O}K$ is computed by using Eq. (1.28) and $\alpha\dot{K}I$ and $\beta\dot{K}I$ are computed by differentiating Eqs. (1.16) and (1.19) as follows:

$$\alpha\dot{K}I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s(x_{3+1}) & -c(x_{3i+1}) \\ 0 & c(x_{3i+1}) & -s(x_{3i+1}) \end{bmatrix} \dot{x}_{3i+1} \tag{2.7}$$

$$\beta\dot{K}I = \begin{bmatrix} -s(x_{3i+2}) & 0 & c(x_{3i+2}) \\ 0 & 0 & 0 \\ -c(x_{3i+2}) & 0 & -s(x_{3i+2}) \end{bmatrix} \dot{x}_{3i+2} \qquad (2.8)$$

The partial rate of change of orientation of B^i in R_o can be obtained from Eq. (2.2) as follows:

$$\bar{\omega}^i_{\dot{x}_r} = \frac{\partial \bar{\omega}^i}{\partial \dot{x}_r} = \omega_{irk}\bar{n}_{ok} \qquad (2.9)$$

To find expressions for $\bar{V}^{Gi}$, $\bar{a}^{Gi}$ and $\bar{V}^{Gi}$ consider writing the expressions for the position vector $\bar{P}_i$ of G_i relative to a fixed point in R_o and then differentiating this expression in R_o with respect to time. From Figure 6, $\bar{P}_i$ can be written as

$$\bar{P}_i = [x_1\delta_{k1} + x_2\delta_{k2} + x_3\delta_{k3} + \sum_u SOK_{k\ell}\xi^u_\ell + SOI_{k\ell}r^i_\ell]\bar{n}_{ok} \qquad (2.10)$$

where ξ^j_ℓ and r^i_ℓ are defined by the expression

$$\bar{\xi}_r = \xi^r_\ell\, \bar{n}_{k\ell} \qquad (2.11)$$

and

$$\bar{r}_i = r^i_\ell\, \bar{n}_{i\ell} \text{ (no sum over i)} \qquad (2.12)$$

where in Eq. (2.11) k is the number of the body in which $\bar{\xi}_r$ is fixed ($k<r$) and where the sum $(\sum_u)$ in Eq. (2.10) is over the bodies that form a chain from B_1 to B_i and where k in Eq. (2.10) is the number of the body in which $\bar{\xi}_r$ is fixed. Note that since $\bar{\xi}_r$ and $\bar{r}_i$ are fixed, bodies B_k and B_i, ξ^r_ℓ and r^i_ℓ are constants.

Hence by differentiating Eq. (2.10) $\bar{V}^{Gi}$ is

$$\overline{V}^{Gi} = (\dot{x}_1\delta_{k1} + \dot{x}_2\delta_{k2} + \dot{x}_3\delta_{k3} + \sum_u \dot{SOK}_{k\ell}\xi^u_\ell + \dot{SOI}_{k\ell}r^i_\ell)\overline{n}_{ok} \qquad (2.13)$$

This equation can be written in the form

$$\overline{V}^{Gi} = V_{ijk}\,\dot{x}_j\overline{n}_{ok} \qquad (2.14)$$

where $V_{ijk} = \delta_{jk}$ for $(i = 1,N;\ j = 1,3;\ k = 1,3)$

and

$$V_{ijk} = \sum_u \frac{\partial SOK_{k\ell}}{\partial \dot{x}_j}\,\xi^u_\ell + \frac{\partial SOI_{k\ell}}{\partial \dot{x}_j}\,r^i_\ell \qquad (2.15)$$

for $(i = 1,15;\ j = 4,3N+3;\ k = 1,3)$

By recalling Eqs. (1.28) and (2.3)

$$V_{ijk} = \sum_u \frac{\partial WOK_{km}}{\partial \dot{x}_j}\,SOK_{m\ell}\xi^u_\ell + \frac{\partial WOI_{km}}{\partial \dot{x}_j}\,SOI_{m\ell}r^i_\ell \quad (j \geq 4) \qquad (2.16)$$

where by Eq. (1.29)

$$\frac{\partial WOK_{im}}{\partial \dot{x}_j} = -e_{imn}\omega_{kjn} \qquad (k = K) \qquad (2.17)$$

Equation (2.16) may be written as

$$V_{ijk} = \sum_u WSK_{jk\ell}\xi^u_\ell + WSI_{jk\ell}\,r^i_\ell \qquad (2.18)$$

where $WSK_{jk\ell} = -e_{imn}\omega_{kjn}\ SOK_{m\ell}$ (2.19)

Now from Eq. (2.14)

$$\overline{V}^{Gi}_{\dot{x}_j} = \frac{\partial \overline{V}^{Gi}}{\partial \dot{x}_j} = V_{ijk}\,\overline{n}_{ok} \qquad (2.20)$$

The acceleration of G^i in R_o is found by differentiating Eq. (2.14), leading to the expression

$$\overline{a}^{Gi} = (V_{ijk}\,\ddot{x}_j + \dot{V}_{ijk}\,\dot{x}_j)\overline{n}_{ok} \qquad (2.21)$$

where from Eqs. (2.15) and (2.19), $\dot{V}_{ijk}$ is

$$\dot{V}_{ijk} = 0 \qquad \text{for } (i = 1,N;\ j = 1,3;\ k = 1,3) \tag{2.22}$$

and $\dot{V}_{ijk} = \sum_u W\dot{S}K_{jk\ell}\xi^u_\ell + W\dot{S}I_{jk\ell}r^i_\ell$

$$\text{for } (i = 1,N;\ j = 4,\ 3N+3;\ k = 1,3)$$

and where by differentiating in Eq. (2.19) $W\dot{S}K_{jk\ell}$ is given by

$$W\dot{S}K_{jk\ell} = -e_{imn}(\dot{\omega}_{kjn}SOK_{m\ell} + \omega_{kjn}S\dot{O}K_{m\ell}) \tag{2.23}$$

To summarize: Algorithms have been developed to find the kinematical quantities (referred to a basis in R_o) needed in the development of the equations of motion. They are recorded below for future reference.

$$\bar{\omega}^i = \omega_{ijk}\ \dot{x}_j\ \bar{n}_{ok} \quad , \tag{2.24}$$

$$\bar{\alpha}^i = (\omega_{ijk}\ \ddot{x}_j + \alpha_{ik})\bar{n}_{ok} \tag{2.25}$$

where from Eq. (2.5) α_{ik} is given by

$$\alpha_{ij} = \dot{\omega}_{ijk}\ \dot{x}_j \tag{2.26}$$

Also,

$$\bar{V}^{Gi} = V_{ijk}\ \dot{x}_j\ \bar{n}_{ok} \tag{2.27}$$

$$\bar{a}^{Gi} = (V_{ijk}\ \ddot{x}_j + a_{ij})\bar{n}_{ok} \tag{2.28}$$

where from equation (2.21)

$$a_{ik} = \dot{V}_{ijk}\ \dot{x}_j \tag{2.29}$$

and where ω_{ijk}, $\dot{\omega}_{ijk}$, V_{ijk}, $\dot{V}_{ijk}$ are given by Eqs. (2.4), (2.6), (2.18) and (2.22).

