Lagrangian Methods in the Analysis of Nonlinear Wave Interactions in Plasma
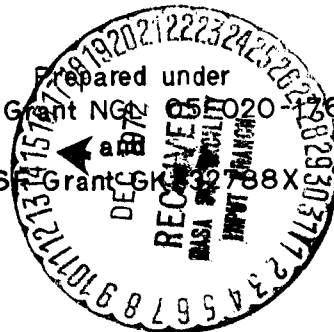
by

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NONLINEAR WAVE INTERACTIONS IN PLASMA

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ABSTRACT

In this dissertation, an averaged-Lagrangian method is developed for obtaining the equations which describe the nonlinear interactions of the wave (oscillatory) and background (non-oscillatory) components which comprise a continuous medium. The method applies to monochromatic waves in any continuous medium that can be described by a Lagrangian density, but it is demonstrated here in the context of plasma physics, where the analysis of nonlinear wave interactions by other methods can be extremely complicated.

The basic approach is that used by Whitham in 1965. It is a perturbation approach, restricted to cases where the perturbation components are small enough that their nonlinear interactions are weak, and the wave amplitudes are only perturbed slowly in space and time. Here, the theory is presented in a more general and unified form by way of a new averaged-Lagrangian formalism which simplifies the perturbation ordering procedure. Earlier theory is extended to deal with a medium distributed in velocity space and to account for the interaction of the background with the waves. The analytic steps are systematized, so as to maximize calculational efficiency.

The method makes direct use of the Lagrangian density and the small-signal relations for the wave parameters, as opposed to the conventional iterative approach, which employs only the nonlinear equations of motion. The method also differs from the quantum mechanical approach, in which a Hamiltonian description is used, and the waves are treated as quasi-particles. For complicated media, the latter approach has come into use as a convenient substitute for the conventional approach, even though the problem is essentially classical. The averaged-Lagrangian method is a

means of improving calculational efficiency without introducing a quantum
formalism.

Although the method itself generates only the interaction equations,
solution procedures are discussed here in connection with the plasma ex-
amples. In one example, a Lagrangian description of a Vlasov plasma is
used to obtain equations for nonlinear wave-wave and wave-particle inter-
actions. For simplicity of exposition, this is done here only in the
quasistatic limit. The random-phase approximation is then applied to
yield equations for the case of a continuous wave spectrum, and these are
shown to be equivalent to the kinetic equations for wave and particle
behavior, as they are encountered in the theory of weak plasma turbulence.

An assessment of the applicability and limitations of the method
shows that it has some definite advantages over other approaches in effi-
ciency and versatility. Provided that a suitable Lagrangian expansion is
known, and the nonlinear coupling is weak, the method provides an effi-
cient classical means of obtaining the interaction equations for the back-
ground and monochromatic waves, or wave spectra. The waves may have arbi-
trary directions of propagation and positive or negative energy. The
background may be anisotropic; it may have any field parameters, and
particle distributions in phase space, which are weakly inhomogeneous in
space and time, and which closely approximate an exact solution to the
full nonlinear equations.

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CONTENTS (Cont.)

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ILLUSTRATIONS

LIST OF SYMBOLS

Symbol		Page where defined on first use
$\mathcal{R}$	Rayleigh dissipation function	136
$\underset{\sim}{r}$	position vector	141
S	surface which encloses a region R in space	125
S	spring constant, in Hook's Law	41
s^i	one of the set $\{s^i : i=1,...M\}$, which are given functions of $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t	7
s_η	sign factor, of value ± 1, used in (2.27) . . .	17
T	normalized, dimensionless, time coordinate . .	100
T	temperature	82
T_j	kinetic energy of element j in a system of discrete elements	41
$\mathcal{T}$	kinetic energy density	41
t	time coordinate	8
U	velocity-dependent scalar potential defined by (3.10)	40
U(n)	potential energy density gained from work done against electron pressure, defined in . .	84
$\mathcal{U}$	potential energy density	41
V	voltage	41
$\underset{\sim}{V}$	normalized, dimensionless, velocity coordinate vector	100
$\underset{\sim}{v}$	velocity coordinate vector (in statistical description)	9
$\underset{\sim}{v}$	perturbed particle velocity (in hydrodynamic description)	83

LIST OF SYMBOLS (Cont.)

(2) Greek Alphabet

Symbol		Page where defined on first use
$\zeta(z)$	particle displacement from equilibrium position z	83
θ_η	complex phase angle of Wave η	11
$\delta\theta_{-\alpha\beta\gamma}$	phase mismatch, defined by (3.38)	58
λ^i	a constant of proportionality in (2.28)	19
$\lambda_\eta, \lambda_{-\alpha\beta\gamma}\ldots$	expansion coefficients defined by (3.35)-(3.37)	57
$\lambda_{\eta,n}$; $\lambda_{-\alpha\beta\ldots,n}$	terms in the secondary expansions of λ_η and $\lambda_{-\alpha\beta\ldots}$, which are of order n in the $\{q_0^i\}$	59
$\mu(i)$	integer denoting the order of q_0^i in powers of ϵ, in accordance with (3.47)	61
σ_η	sign (ϵ_η), the energy parity of Wave η	29
$d\tau$	differential volume element	83
Φ	normalized, dimensionless, electric scalar potential	100
ϕ_η	phase angle corresponding to $\hat{A}_\eta$, defined by (4.28)	89
ϕ'_η	slow-scale component of θ_η	11
$\underline{\Phi}_\eta$	averaged energy flux density of Wave η	27
φ_η	electric scalar potential	9
$X, X_{-\alpha\beta\gamma}$	three-wave coupling coefficient, defined by (3.57)	66
$X_{-\alpha\beta\gamma\delta}$	four-wave coupling coefficient, defined by (4.103)	111

LIST OF SYMBOLS (Cont.)

Symbol		Page where defined on first use
r	denotes real part of a complex quantity	11
s	denotes parameter of particle species s . . .	99
s	denotes one of a set of supplementary variables	128
w	denotes wave component	46
$\alpha,\beta,\gamma,\ldots$	denote quantities of Wave α, Wave β, Wave γ, etc.	16
$(\gamma+\delta)$	denotes a quantity of virtual wave, with frequency and wavevector $(\omega_\gamma+\omega_\delta, \underset{\sim}{k}_\gamma+\underset{\sim}{k}_\delta)$	24
δ	denotes perturbation quantity	43
η	index, denoting quantity of Wave η $(\eta=\alpha,\beta,\ldots)$	11
μ,ν	indices, denoting wave or background components $(\mu,\nu=0,\alpha,\beta,\ldots)$	48
0	denotes slow-scale background component . . .	11
0	denotes parameter of equilibrium state	83
$\underset{\sim}{}$	denotes a vector	9

(5) Operators

Symbol		Page where defined on first use
A_{Op}	nonlinear differential operator	132
$A_{Op}^{ij\cdots}$	differential operator matrices	7
$\left(\frac{\partial}{\partial t}\right)_\eta$, $\left(\frac{\partial}{\partial \underline{x}}\right)_\eta$	differential operators which act only on quantities of Wave η	19
$\left(\frac{D}{Dt}\right)_\eta$	operator defined by (3.58)	66

ACKNOWLEDGMENTS

Finally, the author wishes to express his appreciation to his wife, Martha, for her continued understanding and devotion during the writing of this dissertation and the long period of study which preceded it.

1. INTRODUCTION

This investigation concerns the weak nonlinear effects of waves, in a continuous medium which can be described by a Lagrangian. In the limit considered here, the nonlinearity of the descriptive equations results in weak interactions of the small-signal waves with one another and with the background state, which is defined by the non-oscillatory parameters of the medium. The weak interaction perturbs both the wave and background parameters on a scale which is large in comparison with the wave periods. The primary result of this investigation is an efficient and conceptually useful method for obtaining the equations which describe this behavior. The method makes direct use of the Lagrangian, as opposed to conventional iterative methods, which employ the corresponding equations of motion. The theory is an extended version of that presented by Whitham in 1965.[1-3]

The studies to be described were motivated by problems of current interest in plasma physics, for which the conventional nonlinear analysis is often extremely complicated. The work contributes to plasma physics through applications of the Lagrangian method to specific plasma examples. These examples illustrate characteristic features of the various plasma Lagrangians, and serve to demonstrate the efficacy of the method. However, the method itself is not restricted to the plasma medium. There is current interest in nonlinear wave phenomena in many other areas where Lagrangian techniques may be applied, so in the development of the method the author has taken pains to maintain generality. In the context of physics and mathematics, the content of this study lies primarily in a region of intersection between three disciplines, as indicated by the shaded area in the Venn diagram of Fig. 1.1. The disciplines are (a) the study of nonlinear wave interactions, for which this work develops a useful analytic tool, (b) the mathematics of variational methods, of which Lagrangian mechanics is a part, and (c) plasma physics, to which the method may be usefully applied. To some degree, topics outside the region of intersection are touched upon in this investigation, since some general understanding of all three fields is helpful in understanding and evaluating the method presented here.
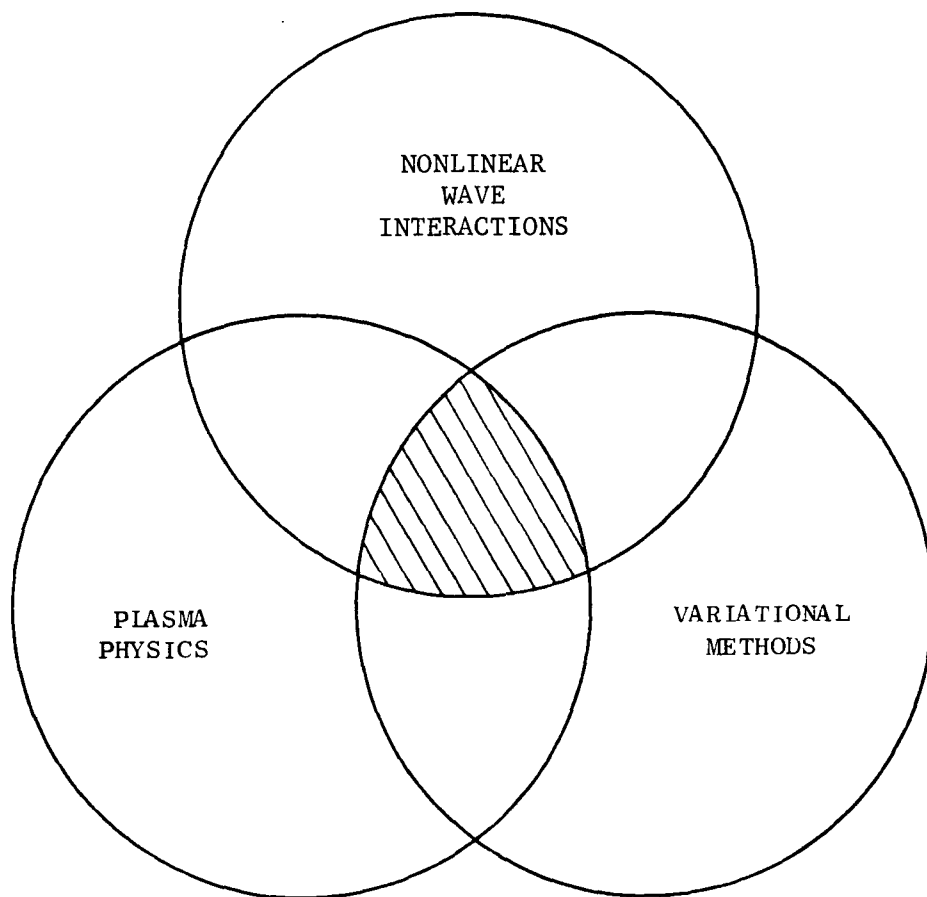
Fig. 1.1 Primary area of investigation, and overlapping
         subjects. The primary area of investigation is shaded.

Interest in the nonlinear interaction of oscillatory modes has its academic origin in the study of simple mechanical systems, such as coupled pendula, but further interest has been stimulated by electrical devices since the 19th century, when Faraday and Rayleigh demonstrated nonlinear interaction in materials which have a variable reactance. Practical application of nonlinear effects began in this century with the development of various traveling wave devices. Since the 1950's, parametric amplifiers have been used at microwave frequencies because of their low noise properties. Coupled-mode theory explains the behavior of these devices, and it is also the most effective means of explaining the behavior of the traveling wave tube (TWT) developed in the 1950's as a high power, high frequency amplifier. These applications of coupled-mode theory are discussed in a book by Louisell (1960),[4] which contains an extensive bibliography on the subject. The development of the laser has stimulated interest in the nonlinear interaction of coherent light waves, and in new devices at optical frequencies, such as tunable parametric oscillators.[5] Some general topics in nonlinear optics are reviewed in a monograph by Bloembergen.[6] The study of acoustic waves in solids at microwave frequencies has led in the last decade to the development of practical microwave delay lines, and the acoustoelectric effect, first demonstrated by D. L. White and others[7,8] has spurred developmental efforts on acoustic traveling wave amplifiers. More recently, efforts have been made toward the study and exploitation of the nonlinear effects of these waves.[9,10]

In plasma physics, the study of nonlinear phenomena has progressed somewhat differently, since the plasma medium is noisy, and has not proved to be well suited to the construction of electronic devices, such as amplifiers, frequency converters, etc. Nonlinear plasma theory now deals with plasma turbulence, especially in relation to controlled thermonuclear fusion devices and astrophysical phenomena. It is also concerned with the heating of the ionosphere by monochromatic waves, and with the scattering of waves by density fluctuations in the ionosphere. Applications of nonlinear theory to plasma turbulence are discussed, for example, in monographs by Kadomtsev[11] and Vedenov.[12]

In plasma theory, the analysis of nonlinear wave phenomena is extremely complicated, except for certain idealized problems.[12-14] Interesting problems have been solved by conventional methods (see texts by Sagdeev and Galeev,[13] and Davidson[14]), but various other approaches have also been employed. Their object has been to improve insight and to reduce analytic complexity. For instance, Sturrock has found it illuminating to treat the interaction of monochromatic waves by use of the Hamiltonian and action-angle variables.[15,16]

More recently, it has become popular to treat interactions between the wave spectra and the plasma particles in a random-phase approximation by methods borrowed from quantum mechanics. Such work has been reviewed by Harris.[17] One disadvantage of that approach is the introduction of a complicated quantum mechanical formalism to solve problems which are essentially classical, e.g., weak plasma turbulence. Another is that the quantum mechanical approach is directed toward the analysis of wave spectra, whereas in electronics, optics, and ionospheric plasma studies, one is more often interested in describing the behavior of monochromatic waves. Tsytovich[18] has recently introduced a semi-quantum mechanical approach, which is classically based, but which relies on the analogy with the quantum concept of induced processes. His approach makes it easier to understand and work with the interaction equations, whereas the Lagrangian method developed here is intended to make it easier to derive the equations themselves.

Early use of Lagrangian methods in the study of nonlinear wave behavior was made by Sturrock,[19] and interest in them was further stimulated by Whitham's articles.[1-3] Whitham used an approximate form of Hamilton's principle to obtain the equations for the slow amplitude variation of a wavetrain due to weak nonlinear effects. The method has since been considerably extended and refined by others. Dougherty[20] extended it to deal with the nonlinear interaction of wavetrains, using a relativistically covariant formalism, and Dysthe[21] has presented and applied a classical version of the method. Early applications of Lagrangian methods to nonlinear wave interactions in plasmas were made by this author in collaboration with Crawford and Kim,[22-24] by Dougherty,[25]

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and by Dewar.[26] These papers all provide useful techniques for the analysis of nonlinear wave evolution in the systematic manner to be presented here.

The present investigation is most closely related to the approach of Dysthe, but goes beyond Lagrangian schemes developed by him and the other investigators in several respects: (a) the analysis accounts simultaneously for wave-wave and wave-background interaction in a unified manner; (b) the analysis allows for the use of velocity as a coordinate, which is essential for applications to plasma problems in which the Lagrangian involves a velocity distribution function; and (c) the analytic steps constitute a method which maximizes the calculational efficiency.

In Section 2, the iterative approach to nonlinear wave analysis is outlined. This serves to introduce the general characteristics of nonlinear phenomena, and to provide a standard with which the Lagrangian approach may be compared.

Section 3 reviews the basic concepts of Lagrangian mechanics, as they apply to a continuous medium. Assuming small perturbations from a slowly-varying reference state, it shows how Hamilton's principle may be specialized to generate a Lagrangian mechanics for interacting modes. A specialized Lagrangian formalism is developed for this purpose, and the analysis results in a Lagrangian method for obtaining the equations for wave and background evolution, in the weak-coupling limit. The section concludes with an outline and flow chart for the method, and a summary of the assumptions upon which the method is based.

In Section 4, the method is applied to two illustrative examples in plasma physics. In the first example, it yields the interaction equations for three ion-acoustic waves and the plasma in which they propagate. The second example deals with three and four waves in a Vlasov plasma, and employs a quasistatic approximation to the electromagnetic behavior. For this problem, the method is applied to a Lagrangian for the Vlasov plasma which is commonly referred to as the Low Lagrangian,[27] although, as we shall show in Section 4, an essentially equivalent result was developed and applied independently by Sturrock. A rigorous

interpretation of this plasma Lagrangian has been given by Galloway and Kim,[24] who employed it in an earlier Lagrangian method for obtaining wave coupling coefficients. The Lagrangian method itself yields only the interaction equations, but Section 4 also discusses solution procedures in connection with the plasma examples. The basic solution procedures are not new; they are presented merely to show how the analytic process may be carried through to a finish, when the Lagrangian approach is used at the start. This generally involves working with a different set of coordinates from those used in a conventional analysis. In general, solutions for nonlinear interaction equations are not known analytically, and in the examples, additional assumptions are introduced in order to put the equations into forms for which solutions are known. In the first example, a solution for the ion-acoustic wave and plasma parameters is stated in terms of Jacobian elliptic functions. In the second example, the random-phase approximation is applied to yield equations for a Vlasov plasma in the presence of a wave spectrum. The interaction equations are shown to be equivalent to the kinetic equations for wave and particle evolution, as they are used in weak turublence theory. Proof of this equivalence involves a transformation from the displacement variables, used by Low, to the parameters of the conventional Vlasov description.

Section 5 is concerned with the applicability of the method, and with possible extensions of the averaged-Lagrangian approach. Related topics in the mathematics of variational methods are discussed.

