An Explicit Form of the Mie Phase Matrix for Multiple Scattering Calculations in the I, Q, U, and V Representation

by

George W. Kattawar, Stephen J. Hitzfelder,

and Judith Binstock

Department of Physics

Texas A&M University

College Station, Texas

Abstract

An explicit expression is obtained for the phase matrix in the I, Q, U, and V Stokes vector representation for a system containing a polydispersion of spherical particles. All of the symmetry relations derived by Hovenier (1969) using general arguments are established explicitly. Convenient algorithms are given for the computation of the phase matrix for a spherical polydispersion. Since this theory is so vitally important in radiative transfer, many researchers will need to compute these functions for realistic aerosols distributions. We have therefore presented results for a haze L distribution so that other researchers will have a way of checking their programs which compute these quantities.

I. Introduction

There is now a considerable amount of interest in the polarization of the radiation reflected from planetary atmospheres. It is certainly clear that a wealth of information can be obtained about planetary atmospheres if more measurements were made of the four component Stokes vector. Recently Kemp, Wolstencroft, and Swedlund (1971) have measured a circular component of polarization namely V/I, from most planets. Now circular polarization as applicable to planetary atmospheres can arise either by scattering from a rough surface (Kemp and Wolstencroft, private communication) or by atmospheric scattering. If the atmosphere is optically thick, as it is for Venus, then if one assumes spherical particles, circular polarization can only arise from at least a double scattering. Single scattering of unpolarized radiation from spherical particles can not produce a circular component. It therefore becomes mandatory to have an accurate method to compute the Mie phase matrix if one is to do multiple scattering calculations employing spherical particles. Virtually all methods to date which deal with this problem use the Fourier decomposition of this matrix to uncouple the equation of transfer. Most methods employ fitting techniques of one form or another to evaluate the expansion coefficients. This method, however, has the unfortunate disadvantage that numerical quadrature must be employed to evaluate the expansion coefficients. This of course implies that one has to use interpolation methods on the phase functions. This technique can have adverse effects since it may require far more Fourier terms to fit the phase matrix than are actually needed. The first person to show that the expansion coefficients for the

Mie phase matrix could be obtained directly from the regular Mie coefficients was Sekera (1955). Later Dave (1970) used this decomposition in his method to compute the intensity and polarization emerging from a plane-parallel atmosphere. However, his calculations were only for a monodispersion of aerosols. Dave also employed the I_ℓ, I_r, U, V representation in his calculations.

There are many symmetry relations that exist for the Mie phase matrix as was demonstrated in an excellent paper by Hovenier (1969) who employed the I, Q, U, V representation. He did not however show these explicitly. It is the purpose of this paper to derive the coefficients of the Fourier decomposed phase matrix in terms of the basic Mie coefficients for spherical polydispersions and to show explicitly all of the symmetry relations derived by Hovenier. The expressions presented will be in a form amenable to numerical computations.

Theory

I. Phase Matrix for Mie Scattering

The well known Mie theory (van de Hulst, 1957) gives expressions for the components of the electric field scattered from a sphere parallel and perpendicular to the plane of scattering, denoted by E_ℓ and E_r respectively. They are expressed as an infinite series involving Legendre polynomials which are functions of the scattering angle Θ. The coefficients of the series, a_n and b_n, are complex functions of the size parameter of the sphere, $x = 2\pi a/\lambda$ and the relative index of refraction m. The components of the electric field are given by:

$$E_r = \frac{1}{kr} S_1(x,m;\Theta)\ e^{-i(kr-\omega t)} \tag{1a}$$

$$E_\ell = \frac{1}{kr} S_2(x,m;\Theta)\ e^{-i(kr-\omega t)} \tag{1b}$$

where

$$S_1(x,m;\Theta) = \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \{a_n \pi_n(\cos\Theta) + b_n \tau_n(\cos\Theta)\} \tag{2a}$$

$$S_2(x,m;\Theta) = \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \{b_n \pi_n(\cos\Theta) + a_n \tau_n(\cos\Theta)\} \tag{2b}$$

The angular functions π_n and τ_n are normally given in the form:

$$\pi_n(\cos\Theta) = \frac{d\ P_n(\cos\Theta)}{d\ \cos\Theta} \tag{3a}$$

$$\tau_n(\cos\Theta) = \pi_n(\cos\Theta)\ \cos\Theta - \sin\Theta\ \frac{d\ \pi_n(\cos\Theta)}{d\ \cos\Theta} \tag{3b}$$

To apply the Mie theory to radiative transfer, the electromagnetic radiation is most easily expressed in terms of a four component Stokes vector. Choosing the representation (I, Q, U, V) the phase matrix can be written as (see van de Hulst, 1957, page 44):

$$\underset{\sim}{P}(\cos\Theta) = \begin{bmatrix} M^{+} & M^{-} & 0 & 0 \\ M^{-} & M^{+} & 0 & 0 \\ 0 & 0 & S_{21} & -D_{21} \\ 0 & 0 & D_{21} & S_{21} \end{bmatrix} \tag{4}$$

where

$$M^{+} = \tfrac{1}{2} (S_1 S_1^* + S_2 S_2^*) \tag{5a}$$

$$M^{-} = \tfrac{1}{2} (S_1 S_1^* - S_2 S_2^*) \tag{5b}$$

$$S_{21} = \tfrac{1}{2} (S_1 S_2^* + S_2 S_1^*) \tag{5c}$$

$$D_{21} = \tfrac{i}{2} (S_1 S_2^* - S_2 S_1^*) \tag{5d}$$

The phase matrix could be computed using Eqs. (2), (3), and (5). Sekera (1952, 1955) expressed the elements of the phase matrix in terms of an infinite Legendre series. Defining the following functions:

$$T_1(x,m;\Theta) = (1-\cos^2\Theta)^{-1} (S_1 - S_2\cos\Theta) \tag{6a}$$

$$T_2(x,m;\Theta) = (1-\cos^2\Theta)^{-1} (S_2 - S_1\cos\Theta) \tag{6b}$$

and

$$R^1(x,m;\Theta) = \tfrac{1}{2}(T_1 T_1^* + T_2 T_2^*) \tag{7a}$$

$$R^2(x,m;\Theta) = \tfrac{1}{2}(T_2 T_2^* - T_1 T_1^*) \tag{7b}$$

$$R^3(x,m;\Theta) = \mathrm{Re}(T_2 T_1^*) \tag{7c}$$

$$R^4(x,m;\Theta) = \mathrm{Im}(T_2 T_1^*) \tag{7d}$$

it can be shown that

$$M^{+} = (1 + \cos^2\Theta)\ R^1 + 2\ \cos\Theta R^3 \tag{8a}$$

$$M^{-} = (\cos^2\Theta - 1)\ R^2 \tag{8b}$$

$$S_{21} = (1 + \cos^2\Theta)\ R^3 + 2\ \cos\Theta R^1 \tag{8c}$$

$$D_{21} = (1 - \cos^2\Theta)\ R^4 \tag{8d}$$

The definitions of S_1 and S_2 and π_n and τ_n are inserted into Eq. (6) yielding:

$$T_1 = \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \{a_n[P_n'(\cos\Theta) + \cos\Theta\ P_n''(\cos\Theta)] - b_n\ P_n''(\cos\Theta)\} \tag{9}$$

T_2 is obtained from Eq. (9) by interchanging a_n and b_n. Recurrence relations are used to express the derivatives of the Legendre polynominals in terms of a finite Legendre series. The resulting series is manipulated to isolate Legendre polynomials of like order. Using the orthogonality of the Legendre polynomials the expressions for T_1 and T_2 are:

$$T_1 = \sum_{n=0}^{\infty} C_n(x,m)\ P_n(\cos\Theta) \tag{10a}$$

$$T_2 = \sum_{n=0}^{\infty} D_n(x,m)\ P_n(\cos\Theta) \tag{10b}$$

where

$$C_n(x,m) = (2n+1) \sum_{j=0}^{\infty} \left\{\frac{(4j+2n+3)(2j+1)(j+n+1)\ a_{2j+n+1}}{(2j+n+1)(2j+n+2)} - \frac{(4j+2n+5)(j+1)\ (2j+2n+3)\ b_{2j+n+2}}{(2j+n+2)(2j+n+3)}\right\} \tag{11}$$

D_n is obtained from Eq. (11) by interchanging a_n and b_n. To obtain the expressions for R^j in Eqs. (7), the products of two infinite Legendre series must be evaluated. Sekera (1952) showed that the product of two infinite Legendre series could be expressed in a relatively simple form as an infinite Legendre series. Dave (1970 a,b) has recently used this technique in obtaining the phase matrix elements in the (I_ℓ, I_r, U, V) Stokes vector representation as Legendre series. Applying this technique to Eqs. (7) it can easily be shown that:

$$R^i(x,m;\Theta) = \sum_{r=0}^{\infty} L_r^i(x,m)\ P_r(\cos\Theta) \quad i = 1,2,3,4 \tag{12}$$

The $L_r^i(x,m)$ are obtained as infinite series, the coefficients of which are defined by recurrence relations, i.e.,

$$L_r^1 = (r+\tfrac{1}{2}) \sum_{n=k}^{\infty} a_n^{(r)} \sum_{i=0}^{k} b_i^{(r)}\ \mathrm{Re}\ \{C_q C_p^* + D_q D_p^*\}/\Delta_i \tag{13a}$$

$$L_r^2 = (r+\tfrac{1}{2}) \sum_{n=k}^{\infty} a_n^{(r)} \sum_{i=0}^{k} b_i^{(r)}\ \mathrm{Re}\ \{D_q D_p^* - C_q C_p^*\}/\Delta_i \tag{13b}$$

$$L_r^3 = (r+\tfrac{1}{2}) \sum_{n=k}^{\infty} a_n^{(r)} \sum_{i=0}^{k} b_i^{(r)}\ \mathrm{Re}\ \{D_q C_p^* + D_p C_q^*\}/\Delta_i \tag{13c}$$

$$L_r^4 = (r+\tfrac{1}{2}) \sum_{n=k}^{\infty} a_n^{(r)} \sum_{i=0}^{k} b_i^{(r)}\ \mathrm{Re}\ \{D_q C_p^* + D_p C_q^*\}/\Delta_i \tag{13d}$$

where

$$k = \begin{cases} \frac{r}{2} & r \text{ even} \\ \frac{r-1}{2} & r \text{ odd} \end{cases}, \tag{13e}$$

$$\Delta_i = \begin{cases} 2, & i = 0 \\ 1, & i > 0 \end{cases}, \tag{13f}$$

$$q = n-i, \tag{13g}$$

$$p = n+i+\delta, \tag{13h}$$

$$\delta = \begin{cases} 0 & r \text{ even} \\ 1 & r \text{ odd} \end{cases}. \tag{13i}$$

The recurrence relations for $a_n^{(r)}$ and $b_i^{(r)}$ are given by Dave (1970a).