As an example, the following expansion illustrates the form of these quantities for B_4 of Figure 6.

$\omega_{4jk} = 0$ for all j, k except the following:

$$\omega_{4,4,1} = 1$$

$$\omega_{4,5,k} = \alpha01_{k2}$$

$$\omega_{4,6,k} = \alpha01_{k\ell}\beta01_{\ell 3}$$

$$\omega_{4,7,k} = SO1_{k1}$$

$$\omega_{4,8,k} = SO1_{k\ell}\alpha12_{\ell 2}$$

$$\omega_{4,9,k} = SO1_{k\ell}\alpha12_{\ell m}\beta12_{m3}$$

$$\omega_{4,13,k} = SO2_{k1}$$

$$\omega_{4,14,k} = SO2_{k\ell}\alpha24_{\ell 2}$$

$$\omega_{4,15,k} = SO2_{k\ell}\alpha24_{\ell m}\beta24_{m3}$$

$$\begin{aligned}\alpha_{4i} = {} & \dot{x}_5\alpha\dot{0}1_{i2} + \dot{x}_6(\alpha\dot{0}1_{i\ell}\beta01_{\ell 3} + \alpha01_{i\ell}\beta\dot{0}1_{\ell 3}) \\ & + S\dot{O}1_{ik}(\dot{x}_7\delta_{k1} + \dot{x}_8\alpha12_{k2} + \dot{x}_9\alpha12_{k\ell}\beta12_{\ell 3}) \\ & + SO1_{ik}(\dot{x}_8\alpha\dot{1}2_{k2} + \dot{x}_9(\alpha\dot{1}2_{k\ell}\beta12_{\ell 3} + \alpha12_{k\ell}\beta\dot{1}2_{\ell 3})) \\ & + S\dot{O}2_{i}(\dot{x}_{13}\delta_{k1} + \dot{x}_{14}\alpha24_{k2} + \dot{x}_{15}\alpha24_{k\ell}\beta24_{\ell 3}) \\ & + SO2_{ik}(\dot{x}_{14}\alpha\dot{2}4_{k2} + \dot{x}_{15}(\alpha\dot{2}4_{k\ell}\beta24_{\ell 3} + \alpha24_{k\ell}\beta\dot{2}4_{\ell 3}))\end{aligned}$$

$V_{4jk} = 0$ for all j,k except the following:

$$V_{4,1,1} = 1$$

$$V_{4,2,2} = 1$$

$$V_{4,3,3} = 1$$

$$V_{4jk} = WSI_{jk\ell}\xi_\ell^2 + WS2_{jk\ell}\xi_\ell^4\ WS4_{jk\ell}r_\ell^4$$

$$a_{4k} = (W\dot{S}1_{jk\ell}\dot{x}_j)\xi_\ell^2 + (W\dot{S}2_{jk\ell}\dot{x}_j)\xi_\ell^4 + W\dot{S}4_{jk\ell}\dot{x}_j)r_\ell^4$$

III. EQUATIONS OF MOTION FOR GENERAL CHAIN SYSTEMS

The equations of motion will be of the form of Eqs. (1.7) that is

$$F_r^* + F_r = 0 \qquad (r = 1,2,\ldots,3N+3) \tag{3.1}$$

Consider first the generalized active force F_r. F_r is given by Eq. (1.2) as

$$F_r = \sum_{j=1}^{N} \left(\frac{\partial \bar{V}^{Gj}}{\partial \dot{x}_r} \cdot \bar{F}_j + \frac{\partial \bar{\omega}^{Bj}}{\partial \dot{x}_r} \cdot \bar{M}_j\right) \qquad (r = 1,2,\ldots,3N+3) \tag{3.2}$$

where $\bar{F}_j$ and $\bar{M}_j$ $(j = 1,2,\ldots,N)$ represent forces and couple torques equivalent to the applied active forces acting on the respective bodies B_j. $\bar{F}_j$ has its line of action passing through point G_j of B_j. Consider $\bar{M}_j$ to be of the form,

$$\bar{M}_j = \overset{\text{ext}}{\bar{M}_j} + \sum_K \overline{MKJ} \tag{3.3}$$

where $\overset{\text{ext}}{\bar{M}_j}$ is the couple torque due to external forces applied to B_j and $\overline{MKJ}$ is an internal couple torque exerted by B_k on B_j as shown in Figure 7. The sum in Eq. (3.3) is over all internal couple torques acting on B_j by adjacent bodies. Hence in view of Eq. (3.3) it is possible to write Eq. (3.2) as

$$F_r = \overset{\text{ext}}{F_r} + \overset{\text{int}}{F_r} \tag{3.4}$$

where
$$\overset{\text{ext}}{F_r} = \sum_{j=1}^{N} \left(\frac{\partial \bar{V}^{Gi}}{\partial \dot{x}_r} \cdot \bar{F}_j + \frac{\partial \bar{\omega}^{Bj}}{\partial \dot{x}_r} \cdot \overset{\text{ext}}{\bar{M}_j}\right) \tag{3.5}$$

and where
$$\overset{\text{int}}{F_r} = \sum_{j=1}^{N} \left(\sum_K \frac{\partial \bar{\omega}^{Bj}}{\partial \dot{x}_r} \cdot \overline{MKJ}\right) \tag{3.6}$$

