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THEORETICAL STUDY OF THE IONOSPHERIC G FACTOR

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by

Keith M. Hagenbuch

March 31, 1972

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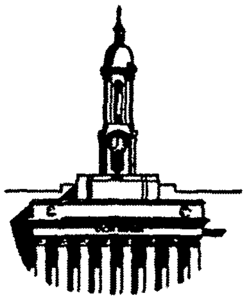
IONOSPHERE RESEARCH LABORATORY



University Park, Pennsylvania

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"Theoretical Study of the Ionospheric G Factor"

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ABSTRACT

A derivation from kinetic theory of the energy balance equation used in wave interaction work is reviewed. G is then defined and two models for G are presented: one, a model based on Gerjuoy-Stein cross sections for both O_2 and N_2; the other based on a more recent model for O_2 cross sections. The two models differ considerably in their temperature dependence. The limits of applicability of the usual energy balance equation (hence of the concept of G) are discussed and it is found that no difficulty arises unless the transmitter power is more than 100 times that now employed at the present operating frequency.

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CHAPTER I

INTRODUCTION

A. Importance of G in Wave Interaction

Cross modulation experiments have become an important technique for determining the parameters of the D-region (such as electron density and collision frequency) [Lee and Ferraro (1969)]. There occurs in the theory of these wave interaction experiments the so-called ionospheric G-factor [Ratcliffe and Shaw (1948)]. G has traditionally represented the fraction of excess energy (in excess of the thermal energy of the neutral background) lost by an electron on the average in a collision with neutral molecules during and immediately after the passage of a heating pulse of radio frequency power through the D-region. G is defined in the assumed form of the thermal energy balance equation [Ratcliffe and Shaw (1948)]

$$\frac{dU}{dt} = - G\nu(U - U_g) + S \qquad 1.1$$

where U is the electron thermal energy, U_g is the neutral background thermal energy, S is the thermal energy input rate from the radio wave, and ν is a collision frequency. A historical review of the development of wave interaction theory and the role that G plays in it is found in Miller (1964).

It is seen in Miller (1964) that the knowledge of G is critical to the reduction and interpretation of wave interaction data. The quantity $G\nu$ is essentially the inverse of the relaxation time of the electron thermal energy. It therefore must be known to determine the electron temperature during relaxation and during the passage of the low power wanted wave. The electron temperature determines the electron collision frequency, and therefore, the amount of interaction.

The model used in present interaction experiments for G is simply a constant [Kissick (1970)]. It is assumed to be sufficiently accurate to consider G constant because the maximum amount of interaction is supposed to take place in a rather narrow height range, over which conditions are thought to be rather uniform. The constant value is determined by a type of wave interaction experiment [Kissick (1970)].

B. Purpose of this Work

An up to date model for G is desirable in wave interaction work. There has been an observed apparent seasonal variation of G correlated with seasonal temperature variation [Kissick (1970)]. This is not consistent with the model discussed above and is also not consistent with a more accurate model [Mentzoni and Row (1963)] that will be discussed later. This work will review the role that various ionospheric parameters play in a G model and will present two models which are consistent with the latest laboratory measurements to determine whether the observed apparent temperature correlation can be explained by an accurate G model. As a result of this work, incorporation of up to date G models in wave interaction will determine whether more precise models significantly affect data reduction results.

The energy relaxation equation is always assumed to be of the form 1.1. Conditions under which this assumption is justified will be discussed. On the basis of this it is possible to determine practical operating frequency and power ranges in which 1.1 is applicable.

C. Methods Used

A review of a derivation of the generalization of 1. 1 using an appropriate kinetic equation will be presented following closely the development in Skharofsky, Johnston and Bachynski (1966). This involves first a spherical harmonic perturbation expansion of the distribution function. The expansion results in a coupled pair of equations involving the spherically symmetric part and a drift part of the distribution function. An asymptotic expansion of these equations will be used to separate the different time scales that are present and to average over the fast time. A resulting equation describing the slow time development of the spherically symmetric part of the distribution function will be averaged over all velocities with respect to kinetic energy to obtain an energy balance equation. Solutions for the spherically symmetric distribution function will be obtained in the steady state for two collision models and used to evaluate terms in the energy balance equation.

D. Results

Two possible models for G are obtained based on the latest available theory and experiment on cross sections for N_2 and O_2. Any later cross section information could be easily used to produce a more accurate model for G. These two models can easily be included in wave interaction theory to determine the sensitivity of data reduction results on the G model.

It is concluded that the temperature dependence of G is critically dependent on what the correct cross section model for O_2 is. However, neither of the presently considered models explains the apparent correlation between G and temperature that results from wave interaction studies.

interaction studies.

It is also concluded that, at the presently used operating frequency for the disturbing, or high power signal, the energy balance equation is, in fact, of the form always assumed unless the power is greater than about two orders of magnitude above that now employed. Only at a lower altitude, or lower operating frequency, or much higher power is the energy balance equation altered from the form commonly assumed.

CHAPTER II

DERIVATION OF LOWEST ORDER DISTRIBUTION FUNCTION EQUATION

In this chapter will be discussed the procedure for obtaining an equation that describes the time relaxation of the spherically symmetric part of the electron distribution. The development is a review of a rather complete analysis in Skharofsky, et. al. (1966) (designated by PKP) and an application specifically to rotation inelastic collisions between ambient electrons and neutral molecules such as O_2 and N_2. The elimination of the short time scales is discussed.

A. Fundamental Assumptions and the Applicable Kinetic Equation

The first assumption made is that the plasma is of sufficiently low density compared with the neutral background, that the only collisions of importance are binary collisions between electrons and neutrals. This is certainly justified in the D-region. Electron-electron and electron-ion collisions can be included by a Fokker-Planck approach. This would be necessary in the F-region. Because of the low density assumption and because of the smallness of the electron mass, it can also be assumed that the neutral particle distributions are not affected by perturbations in the electron distribution.

The kinetic equation that applies in this situation is the Boltzman equation

$$\frac{\partial f}{\partial t} + \vec{u} \cdot \vec{\nabla} f + \vec{a} \cdot \vec{\nabla}_u f + \vec{u} \times \vec{\omega}_b \cdot \vec{\nabla}_u f = \frac{\delta f}{\delta t} \qquad 2.1$$

Here f is the velocity distribution function for electrons, $\vec{a} = -\frac{e\vec{E}}{m}$ where $\vec{E}$ is the externally applied electric field and $\frac{e}{m}$ is the electron charge to mass ratio, $\vec{\omega}_b = e\vec{B}_o/mc$ where $\vec{B}_o$ is the earth's static magnetic field, $\delta f/\delta t$ is the rate of change of f due to binary encounters of electrons with neutrals. The equation is discussed in Chapter 2 of PKP and the expression for the general form for the collision integral, $\delta f/\delta t$, is given by (2-47a) in PKP. Because the analysis and notation is simpler if the static magnetic field is suppressed, it will be neglected. The magnetic field effect might be expected to be small in any case, because, for the problem at hand, the wave frequency is well above the gyrofrequency. The magnetic field can be reintroduced at the end of the analysis if necessary.

B. Further Assumptions and Spherical Harmonic Expansion

Equation 2.1 is much too difficult to work with as it stands. What is often done is to expand the distribution function in orthogonal functions in the hope that the resulting (generally coupled) set of equations can be truncated and solved. When collisions are between electrons and neutrals, the small energy exchanged in an encounter makes a spherical harmonic expansion appropriate. A discussion of this point is given in Chapter 3 of PKP. In the case that differential cross sections depend only on the total scattering angle, not on azimuth angle then the distribution function, when expanded in spherical harmonics, has no azimuth dependence: In fact, especially for one of the models that will later be used, the cross section for inelastic collisions does have azimuthal dependence. But, because the interest here is only in energy relaxation and therefore, as will be seen, only in the spherically symmetric part of the distribution, the presence or absence of aximuthal dependence will not affect the end result. On the other hand azimuthal dependence would ordinarily affect an analysis of momentum transfer effects, but inelastic collisions are unimportant in momentum transfer in the D-region. Therefore the assumption that cross sections do not have azimuthal dependence is justified in the following work.