Section 6 concludes the dissertation with a review of the original contributions of the work, and an assessment of the future utility of the averaged-Lagrangian approach, within and outside the field of plasma physics.

2. CONVENTIONAL THEORY AND BASIC CONCEPTS
OF NONLINEAR WAVE INTERACTION

The conventional approach to nonlinear wave interactions is to work directly with the equations of motion which govern small perturbations from some reference state. In this section, various assumptions are made which simplify the analysis. The Lagrangian analysis of the next section is far less restrictive, but such generality is not needed here. This section is illustrative of the conventional approach, not definitive. A very detailed conventional analysis of nonlinear plasma phenomena may be found in a recent text by Davidson,[14] and Louisell[4] has written a basic reference work on nonlinear interactions in a wider class of media.

2.1 Perturbation Expansion of the Nonlinear Equations

Here we shall consider a general continuous medium, of which a plasma may be an example. The medium is described by M scalar variables $\{q^i: i = 1,\ldots,M\}$ which represent the perturbations from the reference state. The coordinates of the system are $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t, where subscripts 1, 2, and 3 will be used to denote the orthogonal scalar components of $\underset{\sim}{v}$ and $\underset{\sim}{x}$. We shall stipulate that for this medium the descriptive equations for the perturbations may be written in the differential form

$$s^i + A_{Op}^{ij} q^j + A_{Op}^{ijk} q^j q^k + A_{Op}^{ijkl} q^j q^k q^l + \ldots = 0 , \qquad (2.1)$$

where summation over repeated subscripts is implied in this notation, s^i is a function of the coordinates only, and the elements of the matrices $A_{Op}^{ijk\ldots}$ are composed of sums and products of linear differential operators, such as $\partial/\partial t$ or $\partial/\partial x_1$, each acting on a particular component. Thus $A_{Op}^{ijk\ldots}$ may consist of a product of operators $G_j^i H_k^i \ldots$, where a subscript j on a particular factor indicates that it operates only on the scalar variable q^j. Let F_j^i denote any one of these operator-factors; F_j^i must be linear in the sense that, if q^j

is broken into components q_α^j and q_β^j ,

$$F_j^i(q_\alpha^j + q_\beta^j) = F_j^i(q_\alpha^j) + F_j^i(q_\beta^j) \quad . \quad (2.2)$$

Although these factors are linear, the terms of $A_{Op}^{ijk\cdots} q^j q^k \cdots$
are nonlinear in general. For example, one such term is
$A_{Op}^{i11} q^1 q^1$ $(1 \leq i \leq m)$. Let

$$q^1 = q_\alpha^1 + q_\beta^1 \quad , \quad (2.3)$$

then

$$A_{Op}^{i11} q^1 q^1 = G_1^i(q_\alpha^1 + q_\beta^1) \, H_1(q_\alpha^1 + q_\beta^1)$$

$$= \left\{ \begin{array}{c} G_1^i(q_\alpha^1) \, H_1(q_\alpha^1) + G_1^i(q_\beta^1) \, H_1^i(q_\beta^1) \\[2mm] + G_1^i(q_\alpha^1) \, H_1^i(q_\beta^1) + G_1^i(q_\beta^1) \, H_1^i(q_\alpha^1) \end{array} \right\}$$

$$\neq A_{Op}^{i11} q_\alpha^1 q_\alpha^1 + A_{Op}^{i11} q_\beta^1 q_\beta^1 \quad . \quad (2.4)$$

This argument applies to all but the first two terms on the left-hand
side of (2.1). Those two terms are always linear. In general, the
source term, S^i , and the operators, $A_{Op}^{ijk\cdots}$, may be explicitly
dependent on $\underset{\sim}{v}$, $\underset{\sim}{x}$ and t . In this entire work, the $\underset{\sim}{x}$ and t
dependence is required to be on a slow scale, as opposed to the scale
of fluctuations to be found in the solutions for the waves. The slow
scale dependence will be indicated by a single dimensionless parameter
ϵ , as in $S^i(\underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t)$ and $A_{Op}^{ijk\cdots}(\underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t)$. By carrying along
the parameter ϵ in this way it is possible to see the effects of the
separation in scales at any point in the analysis. One simply considers
the limit $\epsilon \to 0$. The use of ϵ facilitates the relative ordering of
terms in the equations. For example

$$\frac{\partial}{\partial t} S^i \sim \epsilon \, S^i \quad , \quad (2.5)$$

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or, in another useful notation,

$$O(\partial s^i/\partial t) = \epsilon \ O(s^i) \quad , \tag{2.6}$$

where $\sim$ translates "is proportionately of the same order as," and $O(s^i)$ denotes the order of s^i in powers of ϵ.

The implications of the ordering scheme will emerge through usage in this investigation. They will not be explored separately, because this sort of procedure is fairly common in the literature. The reader is referred to other sources[27,28] for a more detailed exposition.

Equations of the form (2.1) are capable of describing a rather large class of media. The truth of this assertion is to be found in the many interesting problems of this type which have been studied in the literature. A plasma example will serve to show the versatility of the formalism. Consider the Maxwell-Vlasov equations for an electron plasma with slow scale inhomogeneity. We shall make the quasistatic approximation, in which

$$\frac{\partial}{\partial \underset{\sim}{x}} \times \underset{\sim}{E} = 0 \quad , \quad \underset{\sim}{E} = - \frac{\partial \varphi}{\partial \underset{\sim}{x}} \quad , \quad \underset{\sim}{B} = \underset{\sim}{B}_0 \quad , \tag{2.7}$$

where $\underset{\sim}{E}$ is the electric field, φ is the scalar potential, and $\underset{\sim}{B}$ is the magnetic field, which is approximated by a given static component, $\underset{\sim}{B}_0$. Rationalized MKS units are used here. Under the above approximation, the descriptive equations are

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \frac{e}{m} \left(\frac{\partial \varphi}{\partial \underset{\sim}{x}} - \underset{\sim}{v} \times \underset{\sim}{B}_0 \right) \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0 \quad , \tag{2.8}$$

and

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot \frac{\partial \varphi}{\partial \underset{\sim}{x}} = \frac{e}{\epsilon_0} \int f \ d^3v \quad , \tag{2.9}$$

in which $f(\underset{\sim}{v},\underset{\sim}{x},t)$ is the distribution function; $-e$ is the charge of the electron; m is its mass; and ϵ_0 is the permittivity of free space. One of these equations, (2.9), does not have the form of (2.1)

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because it involves an integral operation on f. However, it may be converted to differential form through the use of a dummy function $\Psi(\underset{\sim}{v})$. This is a well-behaved function of $\underset{\sim}{v}$ which is zero at $|\underset{\sim}{v}| \rightarrow \infty$ and which is arbitrary otherwise except that

$$\int \Psi(\underset{\sim}{v}) d^3 v = 1 \quad . \tag{2.10}$$

Using Ψ, we may rewrite (2.9) as

$$\int \left[\Psi \frac{\partial}{\partial \underset{\sim}{x}} \cdot \frac{\partial \varphi}{\partial \underset{\sim}{x}} - \frac{e}{\epsilon_0} f \right] d^3 v = 0 \quad ; \tag{2.11}$$

and from this we may construct a more restrictive equation by setting the integrand equal to zero:

$$\Psi \frac{\partial}{\partial \underset{\sim}{x}} \cdot \frac{\partial \varphi}{\partial \underset{\sim}{x}} - \frac{e}{\epsilon_0} f = 0 \quad . \tag{2.12}$$

Equations (2.8) and (2.12) together constitute a complete set of differential equations of the form (2.1), but they are more restrictive than the original set. However, once they are solved, the solutions to the original problem may be recovered by integrating over the $\underset{\sim}{v}$ coordinate in such a way that (2.10) may be used to eliminate the dummy functions. Such a technique is used in a plasma example of Section 4, where the solution is found by a Lagrangian method. Our objective, both here and in the Lagrangian approach, is to simplify the formal analysis by casting the equations of motion into a differential form.

2.2 Linear Theory

Hereafter in this section, we shall consider only static, spatially homogeneous reference states. This is done for simplicity of presentation; weak spatial and temporal inhomogeneity are accounted for in the Lagrangian analysis of Section 3.

In the limit of small perturbations, all but the first two terms on the left hand side of (2.1) may be ignored, so that we have a system of

linear inhomogeneous differential equations,

$$A_{Op}^{ij} \, q^j = -S^i \qquad .$$ (2.13)

If the perturbations are somewhat larger, the first few nonlinear terms in (2.1) will still be small, but they may not be negligible. Then the solutions to (2.13) will merely approximate the solution to the complete equations. We are interested only in systems which have small-signal solutions in the form of propagating waves, so that we may expand the $\{q^i\}$ as

$$q^i = q_0^i + \Sigma_\eta \, q_\eta^i \qquad ,$$ (2.14)

where the $\{q_0^i(\underset{\sim}{v})\}$ constitute a particular solution to (2.13), and the $\{q_\eta^i\}$ are wave components which are solutions to the related homogeneous problem $(A_{Op}^{ij} \, q_\eta^j = 0)$. These are expressed in the following complex notation:

$$q_\eta^i \equiv \bar{q}_\eta^i + \bar{q}_{-\eta}^i \quad , \quad \bar{q}_{\pm\eta}^i \equiv \hat{q}_{\pm\eta}^i \exp j\theta_{\pm\eta} \quad , \quad \theta_{-\eta} \equiv -\theta_\eta^* \quad .$$ (2.15)

The quantity $\theta_\eta(\underset{\sim}{x},t)$ is the oscillatory phase of the wave and its first derivatives are

$$\omega_\eta \equiv \partial\theta_\eta/\partial t \quad , \quad \underset{\sim}{k}_\eta \equiv -\partial\theta_\eta/\partial\underset{\sim}{x} \quad .$$ (2.16)

In a spatially homogeneous medium, θ_η is in fact just $(\omega_\eta t - \underset{\sim}{k}_\eta \cdot \underset{\sim}{x} + \phi_\eta')$, where ω_η , $\underset{\sim}{k}_\eta$ and ϕ_η' are all constants. Since the solutions to the problem are to be real, we must have $\hat{q}_{-\eta}^i = \hat{q}_\eta^{i*}$.

The substitution of (2.14) into the linear equations of motion yields

$$A^{ij}(\omega_\eta,\underset{\sim}{k}_\eta) \, \bar{q}_\eta^j = 0 \qquad (i,j = 1,\ldots M) \qquad .$$ (2.17)

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In this equation, A^{ij} is one of a set of matrices

$$\{A^{ijk\cdots}(\omega_\beta, \omega_\gamma, \ldots, \underset{\sim}{k}_\beta, \underset{\sim}{k}_\gamma \cdots)\}$$

which we define by

$$A^{ijk\cdots}{}_{\overline{q}_\beta^j \overline{q}_\gamma^k}\cdots \equiv A_{Op}^{ijk\cdots}{}_{\overline{q}_\beta^j \overline{q}_\gamma^k}\cdots \qquad , \qquad (2.18)$$

where the complex amplitudes $\hat{q}_\beta^j, \hat{q}_\gamma^k$, etc., are regarded as constants. Equation (2.17) is one frequency component of (2.13). It is a secular equation and as such it has solutions

$$\overline{q}_\eta = \left\{\hat{q}_\eta \exp j\theta_\eta\right\} \delta[\omega_\eta(\underset{\sim}{k}_\eta)] \qquad , \qquad (2.19)$$

where $\omega_\eta(k_\eta)$ is a solution of

$$|A^{ij}(\omega_\eta, \underset{\sim}{k}_\eta)| = 0 \qquad . \qquad (2.20)$$

Equation (2.20) is the dispersion relation for the medium. The dispersion function, $|A^{ij}|$, may in general be factored into M factors $D_1^\eta(\omega_\eta, \underset{\sim}{k}_\eta) \cdot D_2^\eta(\omega_\eta, \underset{\sim}{k}_\eta) \cdot D_3^\eta(\omega_\eta, \underset{\sim}{k}_\eta) \ldots$, and (2.20) is satisfied if any one of these factors is zero. Each factor corresponds to a particular eigenmode of the secular equation (2.17), and each represents a branch of the dispersion relation

$$D_\ell^\eta(\omega_\eta, \underset{\sim}{k}_\eta) = 0 \qquad . \qquad (2.21)$$

In this work, to specify a propagating wave component, η , is to specify a corresponding point $(\omega_\eta, \underset{\sim}{k}_\eta)$ on a particular branch. Then the index η alone will serve to identify the branch, $D^\eta(\omega_\eta, \underset{\sim}{k}_\eta)$. Here, and in the rest of the work, we use $(\omega_\eta, \underset{\sim}{k}_\eta)$ to indicate a four-vector

$$(\omega_\eta, \underset{\sim}{k}_\eta) \equiv (\omega_\eta, k_{\eta 1}, k_{\eta 2}, k_{\eta 3}) \qquad . \qquad (2.22)$$

The subscripts r and i will henceforth be used to denote the real

and imaginary parts, respectively, of θ_η , ω_η , k_η , and (ω_η, k_η) .
After solving (2.21), one may substitute $\omega_\eta(\underset{\sim}{k}_\eta)$ into (2.17) to obtain
M scalar equations relating the $\{\hat{q}^i\}$. These are the normal-mode
small-signal relations

$$\overline{q}_\eta^i = c_\eta^i \, \overline{a}_\eta \qquad (i = 1, \ldots, m) \qquad . \qquad (2.23)$$

Here, one convenient scalar field parameter, $\overline{a}_\eta$, is chosen for each
wave, and c_η^i is a complex factor expressed in terms of ω_η, $\underset{\sim}{k}_\eta$, and
reference state parameters. Its algebraic form depends on the particu-
lar dispersion branch considered.

2.3 The Nonlinear Interaction Picture

In the small-signal limit, the $\{q_0^i\}$ and $\{\hat{a}_\eta\}$ are all constant,
and the waves are uncoupled. If the wave amplitudes are gradually
increased, however, a condition will be reached in which appreciable,
but weak, coupling of the small-signal modes and the $\{q_0^i\}$ will occur.
This is the "weak-coupling limit." In this approximation we assume that
the $\{q_0^i\}$ and $\{\hat{a}_\eta\}$ may be slowly varying functions of $\epsilon\underset{\sim}{x}$ and ϵt .
In the nonlinear analysis to be described in Sections 2.4 − 2.5, the
equations will be simplified by means of the small-signal relations
between the wave components $\{\hat{q}_\eta^i = c_\eta^i \, \hat{a}_\eta\}$. There are, however, no
corresponding relations for the slow-scale perturbation components;
$\{q_0^i\}$; so each of them will be treated separately.

Related to the description of the $\{q_0^i\}$ is the concept of a lim-
ited space-time average for functions of $\{q^i\}$, $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t . Let
such a function be $G(q^i, \underset{\sim}{v}, \underset{\sim}{x}, t)$. We define $\overline{G}(\underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t)$ to be an
average of G over intervals of $\underset{\sim}{x}$ and t which are large compared to
the period of oscillation of the uncoupled small-signal waves, but short
compared to the intervals over which significant wave perturbation can
occur. This is meaningful when the coupling is weak. If G is a slow-
scale function of $\epsilon\underset{\sim}{x}$, and ϵt , then G and $\overline{G}$ are equal. The
$\overline{G}$ will be spoken of as the "bar average of G". To illustrate the
averaging we shall consider the case in which G is $(\overline{q}_1^i + \overline{q}_1^{i*})^2$;

then $\overline{G}(\underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t)$ is $2\hat{q}_1^i(\underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t)\hat{q}_1^{i*}(\underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t)$ in the notation of (2.15).

We are now in a position to explain how the equations for the various perturbation components are obtained in the weak-coupling limit. The equation for a monochromatic wave component, α , is obtained by isolating from (2.1) all terms which vary as $\exp j\theta_\alpha$. The small-signal relations are used to simplify the results to a single equation. The equations for the $\{q_0^i\}$, on the other hand, are obtained by taking the bar-average of (2.1).

In the nonlinear analysis, we shall find it useful to consider the waves as perturbations from a "background state," which evolves on the slow scale in space and time. This state, sometimes referred to simply as the "background," is defined by the reference state and the slow-scale perturbation components. Together, the waves and the background define the state of the medium. We may state the relationships symbolically as

$$\text{Medium} \equiv \overbrace{\left\{\overline{q}_\eta^i = c_\eta^i \overline{a}_\eta\right\}}^{\text{Waves}} + \overbrace{\left\{q_0^i\right\} + \text{Reference State}}^{\text{Background}} \quad , \qquad (2.24)$$

where, in this usage, "$\equiv$" means "is defined by," and "+" means "together with." The concept expressed in (2.24) has been employed previously by Dougherty.[20]

The conceptual picture of weak nonlinear interaction is illustrated in Fig. 2.1, which applies to the three-wave case. The waves interact weakly with one another (wave-wave interaction) and with the background (wave-background interaction). The waves affect one another through the nonlinear terms of (2.1). The background state affects the waves, because its parameters appear in the coefficients of the wave equations. The waves affect the background equations, which describe the $\{q_0^1\}$, because products of wave perturbation components contribute to the bar-average of (2.1). The wave-wave interaction can be significant, on the slow scale, only when $\{\omega_\eta\}$ and $\{k_\eta\}$ satisfy certain "synchronism conditions," to be explained in the next subsection.

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WAVE

BACKGROUND
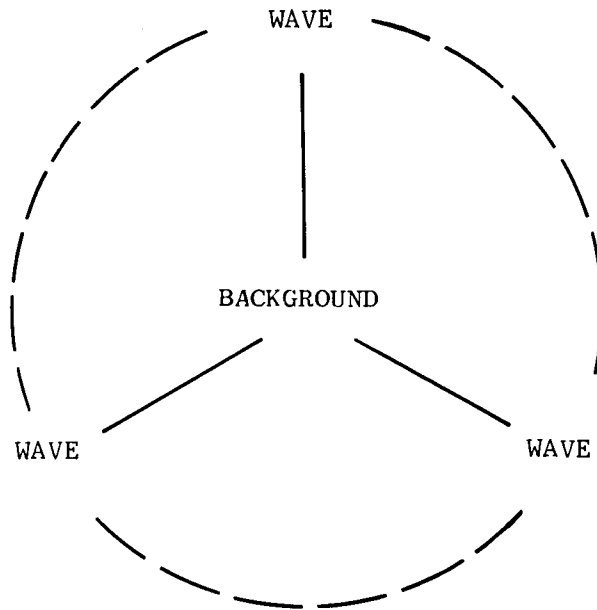
WAVE   WAVE

Fig. 2.1   Lowest order wave-wave and wave-background interaction
           processes. Three synchronous waves interact with one
           another, as indicated by the broken lines, and simul-
           taneously, each wave individually affects the background
           and vice-versa, as indicated by the solid lines. If the
           waves are asynchronous, the wave-wave interaction is neg-
           ligible, but the wave-background interactions persist.