The R^i defined in Eq. (12) are not normalized. The normalization is obtained by redefining the coefficients in Eq. (12). That is:

$$R^i(x,m;\Theta) = \sum_{r=0}^{\infty} \Lambda_r^i\ (x,m)\ P_r(\cos\Theta) \quad i = 1,2,3,4 \tag{14}$$

where

$$\Lambda_r^i(x,m) = L_r^i(x,m)/(\pi Q_s(x,m)x^2) \tag{15}$$

$Q_s(x,m)$ is the efficiency factor for scattering (van de Hulst, 1957). With the $R^i(x,m;\ \theta)$ defined as in Eq. (14), it can be shown that

$$\int_\Omega M^+\ (\cos\Theta)\ d\Omega = 1 \tag{16}$$

II. Mie Scattering Phase Matrix with Rotations

For multiple scattering it is convenient to define the Stokes parameters of the incident and scattered beams using their respective meridian planes as a plane of reference (Chandrasekhar, 1960). The phase matrix defined in Eq. (4) requires that the incident and scattered Stokes vectors be defined with respect to the plane of scattering. It is therefore necessary to perform rotations on the Stokes vectors to use Eq. (4).

Because of the choice of the representation of the Stokes vectors, the matrix for rotation of the Stokes vector axis, through an angle α regarded as positive if clockwise when looking into the beam, has the simple form:

$$\underset{\sim}{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (17)$$

The incident direction is defined by $\mu' = \cos\theta'$, ϕ', the scattered direction is defined by $\mu = \cos\theta$, ϕ, and the scattering angle $\cos\Theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi'-\phi)$ (see Fig. 1). The scattered wave Stokes vector $\underset{\sim}{S}(\mu,\phi)$, is given in terms of the incident wave Stokes vector $\underset{\sim}{I}(\mu',\phi')$ by:

$$\underset{\sim}{S}(\mu,\phi) = \underset{\sim}{M}(\mu,\phi;\mu',\phi')\ \underset{\sim}{I}(\mu',\phi') \qquad (18)$$

where

$$\underset{\sim}{M}(\mu,\phi;\mu',\phi') = \underset{\sim}{L}(\pi - i_2)\ \underset{\sim}{P}(\cos\Theta)\ \underset{\sim}{L}(-i_1) \qquad (19)$$

The elements of $\underset{\sim}{M}$ are easily obtained by matrix multiplication in terms of θ, i_1, and i_2. Spherical trigonometry could then be used to evaluate i_1, i_2, and Θ in terms of μ, μ', and $\Delta\phi = \phi'-\phi$. This is a very tedious task and it is easier to use certain trigonometric results which are also needed for the Rayleigh phase matrix (Chandrasekhar, 1960, page 41). After performing this straightforward but still tedious manipulation the elements of the phase matrix are obtained in the following form:

$$\begin{aligned} M_{11} &= [(1+\mu^2)(1+\mu'^2)/2 + \gamma^2]\, R^1 + 2\mu\mu' R^3 \\ &+ 2\gamma\,[\mu\mu' R^1 + R^3]\cos\Delta\phi \\ &+ \tfrac{1}{2}\gamma^2 R^1 \cos 2\Delta\phi \end{aligned} \qquad (20a)$$

$$\begin{aligned} M_{12} &= \tfrac{1}{2}[2\gamma^2 + (\mu^2+1)(\mu'^2-1)]\, R^2 \\ &+ 2\mu\mu'\gamma R^2 \cos\Delta\phi \\ &+ \tfrac{1}{2}(\mu^2-1)(\mu'^2+1)\, R^2 \cos 2\Delta\phi \end{aligned} \qquad (20b)$$

$$M_{13} = 2\ R^2\ \mu\gamma\ \sin\ \Delta\phi + \mu'\ (\mu^2-1)\ R^2\ \sin 2\Delta\phi \tag{20c}$$

$$M_{14} = 0 \tag{20d}$$

$$M_{21} = \tfrac{1}{2}[2\gamma^2 + (\mu^2-1)(\mu'^2+1)]\ R^2 + 2\mu\ \mu'\ \gamma\ R^2\ \cos\ \Delta\phi + \tfrac{1}{2}(\mu^2+1)(\mu'^2-1)\ R^2\ \cos 2\Delta\phi \tag{20e}$$

$$M_{22} = 3\gamma^2\ R^1/2 + 2\gamma\ (\mu\ \mu'\ R^1 + R^3)\ \cos\ \Delta\phi + \tfrac{1}{2}\ [(\mu^2+1)(\mu'^2+1)\ R^1 + 4\mu\ \mu'\ R^3]\ \cos 2\Delta\phi \tag{20f}$$

$$M_{23} = 2\gamma\ (\mu\ R^1 + \mu'\ R^3)\ \sin\ \Delta\phi + [\mu'(\mu^2+1)\ R^1 + \mu(\mu'^2+1)\ R^3]\ \sin 2\Delta\phi \tag{20g}$$

$$M_{24} = 2\ \mu'\ \gamma\ R^4\ \sin\ \Delta\phi - \mu\ (1-\mu'^2)\ R^4\ \sin 2\Delta\phi \tag{20h}$$

$$M_{31} = -2\mu'\ \gamma\ R^2\ \sin\ \Delta\phi + \mu\ (1-\mu'^2)\ R^2\ \sin 2\Delta\phi \tag{20i}$$

$$M_{32} = -2\gamma\ (\mu'\ R^1 + \mu\ R^3)\ \sin\ \Delta\phi - [\mu\ (\mu'^2+1)\ R^1 + \mu'(\mu^2+1)\ R^3]\ \sin 2\Delta\phi \tag{20j}$$