With this assumption the expansion 3-16 in PKP becomes

$$f = f_0(u) + f_1(u) \cos\theta \qquad 2.2$$

where θ is the angle between the velocity of an electron and the ensemble averaged drift velocity and where u is the electron speed. When the static magnetic field is neglected the drift velocity is opposite the direction of $\vec{E}$. The complete expansion corresponding to 2.2 is

$$f = \sum_{\ell = 0}^{\infty} f_\ell(u, \vec{r}) \, P_\ell(\cos\theta) \qquad 2.3$$

where P_ℓ $(\cos\theta)$ is the Legendre polynomial of order ℓ. The averaged energy is

$$U = \int \frac{1}{2} mu^2 f(\vec{r}, \vec{u}) \, d^3u \qquad 2.4$$

Using the orthogonality property of P_ℓ this becomes

$$U = 4\pi \int_0^{\infty} \frac{1}{2} mu^2 f_0(u, \vec{r}) \, u^2 \, du \qquad 2.5$$

What is therefore needed is an equation describing the time dependence of f_0, since we are only interested in energy relaxation.

The expansion 2.3 converges rapidly, and is therefore useful, when the energy loss in a collision is small and when the applied electric field is not too large. These conditions are satisfied in present wave interaction experiments.

Now $\delta f/\delta t$ must be expanded in Legendre polynomials. This is done in Chapter 3 in PKP in a two step process. First it is assumed that the gas molecules are infinitely heavy so that no energy is lost by an electron in an encounter. This results in the expansion

$$\frac{\delta f}{\delta t} = - \sum_{\ell = 0}^{\infty} \nu_\ell f_\ell P_\ell(\cos\theta) \qquad 2.6$$

where

$$\nu_\ell(u) = Nu \int (1 - P_\ell(\cos\chi))\, I(u, \chi)\, d^2\Omega \qquad 2.7$$

is the ν_ℓth order collision frequency, N is the density of the neutral constituent, I is the differential cross section for an electron collision with a neutral molecule, and χ is the scattering angle in a collision. Notice that ν_0 is zero. ν_1 is the definition of the collision frequency for momentum transfer. But a non zero ν_o is required for an energy relaxation analysis. Therefore the assumption of an infinitely heavy scatterer is relaxed and corrections allowing for scatterer recoil and scatterer motion are made. This is done in Chapter 3 of PKP and the result for elastic collisions is

$$\nu_o f_o = \frac{1}{2u^2} \frac{\partial}{\partial u} \left[\frac{2m}{M-m} \nu_m u^3 \left(f_o + \frac{\kappa T_g}{mu} \frac{\partial f_o}{\partial u}\right) \right] \qquad 2.8$$

where M is the molecular mass, Tg is the neutral temperature, and ν_m is the collision frequency for momentum transfer defined by 2.7 as ν_1. The result for rotation inelastic collisions is

$$\nu_o f_o = \frac{1}{2u^2} \frac{\partial}{\partial u} \left[\frac{2E_{12}^2\, u \nu_1}{m\kappa\, T_g} \left(f_o + \frac{\kappa T_g}{mu} \frac{\partial f_o}{\partial u}\right) \right] \qquad 2.9$$

where E_{12} is the energy transfered to a molecule in an excitation collision in which the molecule is excited from state 1 to state 2, ν_1 is the inelastic collision frequency for the transition 1→2 only

$$\nu_1 = N_1 u \, \sigma_1(u) \qquad 2.10$$

where

$$\sigma_1 = \int I_1 d^2\Omega \qquad 2.11$$

where I_1 is the differential cross section for the transition 1→2, N_1 is the density of molecules in the initial state, and u is the electron speed after the collision. The general result corresponding to 2.9 is just a summation over all possible excitations.

Equation 2.9 can be applied to rotation excitation of quadripole transitions such as occurs in N_2 and O_2. A particular state is characterized by the angular momentum quantum number J. Assume with Mentzoni and Row (1963) that only transitions J→J + 2 are important. Then the generalization of 2.9 for this case is

$$\nu_o f_o = \frac{1}{2u^2} \frac{\partial}{\partial u} \left[\frac{2u^2}{m\kappa T_g} \left(\sum_{J=0}^{\infty} E^2_{J',J+2} N_J \sigma_{J,\ J+2} \right) \left(f_o + \frac{\kappa T_g}{mu} \frac{\partial f_o}{\partial u} \right) \right] \qquad 2.12$$

where 2.10 has been used. Combining elastic and rotation inelastic collisions in 2.9 and 2.12, the result can be written in the compact form

$$\nu_o f_o = \frac{1}{2u^2} \frac{\partial}{\partial u} \left[u^3 \nu_m \, \xi \left(f_o + \frac{\kappa T_g}{mu} \frac{\partial f_o}{\partial u} \right) \right] \qquad 2.13$$

$$\xi = \left[\frac{2m\nu_m}{M-m} + \frac{2}{mu\kappa T_g}\sum_{J=0}^{\infty} E^2_{J,\,J+2}N_J\sigma_{J,\,J+2}\right]\Big/\nu_m \qquad 2.14$$

The generalization to several molecular species is evident.

Now take the applied field to be in a particular direction. This defines the angle θ. Substitution of 2.3 and 2.6 into 2.1, where $\nu_o f$ is given by 2.12 rather than by 2.6, leads to

$$\frac{\partial f_o}{\partial t} + \frac{u}{3}\frac{\partial f_1}{\partial z} - \frac{eE}{3mu^2}\frac{\partial}{\partial u}(u^2 f_1) - \frac{1}{u}\frac{\partial}{\partial u^2}\left[u^3\nu_m\xi\left(f_o + \frac{\kappa T_g}{mu}\frac{\partial f_o}{\partial u}\right)\right] = 0 \qquad 2.15$$

$$\frac{\partial f_1}{\partial t} + u\frac{\partial f_o}{\partial z} - \frac{eE}{m}\frac{\partial f_o}{\partial u} + \nu_m f_1 = 0 \qquad 2.16$$

2.16 is obtained simply by integrating 2.1 over θ and 2.16 by first multiplying by $P_1(\cos\theta)$, then integrating over θ.

These equations are coupled and involve at least three time scales; the wave period, the momentum relaxation time $1/\nu$ and the energy relaxation time $1/(\xi\ \nu_m)$. What is desired is an equation involving only f_o and only the energy relaxation time scale. One way to do this is outlined in Appendix A. What is involved is an asymptotic expansion of f_o and f_1 in multiple time scales. The preceedure results in the equation A29.

$$\frac{\partial f_o}{\partial t} = \frac{1}{2u^2}\frac{\partial}{\partial u}\left[\frac{e^2E_o^2\,\nu_m u^2}{3m^2(\nu_m^2+\omega^2)}\frac{\partial f_o}{\partial u} + \xi\,\nu_m u^3\left(f_o + \frac{\kappa T_g}{mu}\frac{\partial f_o}{\partial u}\right)\right] \qquad 2.17$$

where the temporal behavior is slow on either a wave period or collision time scale. E_o is the amplitude of the applied field and ω is the frequency. If E_o varies it is assumed to vary only on a time scale long compared with $1/\omega$ or $1/\nu_m$.

CHAPTER III

DERIVATION OF THE ENERGY RELAXATION EQUATION

The appropriate equation to use to derive the energy balance equation is

$$\frac{\partial f_o}{\partial t} = \frac{1}{2u^2}\frac{\partial}{\partial u}\left[\frac{e^2 E_o^2 \nu_m u^2}{3m^2(\nu_m^2 + \omega^2)}\frac{\partial f_o}{\partial u} + \xi \nu_m u^3 \left(f_o + \frac{\kappa T_g}{mu}\frac{\partial f_o}{\partial u}\right)\right] \qquad 3.1$$

where f_o is the lowest order (in the spherical harmonic expansion) spherically symmetric part of the electron distribution having time dependence only on the scale long compared with the randomization time and with the radio frequency wave period; e and m are the charge and mass of the electron; E_o is the amplitude of the radio frequency wave; ν_m is the velocity dependent collision frequency for momentum transfer; ω is the radio frequency; κ is the Boltzman constant; and T_g is the temperature of the neutral gas. Furthermore

$$\xi\nu_m = \frac{2m\nu_m}{M-m} + \frac{2}{mu\kappa T_g}\sum_{J=0}^{\infty} E^2_{J,\,J+2} N_J \sigma_{J,\,J+2} \qquad 3.2$$

for one molecular species; M is the molecular mass in cgs units; N_J is the density of molecules in the Jth rotational state; $\sigma_{J,\,J+2}$ is the cross section for the excitation of a molecule originally in state J to state J + 2; $E_{J,\,J+2}$ is the rotational energy gained by the molecule in such an excitation. The generalization to several molecular species is obvious.

In deriving 3.1 it is assumed that spatial gradients can be neglected so that energy lost by the electron gas is lost locally and not through conduction or diffusion.