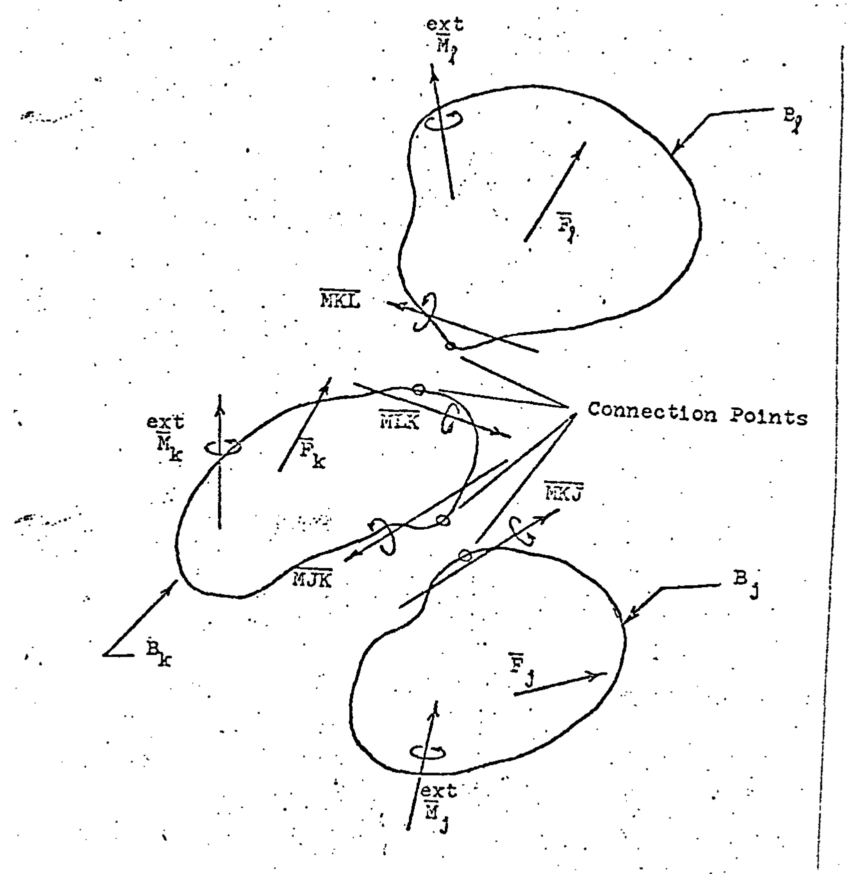

Figure 7.

From Eq. (2.9) and (2.20), Eq. (3.5) can be written as

$$\overset{ext}{F}_r = \sum_{j=1}^{N} (V_{irk} F_{jk} + \omega_{jrk} \overset{ext}{M}_{jk}) \tag{3.7}$$

where F_{jk} and $\overset{ext}{M}_{jk}$ are defined as

$$\overline{F}_j = F_{jk}\overline{n}_{ok} \tag{3.8}$$

and

$$\overset{ext}{\overline{M}}_j = \overset{ext}{M}_{jk}\overline{n}_{ok} \tag{3.9}$$

To find a convenient form for $\overset{int}{F}_r$ consider two adjacent bodies B_j and B_k such as shown in Figure 7. Let B_j be a lower numbered body. Then the generalized coordinates locating B_k relative to B_j are x_{3k+1}, x_{3k+2}, x_{3k+3}. The internal couple moment which B_j exerts on B_k is $\overline{MJK}$, and the internal couple moment exerted by B_k on B_j is $\overline{MKJ}$. (Note, as shown in Figure 7, $\overline{MJK} = -\overline{MKJ}$.) From Eq. (1.24)

$$\overline{\omega}^{B_k} = \overline{\omega}^{B_j} + {}^{B_j}\overline{\omega}^{B_k} \qquad (3.10)$$

where from Eq. (1.23)

$${}^{B_j}\overline{\omega}^{B_k} = (\dot{x}_{3k+1}\delta_{i1} + \dot{x}_{3k+2}\alpha JK_{i2} + \dot{x}_{3k+3}\alpha JK_{i\ell}\beta JK_{\ell 3})\overline{n}_{ji} \qquad (3.11)$$

From Eq. (3.6), the contribution of $\overline{MKJ}$ and $\overline{MJK}$ to F_r is calculated as follows:

$$\overline{\omega}^{B_j}_{\dot{x}_r} \cdot \overline{MKJ} + \overline{\omega}^{B_k}_{\dot{x}_r} \cdot \overline{MJK} = (\overline{\omega}^{B_k}_{\dot{x}_r} - \overline{\omega}^{B_j}_{\dot{x}_r}) \cdot \overline{MJK} \qquad (3.12)$$

Now if $r \neq 3k+1, 3k+2, 3k+3$, then either

$$\overline{\omega}^{B_j}_{\dot{x}_r} = \overline{\omega}^{B_k}_{\dot{x}_r} \qquad \text{or} \qquad \overline{\omega}^{B_j}_{\dot{x}_r} = \overline{\omega}^{B_k}_{\dot{x}_r} = 0 \qquad (3.13)$$

For both cases the contribution to F_r or $\overline{MKJ}$ and $\overline{MJK}$ is zero. To find the contribution to F_r of $\overline{MKJ}$ and $\overline{MJK}$ for $r = 3k+1$, $3k+2$, $3k+3$ note from Eqs. (3.10) and (3.11) that

$$\overline{\omega}^{j}_{\dot{x}_r} = 0 \qquad (r = 3j+1,\ 3j+2,\ 3j+3) \qquad (3.14)$$

and

$$\overline{\omega}^{k}_{\dot{x}_{3k+1}} = \overline{n}_{j1}$$

$$\overline{\omega}^{k}_{\dot{x}_{3k+2}} = \alpha JK_{i2}\overline{n}_{ji} \qquad (3.15)$$

$$\bar{\omega}^{k}_{\dot{x}_{3k+3}} = \alpha JK_{i\ell}\beta JK_{\ell 3}\bar{n}_{ji}$$

Then using Eqs. (3.12), (3.14) and (3.15) the contribution to F_r for r = 3k+1, 3k+2, and 3k+3 due to $\bar{M}KJ$ and $\bar{M}JK$ is

$$
\begin{aligned}
(\bar{\omega}^{B_k}_{\dot{x}_{3k+1}} - \bar{\omega}^{B_j}_{\dot{x}_{3k+1}}) \cdot \bar{M}JK &= MJK_1 \\
(\bar{\omega}^{B_k}_{\dot{x}_{3k+2}} - \bar{\omega}^{B_j}_{\dot{x}_{3k+3}}) \cdot \bar{M}JK &= MJK_i\alpha JK_{i2} \\
(\bar{\omega}^{B_k}_{\dot{x}_{3k+3}} - \bar{\omega}^{B_j}_{\dot{x}_{3k+3}}) \cdot \bar{M}JK &= MJK_i\alpha JK_{i\ell}\beta JK_{\ell 3}
\end{aligned}
\tag{3.16}
$$

where MJK_i is defined by the relation

$$\bar{M}JK = MJK_i\bar{n}_{ji} \tag{3.17}$$

Finally, in view of Eq. (3.13) no internal couple moment can contribute to F_r for r = 3k+1, 3k+2, 3k+3 except $\bar{M}JK$ and $\bar{M}KJ$. Hence,

$$
\begin{aligned}
\overset{int}{F}_{3k+1} &= MJK_1 \\
\overset{int}{F}_{3k+2} &= MJK_i\alpha JK_{i2} \\
\overset{int}{F}_{3k+3} &= MJK_i\alpha JK_{i\ell}\beta JK_{\ell 3}
\end{aligned}
\tag{3.18}
$$

In summary then the generalized active force F_r is considered to be made up of two parts $\overset{ext}{F_r}$ and $\overset{int}{F_r}$. $\overset{ext}{F_r}$ is the contribution to F_r due only to external forces applied to each body and is calculated from Eq. (3.7). $\overset{int}{F_r}$ can be written in a special form where the only term appearing in the internal couple moment acting between the two bodies where x_r is a generalized coordinate used in representing the relative orientation of these two bodies.