$$M_{33} = 3\gamma^2\ R^3/2 + 2\gamma\ (R^1 + \mu\ \mu'\ R^3)\ \cos\ \Delta\phi + \tfrac{1}{2}\ [(\mu^2+1)(\mu'^2+1)\ R^3 + 4\mu\ \mu'\ R^1]\ \cos 2\Delta\phi \tag{20k}$$

$$M_{34} = \tfrac{1}{2}\ [2\gamma^2 + (\mu^2-1)(\mu'^2+1)]\ R^4 + 2\mu\ \mu'\ R^4\ \cos\ \Delta\phi + \tfrac{1}{2}\ (\mu^2+1)(\mu'^2-1)\ R^4\ \cos 2\Delta\phi \tag{20ℓ}$$

$$M_{41} = 0 \tag{20m}$$

$$M_{42} = 2\gamma\mu\, R^4 \sin\Delta\phi + \mu'(\mu^2-1)\, R^4 \sin 2\Delta\phi \tag{20n}$$

$$M_{43} = -\tfrac{1}{2}[2\gamma^2 + (\mu^2+1)(\mu'^2-1)]\, R^4 - 2\mu\mu'\gamma\, R^4 \cos\Delta\phi - \tfrac{1}{2}(\mu^2-1)(\mu'^2+1)\, R^4 \cos 2\Delta\phi \tag{20o}$$

$$M_{44} = [(1+\mu^2)(1+\mu'^2)/2 + \gamma^2)\, R^3 + 2\,\mu\,\mu'\, R^1] + 2\gamma\,[\mu\mu'\, R^3 + R^1] \cos\Delta\phi + \tfrac{1}{2}\gamma^2\, R^3 \cos 2\Delta\phi \tag{20p}$$

where $\gamma = (1-\mu^2)^{\frac{1}{2}}\,(1-\mu'^2)^{\frac{1}{2}}$

In Eqs. (20) the R^i are still explicit functions of cos Θ. The addition theorem for spherical harmonics was applied to Eq. (14) resulting in a more convenient form for the R^i, namely:

$$R^i(x,m;\ \mu,\phi;\ \mu',\phi') = \sum_{n=0}^{\infty} \Lambda_n^i(x,m) \sum_{\ell=0}^{n} (2-\delta_{\ell o})\, Y_n^\ell(\mu)\, Y_n^\ell(\mu') \cos\ell\,\Delta\phi \tag{21}$$

where $Y_n^\ell(\mu) = [\frac{(n-\ell)!}{(n+\ell)!}]^{\frac{1}{2}}\, P_n^\ell(\mu)$. Eqs. (20) and (21) give an exact analytical expression for the phase matrix for Mie scattering in terms of μ, μ', and $\Delta\phi$. Eq. (20) may be shown to be equivalent to a similar expression obtained by Dave (1970b) in another Stokes vector representation.

III. Symmetry Relationships of the Phase Matrix

Symmetry has been used to determine properties of the phase matrix for particles with and without a plane of symmetry (van de Hulst, 1957).

Recently Hovienier (1969) expressed seven symmetry relationships of the phase matrix based on symmetry alone. These relations were derived for particles with a plane of symmetry. For clarity the relationships are reproduced using the notation of this paper.

$$\underset{\sim}{M}\,(\mu',\phi';\ \mu,\phi) = \underset{\sim}{Q}\,\underset{\sim}{M}^{T}\,(\mu,\phi;\ \mu',\phi')\,\underset{\sim}{Q} \tag{22a}$$
$$\underset{\sim}{M}\,(-\mu',\phi';\ -\mu,\phi) = \underset{\sim}{P}\,\underset{\sim}{M}^{T}\,(\mu,\phi;\ \mu',\phi')\,\underset{\sim}{P} \tag{22b}$$
$$\underset{\sim}{M}\,(-\mu,\phi';\ -\mu',\phi) = \underset{\sim}{M}\,(\mu,\phi;\ \mu',\phi') \tag{22c}$$
$$\underset{\sim}{M}\,(\mu,\phi';\ \mu',\phi) = \underset{\sim}{P}\,\underset{\sim}{Q}\,\underset{\sim}{M}\,(\mu,\phi;\ \mu',\phi')\,\underset{\sim}{Q}\,\underset{\sim}{P} \tag{22d}$$
$$\underset{\sim}{M}\,(\mu',\phi;\ \mu,\phi') = \underset{\sim}{P}\,\underset{\sim}{M}^{T}\,(\mu,\phi;\ \mu',\phi')\,\underset{\sim}{P} \tag{22e}$$
$$\underset{\sim}{M}\,(-\mu',\phi;\ -\mu,\phi') = \underset{\sim}{Q}\,\underset{\sim}{M}^{T}\,(\mu,\phi;\ \mu',\phi')\,\underset{\sim}{Q} \tag{22f}$$
$$\underset{\sim}{M}\,(-\mu,\phi;\ -\mu',\phi') = \underset{\sim}{P}\,\underset{\sim}{Q}\,\underset{\sim}{M}\,(\mu,\phi;\ \mu',\phi')\,\underset{\sim}{Q}\,\underset{\sim}{P} \tag{22g}$$

where

$$\underset{\sim}{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \underset{\sim}{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Of these relationships, Eqs. (22b), (22c), and (22d) are singled out for special treatment for two reasons: (1) (22b), (22c), and (22d) have simple explanations in terms of space and time symmetries, and (2) from these the other four relationships can be derived. Simple matrix multiplication yields the following results from Eq. (22b):

$$M_{ii}\,(-\mu',\phi';\ -\mu,\phi) = M_{ii}\,(\mu,\phi;\ \mu',\phi') \tag{23a}$$
$$M_{ij}\,(-\mu',\phi';\ -\mu,\phi) = M_{ji}\,(\mu,\phi;\ \mu',\phi') \quad ij = 12,\ 14,\ 24 \tag{23b}$$
$$M_{ij}\,(-\mu',\phi';\ -\mu,\phi) = -M_{ji}(\mu,\phi;\ \mu',\phi') \quad ij = 13,\ 23,\ 34 \tag{23c}$$