The energy balance equation is obtained from 3.1 by multiplying by 1/2 mu^2 and integrating over all velocity space. The term on the left is just $\frac{dU}{dt} = \frac{d}{dt}(3/2\, \kappa T)$ where T is the electron temperature.

A. The Energy Source Term

The first term on the right is the rate of energy input by the wave, that is, that fraction of the disturbing wave energy converted by collisions into electron thermal energy. After integration by parts this is

$$\frac{dU_E}{dt} = - \frac{4\pi\, e^2 E_o}{6m} \int_o^\infty \frac{\nu_m u^3}{\nu_m^2 + \omega^2} \frac{\partial f_o}{\partial u}\, du \qquad 3.3$$

For a particular model for ν_m this can be written in terms of a semi-conductor integral provided f_o is Maxwellian [Burke and Hara (1963)]. As an example, for the model that will be used to derive G,

$$\nu_m = \alpha N u^2 \qquad 3.4$$

where α is a constant and N is the neutral number density. Substituting 3.4 into 3.3 one obtains

$$\frac{dU_E}{dt} = \frac{5e^2\, E_o^2}{4m\omega} \left(\omega/\hat{\nu}_m \right) \mathcal{C}_{5/2} \left(\omega/\hat{\nu}_m \right) \qquad 3.6$$

where $\hat{\nu}_m = \frac{2\alpha N \kappa T}{m}$ is the collision frequency for the most probable speed.

B. Rotation Inelastic Collisions

The second term on the right of 3.1 for rotation inelastic collisions only is

$$\frac{dU}{dt}\,\text{inel} = -\int_{o}^{\infty} \sum_{J=0}^{\infty} E^2_{J,\,J+2}\, \frac{N_J}{\kappa T_g}\, \sigma_{J,\,J+2}\, u \left(f_o + \frac{\kappa T_g}{mu} \frac{\partial f_o}{\partial u} \right) 4\pi u^2 du \qquad 3.6$$

The reasonable assumption is made that only inelastic collisions with N_2 and O_2 are important. Energy level populations, N_J can be obtained by following Hertzberg (1950) as follows.

N_2 has nuclear spin I = 1. The relative populations of the total number of states with J even to that with J odd is 2 to 1. Assuming a Boltzman distribution of energy levels, take

$$N_{1J}(\text{even}) = \beta N_1 (2J + 1)\, e^{-B_1 \frac{J(J+1)}{\kappa T_g}}$$

$$N_{1J}(\text{odd}) = \gamma N_1 (2J + 1)\, e^{-B_1 \frac{J(J+1)}{\kappa T_g}}$$

where B_1 is the molecular nitrogen rotational constant, N_1 is the number density, and β and γ are to be determined. By summing over all J, requiring the total sum to be N_1 and the ratio of even to odd states to be 2 to 1, and replacing the sums by integrals

$$N_{1J} \cong N_1 \left(\frac{2B_1}{9\kappa T_g} \right) a_{1J}\, (2J + 1)\, e^{-\frac{B_1 J(J+1)}{\kappa T_g}} \qquad a_{1J} = \left\{ \begin{matrix} 6\text{: J even} \\ 3\text{: J odd} \end{matrix} \right\} \qquad 3.7$$

O_2 has nuclear spin I = 0 and all even J states are unpopulated. A similar argument to that used for nitrogen leads to

$$N_{2J} \cong N_2 \left(\frac{2B_2}{9\kappa T_g} \right) a_{2J} (2J + 1)\, e^{-\frac{B_2 J(J+1)}{\kappa T_g}} \qquad a_{2J} = \left\{ \begin{matrix} \text{q: J odd} \\ \text{o: J even} \end{matrix} \right\} \qquad 3.8$$

Furthermore

$$E_J \cong BJ(J+1)$$

so that

$$E_{J,J+2} \cong B(4J+6). \qquad 3.9$$

Substituting 3.7, 3.8, and 3.9 into 3.6

$$\frac{dU_{inel}}{dt} = \sum_{P=1}^{2} \frac{2N_P B_P}{q} \left(\frac{B_P}{kTg}\right)^2 \sum_{J=o}^{\infty} a_{PJ}(2J+1)(4J+6)^2 e^{-B_P J(J+1)/kT_g} \cdot \int_o^{\infty} 4\pi\sigma_{PJ,J+2} u^3 \left(fo + \frac{\kappa T_g}{mu} \frac{\partial fo}{\partial u}\right) du \qquad 3.10$$

This expression will be used to evaluate the inelastic loss rate by using specific cross section models for O_2 and N_2.

The first model to be considered is that of Gerjuoy and Stein (1955). In this model the molecule is considered to be a point quadrupole. Polarization effects, which distort the wave function of the molecule are considered negligible. Also, the Born approximation is used on the electron wave function. The cross section for excitation is

$$\sigma_{J,J+2} = q\frac{(J+2)(J+1)}{(2J+3)(2J+1)} \left(1 - \frac{B(4J+6)}{\frac{1}{2}mu'^2}\right)^{1/2} \qquad 3.11$$

$$q = \frac{8\pi Q^2 a_o^2}{15} \qquad 3.12$$

where Q is the molecular quadrupole moment, a_o is the first Bohr radius of the hydrogen atom, B is the molecular rotation constant, and $\frac{1}{2}mu'^2$ is the incident electron energy.

This model seems to be quite good for N_2 [Dalgarno (1962), Engelhardt Phelps and Risk (1964), Phelps (1968)] in the energy range of interest in wave interaction work, provided Q is chosen to fit the experimental cross section data, rather than as the actual quadrupole moment. At higher energies polarization effects become more important [Sampson and Mjolsness (1966)] and a more sophisticated model must be used. However, it seems simplest and sufficiently accurate for present work to use the Gerjuoy and Stein model for nitrogen at low energies.

The lack of agreement between theory and experiment at higher energies has lead Sampson and Mjolsness (1966), Takayanaki and Geltman (1965) and Geltman and Takayanaki (1966) to improve the nitrogen cross section model by including polarization effects through use of a non-spherically symmetric interaction potential, and by allowing for distortion of the scattered electron wave function. Phelps (1968) indicates that the Geltman and Takayanaki model does not fit experimental data as well as the simpler GS model. The Sampson-Mjolsness model, however, agrees well in a wide energy range with experimental N_2 cross section values, Phelps (1968).

The situation involving O_2 is not so simple because of the difficulty in measuring experimentally the O_2 cross sections at low energies, [Hake and Phelps (1967).]. Such measurements are needed because they will be seen to critically affect the G-factor. The Gerjuoy and Stein and Geltman and Takayanaki models for O_2 give quite different cross sections, particularly in the range of energy in which we are interested. The Gerjuoy and Stein model is almost certainly not correct, but the Geltman and Takayanaki model may also be held in doubt because of the poor agreement between Geltman-Takayanaki theory and experiment for N_2, [Phelps (1968)]. Sampson and Mjolsness have not done calculations for O_2. Phelps (1968) indicates that experimental results are not able to resolve the conflict between the Geltman and Takayanaki and Gerjuoy and Stein results. The considerable difference between cross sections for O_2 predicted by these models is an indication of the state of experimental work on O_2. Because of this situation calculations of G will be done for both Geltman and Takayanaki, and Gerjuoy and Stein models for O_2. For the Gerjuoy and Stein model take Q to have the value indicated by Dalgarno (1962) to be 1.8 in the energy range $0.1 \text{ ev} < Ee < 0.3 \text{ ev}$. The energy of interest to this work is $\sim 2 \times 10^{-2}$ ev. For the Geltman and Takayanaki model a reasonably good fit to the curves in Figure 5 [Geltman and Takayanaki, 1966] is obtained by taking

$$\sigma_{J,J+2} = 4 \times 10^{-4} a_o^2 (2J+3)^2 \left[\frac{\frac{1}{2} m u'^2}{B_2(4J+6)} - 1 \right]^2 \qquad 3.13$$

It should be noted that the curves in this figure can be altered by a different choice of the parameter A. For this reason the closeness of fit 3.13 represents should not be taken too seriously. The real values of the cross sections might be obtained by choosing the numerical factor in 3.13 as much as a factor of ±3 with respect to the 4×10^{-4}.

C. Elastic Collisions

The second term on the right of 3.1 for elastic collisions only is

$$\frac{dU_{el}}{dt} = - \sum_{p=1}^{2} \frac{4\pi m^2}{M_p - m} \int_0^\infty \nu_{mp} u^4 \left(fo + \frac{kT_g}{mu} \frac{\partial f_o}{\partial u}\right) du \qquad 3.14$$

It will turn out that this term is negligible compared with the inelastic loss rate in the energy range of interest in this work. Nevertheless, it can be carried along through the calculation of G for the sake of generality.