For consistency with existing nomenclature, the joint effects
of three or more synchronous waves on the $\{q_0^i\}$, and vice-versa,
are referred to in this work as "nonlinear wave-background inter-
actions." This is to distinguish them from single-wave "quasi-
linear" effects, although not even the latter could exist if the
full equations of motion were linear. In plasma theory, both quasi-
linear and nonlinear wave-background interactions are commonly refer-
red to as "wave-particle interactions." This is because one important
background parameter which is affected by the interactions is the
particle distribution function, $f(\underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t)$. The more general

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"background" label is preferred in this work because (a) the theory developed here is not restricted to plasmas; and (b) even in plasma analyses, the theory may describe the evolution of the slow-scale electric and magnetic field components, as well as the particle distribution functions.

2.4 Nonlinear Wave Analysis

2.4.1 Three-Wave Interactions

For illustrative purposes we now consider the three-wave interactions, which are nonlinear effects of a low order. These interactions are relatively simple, but the same analysis extends to more complicated nonlinear effects. We transform the differential equations (2.1) into algebraic equations by substituting into them the small-signal forms of the solution, (2.15). This procedure is not strictly correct, since we are now considering an amplitude régime where wave coupling is significant, so that the slow space and time dependence of $\{\hat{a}_\eta\}$, $\{\omega_\eta\}$ and $\{\underset{\sim}{k}_\eta\}$ should appear in the equations. However, it will be expedient for calculational purposes to ignore their derivatives for now. The information lost can be recovered to the necessary extent by expansion procedures, used later in this subsection.

In the case of three discrete waves, the index η takes on values $\pm\alpha$, $\pm\beta$, $\pm\gamma$. Let the small signal-forms of the solutions be substituted into the first three terms of (2.1). Then the $(\omega_\alpha, \underset{\sim}{k}_\alpha)$ component of the result is

$$A_\alpha^{ij}\,\hat{q}_\alpha^{j} = \sum_{\substack{\text{sync} \\ \alpha\beta\gamma}} A_{\alpha\beta\gamma}^{ijk}\hat{q}_\beta^{j}\hat{q}_\gamma^{k} \qquad . \qquad (2.25)$$

The subscript "sync $\alpha\beta\gamma$", used in this equation, indicates that the right hand side represents a sum over only those components whose frequencies and wavevectors satisfy the three-wave synchronism conditions

$$(\omega_\alpha, \underset{\sim}{k}_\alpha)_r = (\omega_\beta, \underset{\sim}{k}_\beta)_r + (\omega_\gamma, \underset{\sim}{k}_\gamma)_r \qquad , \qquad (2.26)$$

in which the subscript r denotes the real parts of the $\{(\omega_\eta, \underset{\sim}{k}_\eta)\}$. Unless these conditions are met quite closely by some synchronous combinations, $\alpha\beta\gamma$, the right hand side of (2.25) is zero and we have only the linear problem. We may modify (2.25), and the subsequent analysis, to allow for the case in which there is a small synchronism mismatch, of $\delta\omega$ in frequency and δk in wavevector, provided that $\left|\delta\omega/\omega_\eta\right| \sim \left|\delta k/k_\eta\right| \sim \epsilon$, where $\eta = \alpha, \beta, \gamma \cdots$. However, for simplicity of illustration in this section, we shall consider only the case in which synchronism holds exactly. The effects of small synchronism mismatch are accounted for in the Lagrangian analysis of Section 3.

The general form of the synchronism conditions for the interaction of many waves is

$$\Sigma_\eta s_\eta (\omega_\eta, \underset{\sim}{k}_\eta)_r = (0, \underset{\sim}{0}) \qquad , \qquad (2.27)$$

where $\{s_\eta\}$ is some set of sign factors having value ± 1. Equation (2.27) is a necessary, but not sufficient, condition for significant nonlinear wave coupling. Even if it is satisfied, the strength of the coupling may be negligible in a particular case. Figure (2.2) shows how the synchronism conditions may be satisfied for three collinearly propagating waves in the same mode. Clearly, for certain points on the dispersion curve, synchronism with two other points may be impossible, and in fact, for dispersion curves of certain common forms, such synchronism may be impossible throughout. One may wish to consider synchronism between different branches, and to allow for $\underset{\sim}{k}$'s at various angles with respect to one another. This generally increases the possibilities for synchronism. Because (2.27) is a necessary condition, a search for synchronous combinations is usually the first step in the study of a particular wave interaction problem, and some general discussions may be found in the literature.[11,30] However, in the present work we shall concentrate on rather general methods for finding the equations for the coupled modes, after assuming that synchronism obtains in some sense.
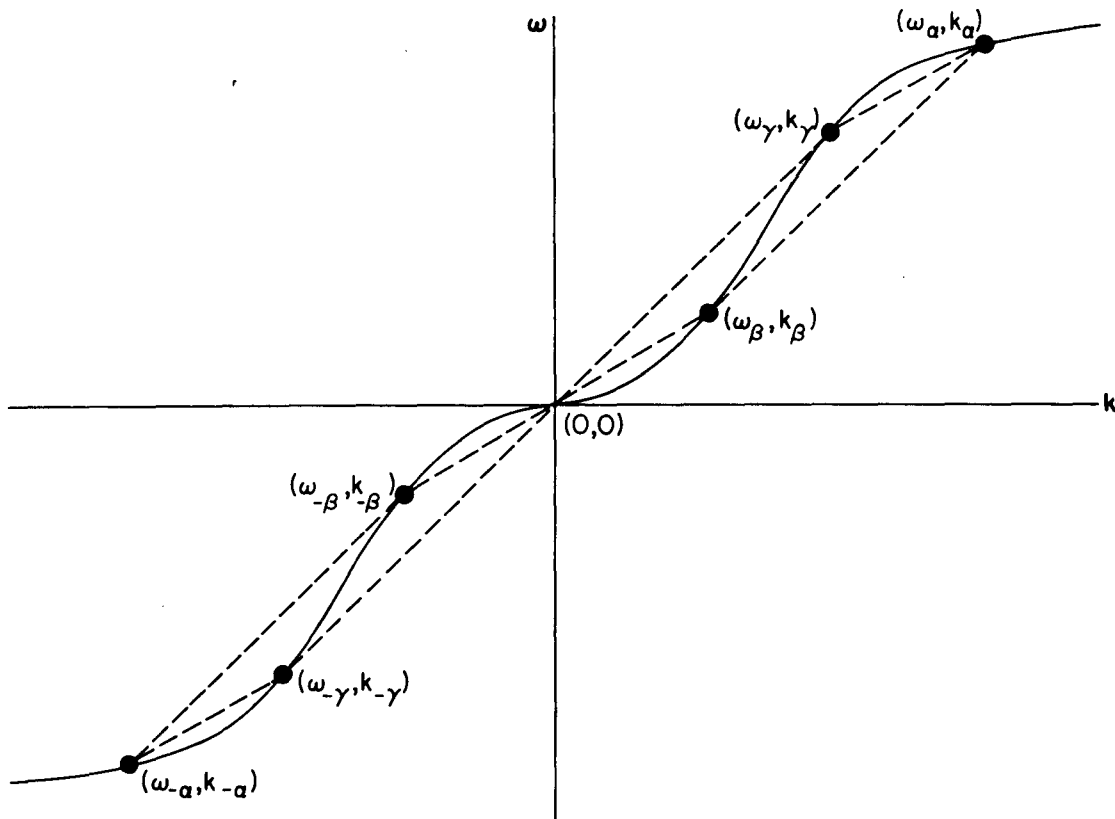
Fig. 2.2   Synchronism between three waves which propagate in the same
            mode. The figure shows a dispersion curve for waves which
            propagate along a common spatial axis with real frequencies
            and wavenumbers. Each wave is represented by two points on
            the dispersion curve, (ω_η , k_η), and $(\omega_{-\eta} , k_{-\eta})$, where
            $\omega_{-\eta} \equiv -\omega_\eta$, and $k_{-\eta} \equiv -k_\eta$ $(\eta = \alpha , \beta , \gamma)$. The synchronism condition,
            necessary for significant wave-wave interaction, is
            $(\omega_\alpha , k_\alpha) = (\omega_\beta , k_\beta) + (\omega_\gamma , k_\gamma)$. The figure shows that the
            waves satisfy this requirement exactly if, and only if, the
            corresponding points on the dispersion curve, together with
            the origin $(0,0)$, comprise the four vertices of a parallelogram.

In the present analysis, which assumes exact synchronism, the next step is to convert (2.25) into a single scalar equation by substituting into it the small-signal relations (2.23). The result is a vector equation, equivalent to M scalar equations, but since the $\{c_\eta^i\}$ of (2.23) represent solutions to the linear equations, the M equations obtained will all be equivalent and of the form

$$\lambda^i D^\alpha(\omega_\alpha, \underset{\sim}{k}_\alpha)\hat{a}_\alpha = \lambda^i \sum_{\substack{sync \\ \alpha\beta\gamma}} D^{\alpha\beta\gamma}(\omega_\alpha, \underset{\sim}{k}_\alpha, \omega_\beta, \underset{\sim}{k}_\beta, \omega_\gamma, \underset{\sim}{k}_\gamma)\hat{a}_\beta \hat{a}_\gamma \quad . \quad (2.28)$$

Here D^α is the dispersion function of Eq. (2.21); λ^i is a multiplicative constant, and $\lambda^i D^{\alpha\beta\gamma}$ belongs to a set of scalars defined by

$$\lambda^i D^{\alpha\beta\gamma\cdots} \equiv A^{ijk\cdots} c_\beta^j c_\gamma^k \cdots \quad . \quad (2.29)$$

As we have seen, (2.28) is not strictly correct, even in a homogeneous medium, because the derivatives of the wave amplitudes have been neglected. However (2.28) can be converted to a valid differential equation by the simple transformation

$$\omega_\eta \rightarrow \omega_\eta - j\left(\frac{\partial}{\partial t}\right)_\eta \quad , \quad \underset{\sim}{k}_\eta \rightarrow \underset{\sim}{k}_\eta + j\left(\frac{\partial}{\partial \underset{\sim}{x}}\right)_\eta \quad . \quad (2.30)$$

To understand this, consider, for example, $(\partial/\partial t)$ operating on a field quantity of Wave η . In the small signal limit it is justified to replace this by $j\omega_\eta$ in order to obtain an algebraic term. If the amplitude of the wave component varies, however, the proper transformation is to replace $(\partial/\partial t)$ by $j\omega_\eta + (\partial/\partial t)_\eta$, where the new operator acts only on the slowly varying amplitude of the component η . If this were done throughout the analysis, a more tedious calculation would yield a differential equation instead of (2.28). The transformation (2.30) is a simpler means of recovering the same differential elements. Application of (2.30) to (2.28) leads to the expansion

$$D^\alpha \hat{a}_\alpha - j\left(\frac{\partial D^\alpha}{\partial \omega_\alpha}\right) \cdot \frac{\partial \hat{a}_\alpha}{\partial t} + j\left(\frac{\partial D^\alpha}{\partial \underset{\sim}{k}_\alpha}\right) \cdot \frac{\partial \hat{a}_\alpha}{\partial \underset{\sim}{x}} + \cdots = D^{\alpha\beta\gamma} \hat{a}_\beta \hat{a}_\gamma + \cdots \quad , \quad (2.31)$$

where the background state of the medium is assumed to be homogeneous, and D^α, $D^{\alpha\beta\gamma}$, and their various derivatives, are all evaluated at $(\omega_\alpha, \underset{\sim}{k}_\alpha)$, $(\omega_\beta, \underset{\sim}{k}_\beta)$, and $(\omega_\gamma, \underset{\sim}{k}_\gamma)$. We require that these points satisfy the linear dispersion relation (2.21), so that the leftmost term of (2.31) vanishes. The result may be cast in the form

$$\frac{\partial \hat{a}_\alpha}{\partial t} + \underset{\sim}{v}_\alpha \cdot \frac{\partial \hat{a}_\alpha}{\partial \underset{\sim}{x}} = j \sum_{\substack{\text{sync} \\ \alpha\beta\gamma}} \frac{D^{\alpha\beta\gamma}}{(\partial D^\alpha / \partial \omega_\alpha)} \hat{a}_\beta \hat{a}_\gamma \quad , \qquad (2.32)$$

where higher order terms in the expansion have been neglected, and

$$\underset{\sim}{v}_\eta \equiv -\frac{(\partial D^\eta / \partial \underset{\sim}{k}_\eta)}{(\partial D^\eta / \partial \omega_\eta)} = \left(\frac{d\omega_\eta}{d\underset{\sim}{k}_\eta}\right)_{D^\eta = 0} \qquad (\eta = \alpha, \beta, \gamma) \quad . \qquad (2.33)$$

The vector $\underset{\sim}{v}_\eta$ is the group velocity for Wave η, i.e., the gradient of the dispersion curve at $(\omega_\eta, \underset{\sim}{k}_\eta)$.

From (2.32), or its higher-order counterparts, one may generate a complete set of equations for the wave components. To see this, consider two alternate forms of the three-wave synchronism conditions:

$$\left(\omega_\beta, \underset{\sim}{k}_\beta\right)_r = \left(\omega_\alpha, \underset{\sim}{k}_\alpha\right)_r + \left(\omega_{-\gamma}, \underset{\sim}{k}_{-\gamma}\right)_r \quad , \qquad (2.34)$$

and

$$\left(\omega_\gamma, \underset{\sim}{k}_\gamma\right)_r = \left(\omega_\alpha, \underset{\sim}{k}_\alpha\right)_r + \left(\omega_{-\beta}, \underset{\sim}{k}_{-\beta}\right)_r \quad . \qquad (2.35)$$

These follow from the original synchronism condition, (2.26), and the requirements on the complex notation, (2.15). If the preceding analysis had begun with the combination (2.34) instead of (2.26) the result would be identical to (2.32) except that the superscripts would be changed. They would be altered by the same substitution required to change (2.26) into (2.34): $\alpha \to \beta$, $\beta \to \alpha$, $\gamma \to -\gamma$. The other substitution, indicated by (2.35), is $\alpha \to \gamma$, $\beta \to \alpha$, $\gamma \to -\beta$.

Equation (2.32), and the equations generated from it by the change of indices, completely describe the behavior of the waves in the case we have considered. They are often referred to in the literature as the "coupled-mode equations," but in this work they will usually be called the "equations of wave evolution." The steps in the wave analysis are summarized schematically in the flow chart of Fig. 2.3. For completeness, the figure also indicates that the application of an approximation procedure, the "random-phase approximation," converts the equations obtained in this manner into another set of equations, the "wave-kinetic" equations. These equations describe the behavior of a wave spectrum, as opposed to a set of monochromatic waves. They are called "kinetic" because the wave variables which appear in them are proportional to the energy density of the waves, rather than to their amplitudes. The random-phase approximation, sometimes called the "RPA", is discussed in detail later in this dissertation (see Section 4.2).

Let us now examine carefully the expansion step used in deriving (2.32). The expansion (2.31) resembles a Taylor series, where $(\partial/\partial t)_\eta$ and $(\partial/\partial \underset{\sim}{x})_\eta$ are manipulated as though they were small perturbations of ω and $\underset{\sim}{k}$ from ω_η and $\underset{\sim}{k}_\eta$, respectively. Since the complex amplitudes vary slowly, their derivatives are small, and the preceding considerations lend credence to this expansion. However, the preceding presentation clearly lacks rigor. For example, our algebraic manipulations might lead to a term like $1/\omega_\alpha$. The expansion procedure would be

$$\frac{1}{\omega_\alpha} \rightarrow \frac{1}{\omega_\alpha - j(\partial/\partial t)_\alpha} \rightarrow \frac{1}{\omega_\alpha} + \frac{j}{\omega_\alpha^2}\left(\frac{\partial}{\partial t}\right)_\alpha + \cdots \tag{2.36}$$

but the meaning of $1/[\omega_\alpha - j(\partial/\partial t)_\alpha]$, and the validity of the expansion process have not been solidly established.