Similarly from (23c)

$$M_{ij}\ (-\mu,\phi';\ -\mu',\phi) = M_{ij}\ (\mu,\phi;\ \mu',\phi') \qquad (24)$$

and from (23d)

$$M_{ii}\ (\mu,\phi';\ \mu',\phi) = M_{ii}\ (\mu,\phi;\ \mu',\phi') \qquad (25a)$$

$$M_{ij}\ (\mu,\phi';\ \mu',\phi) = M_{ij}\ (\mu,\phi;\ \mu',\phi')\ ij = 12,21,34,43 \qquad (25b)$$

$$M_{ij}\ (\mu,\phi';\ \mu',\phi) = -M_{ij}(\mu,\phi;\ \mu',\phi')\ ij = 13,14,23,24,31,32,41,42 \qquad (25c)$$

The matrix elements in Eqs. (20) are easily shown to satisfy Eqs. (23), (24), and (25), noting that $R^i(x,m;\ \mu,\ \phi,\ \mu',\ \phi') = R^i(x,m;\ \mu',\ \phi';\ \mu,\ \phi)$. It is also easily verified that all of Eqs. (20) satisfy the relations of Eqs. (22).

It is easy to see that Eqs. (22) require that M_{13}, M_{14}, M_{23}, M_{24}, M_{31}, M_{32}, M_{41}, M_{42} be odd functions of $\Delta\phi$, and that all other elements be even functions of $\Delta\phi$. This has been pointed out by Hovenier and is easily verified by Eqs. (20).

IV. Computation

In practice there will only be a finite number of the Λ^i_n computed. Thus Eqs. (21) can be written:

$$R^i(x,m;\ \mu,\phi;\ \mu',\phi') = \sum_{\ell=0}^{N} F^i_\ell(x,m;\ \mu,\ \mu')\cos \ell\Delta\phi\ \ i = 1,2,3,4 \qquad (26)$$

where

$$F^i_\ell(x,m;\ \mu,\ \mu') = (2-\delta_{\ell o})\sum_{n=0}^{N}\Lambda^i_n\ (x,m)\ Y^\ell_n(\mu)\ Y^\ell_n(\mu')\ \ i = 1,2,3,4 \qquad (27)$$

N being the number of $\Lambda^i_n(x,m)$ computed. The $Y^\ell_n(\mu)$ are computed by recurrence relations (Dave and Armstrong, 1970). In Eq. (20) there is need for $R^i \cos \Delta\phi$, $R^i \cos 2\Delta\phi$, $R^i \sin \Delta\phi$ and $R^i \sin 2\Delta\phi$. Using trigonometric relations, one can show that

$$R^i \cos q\Delta\phi = \sum_{\ell=0}^{N} F^i_\ell(x,m;\ \mu,\ \mu',\ qc) \cos \ell\Delta\phi \qquad q = 1,2 \tag{28}$$

and

$$R^i \sin q\Delta\phi = \sum_{\ell=0}^{N} F^i_\ell(x,m;\ \mu,\ \mu',\ qs) \sin \ell\Delta\phi \qquad q = 1,2 \tag{29}$$

The $F^i_\ell(x,m;\ \mu,\ \mu',\ qc) = F^i_\ell(qc)$ and $F^i_\ell(x,m;\ \mu,\ \mu',\ qs) = F^i_\ell(qs)$ are given by Dave (1970b). Using Eqs. (28) and (29) we now have the phase matrix elements in a form that is decomposed into a Fourier series in $\Delta\phi$. That is

$$M_{11} = \sum_{\ell=0}^{N} \{[(1+\mu^2)(1+\mu'^2)/2 + \gamma^2]\ F^1_\ell + 2\mu\mu'\ F^3_\ell + 2\gamma\mu\mu'\ F^1_\ell(1c) + 2\gamma F^3_\ell(1c) + (\gamma^2/2)\ F^1_\ell(2c)\} \cos \ell\Delta\phi \tag{30a}$$

$$M_{22} = \sum_{\ell=0}^{N} \{3\gamma^2 F^3_\ell/2\ F^1_\ell + 2\gamma\mu\mu'\ F^1_\ell(1c) + 2\gamma\ F^3_\ell(1c) + ((\mu^2+1)(\mu'^2+1)/2)\ F^1_\ell(2c) + 2\mu\mu'\ F^3_\ell(2c)\} \cos \ell\Delta\phi \tag{30b}$$

$$M_{33} = \sum_{\ell=0}^{N} \{3\gamma^2 F^3_\ell/2 + 2\gamma\mu\mu'\ F^3_\ell(1c) + 2\gamma\ F^1_\ell(1c) + ((\mu^2+1)(\mu'^2+1)/2)\ F^3_\ell(2c) + 2\mu\mu'\ F^1_\ell(2c)\} \cos \ell\Delta\phi \tag{30c}$$

$$M_{44} = \sum_{\ell=0}^{N} \{[(1+\mu^2)(1+\mu'^2)/2 + \gamma^2]\ F^3_\ell + 2\mu\mu'\ F^1_\ell + 2\gamma\mu\mu'\ F^3_\ell(1c) + 2\gamma\ F^1_\ell(1c) + (\gamma^2/2)\ F^3_\ell(2c)\} \cos \ell\Delta\phi \tag{30d}$$