After combining the results 3.3, 3.10, and 3.4 the energy balance equation is

$$\frac{dU}{dt} = \frac{-4\pi e^2 E_o^2}{6m} \int_o^\infty \left(\frac{\nu_m u^3}{\nu_m^2 + \omega^2} \right) \left(\frac{\partial f_o}{\partial u} \right) du$$

$$-\sum_{p=1}^{2} \frac{2N_p B_p}{9} \left(\frac{B_p}{kT_g} \right)^2 \sum_{J=o}^{\infty} a_{pJ} (2J+1)(4J+6)^2 e^{-B_p J(J+1)/kT_g} \int_o^\infty 4\pi\sigma_{J,J+2} u^3 \cdot$$

$$\cdot \left(f_o + \frac{kT_g}{mu} \frac{\partial f_o}{\partial u} \right) du$$

$$-\sum_{p=1}^{2} \frac{4\pi m^2}{M_p - m} \int_o^\infty \nu_{mp} u^4 \left(f_o + \frac{kT_g}{mu} \frac{\partial f_o}{\partial u} \right) du \qquad 3.15$$

CHAPTER IV

DERIVATION OF G

A. Definition of Gν from the Energy Relaxation Equation

Gν is defined by assuming the energy balance equation can be written in the form

$$\frac{dU}{dt} = -G\nu\,(U - U_g) + S \qquad 4.1$$

where ν is an appropriate collision frequency, U_g is the thermal energy of the neutral gas, and S is the energy input rate.

It is obvious that 3. 15 does not have the form 4. 1 in the general case. However, when the electron distribution function is Maxwellian, 3. 15 has the assumed form and at least for this case, Gν is defined.

When the Maxwellian assumption is justified, 3. 15 leads to

$$\frac{dU}{dt} = \frac{e^2 E_o^2}{2m} \frac{\bar{\nu}_m}{\omega^2 + \bar{\nu}_m^2} - \sum_{p=1}^{2} \frac{4\pi m^2}{M_p - m} \left(1 - \frac{T_g}{T}\right) \int_0^\infty \nu_{mp} f_o u^4 du \qquad 4.2$$

$$- \left(1 - \frac{Tg}{T}\right) \sum_{p=1}^{2} \frac{2N_p B_p}{9} \left(\frac{B_p}{kT_g}\right)^2 \sum_{J=o}^{\infty} a_{pJ} (2J+1)(4J+6)^2 e^{\frac{-B_p J(J+1)}{kT_g}} \cdot$$

$$\cdot \int_o^\infty \sigma_{PJ,\,J+2}\, f_o 4\pi u^3 du$$

$$f_o = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{\frac{-mu^2}{2kT}} \qquad 4.3$$

where T is the electron temperature. Comparing 4.1 and 4.2

$$G\nu = \sum_{p=1}^{2} \frac{4\pi m^2}{M_p - m} \frac{2}{3kT} \int_0^\infty \nu_{mp} f_o u^4 \, du$$

$$+ \sum_{p=1}^{2} \frac{4}{27} N_p \left(\frac{B_p}{kT}\right)\left(\frac{B_p}{kT_g}\right)^2 \sum_{J=1}^{\infty} a_{pJ} (2J+1)(4J+6)^2 e^{\frac{-B_p J(J+1)}{kT_g}} \int_0^\infty \sigma_{PJ,\,J+2} f_o 4\pi u^3 \, du \qquad 4.4$$

B. Discussion of Averaged Collision Frequencies

From 4.4 it is seen that G is defined when ν is defined. There are several averaged collision frequencies that can be defined for a given velocity dependent collision frequency model. Three of these are: the averaged collision frequency, the energy-averaged collision frequency (or the collision frequency for energy transfer), and the most probable collision frequency. Any of these can be used to define G and the resultant G's will turn out to differ only by a constant factor from each other. But it seems to be most natural to define G for the purposes of wave interaction in terms of the particular collision frequency that is actually measured in wave interaction experiments. This is the most probable collision frequency.

To be specific take the collision frequency to be

$$\nu_m(u) = \sum_{p=1}^{2} \nu_{mp}(u)$$

$$\nu_{mp}(u) = \alpha_p N_p u^2$$

$$\alpha_1 = 3.7 \times 10^{-23} (\text{cm - sec}) \qquad 4.5$$

$$\alpha_2 = 1.3 \times 10^{-23} (\text{cm - sec})$$

where (1) refers to N_2 and (2) to O_2 and where α_1 has been computed from the data of Pack and Phelps (1961) and α_2 from the results of Mentzoni (1968). Define

$$\alpha N = \sum_{p=1}^{2} \alpha_p N_p \qquad 4.6$$

where N is the total density. $N_1 + N_2$, then

$$\nu_m(u) = \alpha N u^2 \qquad 4.7$$

With this model and using 4. 3 the averaged collision frequency is

$$\bar{\nu} = \int_0^\infty \nu_m(u)\, f_o 4\pi u^2 du = \frac{3\alpha NkT}{m} \qquad 4.8$$

The energy averaged collision frequency is

$$\bar{\nu}_e = \int_0^\infty \left(\frac{\frac{1}{2} mu^2}{\frac{3}{2} kT} \right) \nu_m(u) f_o 4\pi u^2 du = \frac{5\,\alpha NkT}{m} \qquad 4.9$$

Finally the most probable collision frequency is

$$\hat{\nu}_m = \frac{2\alpha NkT}{m} \qquad 4.10$$

In wave interaction experiments the change in the index of refraction due to passage of the high power signal is measured. The index of refraction is related to the plasma parameters through the Sen-Wyller formula (1960). This contains only $\hat{\nu}_m$. In certain limits the formula reduces to the simpler Appleton-Hartree formula (Sen and Wyller, 1960) and in these limits the formula is in its simplest form in terms of either ν or ν_e. However, for the general situation ν_m is the collision frequency measured.

For the sake of completeness, when G is computed it will be computed from $G\nu$ for the above three collision frequencies.

C. Evaluation of G for Two Cross Section Models

With the model given by 4.5, the first term in 4.4 is

$$\frac{10kT}{m} \sum_{p=1}^{2} \frac{m}{M_p - m} \alpha_p N_p$$

Taking Gerjuoy-Stein models for both N_2 and O_2, and using 3.11 and 3.12 in 4.4 it follows that

$$G\nu = \frac{10kT}{m} \sum_{p=1}^{2} \frac{m}{M_p - m} \alpha_p N_p \qquad 4.11$$

$$+ \frac{32}{27}\left(\frac{2kT}{\pi m}\right)^{1/2} \sum_{p=1}^{2} N_p \left(\frac{B_p}{kT}\right)\left(\frac{B_p}{kT_g}\right)^2 \sum_{J=o}^{\infty} a_{pJ} (J+1)(J+2)(2J+3) e^{\frac{-B_p J(J+1)}{kT_g}} K(P,J,T)$$

where

$$K(P,J,T) = q_p \int_o^{\infty} \left[x + \frac{B_p(4J+6)}{kT}\right]^{-1/2} e^{-x} x^{3/2}\, dx \qquad 4.12$$

and the integral is

$$b(P,J)\, e^{b(P,J)} K_1(b(P,J)) \left[1 - 2b(P,J)\right] + 2b^2(P,J)\, K_o(b(P,J)) \qquad 4.13$$

where

$$b(P,J) = \frac{B_p}{kT}(2J+3) \qquad 4.14$$

and where K_1 and K_o are Bessel functions of the third kind.

It is a good approximation to replace 4. 13 by unity, since $B_p << kT$ in the temperature range of interest. Further, because $B\Delta J/kT << 1$ the J-sum in 4. 11 can be replaced by an integral. When this is done equation 5 in Mentzoni and Row (1963) for N_2 only, is obtained. A similar approximation can be done for O_2. The result is

$$G\nu = \frac{10kT}{m}\sum_{p=1}^{2}\frac{m}{M_p - m}\alpha_p N_p + \frac{16}{3}\left(\frac{2kT}{\pi m}\right)^{1/2}\sum_{p=1}^{2} N_p\left(\frac{B_p}{kT}\right) q_p \qquad 4.15$$

Call 4. 15 the GS model.