A rigorous derivation of the result (2.32) can in fact be made by using Fourier transform techniques.[13] The interaction considered is between wavepackets in ω-$\underset{\sim}{k}$ space. These are localized about the points $\pm(\omega_\alpha, \underset{\sim}{k}_\alpha)$, $\pm(\omega_\beta, \underset{\sim}{k}_\beta)$ and $\pm(\omega_\gamma, \underset{\sim}{k}_\gamma)$. The nonlinear equations are Fourier

EQ'S

| LINEAR TERMS | $+$ | FIRST ORDER NONLINEAR TERMS | $\cong 0$ |

ISOLATE COMPONENT $(\omega_\alpha, \underset{\sim}{k}_\alpha)$

S-S RELATIONS

$$\hat{q}^i_\eta = c^i_\eta \hat{a}_\eta$$

SYNCHRONISM

$$\omega_\alpha = \omega_\beta + \omega_\gamma$$
$$\underset{\sim}{k}_\alpha = \underset{\sim}{k}_\beta + \underset{\sim}{k}_\gamma$$

$$D_\alpha \left(\omega_\alpha - j\frac{\partial}{\partial t} , \underset{\sim}{k}_\alpha + j\frac{\partial}{\partial \underset{\sim}{x}} \right) \hat{a}_\alpha \quad + \quad \frac{1}{\lambda_i} \sum_{\substack{\text{sync} \\ \alpha\beta\gamma}} A^{ijk}_{\beta\gamma} \hat{q}^j_\beta \hat{q}^k_\gamma \quad \cong 0$$

EQUATIONS FOR WAVE EVOLUTION

RANDOM-PHASE APPROXIMATION

WAVE KINETIC EQUATIONS
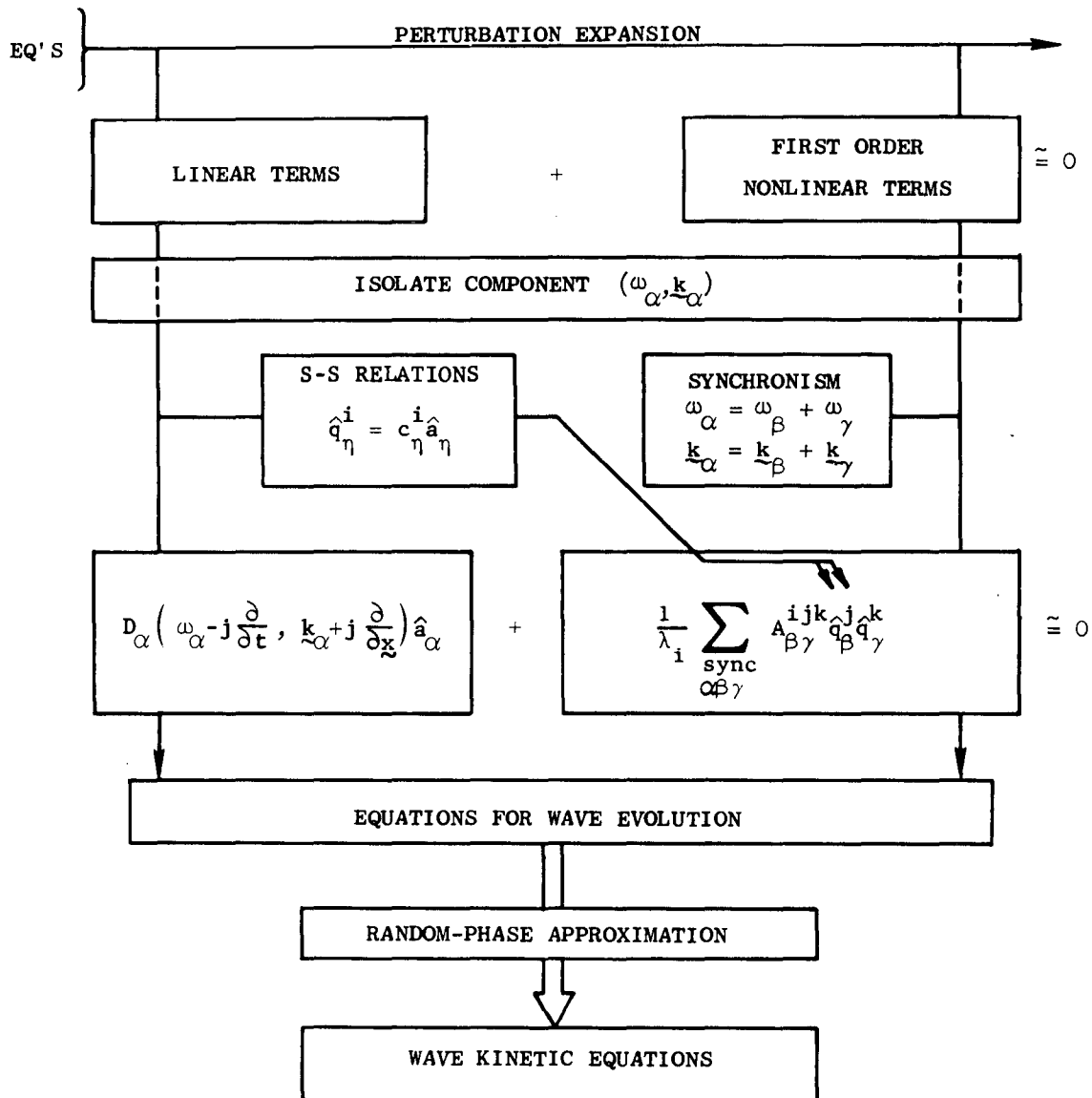
Fig. 2.3 Flow chart for the nonlinear wave analysis. The analysis is
illustrated here for the interaction of three synchronous
waves. Solution of the linearized equations yields the
dispersion relation and S-S (small-signal) relations between
the wave variables. The latter are used to simplify the
nonlinear equations. The resulting equations describe the
evolution of monochromatic waves in the weak-coupling limit.
Application of the random-phase approximation yields the
wave-kinetic equations, which describe the evolution of a
continuous wave spectrum.

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transformed, and the expressions are expanded in Taylor series about these points. Then the transform is inverted and the result is simplified by the substitution of $\hat{q}_\eta^i \exp j(\omega_\eta t - \underset{\sim}{k}_\eta \cdot \underset{\sim}{x})$ for the component $\bar{q}_\eta^i$ associated with each wavepacket $(\eta = \alpha, \beta, \gamma)$. The details of this procedure are tedious and will not be presented here. Our purpose here has been to summarize the conventional approach by providing a simple procedural scheme which maximizes insight rather than rigor.

2.4.2 Higher Orders of Interaction and Virtual Waves

With a few simple modifications, the preceding steps yield the equations for N-wave interaction effects. For example, consider a medium for which the dispersion relation makes three-wave synchronism impossible, but in which there are four propagating waves, α, β, γ, and δ such that

$$(\omega_\alpha, \underset{\sim}{k}_\alpha)_r = (\omega_\beta, \underset{\sim}{k}_\alpha)_r + (\omega_\gamma, \underset{\sim}{k}_\gamma)_r + (\omega_\delta, \underset{\sim}{k}_\delta)_r \quad . \tag{2.37}$$

We shall show later in this work that, for any dispersion relation, it is generally possible to satisfy these four-wave synchronism conditions in the limit of sideband decay (see Section 3.5.3). Here, however, we are concerned with the entire class of nonlinear wave-wave interactions which are characterized by (2.37). For these interactions, it is straightforward to follow the same approach as before to obtain the four-wave extension of (2.28)

$$D^\alpha \hat{a}_\alpha = \sum_{\substack{\text{sync} \\ \alpha\beta\gamma\delta}} D^{\alpha\beta\gamma\delta} \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta \quad . \tag{2.38}$$

Certain terms must be added to convert (2.38) into a valid approximation, however. These terms account for the presence of virtual waves (beats between propagating waves). These phenomena are negligible in the three wave case, but here we must consider their origin and their role in higher order wave interactions. In cases where it is impossible to have three synchronous points on the dispersion diagram, it is still

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Δ

possible for two linear modes, say γ and δ to mix through the quadratic nonlinear terms in the equations. This produces a virtual wave (or beat) at $(\omega_\gamma + \omega_\delta, \underset{\sim}{k}_\gamma + \underset{\sim}{k}_\delta)$. Since this point is not on the dispersion curve, the virtual wave cannot exist independently of the γ- and δ-modes . To lowest order, its amplitude is determined by a three-wave equation like (2.28)

$$D_\ell^{(\gamma+\delta)} \hat{a}_{(\gamma+\delta)} = D_\ell^{(\gamma+\delta)} \hat{a}_\gamma \hat{a}_\delta \qquad (\ell = 1, \ldots, M) \quad . \qquad (2.39)$$

The index $(\gamma+\delta)$ means that the algebraic role played by ω_α and $\underset{\sim}{k}_\alpha$ in (2.28) has been taken over by $\omega_\gamma + \omega_\delta$, and $\underset{\sim}{k}_\gamma + \underset{\sim}{k}_\delta$, respectively. The additional index ℓ , has been employed to identify a particular branch of the dispersion relation, since $(\omega_\gamma + \omega_\delta, \underset{\sim}{k}_\gamma + \underset{\sim}{k}_\delta)$ is not uniquely associated with a particular branch. The beats between δ and γ result in M virtual waves, one for each branch. The expansion process of (2.30) is unnecessary for this case. Since $D_\ell^{(\gamma+\delta)}$ is not zero, (2.39) is already the lowest order approximation, sufficient to determine $\hat{a}_{(\gamma+\delta)}$.

The virtual wave is important to the coupled-mode equations because it can interact with other waves to produce a contribution to the propagating wave being studied. In the four-wave situation the appropriate combination is

$$(\omega_\alpha, \underset{\sim}{k}_\alpha)_r = (\omega_\beta, \underset{\sim}{k}_\beta)_r + (\omega_\gamma + \omega_\delta, \underset{\sim}{k}_\gamma + \underset{\sim}{k}_\delta)_r \quad . \qquad (2.40)$$

The complete four-wave equations are obtained by combining (2.38) and (2.39). The result is

$$D^\alpha \hat{a}_\alpha = \sum_{\substack{\text{sync} \\ \alpha\beta\gamma\delta}} \left[D^{\alpha\beta\gamma\delta} \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta + \sum_\ell D_\ell^{\alpha\beta(\gamma+\delta)} a_\beta a_{(\gamma+\delta)} \right]$$

$$= \sum_{\substack{\text{sync} \\ \alpha\beta\gamma\delta}} \left[D^{\alpha\beta\gamma\delta} + \sum_\ell \left(\frac{D_\ell^{\alpha\beta(\gamma+\delta)} D_\ell^{(\gamma+\delta)\gamma\delta}}{D^{\ell(\gamma+\delta)}} \right) \right] \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta \quad . \qquad (2.41)$$

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The final step is to expand (2.41) according to (2.30). On the right hand side, (2.41) contains a sum over all combinations of small-signal modes β, γ, and δ which satisfy the four-wave synchronism condition. It can therefore describe the evolution of the α component in the presence of an entire series of discrete waves, and it also describes the self-action effects represented in our formalism by the combination $\beta\gamma\delta \rightarrow \alpha\alpha(-\alpha)$. Using this combination, and assuming that there is only one real wave component, α , present in the medium, we may use (2.41) to describe single-wave behavior outside the linear régime.

The foregoing example indicates the procedure for dealing with the effects of virtual waves, which are the only non-algebraic complication in the analysis of the higher order mode interactions. In the case of N interacting modes, the beats between two, three, ..., and (N-1) small-signal modes must be taken into account in obtaining the coupled-mode equations.

2.4.3 The Manley-Rowe Relations

The equations of wave evolution, as they are obtained from the iterative analysis just described, often have symmetry properties which allow them to be cast into much simpler forms. In the three-wave case, for example, some obvious symmetry properties are

$$D^\alpha = (D^{-\alpha})^* , \qquad D^{\alpha\beta\gamma} = (D^{-\alpha-\beta-\gamma})^* , \qquad D^{\alpha\beta\gamma} = D^{\alpha\gamma\beta} . \quad (2.42)$$

These follow from the definition of the D's , the requirement that the $\{q^i_\eta\}$ be real, and the fact that one may switch the indices β and γ at the beginning of the analysis without changing the resulting equations for Wave α . The corresponding symmetries for the higher order interaction coefficients may be deduced in the same manner.

In addition to the symmetries just mentioned, the equations for wave evolution, taken as a whole, often possess symmetries which are not apparent from the iterative analysis, when more than one mode of propagation is involved. These symmetries stem from the fact that for certain large classes of media, the nonlinear equations must be consistent with conservation or loss relations, which Penfield has labeled "frequency-power

formulas" in a book of the same name.[31] These formulas are of four types, which contain information about the transfer of either real or reactive power between modes of oscillation in either dissipative or nondissipative systems. The terms "real power," "reactive power," "dissipative" and "nondissipative," are given formal mathematical definitions by Penfield, and correspond essentially to the physical concepts suggested by the labels. Here, we shall discuss only one type of frequency-power formula; the type which is of the greatest utility in the literature of nonlinear interactions and in the Lagrangian theory of this work. Formulas of this type are called "Manley-Rowe relations" and were originally derived for nonlinear reactive circuits by J. M. Manley and H. E. Rowe in 1956.[32]

Penfield shows that, in generalized forms, these relations hold for any system described by an "energy-state function," and for systems which obey Hamilton's principle. The Lagrangian analysis of this work applies exclusively to distributed systems which obey Hamilton's principle, so the applicability of the Manley-Rowe relations follows. We shall now state the Manley-Rowe relations in the manner of Penfield. We consider a set of coupled oscillatory modes which have frequencies $\{\omega_\eta\}$, each of which is related to a set of J basis frequencies, $\{\omega_a : a = 1,...,J\}$ by the equation

$$\omega_\eta = \sum_{a=1}^{M} n_a^\eta \omega_a \quad , \qquad (2.43)$$

in which the $\{n_a^\eta\}$ are integers. The basis frequencies are incommensurable, i.e., no basis frequency, ω_a , is the sum of integral multiples of any other basis frequencies. Let the power input into each mode, η , be denoted by P_η ; then the Manley-Rowe relations between the $\{P_\eta\}$ are

$$\sum_\eta \frac{P_\eta}{\omega_\eta} n_a^\eta = 0 \qquad (a = 1,...J) \qquad . \qquad (2.44)$$

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There are J such relations, one for each basis frequency. When the interacting modes correspond to wavepackets, earlier investigations by the author and others[16,22,33] have indicated that the $\{P_\eta\}$ of (2.44) should be interpreted as power densities, expressed as

$$P_\eta = \left(\frac{\partial}{\partial t} \epsilon_\eta + \frac{\partial}{\partial x} \cdot \underset{\sim}{\phi}_\eta \right) , \qquad (2.45)$$

where ϵ_η is an energy density component, associated with Wave η, and $\underset{\sim}{\phi}_\eta$ is the corresponding energy flux density. These are related for propagating wavepackets by the group velocity $(\underset{\sim}{\phi}_\eta = \underset{\sim}{v}_\eta \epsilon_\eta)$.

For the three-wave case, characterized by the synchronism conditions (2.26), any two frequencies, say ω_β and ω_γ, may be chosen as the basis set. Using (2.44) − (2.45), we may express the Manley-Rowe relations for this case as

$$\frac{1}{\omega_\alpha} \left(\frac{\partial}{\partial t} \epsilon_\alpha + \frac{\partial}{\partial x} \cdot \phi_\alpha \right) = \frac{-1}{\omega_\beta} \left(\frac{\partial}{\partial t} \epsilon_\beta + \frac{\partial}{\partial x} \cdot \underset{\sim}{\phi}_\beta \right) = \frac{-1}{\omega_\gamma} \left(\frac{\partial}{\partial t} \epsilon_\gamma + \frac{\partial}{\partial x} \cdot \underset{\sim}{\phi}_\gamma \right) \quad (2.46)$$

Given the input power density at one frequency, we may use (2.46) to find the output at the other two.

In general, there is ambiguity over what physical parameters contribute to ϵ_η and $\underset{\sim}{\phi}_\eta$, and no one has yet established a set of conditions which are both necessary and sufficient for the Manley-Rowe relations to hold. Certain sufficient conditions are known, however, and (2.44) − (2.45) always apply to media which have the Lagrangian description defined in the next section.[22,31,33] In that section, the definitions of $\{\epsilon_\eta\}$ and $\{\underset{\sim}{\phi}_\eta\}$ are stated in terms of the Lagrangian formalism, and the relations are derived for the three-wave case.

In the iterative analysis, the Manley-Rowe relations must be derived from the specific form of the equations of motion, once these are obtained for a particular medium. To do so is a common practice, since it serves as a check on the calculation. To this author's knowledge, the generalization of the Manley-Rowe relations to include the effects of background perturbations $\{q_0^i\}$, has not been accomplished for a usefully large class

of continuous media, such as those having a Lagrangian description. Penfield's results for such media do not take into account the energy transfer to or from the background. They do provide a generalization of (2.46) to more than three propagating wave components and beats between them, but as we shall see in the examples of Section 4, the background perturbations are generally significant in comparison with the higher-order wave interaction effects. When this is the case, our power relations should account for the energy exchange with the background.

2.4.4 Explosive Instability

The preceding discussion has employed an interpretation of an energy component, ϵ_η , as the contribution of Wave η to the averaged energy density of the medium, i.e., the quantity ϵ_η is the amount by which the energy density of the medium would be increased by the excitation of Wave η , if all other wave and background parameters were to remain unchanged. This interpretation is in agreement with the more precise definition of the $\{\epsilon_\eta\}$ in terms of the Lagrangian formalism in Section 3, and it suffices for the present discussion of an instability phenomenon which is fundamentally dependent on nonlinear processes. In keeping with plasma terminology, we shall refer to this phenomenon as "explosive instability;" the instability can occur, as we shall demonstrate, only through the nonlinear coupling of positive energy waves with negative energy waves. In this work, a "positive energy wave" is any wave, η , such that $\epsilon_\eta > 0$; a "negative energy wave" is any wave, η , such that $\epsilon_\eta < 0$. A positive value of ϵ_η corresponds to the physically familiar case in which the excitation of Wave η to larger amplitudes requires energy input from some source, internal or external to the medium. However, for some media, the value of ϵ_η may be negative for a particular wave. This is possible, for example, in plasmas.[34] In such a case, the growth of the wave tends to lower the energy of the medium, so that, as the wave grows, the energy released by it must be absorbed by some internal or external sink. If no sink is available, the wave cannot grow, because total energy would not be conserved in the process.

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On the other hand, if the nonlinearities of the medium couple a negative energy wave to a positive energy wave, the latter may act as an energy sink, and, subject to other conditions which we shall discuss, explosive instability may result. In plasmas, early work on the explosive instability was done by Petschek[35] and Sturrock,[34] and the phenomenon has been dealt with in more detail in various recent publications.[36-41]

The implications of the Manley-Rowe relations depend dramatically on the energy parities of the interacting waves. Consequently, we shall employ the Manley-Rowe relations in order to describe more fully the nature of the explosive instability. For clarity, we shall consider the three-wave interaction, to which (2.46) applies. The integration of (2.46) over space and time shows that the Manley-Rowe relations constrain the total energy exchange between wavepackets as follows:

$$E_\alpha + E_\beta = \text{constant}, \qquad E_\alpha + E_\gamma = \text{constant} \qquad , \qquad (2.47)$$

where

$$E_\eta \equiv \int d^3x \; \epsilon_\eta \qquad (\eta = \alpha, \beta, \gamma) \qquad , \qquad (2.48)$$

i.e., the quantity E_η is the total energy perturbation associated with Wavepacket η. From (2.46) and the synchronism conditions, we may obtain the less restrictive relation

$$E_\alpha + E_\beta + E_\gamma = \text{constant} \qquad , \qquad (2.49)$$

which expresses the conservation of total wave energy in the absence of energy input from the background (through linear growth) or external sources.