Next consider the generalized inertia force $F_r{}^*$ of Eq. (3.1). From Eq. (1.3), $F_r{}^*$ may be written as

$$F_r{}^* = \sum_{i=1}^{N} (F_i{}^* \cdot \frac{\partial \overline{V}^{Gi}}{\partial \dot{x}_r} + \overline{T}_1{}^* \cdot \frac{\partial \overline{\omega}^{B_i}}{\partial \dot{x}_r}) \tag{3.19}$$

where from Eqs. (1.4) and (1.5)

$$\overline{F}_i{}^* = -m_i \overline{a}^{Gi} \tag{3.20}$$

and

$$\overline{T}_i{}^* = -\overline{\overline{I}}_i \cdot \overline{\alpha}^{B_i} - \overline{\omega}^{B_i} \times \overline{\overline{I}}_i \cdot \overline{\omega}^{B_i} \tag{3.21}$$

Consider writing the expansion of $\overline{\overline{I}} \cdot \overline{\alpha}^{B_i}$ as follows

$$\overline{\overline{I}}_i \cdot \overline{\alpha}^{B_i} = I_{imn} \overline{n}_{om} \overline{n}_{on} \cdot (\omega_{ijk} \ddot{x}_j + \alpha_{ik}) \overline{n}_{ok} \tag{3.22}$$

where there is no sum on i, and I_{imn} are the components of the inertial dyadic of B_i relative to G_i referred to the $\overline{n}_{oi}$ unit vectors. Note that by performing the dot product this equation takes the form,

$$\overline{\overline{I}}_i \cdot \overline{\alpha}^{B_i} = I_{ikn} (\omega_{ijn} \ddot{x}_j + \alpha_{in}) \overline{n}_{ok} \tag{3.23}$$

where I_{imn} is related to the components of $\overline{\overline{I}}_i$ referred to a basis $\overline{n}_{ij}$ fixed in B_i as follows:

$$I_{imn} = SOI_{m\ell} SOI_{nk} I'_{i\ell k} \tag{3.24}$$

where $I'_{i\ell k}$ are the components of $\overline{\overline{I}}_i$ referred to $\overline{n}_{ij}$. Next, consider the expansion of $\overline{\omega}^{B_j} \times (\overline{\overline{I}}_i \cdot \overline{\omega}^{B_i})$. This may be written as

$$\bar{\omega}^{B_i} \times (\bar{\bar{I}} \cdot \bar{\omega}^{B_i}) = \bar{\omega}^{B_i} \times (I_{imn}\bar{n}_{om}\bar{n}_{on} \cdot \omega_{ijk}\dot{x}_j\bar{n}_{ok})$$

$$= \bar{\omega}^{B_i} \times (I_{imn}\bar{n}_{om}\omega_{ijk}\dot{x}_j)$$

$$= \omega_{isn}\dot{x}_s\bar{n}_{on} \times I_{imk}\bar{n}_{om}\,\omega_{ijk}\dot{x}_j$$

$$= e_{nm\ell}\omega_{isn}\omega_{ijk}I_{imk}\dot{x}_s\dot{x}_j\bar{n}_{o\ell}$$

or finally as

$$\bar{\omega}^{B_i} \times (\bar{\bar{I}}_i \cdot \bar{\omega}^{B_i}) = e_{nmk}\omega_{isn}\omega_{ij\ell}I_{im\ell}\dot{x}_s\dot{x}_j\bar{n}_{ok} \tag{3.25}$$

Combining (3.23), (3.25) and using Eq. (2.28) for $\bar{a}^{G_i}$, F_r^* may be written as

$$F_r^* = -\{m_i(V_{ijk}\ddot{x}_j + a_{ik})V_{irk} + I_{ikn}(\omega_{ijn}\ddot{x}_j + \alpha_{in})\omega_{irk} + e_{nmk}\omega_{isn}\omega_{ij\ell}I_{im\ell}\dot{x}_s\dot{x}_j\omega_{irk}\} \tag{3.26}$$

Finally using Eq. (3.4) and (3.26) for F_r and F_r^* the equations of motion may be written in the form

$$F_{rj}\ddot{x}_j = f_r + F_r^{ext} + F_r^{int} \tag{3.27}$$

where

$$A_{rj} = (m_iV_{ijk}V_{irk} + I_{ikn}\omega_{ijn}\omega_{irk}) \tag{3.28}$$

and where

$$f_r = -(m_ia_{ik}V_{irk} + I_{ikn}\alpha_{in}\omega_{irk} + e_{nmk}\omega_{isn}\omega_{im\ell}I_{im\ell}\dot{x}_s\dot{x}_j\omega_{irk}) \tag{3.29}$$

Equations (3.27) are 3N+3 second order nonlinear differential equations. These equations may be solved numerically for the generalized coordinates x_r ($r = 1,2,\ldots,3N+3$) as functions of time. The quantities A_{rj}, f_r, $\overset{ext}{F_r}$, $\overset{int}{F_r}$ may be computed by using the algorithms developed in Chapters I and II.