The second model is obtained by using a Gerjuoy-Stein model for the N_2 cross section, but a Geltman-Takayanaki model for O_2. In this case $G\nu$ can be written in the form of 4. 11 provided

$$K(2, J, T) = 3 \times 10^{-4} a_o^{\,2}\, \frac{(2J+1)(2J+3)}{(J+1)(J+2)}\left(\frac{kT}{B_2}\right)^2 \qquad 4.16$$

The analytical approximation for this model, derived in the same way as 4. 15 is

$$G\nu = \frac{10kT}{m}\sum_{p=1}^{2}\frac{m}{M_p - m}\alpha_p N_p + \frac{16}{3}\left(\frac{2kT}{\pi m}\right)^{1/2}\left[N_1\left(\frac{B_1}{kT}\right)q_1 + 2.4 \times 10^{-3} a_o^{\,2} N_2\left(\frac{kT}{B_2}\right)\right] \qquad 4.17$$

Call 4. 17 the GT model

The expression for G is obtained by dividing 4. 11 or 4. 17 by the appropriate collision frequency. In Figure 1 are plotted G-factors as a function of the electron temperature for the GS and the GT models and for the three collision frequencies. There are differences which occur because of the different collision frequencies used. But there are also considerable differences between G's corresponding to the same collision frequency but to different cross section models.

The latter differences are due solely to the Geltman-Takayanaki model for O_2. Specifically they arise because, in the energy range of interest, the Gerjuoy-Stein cross section for oxygen is practically independent of energy, whereas the Geltman-Takayanaki model varies as energy squared. This in itself would not result in such a large difference were it not the case that the cross section for O_2 in the GT model begins to dominate the cross section for N_2 in the GS model.

It should be emphasized again that the GT model has not been verified experimentally. Nevertheless, if it does turn out to be a better model than the GS model for O_2, then G, and therefore $G\nu$, is larger than has been assumed and measured in wave interaction work.

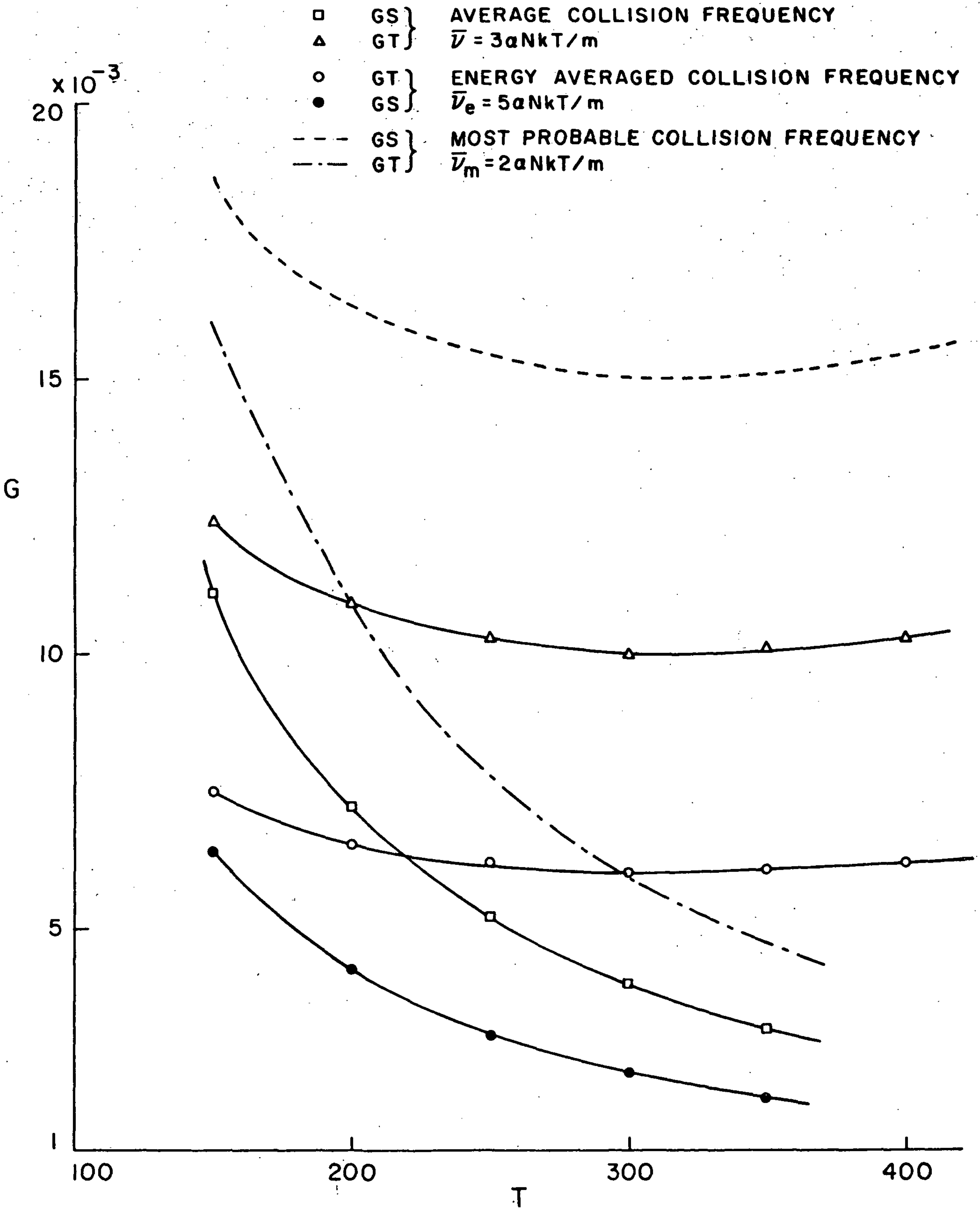

FIGURE 1: Ionospheric G Factor for Two Cross Section Models and Three Collision Frequencies.

CHAPTER V

THE DISTRIBUTION FUNCTION IN THE STEADY STATE

It was seen in Chapter IV that G is defined only when equation 4.1 can be written. This seems to rest on the assumption that f_o is Maxwellian. In this chapter some of the limiting forms of the actual distribution function will be considered.

A. The General Solution

Skharofsky, et al., (1966) derive an expression for the steady state electron distribution function. This can also be obtained by using equation 3.1 in the steady state limit. The solution, in the absence of a static magnetic field, is

$$f_o = Ae^{-w} \qquad 5.1$$

$$w = \int_0^u \frac{mudu}{kT_g + \dfrac{e^2 E_o^2}{3m\xi(\nu_m^2 + \omega^2)}} \qquad 5.2$$

Throughout the following calculations take ν_m to be

$$\nu_m = \alpha N u^2 \text{ (sec}^{-1}) \qquad 4.7$$

where, from 4.5 and 4.6 α has the approximate value

$$\alpha \cong 3.0 \times 10^{-23} \text{ (sec - cm)}$$

and N lies in the range

$$3.8 \times 10^{14} \leq N \leq 6.5 \times 10^{15} \text{ (cm}^{-3})$$

between 80 km. and 60 km. In the same height range the temperature range is

$$186^\circ \leq T_g \leq 243^\circ$$

Again define $\hat{\nu}_{mg}$ as in 4.10, but replacing T with T_g. Then $\hat{\nu}_{mg}$ lies in the range

$$6.2 \times 10^5 \leq \nu_{mg} \leq 1.5 \times 10^7 \ (\text{sec}^{-1})$$

in the height range 80 km. to 60 km.

Distribution functions will be evaluated for two models: one in which ξ is constant and the other in which $\xi = \beta/u^2$. These may be thought to correspond to the GT and GS models, respectively, discussed in Chapter IV. One can make an actual computation of ξ by using equation 3.2, or by defining $G\nu$ in terms of $\xi\nu_m$ through the use of 3.1 and an assumption that f_o is Maxwellian. Using the former method one gets for the GS model

$$\xi\nu_m = \sum_{P=1}^{2} \left(\frac{2m\nu_{mP}}{M_p - m} + \frac{8q_pB_pN_p}{mu} \right) \qquad 5.3$$

and for the GT model

$$\xi\nu_m = \sum_{P=1}^{2} \frac{2m\nu_{mp}}{M_p - m} + \frac{8q_1B_1N_1}{mu} + \frac{8 \times 10^{-4} a_o^2 N_2 mu^3}{B_2} \qquad 5.4$$

In the energy range of interest in the D-region the elastic contribution can be neglected. Now although ξ as defined by 5.3 varies as $1/u^3$ and as defined by 5.4 as a more complicated function, these can be fitted fairly well by the above representation in the energy range of interest, provided we take $\beta = 7.7 \text{ X } 10^{11}$ and the constant value equal to 1.07×10^{-2}.