We shall now divide all possible three-wave interactions into three mutually exclusive classes, and examine the implications of (2.47) — (2.49) for each class. To this end we define a set of energy parity factors, $\{\sigma_\eta\}$, defined by

$$\sigma_\eta \equiv \text{sign}(\epsilon_\eta) \qquad (\eta = \alpha, \beta, \gamma) \qquad . \qquad (2.50)$$

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The wave interactions will be classified according to the parity permutations, $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma)$, involved.

The interactions of Class A are defined to be those for which all three interacting wavepackets have the same energy parity. The parity permutations in this class are $(+,+,+)$ and $(-,-,-)$. For Class A interactions, the relations $(2.47) - (2.49)$ clearly show that the amount of energy which can be transferred from any wavepacket to the others, and vice-versa, is limited by the initial conditions.

The interactions of Class B are defined to be those for which the parities of the two lower frequency waves are different. The parity permutations in this class are $(+,+,-)$, $(+,-,+)$, $(-,+,-)$, and $(-,-,+)$. For Class B interactions, the conservation of total energy, (2.49), does not limit the magnitude of the energy exchange in any way. The Manley-Rowe equations (2.47) still impose a limit on the energy exchange, however, because the parity of the high frequency wave, σ_α , must match the parity of at least one of the other waves, σ_β or σ_γ .

The interactions of Class C are defined to be those for which the parities of the lower frequency waves are both opposite to the parity of the high frequency wave. The parity permutations in this class are $(+,-,-)$, and $(-,+,+)$. For Class C interactions, neither (2.47) nor (2.49) place any upper bounds on the magnitudes $|E_\alpha|$, $|E_\beta|$, or $|E_\gamma|$. This is the case of explosive instability, in which the amplitudes of all three waves grow as a result of the nonlinear interaction, until they are limited by other physical processes. In this class of interactions, the only constraints expressed by the Manley-Rowe relations involve the relative rates at which $|E_\alpha|$, $|E_\beta|$, and $|E_\gamma|$ may change.

This completes the analysis of parity effects on the Manley-Rowe relations for the three-wave case. For higher orders of wave interaction, similar arguments lead to other parity conditions which are necessary for the existence of the explosive instability.

The phenomenon of explosive instability is purely nonlinear in nature, since it relies on the existence of nonlinear wave coupling and it can occur for waves which would have no significant growth or damping in the linear limit. Of course, the parity conditions of Class C do not guarantee

that explosive instability will occur. They are merely part of a set of necessary conditions. When the waves are damped, or they are not in perfect synchronism, it is also necessary that the strength of the wave coupling exceed certain "threshold values" for the instability.[41]

2.5 Nonlinear Background Analysis

Once the wave evolution is described, M background equations involving the purely slowly varying components $\{q_0^i\}$ are needed to completely describe the system. In the small-signal limit these may be constants, they need not be of a wave nature. It is thus not generally possible to describe them in terms of linear modes, as in (2.23). However, some systems have branches which pass through the origin of the dispersion diagram, where

$$(\omega, \underset{\sim}{k}) = (0, \underset{\sim}{0}) \quad . \tag{2.51}$$

In this case the $\{q_0^i\}$ may be fields of a wavepacket, which has a Fourier spectrum localized about the origin. This is possible in the example of coupled ion-acoustic waves in Section 4. In contrast, electrostatic (Langmuir) oscillations in a plasma do not occur below the electron plasma frequency, ω_p . The background plasma potential, φ_0 , is therefore not a perturbed wave, even though it may be affected by Langmuir waves through nonlinear terms in the equations of motion.

Because of these considerations, we shall not attempt to modify an expansion process like (2.30) to describe the background evolution. Instead, we shall take the bar average of the differential equations (2.1), after employing (2.14) and substituting the small-signal mode structure (2.23) for the wave components, with the $\{\hat{a}_\eta\}$ allowed to vary in $\epsilon \underset{\sim}{x}$ and ϵt . The result is a system of M differential equations, involving the components $\{q_0^i , i=i,\ldots M\}$ and the $\{\hat{a}_\eta\}$. These equations are referred to in this work as the "equations for background evolution," or more simply as the "background equations." The evolution of the background components $\{q_0^i\}$ is determined by the wave components through these equations, and the interaction process will be referred to as a

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"wave-background interaction." The specific forms of the background equations will be developed for the particular problems considered, such as the examples of Section 4. The steps of the analysis to lowest order in the wave-background interaction are shown in the flow chart of Fig. 2.4. The synchronism concept introduced for the wave-wave interactions applies to the wave-background interactions too. To show this we shall consider two variants of (2.26):

$$(0,\underset{\sim}{Q}) = (\omega_\eta, \underset{\sim}{k}_\eta)_r + (\omega_{-\eta}, \underset{\sim}{k}_{-\eta})_r \quad , \tag{2.52}$$

and

$$(0,\underset{\sim}{Q}) = (\omega_\alpha, \underset{\sim}{k}_\alpha)_r + (\omega_{-\beta}, \underset{\sim}{k}_{-\beta})_r + (\omega_{-\gamma}, \underset{\sim}{k}_{-\gamma})_r \quad . \tag{2.53}$$

Equation (2.52) is satisfied identically for any real wave component $(q_\eta^i + q_{-\eta}^i)$ because of the constraints on the complex notation (see Fig. 2.2). Equation (2.53) is equivalent to the three-wave synchronism condition (2.26). The extension to combinations of more than three waves is obvious. Equations (2.52) and (2.53) show how products of complex quantities, like $q_\eta^i q_{-\eta}^j$ or $q_\alpha^i q_{-\beta}^j q_{-\gamma}^k$, may result in a quantity which has no fast scale variation. This is represented by the vector $(0,\underset{\sim}{Q})$. The resulting quantity, say p, may be written as $|p| \exp j\theta$ where θ is fixed. Although the slowly varying quantities represented by $(0,\underset{\sim}{Q})$ result from products of wave components, they are not necessarily wave-like perturbations. They are slowly varying quantities to be determined by the background equations of motion. Since these are obtained by a bar averaging process, the only combinations of wave components which can contribute to them are those which satisfy (2.52), (2.53), or their counterparts for higher-order effects. For the examples considered in the present work, the highest order nonlinear terms are quadratic in the $\{q^i\}$ so the wave-background effects are adequately characterized by (2.52).

EQ'S

PERTURBATION EXPANSION

| SLOW SCALE SOURCE TERM | + | LINEAR TERMS | + | LOWEST ORDER NONLINEAR TERMS | $\cong 0$ |

ISOLATE THE SLOW COMPONENT $(0,\underset{\sim}{0})$

S-S RELATIONS

$$\hat{q}^i_\eta = c^i_\eta \hat{a}_\eta$$

| $S^i(\underset{\sim}{v}, \underset{\sim}{ex}, \epsilon t)$ | + | $A^{ij}_{0p} q^j_0$ | + | $\displaystyle\sum_\eta A^{ijk}_{0p} \hat{q}^j_\eta \hat{q}^k_{-\eta}$ | $\cong 0$ |

EQUATIONS FOR BACKGROUND EVOLUTION

RANDOM-PHASE APPROXIMATION

BACKGROUND KINETIC EQUATIONS
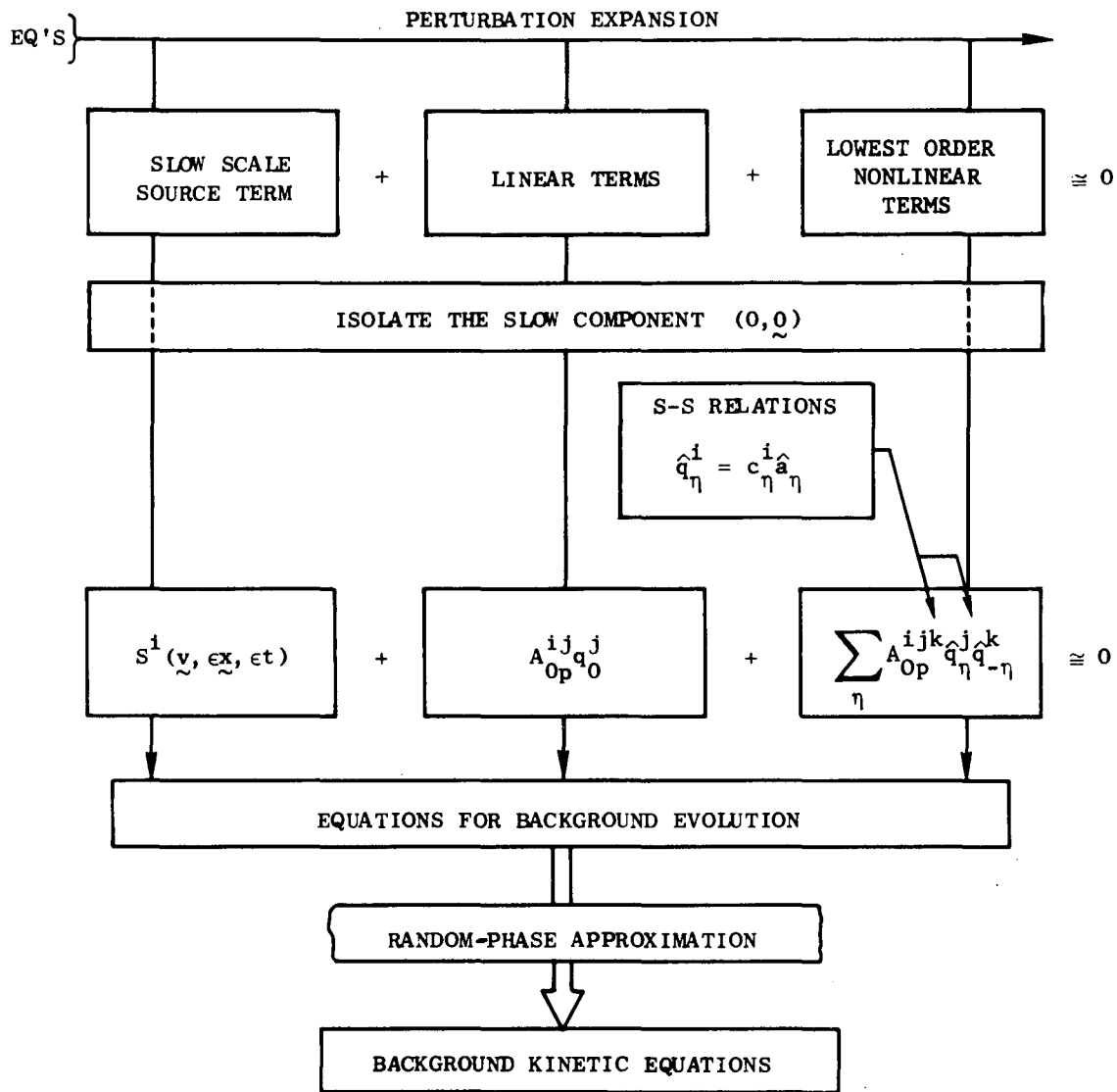
Fig. 2.4 Flow chart for the nonlinear background analysis. The
analysis is illustrated here for the quasilinear wave-
background interactions. The non-oscillatory $(0,\underset{\sim}{0})$ compo-
nents of the equations are isolated through the bar-
averaging process. The waves contribute individually to
the background equations through the nonlinear terms, since
the products of oscillatory quantities may have a non-zero
average. The S-S relations are used to reduce the number
of wave variables in the equations. In general, the appli-
cation of the RPA, to the equations for background evolu-
tion, yields the kinetic equations, which describe the
background evolution in the presence of a wave spectrum.

2.6 Scope and Complexity of the Conventional Approach

This completes the basic outline of the conventional approach. We have given a prescription for obtaining a complete set of coupled-mode and background equations which govern small perturbations from a static, homogeneous reference state. In the Lagrangian analysis of Section 3, additional important properties and useful standard forms of these equations will be discussed.

2.6.1 Scope of Application

The foregoing prescription can be extended to account for a weakly inhomogeneous reference state by adding terms to the expansion (2.23). One must search elsewhere[11] for justification of the process, however, since the Fourier analysis breaks down in that case. Weak inhomogeneity will be dealt with by way of the Lagrangian method in later sections, so it will not be discussed in more detail here. Among other cases which will be considered in the Lagrangian analysis are those in which the most convenient reference state is not a particular solution to the full nonlinear equations.

An advantage of the conventional method is that it can deal with dissipative media, such as collisional plasmas, so long as the equations are of the form (2.1). In such cases the Lagrangian formalism does not apply without further elaboration. This point is discussed further in Section 5.

2.6.2 Algebraic Complexity

Up to this point the discussion has concentrated on methodology and conceptual simplicity, and this has obscured the considerable algebraic complexity encountered in many applications of the iterative method. Often, too, this complexity is obscured in the nonlinear analyses to be found in the literature. There, except for the simplest problems, many algebraic steps are commonly omitted from the presentations; the results are simply stated.

Virtual waves obviously complicate the analysis of high-order wave interactions, and other major complications arise in the calculation of expressions like $D^{\alpha\beta\gamma\cdots}$. The complexity of this expression increases both with the number of superscripts, N , and with M , the number of scalar variables in the problem. If the coupling is between N modes and the variables are $\{q^i: i=1,\ldots M\}$, $D^{\alpha\beta\gamma\cdots}$ is calculated according to (2.29). The number of algebraic terms generated by that scalar equation is M^{N-1} , and these terms may themselves be associations of other terms. The need for simplification becomes quite evident in plasma theory where, as we shall see, the lowest order wave-wave coupling may be of the four-wave type, and seven scalar variables may be involved.

The complexities just discussed are in the algebra: the crude output of the method before cancellation of terms and manipulation results in a convenient form. In practice, it is common that after tedious manipulation the expressions for the wave coupling parameters become relatively simple. This is to be explained in part by the Manley-Rowe relations, which imply that the coupling equations have symmetry properties which are not obvious from the iterative analysis when more than one mode of wave propagation is involved. Another reason for the simplification is, as we shall see in the following sections, that many systems which have complicated descriptive equations may be described by Lagrangians of relatively simple algebraic forms.

Having established empirically that the results of the iterative method can often be recast into a much simpler algebraic form, it is natural that we should look for a method of obtaining a simplified set of equations more directly. Such a method is developed in the next section.

3. THE AVERAGED-LAGRANGIAN APPROACH TO
NONLINEAR WAVE INTERACTIONS

This section develops a formalism and a method for the analysis of nonlinear wave interactions in a continuous medium which can be described by a Lagrangian density, $\mathcal{L}$. Section 3.1 reviews Hamilton's principle and the corresponding classical Lagrangian mechanics for a continuous medium, and introduces terminology which is essential to the presentation of later sections. To apply this mechanics, it will prove convenient to employ an expansion of $\mathcal{L}$ about some reference state, and this section considers which choice of reference state will be best suited to the analysis. Conventionally, a Lagrangian density is associated with the equations of motion for the medium as a whole, but Section 3.2 shows how Lagrangian descriptions may be developed for separate interacting modes, each mode having its own set of variables. Here a "dual expansion" of $\mathcal{L}$ is introduced which is helpful in separating the study of fast-scale wave perturbations from that of slow-scale perturbations, which comprise part of the background in which the waves propagate.

The Lagrangian mechanics of wave interaction is used in an averaged-Lagrangian approach presented in Sections 3.3 through 3.5. There, the Lagrangian density is averaged and simplified by means of the small-signal mode structure obtained from the linear theory. This is done <u>before</u> Hamilton's principle is applied. The averaging technique has sometimes been referred to as Whitham's method, but it has substantial antecedents in the literature.[19] Since the papers of Whitham,[1-3] the method has been used and refined considerably by others.[20,21] This section follows to a considerable extent the approach of Dysthe.[21] However, his analysis is extended, to account for background evolution and the use of velocity as a coordinate. Section 3.6 summarizes the theoretical development. It systematizes the analytic steps in a form which efficiently yields the coupled-mode equations for a given type of interaction in a medium which has a Lagrangian expansion in small perturbations about a quasistationary and weakly inhomogeneous reference state.

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3.1 The Lagrangian Formalism

The Lagrangian approach presented here is restricted to systems which are described by a variational principle of the Hamilton type

$$\delta \int_{t_1}^{t_2} dt\ L = 0 \quad . \tag{3.1}$$

The integral in this equation is commonly referred to as the "action integral." In the present work, the variational principle will be modified from the form usually encountered in classical mechanics, in order to allow for the use of velocity as a coordinate. This is convenient for plasma analysis. For our purposes it is necessary that the Lagrangian, L , be expressed as an integral, over all positional and velocity coordinates, of a Lagrangian density, $\mathcal{L}$, i.e.

$$L = \int d^3x \int d^3v\ \mathcal{L}(q^i,\underset{\sim}{v},\underset{\sim}{x},t) \quad , \tag{3.2}$$

where $\{q^i(\underset{\sim}{v},\underset{\sim}{x},t),\ i=1,\ldots,M\}$ is a complete set of scalar generalized variables representing perturbations from a reference state, and $\mathcal{L}(q^i,\underset{\sim}{v},\underset{\sim}{x},t)$ is a suitably brief notation for the Lagrangian density, which is a function of the coordinates, $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t, the $\{q^i\}$, and their partial derivatives with respect to the coordinates. We stipulate that (3.1) applies to any region of the medium that is bounded in $\underline{x}$-$\underset{\sim}{v}$ space, provided that the δ-variations of the $\{q^i\}$ are such as to vanish at t_1, t_2 , and on the bounding surface. To obtain the equations of motion from Hamilton's principle, one makes a small variation δq^i, of one variable. Then

$$\delta \int_{t_1}^{t_2} dt\ L = \int_{t_1}^{t_2} dt \int d^3x \int d^3v \left\{ \frac{\partial \mathcal{L}}{\partial q^i}\ \delta q^i + \frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial t)}\ \frac{\partial}{\partial t}\ (\delta q^i) \right.$$

$$\left. + \sum_k \frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial x_k)}\ \frac{\partial}{\partial x_k}(\delta q^i) + \sum_k \frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial v_k)}\ \frac{\partial}{\partial v_k}\ (\delta q^i) + \ldots \right\} \quad . \tag{3.3}$$

The dots signify terms which would result from higher order derivatives of q^i in $\mathcal{L}$. Because of the restrictions on the delta variations, the right hand side of (3.3) may be integrated by parts, without the introduction of surface terms. Next, δq^i is taken outside the brackets. Then, since δq^i is arbitrary in shape within the region of integration, (3.1) can hold if, and only if, the quantity remaining inside the brackets is itself zero for all $\underline{v}$, all $\underline{x}$, and for all t within the region. This result is the Euler-Lagrange equation for q^i

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial t)}\right] - \sum_k \frac{d}{dx_k}\left[\frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial x_k)}\right] - \sum_k \frac{d}{dv_k}\left[\frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial v_k)}\right] + \ldots = 0.$$

$$(3.4)$$

The dots again indicate terms which arise if $\mathcal{L}$ is a function of higher order derivatives of q^i . If these are included, the Euler-Lagrange equations have the general form

$$\sum_{\substack{j,k,\ell,\\ m(j),n(k)}} (-1)^{\ell+m(j)+n(k)} \left(\frac{\partial}{\partial t}\right)^\ell \left(\frac{\partial}{\partial x_j}\right)^{m(j)} \left(\frac{\partial}{\partial v_k}\right)^{n(k)} \left\{\partial \mathcal{L}/\partial\left[\left(\frac{\partial}{\partial t}\right)^\ell \left(\frac{\partial}{\partial x_j}\right)^{m(j)} \left(\frac{\partial}{\partial v_k}\right)^{n(k)} q^i\right]\right\}$$
$$= 0 \quad , \qquad (3.5)$$

where ℓ, $m(j)$ and $n(k)$ are positive integers or zero, and j and k are of value 1, 2, or 3, to indicate the cartesian position or velocity axes. In the literature, it is often assumed that $\mathcal{L}$ contains only the $\{q^i\}$ and their first-order derivatives. Then the name "Euler-Lagrange equations" refers to (3.4) where the terms represented by dots are zero. However, in this investigation, the name Euler-Lagrange equations will refer to the complete equations (3.5), unless otherwise indicated.