Consider now the possibility of specifying the relative motion between a number of adjacent bodies (that is, specifying some of the generalized coordinates) and solving for the unknown internal moments between these bodies as well as for the other unknown generalized coordinates as functions of time. This is accomplished by numerically integrating the reduced set of equations which do not involve the unknown moments for the unknown generalized coordinates as functions of time. Then the components of the unknown moments are found from the remaining set of equations. As an example, suppose the relative motion between B_2 and B_4 of Figure 6 is specified and $\overline{M}24$ is to be found along with the other generalized coordinates which specify the position of the system. The equations to be solved are then

$$\sum_{j=1}^{12} A_{rj}\ddot{x}_j + \sum_{j=16}^{N} A_{rj}\ddot{x}_j = f_r + \overset{ext}{F_r} + \overset{int}{F_r} - \sum_{j=13}^{15} A_{rj}\ddot{x}_j \qquad (3.30)$$

$$(r = 1,2,\ldots,12;\ 16,17,\ldots,3N+3)$$

and (Eqs. (3.18))

$$M24_1 = -\sum_{j=1}^{N} A_{13\,j}\ddot{x}_j + f_{13} + \overset{ext}{F_{13}}$$

$$M24_1\alpha 24_{12} + M24_2\alpha 24_{22} + M24_3\alpha 24_{32}$$
$$= -\sum_{j=1}^{N} A_{14\,j}\ddot{x}_j + f_{14} + \overset{ext}{F_{14}} \qquad (3.31)$$

$$M24_1\alpha24_{1\ell}\beta24_{\ell3} + M24_2\alpha24_{2\ell}\beta24_{\ell3} + M24_3\alpha24_{3\ell}\beta24_{\ell3}$$

$$= -\sum_{j=1}^{N} A_{15j}\ddot{x}_j + f_{15} + \overset{ext}{F}_{15}$$

In a similar fashion it is possible to find unknown external forces when motion is prescribed.

IV. SPECIALIZATION FOR IMPULSIVE FORCES

When the N-body system is subjected to impulse forces, forces which become very large over short periods of time, it is possible to derive equations which provide the change in the first derivative of the generalized coordinates. Following Kane (1), if the generalized impulse "I_r" is defined as

$$I_r = \sum_{i=1}^{N} \left(\bar{V}_{\dot{x}_r} \cdot \int_{t_2}^{t_1} \bar{F}_i dt + {}^{B}\bar{\omega}^{i}_{x_r} \cdot \int_{t_2}^{t_2} \overset{ext}{\bar{M}} dt + {}^{B}\bar{\omega}^{i}_{x_r} \cdot \sum_{K} \int_{t_1}^{t_2} \bar{M}KJ dt\right) \tag{4.1}$$

and the generalized momentum p_r is defined as

$$p_r(t) = \sum_{i=1}^{N} m_i {}^{G}\bar{V}^{i} \cdot {}^{G}\bar{V}^{i}_{x_r} + \sum_{i=1}^{N} {}^{B}\bar{\omega}^{i} \cdot \bar{\bar{I}}_i \cdot {}^{B}\bar{\omega}^{i}_{\dot{x}_r} \tag{4.2}$$

Then it is possible to show that (1)

$$p_r(t_2) - p_r(t_1) \approx I_r \qquad (r = 1, \ldots, 3N+3) \tag{4.3}$$

Equations (4.3) are the generalized impulse and momentum equations. Comparing Eq. (4.1) with Eqs. (3.2), (3.6) and (3.18) it is seen that like F_r, I_r can be broken up into two parts as

$$I_r = \overset{ext}{I_r} + \overset{ext}{I_r} \tag{4.4}$$

where $$I_r = \sum_{i=1}^{N} \left({}^{G}\bar{V}^{i}_{\dot{x}_r} \cdot \int_{t_1}^{t_2} \bar{F}_i dt + {}^{B}\bar{\omega}^{i}_{\dot{x}_r} \cdot \int_{t_1}^{t_2} \overset{ext}{\bar{M}_i} dt\right) \tag{4.5}$$

and where $$\overset{int}{I_{3j+1}} = \int_{t_1}^{t_2} MKJ_1 dt \tag{4.6}$$

$$\overset{int}{I_{3j+2}} = \alpha KJ_{i2} \int_{t_1}^{t_2} MKJ_i dt$$

$$\overset{int}{I_{3j+3}} = \alpha KJ_{i\ell} \beta KJ_{\ell 3} \int_{t_1}^{t_2} MKJ_i dt$$

The expression for the generalized momentum equation (4.2) can be written using Eqs. (2.24) and (2.27) as

$$p_r(t) = \left(\sum_{i=1}^{N} m_i V_{ijk} V_{irk} + \omega_{ijn} I_{ikn} \omega_{irk}\right)\dot{x}_j \tag{4.7}$$

and then using (3.28)

$$p_r(t) = A_{rj}\dot{x}_j \tag{4.8}$$

Hence using Eqs. (4.4) and (4.8) in (4.3), equation (4.3) can be written as

$$A_{rj}(\Delta\dot{x}_j) = \overset{\text{ext}}{I_r} + \overset{\text{int}}{I_r} \tag{4.9}$$

where

$$\Delta\dot{x}_j = \dot{x}_j(t_2) - \dot{x}_j(t_1) \tag{4.10}$$

Hence using Eqs. (4.9) it is possible to algebraically solve for the change in the first derivatives of the generalized coordinates.

V. ILLUSTRATIVE APPLICATION

To obtain a simple illustration of the analysis, consider the dynamics of the manipulator system shown in Figure 8. The system

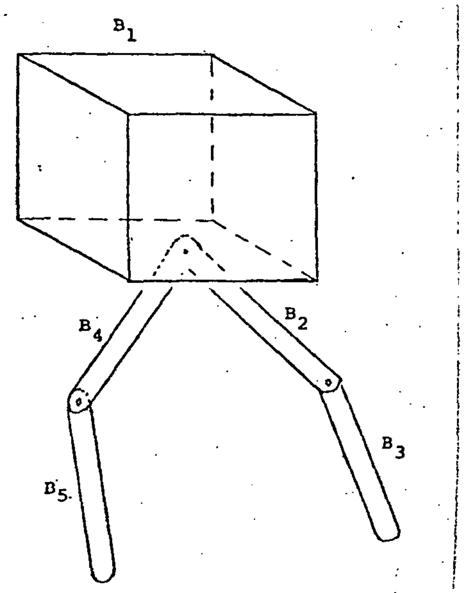

Figure 8.

consists of a main body B_1 and two manipulator arms, each containing two members. The manipulator arms are considered to be connected to the main body by ball and socket joints. The lower part of each arm is connected to the upper part by a hinge joint. The system is located relative to an inertia frame by 14 generalized co-ordinates as follows: x_i $(i = 1,6)$ locate the position of the center of mass of B_1 and the orientation of B_1 relative to the inertia frame, x_i $(i = 7,9)$ locate B_2 relative to B_1, x_{10} locates B_3 relative to B_2, x_i $(i = 11,13)$ locates B_4 relative to B_1 and finally x_{14} locates B_5 relative to B_4. The above orientation angles are developed in a dextral sense from the reference configuration of Figure 9.