$$M_{12} = \sum_{\ell=0}^{N} \{[\gamma^2 + (\mu^2+1)(\mu'^2-1)/2]\ F^2_\ell + 2\mu\mu'\gamma\ F^2_\ell(1c) + ((\mu^2-1)(\mu'^2+1)/2) + F^2_\ell(2c)\} \cos \ell\Delta\phi \tag{30e}$$

$$M_{21}\ (\mu,\ \mu') = M_{12}\ (\mu',\ \mu) \tag{30f}$$

$$M_{13} = \sum_{\ell=0}^{N} \{2\mu\gamma\ F_\ell^2(1s) + \mu'(\mu^2-1)\ F_\ell^2(2s)\}\ \sin\ \ell\Delta\phi \tag{30g}$$

$$M_{31}\ (\mu,\ \mu') = -M_{13}(\mu',\mu) \tag{30h}$$

$$M_{23} = \sum_{\ell=0}^{N} \{2\gamma\mu\ F_\ell^1(1s) + 2\gamma\mu'\ F_\ell^3(1s) + \mu'(\mu^2+1)F_\ell^1(2s) + \mu(\mu'^2+1)\ F_\ell^3(2s)\}\ \sin\ \ell\Delta\phi \tag{30i}$$

$$M_{32}\ (\mu,\mu') = -M_{23}(\mu',\mu) \tag{30j}$$

$$M_{24} = \sum_{\ell=0}^{N} \{2\mu'\gamma\ F_\ell^4(1s) + \mu(\mu'^2-1)\ F_\ell^4(2s)\}\ \sin\ \ell\Delta\phi \tag{30k}$$

$$M_{42}\ (\mu',\mu) = M_{24}(\mu,\mu') \tag{30\ell}$$

$$M_{34} = \sum_{\ell=0}^{N} \{[\gamma^2 + (\mu^2-1)(\mu'^2-1)/2]\ F_\ell^4 + 2\mu\mu'\gamma\ F_\ell^4(1c) + ((\mu^2+1)(\mu'^2-1)/2)\ F_\ell^4(2c)\}\ \cos\ \ell\Delta\phi \tag{30m}$$

$$M_{43}\ (\mu,\mu') = -M_{34}(\mu',\mu) \tag{30n}$$

With the phase matrix elements in this form Chandrasekhar showed that the ϕ dependence can be decoupled in the equation of transfer.

The value of N is of considerable importance. The amount of computation required will be proportional to N. Downward recurrence is used to compute the Mie coefficients, a_n and b_n, of Eqs. (2) (Kattawar and Plass, 1967). A limit has been placed on this series, terminating it when $|a_n|\ /\ \max_n\ |a_n| \leq 10^{-9}$. This determines the value of N, as can be seen in Eq. (11). Thus no significant error could arise from terminating the series.

The phase matrix elements are computed for $(\mu,\ \mu') = (\mu_i,\ \mu_j)$, $i,j = 1,\ N_\mu$ where the μ_i and μ_j are abscissas for a N_μ point Lobatto quadrature. Full advantage is taken of the symmetry relations expressed in Eqs. (22) to prevent duplicate computation and storage. Lobatto quadrature was chosen so as to include the forward and backscattering points ($\cos\Theta = \pm 1$).

To conserve flux, it is important that $M_{11} = M^+$ be properly normalized. That is from Eq. (16):

$$\int_0^{2\pi} \int_{-1}^{+1} M_{11} \, d\mu \, d\phi = 1 \qquad (31)$$

We see from Eq. (30) that only the first term of the Fourier series will contribute. That is:

$$2\pi \int_{-1}^{+1} M_{11}^{o} \, d\mu = 1 \qquad (32)$$

Since M_{11}^{o} is computed for Lobatto abscissas, the integral in Eq. (32) is replaced by a sum

$$2\pi \sum_{i=1}^{N_\mu} \{C_i \, M_{11}^{o} \, (\mu_i, \mu_j)\} - 1 = \delta_j , \; j = 1, N_\mu \qquad (33)$$

where the C_i are the Lobatto quadrature weights. Because of roundoff error, the δ_j in Eq. (33) will not be zero. They are used to compute a correction matrix which is made symmetric to preserve symmetry. The number of Lobatto abscissas chosen is large enough to insure that the corrections applied are small. This method will in general require fewer Lobatto points to achieve the same normalization than conventional methods of fitting the phase function.

V. Results

A program has been written to compute the phase matrix for a single particle and for a spherical polydispersion. The phase matrix elements were computed for a particle size in the Rayleigh limit. The computed phase matrix elements agreed with the Rayleigh phase matrix elements computed from the power series expansion given by van de Hulst (1957). A set of Λ_n^i computed by this program is given in Table I. This set of Λ_n^i was computed for a haze L distribution (Diemendjian, 1969). The index of refraction was real, $m = 1.55$, and the wavelength $\lambda = 0.7\mu$. The integration over the size distribution was performed using a 25 point Gauss Legendre quadrature, with initial and final size parameters given by 0.05 and 26 respectively.

Conclusion

We have explicitly shown all of the symmetry relations which exist for the phase matrix of a system containing a polydispersion of spherical particles. These symmetry relations are highly essential if one is to perform multiple scattering calculations. These relations can be used to significantly reduce the amount of core storage and increase the efficiency of the computer program. Convenient and stable numerical algorithms have been presented so that computations involving a spherical polydispersion can be performed. The method has been thoroughly tested and yields highly accurate results which are essential if one is to perform radiative transfer calculations including polarization.

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Table I. Λ_n^i computed for a haze L distribution.