3.1.1 Conservation of Generalized Energy

An outgrowth of the Lagrangian description is a conservation law for generalized energy. This relation is introduced here and used later in the interpretation of the wave interaction equations. For this presentation we shall assume that $\mathcal{L}$ involves only the $\{q^i\}$ and their first order derivatives.

When the medium is governed by the Lagrangian formalism one may define a corresponding Hamiltonian density. We shall be interested in this quantity only when it is expressed entirely in terms of the scalar variables, with no generalized momenta introduced. In this case it will be called the generalized energy density, $\mathcal{E}$, defined by

$$\mathcal{E}(q^i, \underset{\sim}{v}, \underset{\sim}{x}, t) \equiv \sum_i \frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial t)} \frac{\partial q^i}{\partial t} - \mathcal{L} \quad . \tag{3.6}$$

When $\mathcal{L}$ has its traditional dimensions of energy, so does $\mathcal{E}$. However, $\mathcal{L}$ is sometimes not proportional to ($\mathcal{T}-\mathcal{U}$), as pointed out in Section 1. In these instances, $\mathcal{E}$ is not equal to the total energy ($\mathcal{T}+\mathcal{U}$) associated with the perturbations. For this reason, Sturrock[34] has called $\mathcal{E}$ the "pseudo-energy" density.

Corresponding to $\mathcal{E}$, one may define a generalized energy flux, $\underset{\sim}{\mathcal{F}}$, in phase space. The flux vector is divided for convenience into spatial and velocity components, $\underset{\sim}{\mathcal{F}}^s$ and $\underset{\sim}{\mathcal{F}}^v$, respectively. The definitions are

$$\mathcal{F}_k^s \equiv \sum_i \frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial x_k)} \frac{\partial q^i}{\partial t} \quad , \qquad \mathcal{F}_k^v \equiv \sum_i \frac{\partial \mathcal{L}}{\partial(\partial q^i/\partial v_k)} \frac{\partial q^i}{\partial t} \quad . \tag{3.7}$$

From these and (3.4), it follows that $\mathcal{E}$ and $\underset{\sim}{\mathcal{F}}$ satisfy the conservation relation,

$$\frac{d\mathcal{E}}{dt} + \sum_k \frac{d}{dx_k} \mathcal{F}_k^s + \sum_k \frac{\partial}{\partial v_k} \mathcal{F}_k^v = - \frac{\partial \mathcal{L}}{\partial t} \quad , \tag{3.8}$$

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in which $\partial/\partial t$ refers to explicit time dependence. If this result is applied to an $\mathcal{L}$ which has no explicit time dependence, the right-hand side vanishes, leaving a conservation relation in terms of the total perturbations $\{q^i\}$.

The result (3.8) is one part of a stress-energy-momentum tensor relation of Lagrangian mechanics.[42] This contains similar equations involving the rate of change of generalized momenta, but they are not essential to this discussion. Sturrock[34,43] has discussed certain properties of the full tensor relation applied to waves in the small-signal limit.

3.1.2 Construction of the Lagrangian Density

The well-established approach for constructing an appropriate Lagrangian density is to find L for a system of discrete elements and then take the limit of a continuum. The classical prescription for finding L is restricted to mechanical systems having holonomic constraints, which are eliminated by transformation to a set of independent generalized variables, $\{q^i_j : i=1,...M\}$, where j refers to a particular particle or body.[44] After the transformation, the only independent variable is the time, t . The state of a body j at a particular instant of time is described completely by the dependent scalar variables, $\{q^i_j\}$. The force on a body, $\underset{\sim}{F}_j$, defines a generalized force, $\underset{\sim}{G}_j$, according to

$$G^i_j = \sum_k F_{jk}\left(\partial x_k/\partial q^i_j\right) \qquad (k = 1,2,3, \ i=1,...,M) \qquad . \qquad (3.9)$$

As defined here, $\underset{\sim}{G}_j$ is a vector with M scalar components, $\{G^i_j\}$. If the $\{G^i_j\}$ can be obtained from some velocity-dependent potential function $U_j(q^i_j, \partial q^i_j/\partial t)$ according to

$$G^i_j = -\frac{\partial U}{\partial q^i} + \frac{d}{dt}\left[\frac{\partial U}{\partial\left(\partial q^i_j/\partial t\right)}\right] \qquad , \qquad (3.10)$$

then L for the entire system is given by

$$L = \sum_j T_j - U \quad , \tag{3.11}$$

where T_j represents the kinetic energy of particle j .

The next step in the description of a continuum is to convert the summation over particles into an integral over coordinate or phase space by introducing a density function $n(\underset{\sim}{x},t)$ or $f(\underset{\sim}{x},\underset{\sim}{v},t)$. In the latter case, the result is a Lagrangian of the form (3.2), where the $\{q_j^i\}$ are replaced by $q^i(\underset{\sim}{x},\underset{\sim}{v},t)$. In this scheme, a particle j is identified by its location in phase space. Thus the classical prescription for a continuous medium is

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \quad . \tag{3.12}$$

The kinetic energy density is $\mathcal{T}$, and the potential energy density is $\mathcal{U}$.

Unfortunately, for many non-mechanical systems of interest, it is not clear which part of the energy of the system is to be classified as kinetic, and which as potential. Nevertheless, Lagrangians exist for many such systems. An example is a network of inductors and capacitors. For such a system a valid Lagrangian can always be obtained through (3.11) if the energy stored in the inductors is identified with T , and the energy stored in the capacitors is identified with U . The generalized variables in such a system are $\left\{ Q_j^i \equiv \int_0^t I_j^i dt \right\}$, where i denotes a particular current loop of the circuit and j denotes a particular circuit element. In this case L is easy to find because an L-C network is analogous to a mechanical system of masses and springs, each having one degree of freedom. The governing equations have the same form if we make the identification $Q_j^i \to$ displacement component, ζ_j^i ; $I_j^i \to$ velocity component, $\partial \zeta_j^i / \partial t$; $V_j^i \to$ force component, F_j^i ; $L_j \to$ mass, M_j ; and $1/C_j \to$ spring constant, S_j . An L-C network and its mechanical analog are shown in Fig. 3.1.
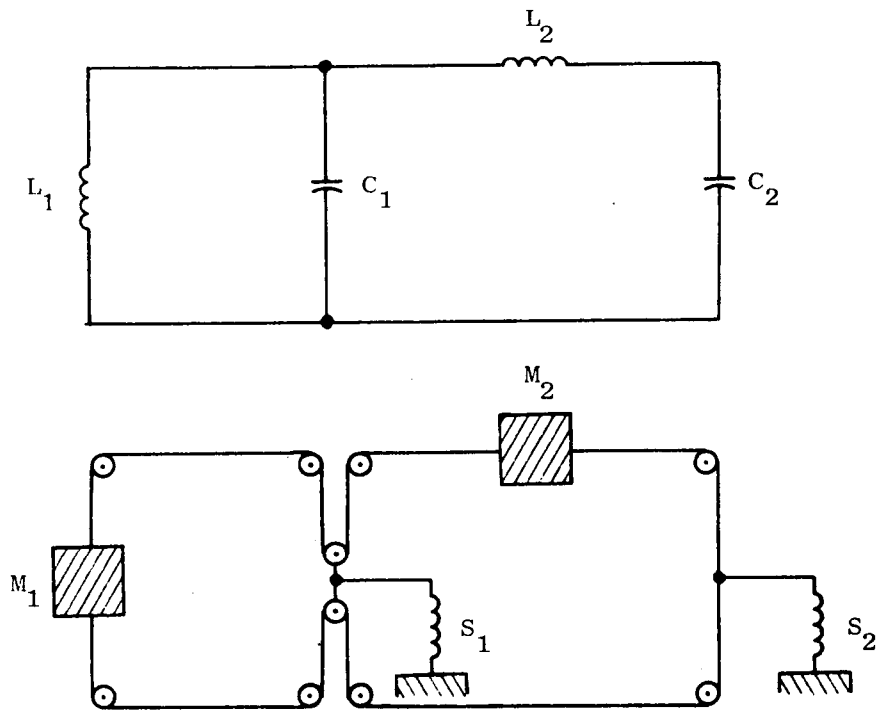
Fig. 3.1 Analogous electrical and mechanical networks. The Lagrangian for the electrical network is obtained by comparison with the mechanical network, for which we may employ the classical prescription L = (kinetic energy) - (potential energy)

Other Lagrangians, which are less firmly based on analogies, are to be found in the Lagrangian descriptions for fluids and plasmas where both the particle equations of motion and Maxwell's field equations are to be derived from $\mathcal{L}$. The approach commonly found in the literature, and which is used in this investigation as well, is to obtain a Lagrangian heuristically, either by guesswork or loose analogy with (3.11) or (3.12). The result is then verified by a demonstration that it yields the correct equations of motion. The present investigation, and a large body of other work, show that the Lagrangian formulation may be very useful in a direct sense, as opposed to being a source of equations of motion for a system. It would be useful to develop a method for determining when and how L may be constructed from a particular system of equations. The problem has received some attention in the literature, for instance by Penfield and Haus,[45] but it is by no means solved. The problem will be discussed further in Section 5.

3.1.3 Expansion of the Lagrangian

In order to proceed further, we shall consider the medium only in the limit of small perturbations from some reference state. This may, for example, be described by some set of generalized variables $\{q_r^i(\underset{\sim}{x},\underset{\sim}{v},t)\}$, in which case $\mathcal{L}$ may be expanded about this state by the substitution

$$q^i = q_r^i + q_\delta^i \qquad (i=1,\ldots,M) \qquad . \qquad (3.13)$$

Here the perturbations are the $\{q_\delta^i\}$. In our later plasma problems, the reference state may be described by a velocity distribution function and appropriate φ and $\underset{\sim}{A}$ fields. In these cases, the perturbation assumption is incorporated into the derivation of $\mathcal{L}$, so that the $\{q^i\}$ in (3.2) automatically represent perturbations from the reference state. Henceforth the notation will be adjusted so that the $\{q^i\}$ will always denote perturbed generalized variables. Together with the reference state, they completely describe the state of the system. The total Lagrangian density may be expanded about the reference state in joint

powers of the perturbations:

$$\mathcal{L} = \mathcal{L}_{(0)} + \mathcal{L}_{(1)} + \mathcal{L}_{(2)} + \mathcal{L}_{(3)} + \ldots \qquad . \qquad (3.14)$$

In this notation, a given term, $\mathcal{L}_{(m)}$, contains all terms in the expansion which are products of m factors, each of which is one of the $\{q^i\}$ or their various derivatives. In other words, $\mathcal{L}_{(m)}$ is of "joint power m in the $\{q^i\}$." Since the perturbations are small, it is possible to obtain an approximate form for $\mathcal{L}$ by terminating the expansion at the desired order.

In this investigation, the expansion will always be made about a reference state which is nearly homogeneous in $\underset{\sim}{x}$ and t. This means that the explicit space and time fluctuations of $\mathcal{L}$ are very small if measured over the oscillation period of any wave disturbances in the medium. The fast-scale wave components are implicit in the $\{q^i\}$. An equivalent interpretation is that the explicit $\underset{\sim}{x}$ and t dependences of $\mathcal{L}$ are on scales which are slow compared to those of the wave oscillations. In keeping with the notation of Section 2.1, the slow-scale dependences will be identified by a factor ϵ, as in $\mathcal{L}(q^i, \underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t)$.

Now let us consider the effects of the various terms $\{\mathcal{L}_{(m)}\}$ on the Euler-Lagrange equations of motion. The effects are illustrated in Fig. 3.2, which shows that each functional in the expansion contributes terms of a particular type to the left hand side of (3.5). Since $\mathcal{L}_{(0)}$ contains only reference state parameters it clearly cannot contribute to the equations of motion for any particular variable q^i. The term $\mathcal{L}_{(1)}$ is of first order in the $\{q^i\}$, so its effect will be to contribute a source term, $S^i(\underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t)$, which has a purely explicit dependence on $\underset{\sim}{v}, \underset{\sim}{x}$, and t; it does not involve the $\{q^i\}$. The functional $\mathcal{L}_{(2)}$ generates a linear differential function involving the $\{q^i\}$, while the terms $\mathcal{L}_{(3)}$ and higher contribute nonlinear differential functions of the variables. From the order of the corresponding nonlinear functions it is clear that the term $\mathcal{L}_{(3)}$ describes the three-wave interactions, and that the expansion of $\mathcal{L}$ through $\mathcal{L}_{(N)}$ must be used
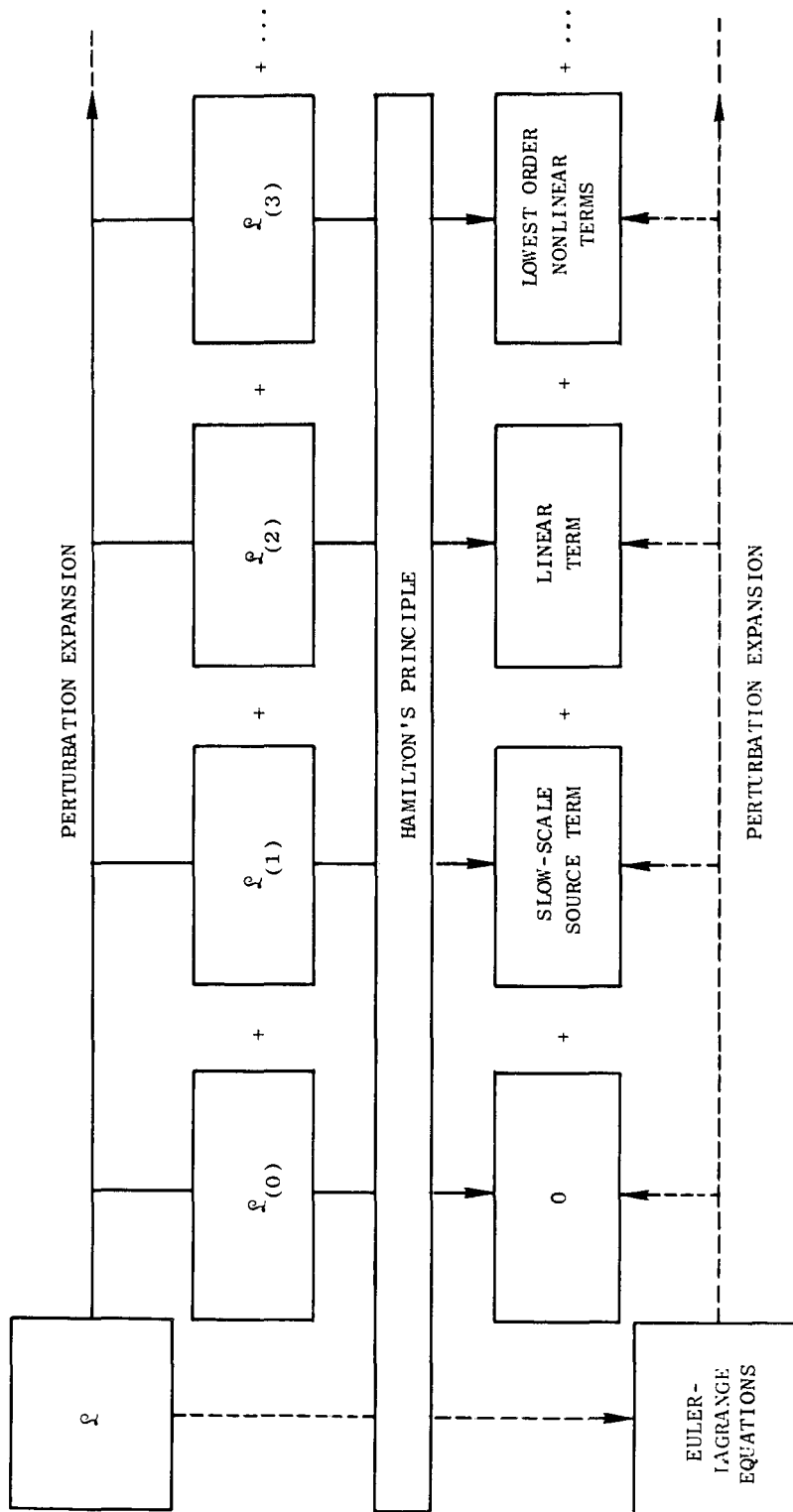
Fig. 3.2   Perturbation expansion of the Lagrangian density, and the nonlinear equations of motion. The expansions are about a reference state which is nearly homogeneous in space and time. Through Hamilton's principle, each term in the expansion of $\mathcal{L}$ generates a particular class of terms in the equations of motion. The Lagrangian perturbation description is indicated by solid lines, the direct expansion of the nonlinear equations is indicated by dashed lines.

to describe interactions of N waves. In all, the expansion of $\mathcal{L}$ generates M Euler-Lagrange equations, which have the same general form as (2.1). They are

$$S^i + A_{Op}^{ij} q^j + \text{higher order terms} = 0 \quad (i,j=1,\ldots M) \quad . \quad (3.15)$$

As in Section 2, we are interested only in systems for which the descriptive equations (3.15) have solutions which involve propagating waves; we assume that

$$q^i = q_0^i + q_w^i \quad \left(q_w^i = \sum_\eta q_\eta^i\right) \quad , \quad (3.16)$$

where the $\{q_0^i(\underline{y},\epsilon\underline{x},\epsilon t)\}$ denote background perturbation components, and the components associated with a particular wave, η , are denoted by $\{q_\eta^i\}$. The complex notation of (2.15) – (2.16) will be employed to describe these components, and their small-signal mode structure. In solving (3.15), we shall make the weak-coupling approximation, in which the mode parameters $\{\hat{q}_\eta^i\}$, $\{\omega_\eta\}$, and $\{\underline{k}_\eta\}$, evolve on the slow scale, with $\epsilon\underline{x}$ and ϵt . So do the slow-scale perturbations, $\{q_0^i\}$, which, in the linear limit, constitute a particular solution to the inhomogeneous problem (3.15).