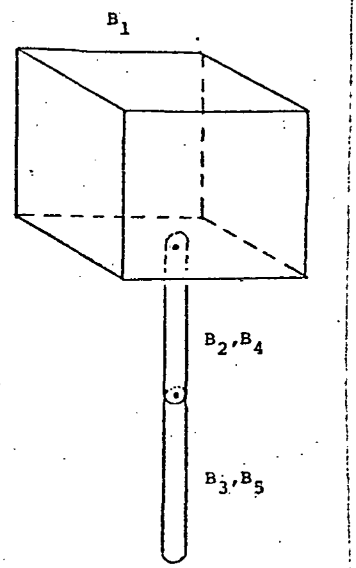

Figure 9.

The governing dynamical equations of motion as developed in the foregoing chapters were used to study the dynamics of this system. The equations were programmed to be developed and solved on a digital computer. This procedure employed an IBM 360-65 computer together with a fourth order Runge-Kutta integration scheme. The physical data of the manipulator is shown in the table of Figure 10.

Body	Mass	Mass Center Location	Reference Point Location	Inertia
1	10 slugs	$\bar{r}_1=0$	$\bar{\xi}_1=0$	$\bar{\bar{I}}_1=240\bar{n}_{11}\bar{n}_{11}+240\bar{n}_{12}\bar{n}_{12}+240\bar{n}_{13}\bar{n}_{13}$ slug in^2
2	.5 slugs	$\bar{r}_2=-18\bar{n}_{23}$ in	$\bar{\xi}_2=-6\bar{n}_{13}$ in	$\bar{\bar{I}}_2=54\bar{n}_{21}\bar{n}_{21}+54\bar{n}_{22}\bar{n}_{22}$ slug in^2
3	.5 slugs	$\bar{r}_3=-18\bar{n}_{33}$ in	$\bar{\xi}_3=-36\bar{n}_{23}$ in	$\bar{\bar{I}}_3=54\bar{n}_{31}\bar{n}_{31}+54\bar{n}_{22}\bar{n}_{22}$ slug in^2
4	.5 slugs	$\bar{r}_4=-18\bar{n}_{43}$ in	$\bar{\xi}_4=-6\bar{n}_{13}$ in	$\bar{\bar{I}}_4=54\bar{n}_{41}\bar{n}_{41}+54\bar{n}_{42}\bar{n}_{42}$ slug in^2
5	.5 slugs	$\bar{r}_5=-18\bar{n}_{53}$ in	$\bar{\xi}_5=-36\bar{n}_{43}$ in	$\bar{\bar{I}}_5=54\bar{n}_{51}\bar{n}_{51}+54\bar{n}_{52}\bar{n}_{52}$ slug in^2
Disk 1	.25 slugs	$\bar{r}_{D1}=0$	$\bar{\xi}_{D1}=-36\bar{n}_{33}$ in	$\bar{\bar{I}}_{D1}=.5\bar{n}_{D11}\bar{n}_{D11}+.5\bar{n}_{D12}\bar{n}_{D12}+\bar{n}_{D13}\bar{n}_{D13}$ slug in^2
Disk 2	.25 slugs	$\bar{r}_{D2}=0$	$\bar{\xi}_{D2}=-36\bar{n}_{53}$ in	$\bar{\bar{I}}_{D2}=.5\bar{n}_{D21}\bar{n}_{D21}+.5\bar{n}_{D22}\bar{n}_{D22}+\bar{n}_{D23}\bar{n}_{D23}$ slug in^2

Figure 10.

Three types of problems were considered: In the first the manipulator was used to bring two circular plates into coincidence with each other. The plates and manipulator system were considered to be in a weightless environment. During the motion the main body was free to rotate and translate. The manipulator arm motion is shown in Figure 11 and was specified with input functions of the form

$$\theta(t) = \theta_o + (\theta_1 - \theta_o)[(t/T) - (1/2\pi) \sin (2\pi t/T)] \tag{5.1}$$

where T is the time of motion duration and θ_o and θ_1 are the values of θ for t=0 and t=T. The output displacement and rotation of the main body was determined and is shown in the graphs of Figures 12 and 13.

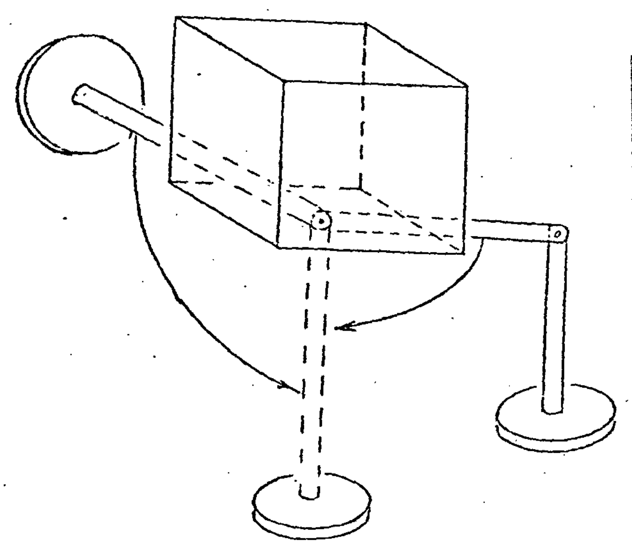

Figure 11.

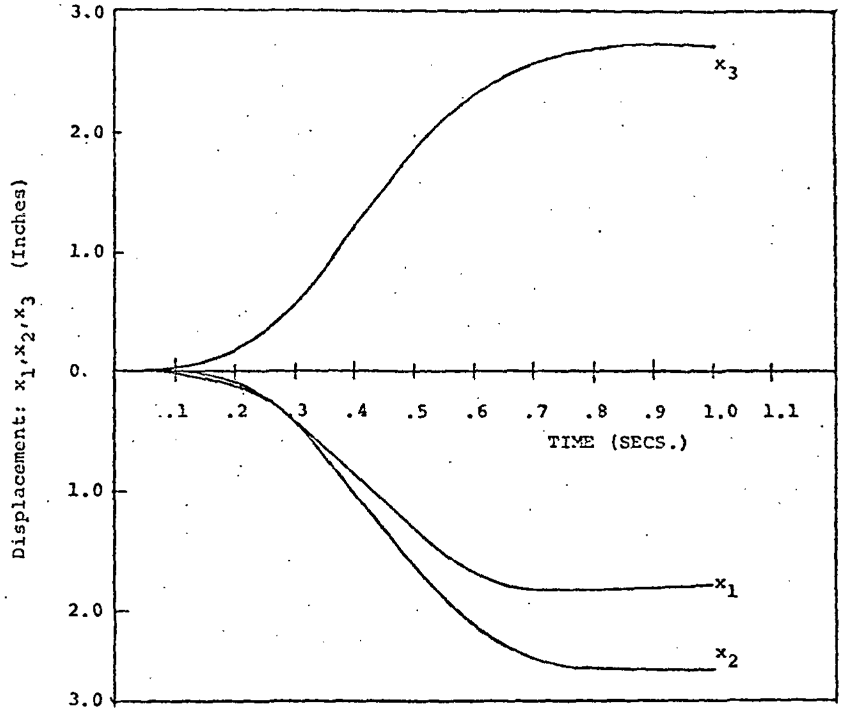

Figure 12.