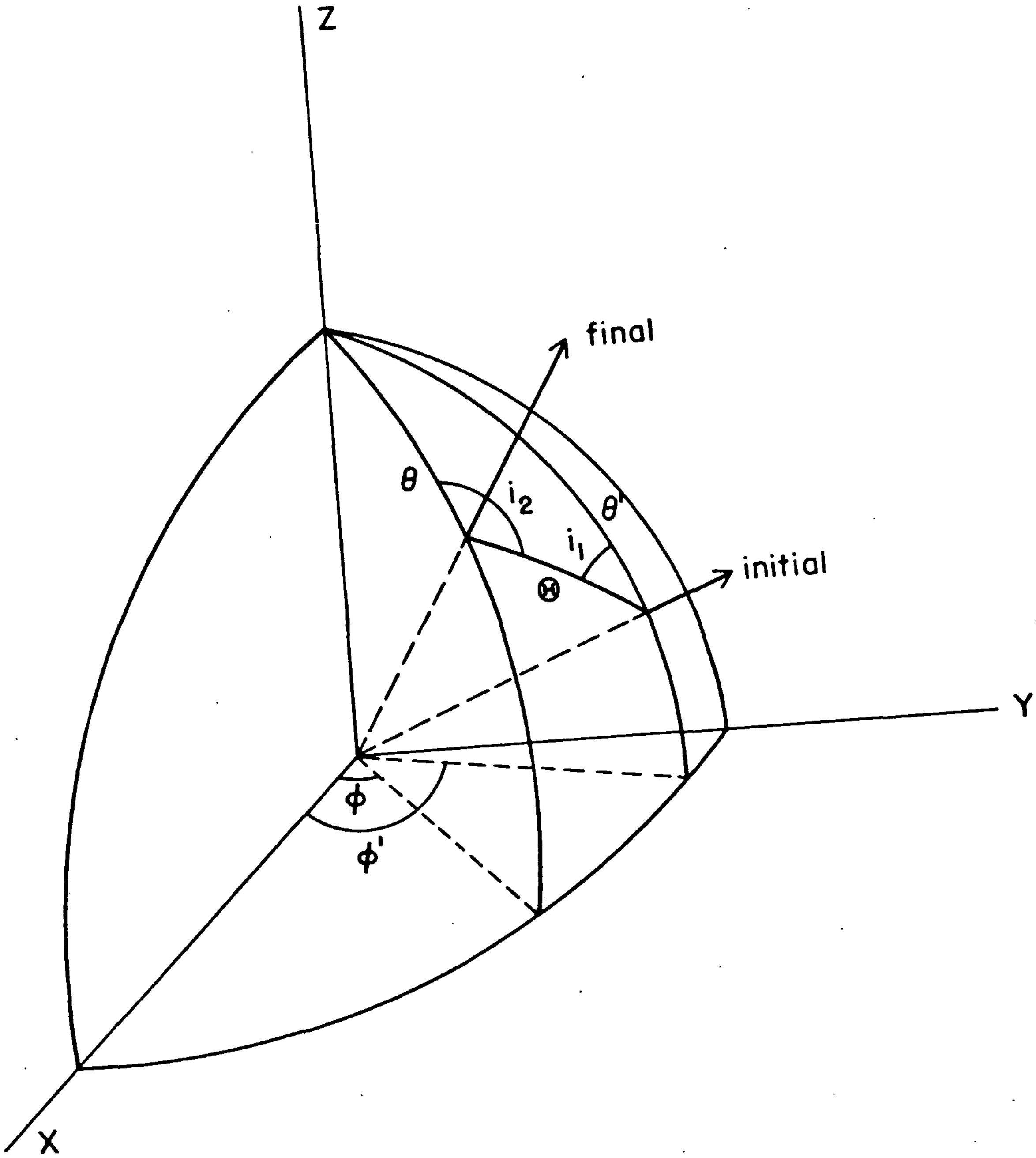
n	Λ^1_{n-1}	Λ^2_{n-1}	Λ^3_{n-1}	Λ^4_{n-1}
1	5.87198D-01	1.99920D-02	4.61950D-01	1.75102D-02
2	-1.21721D 00	-3.42603D-02	-1.55376D 00	4.67879D-02
3	2.49366D 00	6.41209D-02	1.99400D 00	7.41132D-02
4	-2.42885D 00	-6.56830D-02	-3.02556D 00	4.76663D-02
5	3.51320D 00	5.27010D-02	2.84993D 00	9.23360D-02
6	-2.95277D 00	-8.05196D-02	-3.64407D 00	8.90394D-03
7	3.76171D 00	1.57731D-02	3.05042D 00	8.70745D-02
8	-2.91541D 00	-8.85832D-02	-3.63896D 00	-3.55819D-02
9	3.54658D 00	-3.86154D-03	2.80852D 00	5.99563D-02
10	-2.58469D 00	-7.12539D-02	-3.32244D 00	-3.81840D-02
11	3.13487D 00	-9.75274D-03	2.40451D 00	4.89202D-02
12	-2.17654D 00	-5.13186D-02	-2.88313D 00	-1.96974D-02
13	2.67786D 00	-1.75891D-02	1.99796D 00	3.36772D-02
14	-1.78370D 00	-3.18462D-02	-2.43131D 00	-2.10512D-02
15	2.22449D 00	-5.04434D-03	1.61188D 00	2.60111D-02
16	-1.43252D 00	-2.29176D-02	-2.00341D 00	-8.12539D-03
17	1.81405D 00	-4.36990D-03	1.28436D 00	1.69417D-02
18	-1.12933D 00	-1.39860D-02	-1.61774D 00	-7.82308D-03
19	1.44884D 00	5.88121D-04	1.00122D 00	1.37131D-02
20	-8.74186D-01	-1.22472D-02	-1.27900D 00	-3.40970D-03
21	1.12913D 00	-7.22753D-04	7.66091D-01	9.94977D-03
22	-6.59876D-01	-8.75785D-03	-9.82548D-01	-3.42213D-03
23	8.54416D-01	-1.29412D-03	5.69968D-01	8.55098D-03
24	-4.83794D-01	-7.74472D-03	-7.31823D-01	-3.41107D-03
25	6.24761D-01	-2.57938D-03	4.10339D-01	6.15810D-03
26	-3.40816D-01	-5.48696D-03	-5.25016D-01	-4.19643D-03
27	4.39881D-01	-3.10403D-03	2.82034D-01	4.24877D-03
28	-2.27975D-01	-3.79840D-03	-3.62786D-01	-4.08418D-03
29	2.97623D-01	-3.41556D-03	1.83156D-01	3.14030D-03
30	-1.43541D-01	-2.61613D-03	-2.39510D-01	-2.87051D-03
31	1.91263D-01	-3.24428D-03	1.11875D-01	2.68946D-03
32	-8.47754D-02	-1.81207D-03	-1.49453D-01	-1.84933D-03
33	1.15825D-01	-2.65986D-03	6.39933D-02	2.25658D-03
34	-4.71921D-02	-1.20627D-03	-8.79169D-02	-8.80125D-04
35	6.65080D-02	-1.78121D-03	3.49723D-02	1.60524D-03
36	-2.51572D-02	-6.16232D-04	-4.93151D-02	-3.67852D-04
37	3.67826D-02	-9.05921D-04	1.83486D-02	1.23640D-03
38	-1.30096D-02	-5.17582D-04	-2.69684D-02	1.49957D-04