The two models, when used in 5.2 lead to

$$W_{GS} = \int_0^x \frac{dx}{1 + \frac{cx}{1 + a x^2}} \qquad 5.5$$

$$W_{GT} = \int_{o}^{x} \frac{dx}{1 + \frac{b}{1+ax^2}} \qquad 5.6$$

where

$$x = mu^2/2kT_g \qquad 5.7$$

$$c = 2e^2 E_o^{\,2}/(3m^2 \beta \omega^2) \qquad 5.8$$

$$b = e^2 E_o^{\,2}/(3m\, \xi\, kT_g \omega^2) \qquad 5.9$$

$$a = (\hat{\nu}_{mg}/\omega)^2 \qquad 5.10$$

$$\xi = 1.07 \times 10^{-2} \qquad 5.11$$

$$\beta = 7.7 \times 10^{7}\ (m^2/sec^2) \qquad 5.12$$

5.8 and 5.9 can be rewritten

$$c = 4.04 \times 10^{14}\ P/\ (h^2 f^2) \qquad 5.13$$

$$b = .95 \times 10^{17}\ P/(h^2 f^2 T_g) \qquad 5.14$$

where P is in megawatts, h in kilometers, and f in cycles per second. In interaction work at Penn State $f = 4.5 \times 10^6$ and $P = 15$ for which

$$\begin{aligned} 4.8 \times 10^{-4} &\leq a \leq .28 \\ .047 &\leq c \leq .084 \\ .060 &\leq b \leq .081 \end{aligned} \qquad 5.15$$

in the height range 80 to 60 kilometers.

All of the values in 5.15 can be increased two orders of magnitude by decreasing f to 1×10^6 and by increasing P to 150 megawatts. It is of interest in wave interaction work to know whether results obtained from the energy balance equation 4.1 are meaningful for a fairly wide range of frequencies and powers. In other words, are conditions easily arranged so that the energy balance equation 4.1 must be modified? To at least partially answer this question we want to determine whether 4.1 is modified in any essential way when the distribution is given by 5.1 and 5.2.

B. Solution When ξ is Constant

5.6 can be integrated to give

$$W_{GT} = x - ba^{-1/2}(1+b)^{-1/2} \tan^{-1}(a^{1/2}x(1+b)^{-1/2}) \qquad 5.16$$

Distribution function averages cannot be evaluated analytically using 5.16. For this reason consider limiting cases in which 5.16 can be approximated by a simpler dependence on x. First let the argument of $\tan^{-1}$ be large. Then

$$W_{GT} \cong -\frac{\pi}{2} + ba^{-1/2}(1+b)^{-1/2} + x + ba^{-1}x^{-1} - \frac{1}{3}a^{-2}b(1+b)x^{-3} \qquad 5.17$$

When $b/a << 1$ and $a >> 1$ this can be replaced by

$$W_{GT} \cong x + ba^{-1}x^{-1} \qquad 5.18$$

This corresponds to a very high collision frequency or low operating frequency. The second case is that where the argument of $\tan^{-1}$ is small. Then

$$W_{GT} = x(1+b)^{-1} + \frac{1}{3}abx^3(1+b)^{-2} \qquad 5.19$$

It is assumed here that the peak in the speed distribution lies near $x = 1$, so we are interested in approximations valid in this region. However, when b is of order 100 the electron temperature is on the order of 100 times the neutral temperature. When this is the case the second term in 5.19 is the significant one. However, such values of b are practically unattainable. Then for all practical situations in which the approximation 5.19 is valid, or when $a^{1/2} << 1$

$$W_{GT} = x(1+b)^{-1} \qquad 5.20$$

which is a Maxwellian with temperature

$$T = T_g (1 + b)$$

C. Solution When $\xi = \beta / v^2$

Again 5.5 can be integrated exactly but the result cannot be used to evaluate distribution function averages. Consider first the case in which a >>1

$$W_{GS} \cong x - \frac{c}{a} \ln (1 + ax/c) \qquad 5.21$$

This again applies for high collision frequencies or low operating frequencies. The other case is that in which a <<1

$$W_{GS} = \frac{1}{c} \ln (1 + cx) \qquad 5.22$$

This applies to most of the D-region high wave frequencies for values of c not above about 10.

CHAPTER VI

EFFECT ON ENERGY RELAXATION BY USING NON-MAXWELLIAN DISTRIBUTION FUNCTIONS

The definition of $G\nu$ (4. 4) obtained from (4. 1) and 3. 15 rests on the assumption that fo is Maxwellian. It was seen in Chapter V that the distribution does not have the form of a Maxwellian in general. However, for small applied fields it is natural to expect that the actual distribution would be nearly Maxwellian and that (4. 2) and (4. 4) are good approximations. In this chapter the energy relaxation equation will be obtained using approximations to the actual distributions that converge uniformly to a Maxwellian as the applied field becomes small. A comparison will be made with (4. 2).

A. The Energy Balance Equation for Two Models

The energy balance equation that follows from 3. 1 with 3. 2 is

$$\frac{dU}{dt} = -\frac{kT_g a^{1/2}}{2}\left(\frac{2kT_g}{m}\right)^{3/2}\left[\int_0^\infty \frac{e^2 E_o^2}{3m\omega^2 kT_g}\left(\frac{x^{5/2}}{1+ax^2}\right)4\pi\,\frac{df_o}{dx}dx + \int_0^\infty \xi x^{5/2} 4\pi\, f_o + \left(\frac{df_o}{dx}\right) dx\right] \qquad 6.1$$

where all constants are defined in Chapter V and where the following change of variables has been made

$$x = mu^2/2kT_g . \qquad 6.2$$

This equation is identical with 3. 15 when 3. 2 is used. With the approximations, discussed in Chapter V, to the Geltman and Takayanaki and to the Gerjuoy and Stein models 6. 1 becomes

$$\frac{dU}{dt} = -\frac{kT_g \omega a^{1/2} \xi}{2} \left(\frac{2kT_g}{m}\right)^{3/2} \left[\int_o^\infty \frac{bx^{5/2}}{1+ax^2} 4\pi \frac{df_o}{dx} dx + \int_o^\infty x^{5/2} 4\pi \left(f_o + \frac{df_o}{dx}\right) dx \right] \qquad 6.3$$

for the model where O_2 cross section is Geltman and Takayanaki and the N_2 cross section is Gerjuoy and Stein where ξ and b are given by 5. 11 and 5. 9 respectively. Call this model GT. Similarly 6. 1 becomes

$$\frac{dU}{dt} = -\frac{kT_g \omega a^{1/2}}{2} \left(\frac{\beta m}{2kT_g}\right) \left(\frac{2kT_g}{m}\right)^{3/2} \left[\int_o^\infty \frac{cx^{5/2}}{1+ax^2} 4\pi \frac{df_o}{dx} dx + \int_o^\infty x^{3/2} 4\pi \left(f_o + \frac{df_o}{dx}\right) dx \right] \qquad 6.4$$

for the model where both N_2 and O_2 cross sections are Gerjuoy-Stein with β and c given by 5. 12 and 5. 8 respectively. Call this model GS. In the steady state 6. 3 and 6. 4 are satisfied by distributions derived from 5. 6 and 5. 5 respectively. Therefore, when discussing the effects of various non-Maxwellian distributions 5. 5 and 5. 6 should not be used in non-steady state conditions, since they satisfy the steady state parts of 6. 3 and 6. 4 alone. What will be done is to use distributions derived from expressions identical in form to 5. 5 and 5. 6, but with constants different from c, b, a.

B. Comparison of Relaxation Equations Derived Using a Maxwellian with Those Derived Using Non-Maxwellian Distributions.

Consider first the GT model and 6. 3. When a <<1, which is satisfied at a frequency of 4. 5 MHz throughout the D-region above 60 km. the distribution is Maxwellian, as is implied by 5. 20. The next higher order term in the expansion of 5. 16 becomes important only when b is several orders of magnitude larger than indicated in 5. 15. It is therefore clear that at the presently used frequency and power, the electron distribution is Maxwellian and the energy balance equation has the assumed form 4.1 provided the GT model is correct.

When a>>1 the simplest approximation to the distribution is derived from 5.18. But the result does not coverge uniformly for all x to a Maxwellian for small $\frac{b}{a}$. If a better approximation is used then the required integrals in 6.3 cannot be performed in closed form. The condition a<<1 applies only below 60 km. or when a lower operating frequency is used.