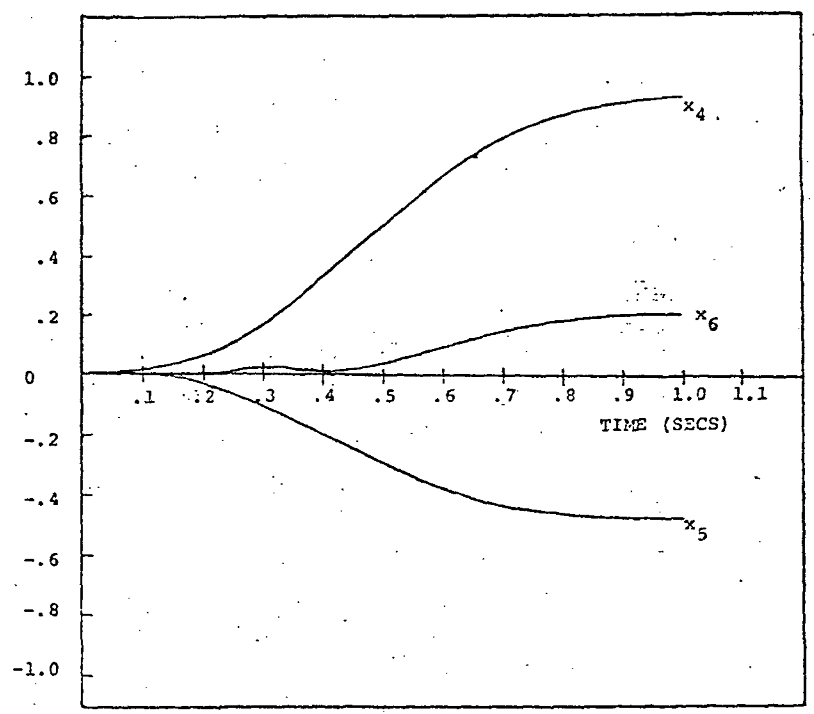

Figure 13.

In the second problem the forces and moments required to hold the main body fixed in space were determined for the manipulator motion of the first problem. The resulting forces and moments are shown in Figures 14, 15.

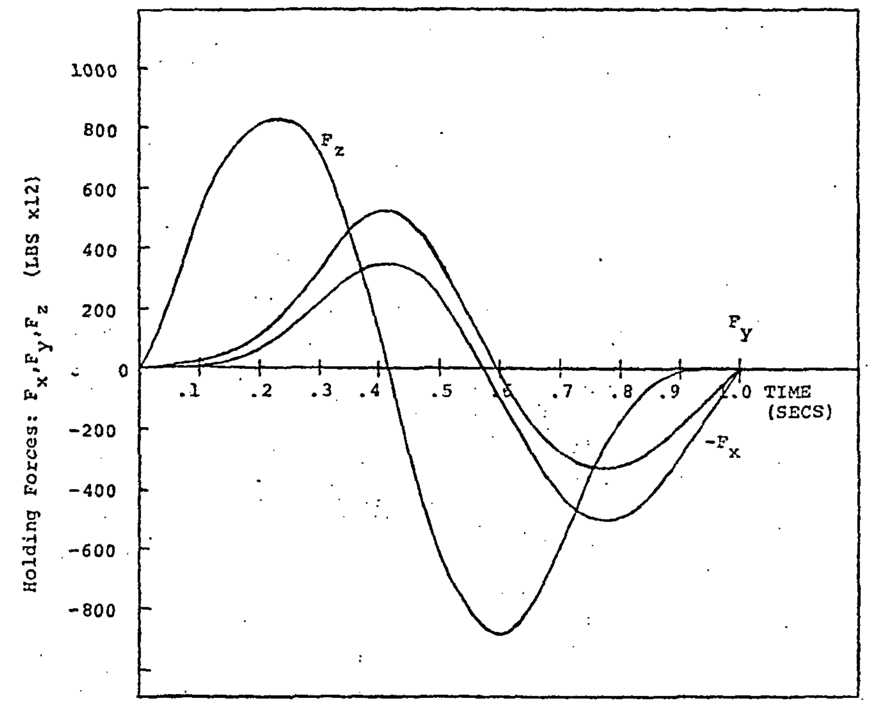

Figure 14.

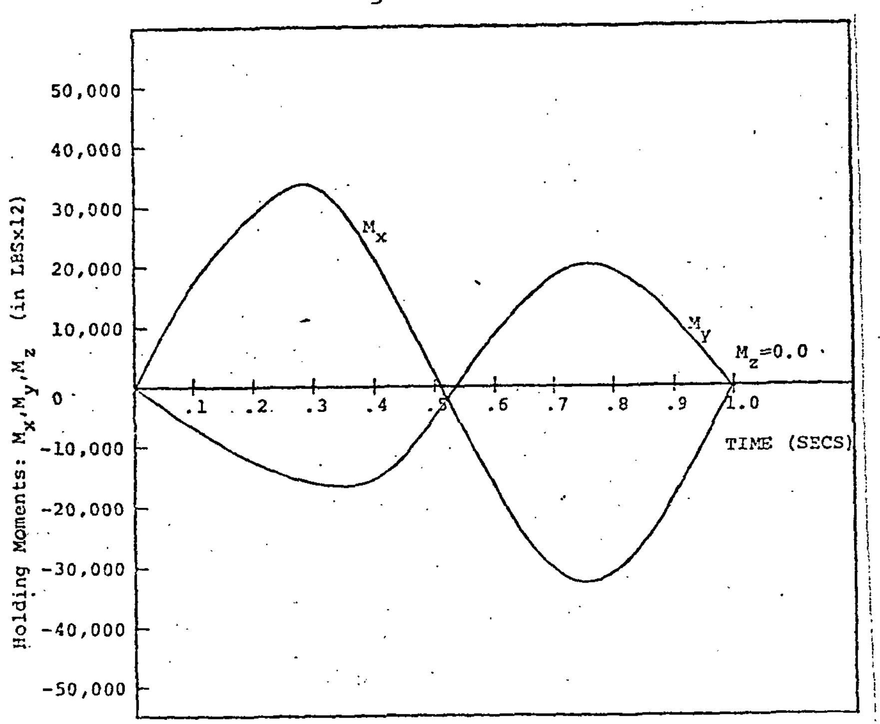

Figure 15.