Table I. (continued)

n	Λ^{1}_{n+1}	Λ^{2}_{n+1}	Λ^{3}_{n+1}	Λ^{4}_{n+1}
39	1.98892D-02	-5.79126D-04	9.27671D-03	8.71619D-04
40	-6.29980D-03	-3.05663D-04	-1.43985D-02	1.21393D-04
41	1.05096D-02	-1.88660D-04	4.30147D-03	6.01165D-04
42	-2.79941D-03	-1.93617D-04	-7.42623D-03	1.13256D-04
43	5.19699D-03	-1.33892D-04	1.82590D-03	3.29851D-04
44	-1.05266D-03	-7.59632D-05	-3.46481D-03	2.60925D-05
45	2.30946D-03	-3.50806D-05	5.98164D-04	1.59653D-04
46	-2.73341D-04	-1.45872D-05	-1.46369D-03	2.86345D-05
47	9.26912D-04	-2.98303D-06	1.13593D-04	9.44095D-05
48	-2.16228D-05	-5.83553D-06	-5.61854D-04	2.12885D-05
49	3.50766D-04	1.58391D-05	-1.81393D-06	4.64082D-05
50	1.28078D-05	-3.67674D-06	-2.03372D-04	8.02355D-06
51	1.15389D-04	3.53837D-06	-9.66379D-06	1.44663D-05
52	6.93699D-06	5.02745D-07	-5.99067D-05	1.42730D-06
53	3.02653D-05	-4.22105D-06	-2.06796D-06	2.06878D-06
54	6.11859D-07	3.47942D-07	-1.24725D-05	-1.21166D-07
55	4.33266D-06	-2.02721D-06	-9.22947D-08	2.14390D-07
56	-2.19452D-07	-5.65978D-08	-1.09898D-06	-2.87322D-08
57	5.12091D-07	8.70223D-08	1.61478D-07	1.57304D-08
58	-6.69967D-08	-1.83192D-09	-1.56767D-07	-1.00498D-08
59	2.97352D-08	2.12148D-08	1.89776D-08	2.94021D-09
60	-1.16068D-08	-2.49074D-09	-2.66790D-09	-6.81839D-10
61	1.04883D-09	-1.62713D-09	4.77566D-09	-1.19632D-10
62	-1.23888D-09	-8.77416D-11	3.37665D-10	-7.71071D-12
63	-5.11226D-12	1.57982D-11	2.99329D-10	-8.90575D-12
64	-3.82335D-11	-1.11644D-11	-7.70559D-11	-1.63240D-13
65	2.93931D-11	-7.02785D-12	7.34807D-12	5.66831D-14
66	-1.14262D-12	3.47768D-13	-7.31043D-12	7.49369D-15
67	1.83695D-12	-7.24410D-13	3.15561D-13	1.56659D-15
68	-8.32212D-14	-8.40650D-15	-2.10404D-13	1.63335D-17
69	6.57836D-15	1.46748D-14	-1.60204D-15	-6.23746D-18
70	3.03898D-16	1.29124D-15	2.42402D-16	-5.93870D-19
71	4.05677D-17	1.44051D-16	7.69351D-17	-5.76496D-20
72	6.43026D-18	1.41514D-17	1.31078D-17	-3.29041D-21
73	2.81148D-18	2.83633D-19	3.50820D-18	1.33470D-22
74	2.16515D-19	-5.90804D-20	2.18931D-19	1.73371D-25
75	2.02899D-20	-8.49679D-21	1.87391D-20	1.00051D-27
76	1.56446D-21	-7.90126D-22	1.35729D-21	2.98913D-30

FIGURE CAPTIONS

Figure 1. Illustration showing the rotation angles i_1 and i_2 needed to refer the final Stokes vector to the proper meridian plane.


Z
final
θ
i2
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