Consider next the GS model and 6.4. When a<<1 the distribution is obtained from 5.22 and has the form

$$f_o = \left(\frac{m}{2\pi kT_g}\right)^{3/2} \gamma^{3/2} \frac{\Gamma(\gamma^{-1})}{\Gamma(\gamma^{-1} - 3/2)} (1 + \gamma x)^{-\gamma^{-1}} \qquad 6.5$$

Defining the temperature by

$$\frac{3}{2}kT \equiv \overline{\frac{1}{2}mu^2} \qquad 6.6$$

and using 6.5 leads to

$$\gamma = \frac{2}{5}(1 - T_g/T) \qquad 6.7$$

Using 6.5 and 6.7 and carrying out the integrations in 6.4 one obtains

$$\frac{dU}{dt} = - \frac{kT_g \omega a^{1/2}}{2} \left(\frac{\beta m}{2kT_g}\right) \left[- \frac{15}{2} c \left(\frac{T}{T_g}\right) + 3\left(\frac{T}{T_g}\right)\left(1 - \frac{T_g}{T}\right)\right] \qquad 6.8$$

which is identical to the result obtained by using a Maxwellian in 6.4.

Therefore the balance equation has the assumed form 4.1 and $G\nu$ is the same whether evaluated using a Maxwellian or 6.5. These conclusions follow if the GS model is correct and provided c is not larger than about 10.

When a>>1 then the distribution is obtained from 5.21 and has the form

$$f_o = \frac{1}{2\pi}\left(\frac{m}{2kT_g}\right)^{3/2} \frac{e^{-x}(x+\alpha)^{\alpha}}{\int_0^{\infty} x^{1/2} e^{-x}(\alpha+x)^{\alpha}\,dx} \qquad 6.9$$

where the integral can be expressed in terms of a Whittaker function (Abramowitz and Stegun, 1964). Defining the temperature as in 6.6, one finds

$$T/T_g = \frac{2}{3}\,\frac{\int_0^{\infty} x^{3/2} e^{-x}(\alpha+x)^{\alpha}\,dx}{\int_0^{\infty} x^{1/2} e^{-x}(\alpha+x)^{\alpha}\,dx} \qquad 6.10$$

Using 6.4 it follows that

$$\frac{dU}{dt} = \frac{kT_g\omega a}{2}\,{}^{1/2}\left(\frac{\beta m}{2kT_g}\right)\left(\frac{c}{a} - \alpha\right)\frac{\int_0^{\infty} x^{-1/2} e^{-x}(\alpha+x)^{\alpha}\,dx}{\int_0^{\infty} x^{1/2} e^{-x}(\alpha+x)^{\alpha}\,dx} \qquad 6.11$$

The evaluation of the integrals results in expressions which do not reduce 6.11 to the form of 4.1. However, considering the limit that $c/a<<1$ and $\alpha<<1$, by expanding the Whittaker functions in powers of α one can show

$$T/T_g \cong 1 + \frac{2}{3}\alpha \qquad 6.12$$

plus higher order terms and

$$\frac{dU}{dt} = \frac{kT_g\omega a}{2}\left(\frac{\beta m}{2kT_g}\right)^{1/2}\frac{\left(\frac{c}{a} - \alpha\right)}{\frac{1}{2} + \alpha} \qquad 6.13$$

accurate to first order in α. This gives an energy balance equation

$$\frac{dU}{dt} = \frac{kT_g\omega a}{2}\left(\frac{\beta m}{2kT_g}\right)^{1/2}\left[\frac{c}{a}(2 - 4\alpha) - 2\alpha\right] \qquad 6.14$$

with α given by 6.12. The equivalent equation for a Maxwellian derived from 6.4 is

$$\frac{dU}{dt} = \frac{kT_g\omega a}{2}\left(\frac{\beta m}{2kT_g}\right)^{1/2}\left[\frac{2c}{a}\,\frac{T_g}{T} - 3\left(\frac{T}{T_g} - 1\right)\right] \qquad 6.15$$

6.14 and 6.15 are identical to lowest order in $\frac{T}{T_g} - 1$. But they begin to differ significantly, particularly in the energy input term, when $\alpha \sim 1$ which is essentially the same as $c/a \sim 1$. Therefore the balance equation has the assumed form 4.1 again provided $c/a \ll 1$ and the GS model is correct. The inequality would be satisfied for powers up to 100 times that presently used.

C. Discussion

It can be seen that the assumption that f_o is Maxwellian for the purpose of obtaining the energy balance equation is well justified at least in the cases considered. Apparently this is so because the form 4.1 is somewhat insensitive to the exact shape of the electron distribution. For both the GS and GT models the distribution can be taken Maxwellian whenever the collision frequency is small compared with the wave frequency and the operating power is not more than 100 times that now used. The distribution can also be taken Maxwellian, at least for the GS model, when the collision frequency is large compared with the wave frequency and the operating power is not more than 100 times that now used.

It should be noted that c and b are increased as the operating frequency is decreased. The distribution might be sufficiently non-Maxwellian, if the wave frequency were 450 kHz and if the operating power were the same as now used, that 4. 1 is not justified. But at such low operating frequencies static magnetic effects most certainly would have to be included in a theoretical analysis.

CHAPTER VII

DISCUSSION AND CONCLUSIONS

A kinetic theory derivation of the energy balance equation (1.1) appropriate to D-region wave interaction studies has been reviewed. The energy balance equation has the assumed form 4.1 and 4.2 whenever the electron velocity distribution is Maxwellian. Then $G\nu$ is defined in 4.4 and G is defined when an appropriate collision frequency is defined.

Two models for G have been discussed, based on two theoretical models for the interaction between low energy electrons and molecular oxygen and molecular nitrogen. The first (GS), is based on Gerjuoy-Stein cross sections for both N_2 and O_2 and has been considered by Mentzoni. (In this case $G\nu$ is given by 4.13 and 4.14 or by 4.17). The Gerjuoy-Stein model for N_2, although very simple, is almost certainly reasonably accurate in the energy range of interest here [Phelps (1968)]. However, using the Gerjuoy-Stein model for O_2 may not be justified [Phelps (1968)]. In the GS model, G has very nearly $T^{-3/2}$ dependence. Furthermore, G depends on neutral particle densities only through the percentage compositions, which are essentially constant throughout the D-region. Thus, G really depends only on the electron temperature and only very weakly on the neutral temperature.

The second model (GT) is based on Gerjuoy-Stein cross sections for N_2, but on Geltman-Takayanaki cross sections for O_2. In this case $G\nu$ is given by the approximation 4.18. Through the electron temperature range of interest (Fig. 1) G is practically a constant and is about twice the value used and obtained in wave interaction studies [Kissick (1970)] .

When better experimental data on O_2 cross sections become available, and if the cross sections are neither of the Gerjuoy-Stein or Geltman-Takayanaki forms, then 4.4 can be used to obtain the corresponding $G\nu$.

It is seen from Fig. 1 that the apparent temperature dependence of G ($\sim T^{-5/2}$) obtained from wave interaction studies [Kissick (1970)] is not explained on the basis of the two models discussed here. In order to have this temperature dependence either the N_2 or the O_2 cross section, or both, would have to be inversely proportional to the electron energy. This would seem to be very unlikely.

Approximate limits have been determined on the practical use of the energy balance equation in the form 4.1, and therefore on the definition of G using this equation. This has been done by using approximations to the steady state form for the actual electron distribution function in 3.1 in place of a Maxwellian. The resulting equation is compared with the energy balance equation obtained using a Maxwellian distribution.

This comparison was possible in two limiting cases. First is the limit $a \ll 1$, where (a) is defined by 5.10. At the wave frequency now used at Penn State, this is satisfied throughout the range of 60 to 80 kilometers. Then, even though the actual distribution is not Maxwellian for the GS model, the energy balance equation is the same as though it were Maxwellian. In the GT model the actual distribution is Maxwellian in the above limit. Thus, provided the incident wave power is not more than 100 times that now used, the energy balance equation has the form 4.1 to zero order in (a).

The other limiting case is that where ($a \gg 1$ and $c/a \ll$). Now $c/a \ll 1$ is satisfied throughout the height range 60 to 80 kilometers with the present operating frequency and power. This ratio is independent of frequency. The above conditions can be thought to apply at the present or lower operating power, but at lower frequency (<500 κ c). For the GS model the energy balance equation has the form 4.1, even when the actual distribution is not Maxwellian, to lowest order in c/a. However, when c/a is of order one this is no longer so.

At the present operating frequency it can be concluded that the energy balance equation is of the form 4.1 and G is well defined provided the power used is not greater than about 100 times that now used.