In the third problem the main body and manipulator system were initially motionless in space in the configuration is had at time zero of the first problem. It was then struck at the mass center of the main body by an impulse of 100 lb. sec. as shown in Figure 16.

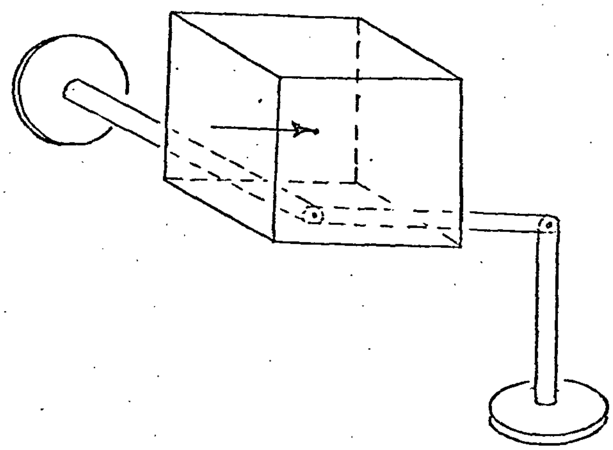

Figure 16.

The bodies of the manipulator arms were left free to rotate relative to their adjoining members. The resulting output increments in the rotation and translation speeds are shown in the table of Figure 17.

i	x_i at t=0.0	Δx_i	i	x_i at t=0.0	Δx_i
1	0.0	112 in/sec	8	90°	3.51
2	0.0	0.0	9	0.0	1.99
3	0.0	-1.99×10^{-5}	10	90°	-4.39
4	0.0	-5.74×10^{-6} rad/sec	11	90°	3.02
5	0.0	1.99	12	0.0	-1.99
6	0.0	1.11×10^{-5}	13	0.0	3.02
7	0.0	6.25×10^{-6}	14	0.0	3.33

Figure 17.

These examples are not meant to be exhaustive studies of manipulators or even of the relatively simple system of Figure 8. Instead they are intended to be simple illustrations of the kind of analyses made possible through the theoretical developments of the foregoing chapters. They show that : (1) given the forces on the system, the resulting displacements and velocity of the members of the system are determined; (2) given the displacements and velocities of the members of the system, the resulting forces are determined; and finally, (3) given a combination of forces, displacements and velocities, the unknown resulting velocities, displacements, and forces are determined.

VI. CONCLUDING REMARKS

The analysis presented represents a new kind of finite-element analysis applicable with a broad class of chain-like dynamical systems. It is computer oriented and designed so that non-working constraint forces are automatically eleminated. Furthermore, the analysis is developed in a way that allows for either forces or displacements to be specified with the unknown resulting displacements or forces then determined.

The method is applicable with any dynamical system which can be modelled by a series of connected rigid bodies provided only that no closed loops are formed by the bodies. Manipulator systems and teleoperators are thus prime candidates for anlaysis by this method. The method is also directly applicable with human body models and cable problems. Finally, by introducing spring and torsion forces at the joints the analysis becomes a nonlinear finite-element elastic analysis.

REFERENCES

1. Kane, T.R., Dynamics, Holt, Rinehart, and Winston, New York, 1967.

2. Pipes, L.A., Applied Mathematics for Engineers and Physicists, McGraw Hill, 2nd Edition, New York, 1958, pp. 437-440.

3. Coughey, T.K., "Whirling of a Heavy Chain," Proceedings of the Third U.S. National Congress of Applied Mechanics, ASME 1958, pp. 101-108.

4. Huang, T. and D.W. Dareing, "Frequencies of a Hanging Chain," Journal of the Acoustical Society of America, Vol. 45, No. 4, April 1969, pp. 1046-1049.

5. Woodward, J.H., "Frequencies of a Hanging Chain Supporting an End Mass," Journal of the Acoustical Society of America, Vol. 49, No. 5. (part 2), May 1971, pp. 1675-1677.

6. Elnan, O., and C. Evert, "Dynamics of a Tethered Balloon in the Longitudinal Plane," Goodyear Aerospace Report-GER-12901, Oct. 1966.

7. Turner, M.J., R.W. Clough, H.C. Martin, and L.J. Topp, "Stiffness and Deflection Analysis of Complex Structures," Journal of the Aeronautical Sciences, Vol. 23, 1956, pp. 805-823.

8. Zienkiewicz, O.C., The Finite Element Method in Structural and Continuum Mechanics, McGraw Hill, London, 1967.

9. Oden, J.T., "Finite Elements of Nonlinear Elastic Continua," McGraw Hill, New York, 1972.

10. Roberts, V.L. and D.H. Robbins, "Multidimensional Mathematical Modelling of Occupant Dynamics under Crash Conditions," Society of Automotive Engineers, International Automotive Engineering Congress, Detroit 1969, paper No. 690248.

11. Kane, T.R. and M.P. Scher, "Human Self-Rotation by Means of Limb Movements," Journal of Biomechanics, Vol. 3, 1970, pp. 39-49.

12. Passerello, C. and R.L. Huston, "Human Attitude Control," Journal of Biomechanics, Vol. 4, 1971, pp. 95-102.

13. Huston, R.L. and C. Passerello, "On the Dynamics of a Human Body Model," Journal of Biomechanics, Vol. 4, 1971, pp. 369-378.

14. Young, R.D., "A Three-Dimensional Mathematical Model of an Automobile Passenger," Texas Transportation Institute Research Report 140-2, August 1970.

15. Chace, M.A. and Y.O. Bayazitogla, "Development and Application of a Generalized d'Alembert Force for Multifreedom Mechanical Systems," Trans. ASME, J. of Engineering for Industry, 1970.

16. Hooker, W.W., "A Set of r Dynamical Attitude Equations for an Arbitrary n-Body Satellite Having r Rotational Degrees of Freedom," AIAA Journal, Vol. 8, pp. 1205-7, July 1970.

17. Fleischer, G.E., "Multi-Rigid-Body Attitude Dynamics Simulator," JPL Technical Report No. 32-1516, February 15, 1971.