3.1.4 Choice of Reference State

It has already been stipulated that the reference states used in the perturbation expansion of $\mathcal{L}$ must be nearly homogeneous in $\underline{x}$ and t . However, the analysis of nonlinear effects may be simplified considerably when the reference states meet certain additional requirements. In the present work, the reference states used are of two types; their characteristics are outlined below.

The first type of reference state we shall consider is an approximate solution to the full nonlinear descriptive equations for the medium. If this state is very close to an exact solution, the perturbations may be small enough that the higher order (nonlinear) terms in (3.15) may be

ignored. This leaves a system of inhomogeneous linear differential equations, which are equivalent in form to (2.13). A conventional frequency analysis of the equations of motion shows that the slowly fluctuating source term, s^i , cannot excite the fast-scale wave disturbances, $\{q^i_\eta\}$. Therefore, when we are interested only in the evolution of the waves in the presence of nonlinear effects, the term s^i may be dropped in (2.1), and the expansion for $\mathcal{L}$ may be simplified to

$$\mathcal{L} = \mathcal{L}_{(2)} + \mathcal{L}_{(3)} + \cdots \quad . \tag{3.17}$$

However, since the reference state is only an approximate solution, (3.17) is not necessarily a valid approximation for the description of the slow-scale perturbation components, $\{q^i_0\}$.

To complete the perturbation analysis, one must describe the non-linear evolution of the slow-scale perturbations from some known state. For this, it is convenient to pick a reference state which is an exact solution to the full set of nonlinear equations. This may be an equilibrium state, or it may involve slow-scale time dependence. In either case it is physically possible for all the perturbations to approach zero: $q^i \to 0$ $(i=1,\ldots M)$. Then it is clear from (2.9) that each s^i is zero $(i=1,\ldots M)$, and that $\mathcal{L}_1$ has no effect whatsoever on the equations which describe the perturbations from this reference state. Again, the approximation (3.17) is justified, but this time for the complete perturbations, including the slow components.

3.2 The Lagrangian Mechanics of Wave Interaction

Up to now, the equations of motion have been studied for the total perturbation variables, $\{q^i\}$. For convenience in analysis, these have been divided into components $\{q^i_0\}$ and $\{q^i_\eta\}$ in accordance with (3.16). However, under certain conditions, it is possible to treat these components as variables in their own right. Then, when they are varied separately in accordance with Hamilton's principle, $\mathcal{L}$ yields a separate equation of motion for each component.

We shall consider the case of $N+1$ components, N of them waves. Their sum is the total perturbation from the reference state, which we shall require to be an exact solution to the equations of motion. Hamilton's principle holds for an infinitesimal variation of q^i which is of arbitrary shape within the interval $t_1 < t < t_2$, but which is zero at t_1 and t_2. By definition, Hamilton's principle must also apply to any subset of these allowable variations. Let us consider those in which we vary only one component, q_μ^i, of the expansion

$$q^i = \sum_\mu q_\mu^i \qquad (i=1,\ldots,M; \; \mu=0,\ldots,N \text{ or } 0,\alpha,\beta,\gamma,\ldots) \qquad . \quad (3.18)$$

The subscript zero corresponds to the slow (or background) component, any other value denotes a particular mode of oscillation. The wave subscripts may be integers one through N or one of the series α, β, γ, ... when convenient.

Under the variation of q_μ^i, Hamilton's principle yields an equation of motion for that component, in which the other components play the role of coefficients. In the Lagrangian, the $\{q_\nu^i : \nu \neq \mu\}$ contribute to the explicit dependence on $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t. To indicate this, the Lagrangian density appropriate to the variation of q_μ^i will be denoted by $\mathcal{L}^\mu$, where

$$\mathcal{L}^\mu\left(q_\mu^i, \underset{\sim}{v}, \underset{\sim}{x}, t\right) \equiv \mathcal{L}\left(\left[q_\mu^i + \sum_{\nu \neq \mu} q_\nu^i(\underset{\sim}{v},\underset{\sim}{x},t)\right], \underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t\right) \qquad . \quad (3.19)$$

The algebraic expression for $\mathcal{L}^\mu$ is obtained by making the substitution $q^i \to q_\mu^i + \sum_{\nu \neq \mu} q_\nu^i(\underset{\sim}{v},\underset{\sim}{x},t)$ in the perturbation expansion of $\mathcal{L}$. When q_μ^i is varied in this expression, the other components are regarded as given functions of $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t. Thus, $\mathcal{L}^\mu$ takes on additional explicit dependence on the coordinates. Clearly, some terms in the expansion of $\mathcal{L}^\mu$ do not contain any elements of $\{q_\mu^i : i=1,\ldots,M\}$. They have no effect on the equations of motion for component μ, and may be omitted for purposes of calculation.

At this point, two courses of action are available for the application of Hamilton's principle. One is to make an arbitrary variation of each component q_μ^i , without previously making any assumptions about the form of the perturbation components. The other procedure is to use the assumed form of the components, (2.15), to simplify the action integral of (3.1) at the outset. The variation of q_μ^i must then be restricted, to be consistent with those assumptions. The first approach will be outlined briefly below for the insight it provides into Lagrangian Mechanics. It has been applied in earlier work[22-24] to obtain an energy-transfer relation for three-wave interactions. The second procedure will be discussed in the remainder of this section. It forms the basis of the primary result of this investigation: the averaged-Lagrangian method for obtaining the equations for wave and background evolution.

If an arbitrary variation is made in q_μ^i , Hamilton's principle yields an Euler-Lagrange equation such as (3.5), with q^i replaced by q_μ^i , and with $\mathcal{L}(q^i, \underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t)$ and its expansion replaced by

$$\mathcal{L}^\mu \equiv \sum_m \mathcal{L}_{(m)}^\mu \quad , \qquad \mathcal{L}_{(m)}^\mu \equiv \mathcal{L}_{(m)} \left(\left[q_\mu^i + \sum_{\nu \neq \mu} q_\nu^i(\underset{\sim}{v}, \underset{\sim}{x}, t) \right], \ \underset{\sim}{v}, \ \epsilon \underset{\sim}{x}, \ \epsilon t \right) \quad ,$$

$$(i=1,\ldots,M; \ \mu,\nu = 0,\ldots,N \ \text{ or } \ 0,\alpha,\beta,\gamma,\ldots) \quad . \qquad (3.20)$$

When q_μ^i is varied, the other components are regarded as given functions of $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t . Thus, $\mathcal{L}^\mu$ takes on additional explicit dependence on the coordinates. In this scheme of analysis, the number of variables is $(N+1)M$, one for each permutation (i,μ) . The Euler-Lagrange equation for q_μ^i contains less information than (3.5), which illustrates a rule of variational calculus: the result of considering a more restrictive class of variations is always a loss of information.[46] However, the process may be repeated for each of the $\{q_\mu^i\}$, thus producing a system of $(N+1)M$ equations of motion. These completely describe the physical system.

In making the variation δq^i_μ , we dropped the assumption that q^i_μ corresponds to a single background or wave component of the form specified in (2.15). However, it is permissible to introduce such assumptions later in the Euler-Lagrange equations of motion, if this leads to a simultaneous solution of the equations. This procedure has been applied in earlier work by the author to the interactions of three monochromatic waves, with background perturbations and virtual waves neglected. One very useful result is an energy transfer relation, closely related to the conservation relation for generalized energy (3.8). It involves energy components associated with each wave, and leads to the set of coupled-mode equations for the three-wave problem. The same energy-transfer relation will be derived later in this investigation by the more versatile averaged-Lagrangian approach.

3.3   The Averaged-Lagrangian Approach

The averaged-Lagrangian approach is based on a specialized form of Hamilton's principle which we shall call the "averaged Hamilton's principle." In this subsection, the averaged Hamilton's principle is introduced, and a formalism is developed which facilitates the application of the principle to nonlinear wave analysis.

3.3.1   Restricted Variation of a Perturbation Component

If we assume in evaluating the action integral that the perturbation components correspond to various frequencies of oscillation, the variations applied in Hamilton's principle must be restricted to be consistent with that assumption. We shall stipulate that the $\{\hat{q}^i_\eta\}$, $\{\omega_\eta\}$ and $\{\underset{\sim}{k}_\eta\}$ of (2.15) and (2.16) are parameters of slowly evolving waves, i.e., they are functions of $\epsilon \underset{\sim}{x}$ and ϵt . Therefore, the allowable variations, $\overline{\delta} q^i_\eta$, are those produced by a slow-scale variation of either complex amplitude $\overline{\delta}\hat{q}^i_\eta$ or $\overline{\delta}\hat{q}^i_{-\eta}$, or a variation $\overline{\delta}\theta_\eta$ in the phase which involves small, slow-scale changes in the frequency and the wavevector. The equations of motion are obtained from Hamilton's principle. One finds the algebraic form of the perturbation of the action integral, and infers the conditions on $\hat{q}^i_\eta$, $\hat{q}^i_{-\eta}$, and θ_η which are necessary to make the

perturbation identically zero. Such an approach will be used here.

The assumed form of the components, with its separation of fast and slowly fluctuating quantities, implies that

$$\int dt \int d^3x \int d^3v \; \mathcal{L} = \int dt \int d^3x \int d^3v \; \overline{\mathcal{L}} \quad . \tag{3.21}$$

This fact, together with (3.1), implies the "averaged Hamilton's principle,"

$$\overline{\delta} \int_{t_1}^{t_2} dt \int d\underset{\sim}{x} \int d\underset{\sim}{v} \; \overline{\mathcal{L}} = 0 \quad . \tag{3.22}$$

The variation symbol $\overline{\delta}$ denotes variations $\{\overline{\delta}q^i_\mu : \mu = 0, \ldots, N\}$. They are constrained in such a way as to be consistent with the assumptions used to evaluate $\overline{\mathcal{L}}$, i.e., only slow-scale variations in the $\{q^i_0\}$, $\{\hat{q}^i_\eta\}$, and $\{\theta_\eta\}$ are allowed. Since this constitutes a more restrictive class of variations than those allowed in the exact Hamilton's principle, one might suspect that information is lost in the averaging process. However, this is just the information which has been incorporated into the assumed form of the solution (2.15) and, as long as that form is valid, the averaged Hamilton's principle will suffice to determine the additional information needed: a set of equations for the $\{\overline{q}^i_\mu(\underset{\sim}{v},\epsilon\underset{\sim}{x},\epsilon t)\}$.

To illustrate the implications of (3.22), consider problems in which $\overline{\mathcal{L}}$ involves only the $\{\hat{q}^i_\mu\}$, $\{\theta_\eta\}$, and their first derivatives in $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t . The equations of motion then take the form of Euler-Lagrange equations, in which the variables are the $\{q^i_0\}$, the $\{\hat{q}^i_\eta\}$, and the $\{\theta_\eta\}$. The equations for the wave components are

$$\frac{\partial}{\partial t}\left[\frac{\partial\overline{\mathcal{L}}}{\partial(\partial\hat{q}_\eta/\partial t)}\right] + \sum_k\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial\overline{\mathcal{L}}}{\partial(\partial\hat{q}^i_\eta/\partial x_k)}\right] + \frac{\partial}{\partial v_k}\left[\frac{\partial\overline{\mathcal{L}}}{\partial(\partial\hat{q}^i_\eta/\partial v_k)}\right]\right\} - \frac{\partial\overline{\mathcal{L}}}{\partial\hat{q}^i_\eta} = 0 \quad ,$$

$$\tag{3.23}$$

F

and

$$\frac{\partial}{\partial t}\left(\frac{\partial \bar{\mathcal{L}}}{\partial \omega_\eta}\right) - \sum_k \frac{\partial}{\partial x_k}\left(\frac{\partial \bar{\mathcal{L}}}{\partial k_{\eta k}}\right) - \frac{\partial \bar{\mathcal{L}}}{\partial \theta_\eta} = 0 \quad , \qquad (3.24)$$

where we have made use of equation (2.16). The Euler-Lagrange equations for the background components are simply

$$\frac{\partial}{\partial t}\left[\frac{\partial \bar{\mathcal{L}}}{\partial(\partial q_0^i/\partial t)}\right] + \sum_k \left\{\frac{\partial}{\partial x_k}\left[\frac{\partial \bar{\mathcal{L}}}{\partial(\partial q_0^i/\partial x_k)}\right] + \frac{\partial}{\partial v_k}\left[\frac{\partial \bar{\mathcal{L}}}{\partial(\partial q_0^i/\partial v_k)}\right]\right\} - \frac{\partial \bar{\mathcal{L}}}{\partial q_0^i} = 0 \quad .$$

$$(3.25)$$

If $\bar{\mathcal{L}}$ involves higher order derivatives of the variables, the equations of motion are more complicated, but they are still obtained from (3.22).

3.3.2   The Dual Expansion of $\mathcal{L}$

It is apparent that the wave components, $\{q_\eta^i\}$, are qualitatively different from the background perturbation components, $\{q_0^i\}$; the former are related by a mode structure in the small-signal limit, the latter are not. Because of this, different techniques will be used in the present investigation to obtain the equations of wave and background evolution. The simple perturbation expansion of $\mathcal{L}$ in (3.14) is not well-suited to this purpose; it is more convenient to order the wave and background components separately in a dual expansion of $\mathcal{L}$. The benefits of this procedure are made obvious by the subsequent applications.

To make the dual expansion, substitute $(q_0^i + q_w^i)$ for q^i in the Lagrangian expansion (3.14), and reorder the terms into the series

$$\mathcal{L} = \sum_m \mathcal{L}_m(q_w^i, q_0^i, \underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t) \quad , \qquad (3.26)$$

where a term $\mathcal{L}_m$ is of joint power m in the wave components $\{q_w^i\}$ and their various derivatives, i.e., each term of $\mathcal{L}_m$ may be written as the

product of m factors, each of which is proportional to one of the $\{q_w^i\}$ or their various derivatives. Now expand each term $\mathcal{L}_m$ in joint powers of the $\{q_0^i\}$ and their derivatives. The result must have the form

$$\mathcal{L}_m = \sum_n \mathcal{L}_{m,n}(q_w^i, q_0^i, \underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t) \quad , \tag{3.27}$$

where the index n denotes the order of the term in joint powers of the $\{q_0^i\}$. The dual expansion for $\mathcal{L}$ is thus

$$\mathcal{L} = \sum_m \mathcal{L}_m = \sum_m \sum_n \mathcal{L}_{m,n} \quad . \tag{3.28}$$

The analysis of mode interaction is to be based on the averaged Hamilton's principle, so only the bar average of the dual expansion will be used. This averaged dual expansion is relatively simple because all oscillatory components are dropped from each term of the dual expansion. Under the bar average, some of the $\{\mathcal{L}_{m,n}\}$ vanish, and others affect the equations of motion only for certain components. Neither the wave nor the background components are affected by $\overline{\mathcal{L}}_{00}$, since this term contains only reference state parameters. The terms $\overline{\mathcal{L}}_{0,n}$ $(n \geq 1)$ affect only the background evolution, since they involve only the $\{q_0^i\}$. Terms $\overline{\mathcal{L}}_{1,n}$ $(n \geq 0)$ vanish identically, since they are first order in the wave perturbations. They can affect neither the wave nor the background evolution.

The foregoing considerations show that, for the application of the averaged Hamilton's principle to the wave components, the expansion of $\overline{\mathcal{L}}$ may be simplified to

$$\overline{\mathcal{L}}^w = \sum_{m=2}^{\infty} \sum_{n=0}^{\infty} \overline{\mathcal{L}}_{m,n} \quad , \tag{3.29}$$

in which the superscript w indicates that only the wave components $\{q_\eta^i\}$ are to be regarded as variables. In this case, the $\{q_0^i\}$ are treated as given functions of $\underset{\sim}{v}, \underset{\sim}{x},$ and t .

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For the variation of the background components, the equivalent expansion is

$$\overline{\mathcal{L}}^b = \sum_{n=1}^{\infty} \mathcal{L}_{0,n} + \sum_{m=2}^{\infty} \sum_{n=1}^{\infty} \overline{\mathcal{L}}_{m,n} \quad , \qquad (3.30)$$

in which the superscript b indicates that only the background perturbation components, $\{q_0^i\}$, are to be regarded as variables, and the $\{q_\eta^i\}$ are to be treated as given functions of $\underset{\sim}{v}$, $\underset{\sim}{x}$, and t .

An inspection of the algebraic form of the $\{\overline{\mathcal{L}}_{m,n}\}$ reveals that, through the averaged Hamilton's principle, each term in the dual expansion generates a particular class of terms in the equations of wave and background evolution. The relationships are shown in Fig. 3.3, and they will be described below. Their origins will become readily apparent later in the work, when procedures for obtaining the equations are established and applied.