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APPENDIX A

DECOUPLING THE PERTURBATION EQUATIONS BY AN ASYMPTOTIC EXPANSION

Begin with the equations

$$\frac{\partial f_o}{\partial t} = \frac{1}{2v^2}\frac{\partial}{\partial v}\left[\frac{2eE_o\phi(t)}{3m} v^2 f_1 + \xi \nu_m v^3 \left(f_o + \frac{kT_g}{mv}\frac{\partial f_o}{\partial v}\right)\right] \qquad \text{A 1}$$

$$\frac{\partial f_1}{\partial t} = -\nu_m f_1 + \frac{eE_o\phi(t)}{m}\frac{\partial f_o}{\partial v} \qquad \text{A 2}$$

Define the dimensionless constants and variables

$$u = \left(\frac{2kT_g}{m}\right)^{-1/2} v$$

$$\tau = wt \qquad \text{A 3}$$

$$\alpha = eE_o/(2mkT_g\,\omega^2)^{1/2}$$

$$g_o(\tau) = f_o(t)$$

$$g_1(\tau) = f_1(t)$$

Then A 1 and A 2 become

$$\frac{\partial g_o}{\partial t} = \frac{1}{2u^2}\frac{\partial}{\partial u}\left[\frac{2\alpha\phi(\tau)}{3} u^2 g_1 + \frac{\xi\nu_m}{\omega}\left(u^3 g_o + \frac{u^2}{2}\frac{\partial g_o}{\partial u}\right)\right] \qquad \text{A 4}$$

$$\frac{\partial g_1}{\partial \tau} = -\frac{\nu_m}{\omega} g_1 + \alpha\phi(\tau)\frac{\partial g_o}{\partial u} \qquad \text{A 5}$$

Assume two distinct time scales [Cole (1968)] τ_o and τ_1 where τ_1 is long and τ_o is short. Assume further that

$$\tau = \tau(\tau_o, \tau_1) \qquad \text{A 6}$$

such that $\frac{\partial \tau_o}{\partial \tau} = 1$ and $\frac{\partial \tau_1}{\partial \tau} = \epsilon$ where ϵ is a small parameter. τ_1 is associated with $\xi \nu_m / \omega$ and τ_o with ν_m / ω. Accordingly assume

$$\xi \nu_m / \omega = \epsilon \zeta (v) \qquad \text{A7}$$

$$\nu_m / \omega = \eta (v) \qquad \text{A8}$$

where η and ζ are of order 1. Now assume the expansion

$$g_o = \sum g_o^i \epsilon^i \qquad \text{A9}$$

$$g_1 = \sum g_1^i \epsilon^i \qquad \text{A 10}$$

to be valid in an asymptotic sense [Cole, 1968] .

Substitution of A 7 - A 10 in A 4 and A 5 leads to

$$\frac{\partial g_o^o}{\partial \tau_o} = \frac{1}{u^2} \frac{\partial}{\partial u} \left(\frac{\alpha \phi (\tau_o)}{3} \right) u^2 g_1^o \qquad \text{A 11}$$

$$\frac{\partial g_1^o}{\partial \tau_o} = - g_1^o + \alpha \frac{\partial g_o^o}{\partial u} \qquad \text{A 12}$$

to lowest order in ϵ, where ϕ has been taken to depend only on the fast time scale for simplicity. To first order in ϵ

$$\frac{\partial g_o^o}{\partial \tau_o} + \frac{\partial g_o^1}{\partial \tau_o} = \frac{\alpha \phi (\tau_o)}{3u^2} \frac{\partial}{\partial u} (u^2 g_1^1) + \frac{1}{2u^2} \frac{\partial}{\partial u} \left[\zeta (u^3 g_o^o + \frac{u^2}{2} \frac{\partial g_o^o}{\partial u}) \right]$$

$$\frac{\partial g_1^o}{\partial \tau_1} + \frac{\partial g_1^1}{\partial \tau_o} = \eta g_1^1 + \alpha \phi (\tau_o) \frac{\partial g_o^1}{\partial u} \qquad \text{A 14}$$

Now these expansions still contain both time scales. What will be done is to take averages over the short time scale τ_o. But this requires solving for the short time dependence. However, A 11 and A 12 are no more easily solved than before. Therefore expand in the small parameter α.

$$g_o^o = a_o + \alpha a_1 + \alpha^2 a_2 \qquad \text{A 15}$$

$$g_1^o = b_o + \alpha b_1 + \alpha^2 b_2$$

The initial conditions are that f_o is Maxwellian with temperature T and f_1 = o and α = o at time t = o. Thus a_o is initially Maxwellian and b_o is zero. To lowest order in α, A 11 and A 12 give

$$\frac{\partial a_o}{\partial \tau_o} = o \qquad \text{A 16}$$

Thus a_o can depend only on the long time scale. To first order in α

$$\frac{\partial a_1}{\partial \tau_o} = o \qquad \text{A 17}$$

$$\frac{\partial b_1}{\partial \tau_o} = - \eta b_1 + \phi (\tau_o) \frac{\partial a_o}{\partial u}$$

which implies that

$$b_1 = \frac{\partial a_o}{\partial u} \left[\frac{\eta \sin \tau_o}{1 + \eta^2} - \frac{\cos \tau_o}{1 + \eta^2} \right] + F_1 e^{-\eta \tau_o} \qquad \text{A 18}$$

where F_1 is an arbitrary function having no short time dependence and where $\phi(\tau_o) = \sin \tau_o$. To second order in α

$$\frac{\partial a_2}{\partial \tau_o} = \frac{\sin \tau_o}{3u^2} \frac{\partial}{\partial u} (u^2 b_1) \qquad \text{A 19}$$

$$\frac{\partial b_2}{\partial \tau_0} = - \eta b_2 + \sin \tau_0 \frac{\partial a_1}{\partial u}$$

which can be integrated, solving for a_2, b_2. Expand A 13 and A 14 by taking

$$g_0^1 = c_0 + \alpha c_1 + \alpha^2 c_2$$

A 20

$$g_1^1 = d_0 + \alpha d_1 + \alpha^2 d_2$$

To lowest order in α, A13 and A 14 give

$$\frac{\partial a_o}{\partial \tau} + \frac{\partial c_o}{\partial \tau_o} = \frac{1}{2u^2} \frac{\partial}{\partial u} \left[\zeta \left(u^3 a_o + \frac{u^2}{2} \frac{\partial a_o}{\partial u} \right) \right] \qquad \text{A21}$$

$$\frac{\partial a_o}{d\tau_o} = - \eta d_o \qquad \text{A22}$$

By A16, a_o is independent of τ_o. Integrating A21 on τ_o it is seen that in order that c_o be finite it must be required that

$$\frac{\partial a_o}{\partial \tau_1} = \frac{1}{2u^2} \frac{\partial}{\partial u} \left[\zeta \left(u^3 a_o + \frac{u^2}{2} \frac{\partial a_o}{\partial u} \right) \right] \qquad \text{A23}$$

and $$\frac{\partial c_o}{\partial \tau_o} = 0 \qquad \text{A24}$$

To reconstruct A4 and A5 with the short time scale absent expand each term using the results A6 to A24 and average over τ_o. Assume $\alpha^2 \ll \epsilon$. To lowest order the first term in A4 is

$$\overline{\frac{\partial g_o}{\partial \tau}} = \epsilon \frac{\partial g_o}{\partial \tau_1} \qquad \text{A25}$$

and next

$$\overline{\phi(\tau_o) g_1} = \frac{\alpha}{2} \frac{\eta}{1 + \eta^2} \frac{\partial a_o}{\partial u} \qquad \text{A26}$$

and finally

$$\frac{\zeta \nu_m}{\omega}\left(u^3 g_o + \frac{1}{2u^2}\frac{\partial g_o}{\partial u}\right) = \epsilon\, \zeta\left(u^3 a_o + \frac{u^2}{2}\frac{\partial a_o}{\partial u}\right) \qquad \text{A27}$$

Therefore to lowest order

$$\frac{\partial g_o}{\partial \tau} = \frac{1}{2u^2}\frac{\partial}{\partial u}\left[\frac{\alpha^2}{3}\frac{\eta u^2}{1+\eta^2}\frac{\partial g_o}{\partial v} + \frac{\xi \nu_m}{\omega}\left(u^3 g_o + \frac{1}{2}u^2\frac{\partial g_o}{\partial u}\right)\right] \quad \text{A28}$$

where τ refers only to the long time scale. Substituting the original variables

$$\frac{\partial f_o}{\partial t} = \frac{1}{2v^2}\frac{\partial}{\partial v}\left[\frac{e^2 E_o{}^2 \nu_m v^2}{3m^2(\nu_m{}^2 + \omega^2)}\frac{\partial f_o}{\partial v} + \xi\, \nu_m\, v^3\left(f_o + \frac{kT_g}{mv}\frac{\partial f_o}{\partial v}\right)\right] \quad \text{A29}$$