The top line of Fig. 3.3 shows the equations of background evolution, which are obtained through the variation of the $\{q_0^i\}$ in the averaged-Hamilton's principle. All terms of these equations involve the $\{q_0^i\}$ and the reference state parameters; they differ in the extent to which they involve the wave components, $\{q_\eta^i\}$. The background terms do not involve the $\{q_\eta^i\}$ at all, and would thus describe the evolution of the background in the absence of waves. The quasilinear terms describes the effects of individual waves on the $\{q_0^i\}$. These terms involve quadratic products, $\hat{q}_\eta^i \hat{q}_{-\eta}^j$ of the components associated with each wave, η . They are called "quasilinear" because the effects they describe can take place even when the waves propagate without wave-wave coupling, according to the linear theory. The nonlinear W-B (wave-background) interaction terms, generated by $\overline{\mathcal{L}}_3$, $\overline{\mathcal{L}}_4$, etc., describe the collective nonlinear effects of the waves on the background. They involve the wave components through products of the form $\hat{q}_{-\alpha}^i \hat{q}_\beta^j \hat{q}_\gamma^k \cdots$, where $\alpha\beta\gamma\ldots$ is a synchronous mode combination.

The bottom line of Fig. 3.3 shows the equations of wave evolution, which are obtained through the variation of the $\{\hat{q}_\eta^i\}$ in the averaged-Hamilton's principle. The quasilinear terms describe the propagation of

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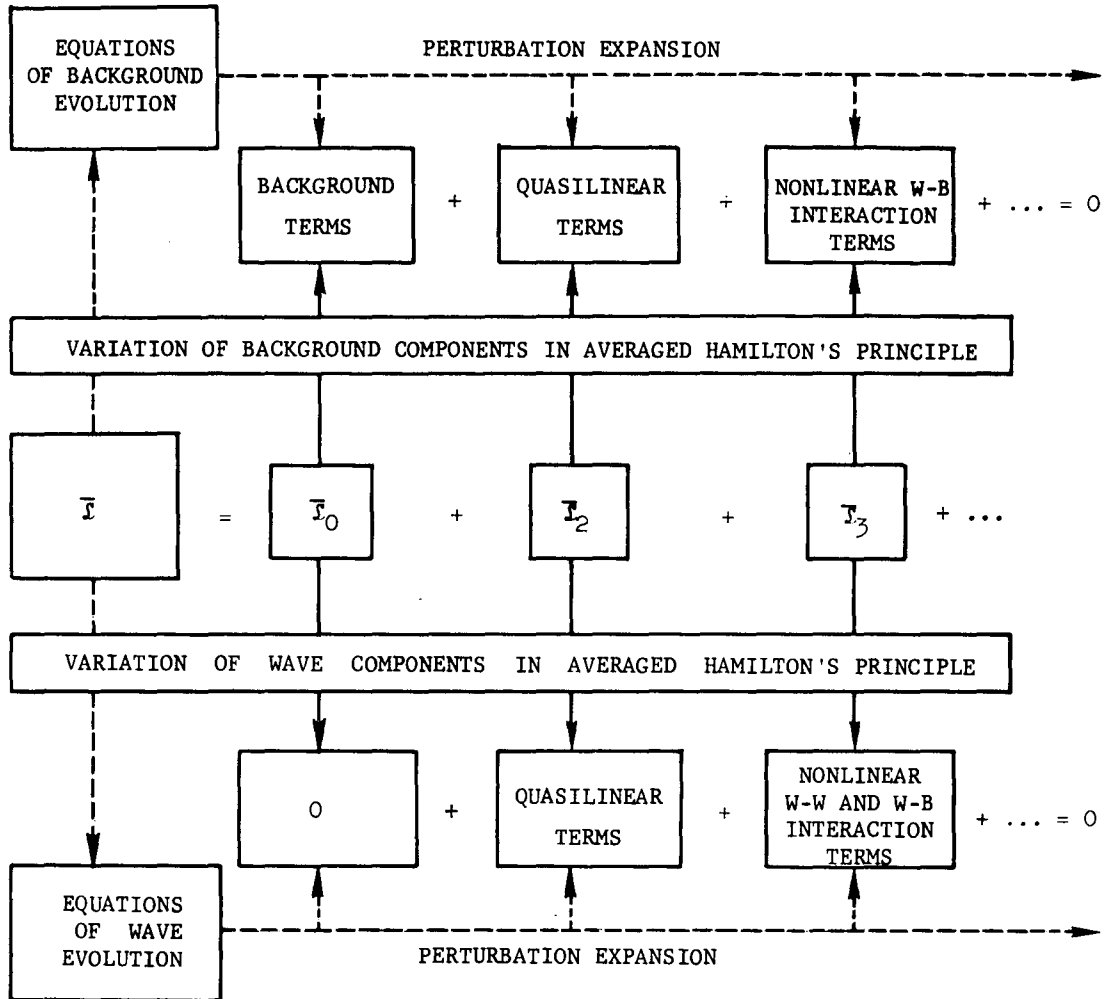
Fig. 3.3 Perturbation expansion of the averaged Lagrangian density, and the interaction equations. The primary expansion, in joint powers of the wave components, is shown. Each term in the primary expansion has a secondary expansion, in joint powers of the background (non-oscillatory) perturbation components. Through the averaged Hamilton's principle, each term in the primary expansion of $\bar{\mathcal{L}}$ generates a particular class of terms in the wave and background equations. The Lagrangian perturbation analysis is indicated by solid lines; the direct expansion of the equations is indicated by dashed lines.

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individual wave components in the absence of wave-wave coupling effects; they account for any wave growth or damping which results from the characteristics of the background state. These terms are labeled "quasilinear" because, although they are linear in the $\{\hat{q}_\eta^i\}$, they also involve the variables $\{q_0^i\}$, which fluctuate on the slow scale. The nonlinear W-W and W-B interaction terms, generated by $\overline{\mathcal{L}}_3$, $\overline{\mathcal{L}}_4$, etc., involve the $\{\hat{q}_\eta^i\}$ through products of the form $\hat{q}_\beta^i \hat{q}_\gamma^j \ldots$, where $\beta, \gamma, \ldots$ are part of a synchronous combination, $\alpha\beta\gamma\ldots$. These terms describe the nonlinear interaction of the waves with one another, and they account for the effects of the slowly evolving background through their dependence on the $\{q_0^i\}$ and the reference state parameters.

Henceforth in this work, when the dual expansion is used in perturbation analyses, $\overline{\mathcal{L}}$ will be approximated by truncation of the series $\sum_m \overline{\mathcal{L}}_m$ at some order, and each term, $\overline{\mathcal{L}}_m$, in this approximation will itself be approximated by a partial expansion, $\sum_n \mathcal{L}_{m,n}$ in powers of the $\{q_0^i\}$. This will be done in such a manner that the approximation is self-consistent to a given accuracy in powers of ϵ .

For the averaged-Lagrangian analysis, it is necessary to develop a formal notation for the expansion terms of $\overline{\mathcal{L}}$. In order to determine the algebraic form of those terms, we shall begin by considering how they may be evaluated in the small-signal limit, where the small-signal relations may be used to simplify the expressions.

The small-signal relations themselves are established in the Lagrangian formalism through the solution of the quasilinear Euler-Lagrange equations generated by the term $\mathcal{L}_2\left(q_w^i, q_0^i, \underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t\right)$ in the dual expansion of $\mathcal{L}$. The equations are linear in the variables, $\{q_w^i\}$; the $\{q_0^i\}$ are taken as given functions of the coordinates. When the equations are solved, the resulting small-signal relations will have the same general forms, (2.21), and (2.23), encountered in the iterative analysis, although the variables in the Lagrangian description of the problem may be different. The significant expressions of the small-signal relations are the dispersion functions, $\{D_\ell^\eta\}$, and the mode structure coefficients, $\{c_\eta^i\}$. These may, in accordance with the dual expansion scheme, be regarded as expansions

in joint powers of the $\{q_0^i\}$, i.e.,

$$D_\ell^\eta(\omega_\eta, k_\eta, q_0^i, \underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t) = D_\ell^{\eta,0} + D_\ell^{\eta,1} + \cdots \quad , \qquad (3.31)$$

$$c_\eta^i(\omega_\eta, k_\eta, q_0^i, \underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t) = c_{\eta,0}^i + c_{\eta,1}^i + \cdots \quad . \qquad (3.32)$$

Once established, the small-signal relations may be used to pro-
duce an approximate expression for $\overline{\mathcal{L}}$. To this end, each wave variable
in the dual expansion of $\overline{\mathcal{L}}$ is expressed in terms of the complex nota-
tion of (2.15) – (2.16). The small-signal relations are then used to
express the results in terms of the $\{q_0^i\}$, the $\{\theta_\eta\}$, and a set of
complex amplitude parameters, $\{\hat{a}_\eta\}$, one for each wave present in the
medium. Finally, the bar average is taken; the result is the desired
expression for $\overline{\mathcal{L}}$. From the previous discussion of the dual expansion
it is obvious that

$$\overline{\mathcal{L}}_0 = \mathcal{L}(q_0^i, \underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t) \quad , \qquad (3.33)$$

and

$$\overline{\mathcal{L}}_1 = 0 \quad . \qquad (3.34)$$

From the structure of the $\{\mathcal{L}_m : m \geq 2\}$ we see that the bar averages of
those terms may be expressed in the forms

$$\overline{\mathcal{L}}_2 = \sum_\eta \lambda_\eta \hat{a}_\eta \hat{a}_{-\eta} \exp\left(-2\theta_\eta i\right) + (\text{terms of higher order in } \epsilon) \qquad (3.35)$$

$$\overline{\mathcal{L}}_3 = \sum_{\substack{\text{sync} \\ \alpha\beta\gamma}} \left\{ \lambda_{-\alpha\beta\gamma} \hat{a}_{-\alpha} \hat{a}_\beta \hat{a}_\gamma \exp j\delta\theta_{-\alpha\beta\gamma} + \lambda_{\alpha-\beta-\gamma} \hat{a}_\alpha \hat{a}_{-\beta} \hat{a}_{-\gamma} \exp\left(-j\delta\theta_{-\alpha\beta\gamma}^*\right) \right\}$$

$$+ (\text{terms of higher order in } \epsilon) \quad , \qquad (3.36)$$

$$\overline{\mathcal{L}}_4 = \sum_{\substack{\text{sync} \\ \alpha\beta\gamma\delta}} \left\{ \lambda_{-\alpha\beta\gamma\delta} \hat{a}_{-\alpha} \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta \exp j\delta\theta_{-\alpha\beta\gamma\delta} + \lambda_{\alpha-\beta-\gamma-\delta} \hat{a}_\alpha \hat{a}_{-\beta} \hat{a}_{-\gamma} \hat{a}_{-\delta} \exp\left(-j\delta\theta_{-\alpha\beta\gamma}^*\right) \right\}$$

$$+ (\text{terms of higher order in } \epsilon) \quad , \qquad (3.37)$$

etc.,

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in which the algebraic form of each coefficient,

$$\lambda_{-\alpha\beta\gamma\ldots}(\omega_{-\alpha},\underset{\sim}{k}_{-\alpha},\omega_{\beta},\underset{\sim}{k}_{\beta},\omega_{\gamma},\underset{\sim}{k}_{\gamma}\ldots,q_0^i,\underset{\sim}{v},\epsilon\underset{\sim}{x},\epsilon t)\quad,$$

is determined through the approximate evaluation of the $\{\bar{\mathcal{L}}_m\}$, using the small-signal relations. Corrections to that approximation appear in the terms of higher order in ϵ , which will be dealt with later in the analysis (see Section 3.3.3). In the expressions (3.35) $\theta_{\eta i}$ denotes the imaginary part of θ_η . The effects of synchronism mismatch are accounted for in (3.36) − (3.37) through the real parts of the quantities $\{\delta\theta_{-\alpha\beta\gamma\ldots}\}$, defined by

$$\delta\theta_{-\alpha\beta\gamma\ldots}(\epsilon\underset{\sim}{x},\epsilon t) \equiv \theta_{-\alpha} + \theta_\beta + \theta_\gamma + \ldots \quad . \tag{3.38}$$

The notation "sync", used in (3.36) − (3.37), indicates that the summations are over mode combinations for which

$$\frac{1}{\omega_\eta}\frac{\partial}{\partial t}\,\mathrm{Re}\!\left(\delta\theta_{-\alpha\beta}\ldots\right) \sim \frac{1}{\underset{\sim}{k}_\eta}\cdot\frac{\partial}{\partial\underset{\sim}{x}}\,\mathrm{Re}\!\left(\delta\theta_{-\alpha\beta\gamma}\ldots\right) \sim \epsilon\ ,\quad (\eta=\alpha,\beta,\gamma,\ldots) \tag{3.39}$$

in which the symbol "$\mathrm{Re}(\delta\theta_{-\alpha\ldots})$" denotes the real part of $\delta\theta_{-\alpha\ldots}$. Equation (3.39) is the generalization of the synchronism conditions (2.26) to account for slight synchronism mismatch between frequencies and wavevectors which are complex quantities. Mode combinations which do not meet this requirement are termed "asynchronous;" they do not contribute to the $\{\bar{\mathcal{L}}_m\}$.

In keeping with the dual expansion scheme, we associate each λ coefficient of (3.35) − (3.37) with a series expansion

$$\lambda_{-\alpha\beta\gamma\ldots} = \lambda_{-\alpha\beta\gamma\ldots,0} + \lambda_{-\alpha\beta\gamma\ldots,1} + \ldots \quad, \tag{3.40}$$

in which the second set of subscripts indicates the order of the term in joint powers of the $\{q_0^i\}$. The λ coefficients have algebraic and symmetry properties which follow from the synchronism conditions

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and the fact that $\bar{\mathcal{L}}$ is real. For the $\{\lambda_\eta\}$ these are

$$\lambda_\eta = \lambda_\eta^* \quad , \qquad \lambda_\eta = \lambda_{-\eta} \quad . \tag{3.41}$$

For the other λ coefficients we have

$$\lambda_{-\alpha\beta\gamma\ldots} = \lambda_{\alpha-\beta-\gamma\ldots}^* \quad , \tag{3.42}$$

and the property that the λ coefficients are completely symmetric with respect to permutation of the indices. For example,

$$\lambda_{-\alpha\beta\gamma} = \lambda_{-\alpha\gamma\beta} = \lambda_{\beta-\alpha\gamma} = \lambda_{\beta\gamma-\alpha} = \lambda_{\gamma-\alpha\beta} = \lambda_{\gamma\beta-\alpha} \quad . \tag{3.43}$$

3.3.3 Expansion in ϵ

So far, each term in the expansion of $\bar{\mathcal{L}}$ has been expressed only to lowest order in ϵ ; the small-signal relations have been assumed to hold exactly. According to the weak-coupling approximation, however, the linear mode amplitudes may depend on $\epsilon\underset{\sim}{x}$ and ϵt , and this dependence should be taken into account in the evaluation of $\bar{\mathcal{L}}$. As in Section 2, the information will be recovered by the expansions $\omega_\eta \to \omega_\eta - j(\partial/\partial t)_\eta$, $\underset{\sim}{k}_\eta \to \underset{\sim}{k}_\eta + j(\partial/\partial\underset{\sim}{x})_\eta$ in each term, $\bar{\mathcal{L}}_m$, originally evaluated according to the small-signal relations. For example, $\bar{\mathcal{L}}_2$ becomes

$$\bar{\mathcal{L}}_2 = \sum_\eta \left[\lambda_\eta \, \hat{a}_\eta \hat{a}_{-\eta} - \frac{j}{2} \frac{\partial\lambda_\eta}{\partial\omega_\eta} \left(\frac{\partial\hat{a}_\eta}{\partial t} \hat{a}_{-\eta} - \hat{a}_\eta \frac{\partial\hat{a}_{-\eta}}{\partial t} \right) \right.$$
$$\left. + \frac{j}{2} \frac{\partial\lambda_\eta}{\partial\underset{\sim}{k}_\eta} \cdot \left(\frac{\partial\hat{a}_\eta}{\partial\underset{\sim}{x}} \hat{a}_{-\eta} - \hat{a}_\eta \frac{\partial\hat{a}_{-\eta}}{\partial\underset{\sim}{x}} \right) \right] \exp\left(-2\theta_{\eta i}\right) \quad . \tag{3.44}$$

The form given for the expansion in (3.44) ignores the derivatives of ω_η and $\underset{\sim}{k}_\eta$. This is an approximation, since the operators $(\partial/\partial x)_\eta$ and $(\partial/\partial t)_\eta$ act on both wave variables, $\hat{a}_\eta$ and θ_η , and their derivatives. A full expansion, which takes this fact into account, is

much more complex than that presented here, and it interferes with our use of the small-signal relations to express $\bar{\mathcal{L}}$ in terms of a single parameter for each component. For these reasons, only θ_η and its first derivatives, ω_η and $\underset{\sim}{k}_\eta$, are accounted for in the subsequent averaged-Lagrangian analysis. Such a procedure by no means restricts the analysis to a homogeneous reference state. If the inhomogeneity is sufficiently weak, we may proceed with the Lagrangian analysis as though $\omega_\eta(\epsilon\underset{\sim}{x},\epsilon t)$ and $k_\eta(\epsilon\underset{\sim}{x},\epsilon t)$ were constant. The resulting equations are still valid for very small, but finite, values of ϵ . Therefore, the effects of a slowly fluctuating, weakly inhomogeneous background may be described, to lowest order, by means of the equations already obtained and the properties of the phases, $\{\theta_\eta\}$. This is a version of the classical Eikonal analysis for ray optics in inhomogeneous media.[44]

In (3.44), consider the terms involving derivatives of $\hat{a}_\eta$ and $\hat{a}_{-\eta}$ which are jointly of some integral order ℓ . Clearly, these terms are of a higher order than $\lambda_\eta \hat{a}_\eta \hat{a}_{-\eta}$ by a factor ϵ^ℓ . In approximating $\bar{\mathcal{L}}$ by the series $\bar{\mathcal{L}}_2 + \bar{\mathcal{L}}_3 + \ldots + \bar{\mathcal{L}}_n$ it is important that each term be expanded up to the same order in ϵ . Since ϵ is related to the mode amplitudes, we expect that, when taken to lowest order in ϵ , $\bar{\mathcal{L}}_n$ will be of higher order than the lowest order approximation to any preceding term $\bar{\mathcal{L}}_m$ (m < n) . Each of these terms must be expanded up to an equivalent order if the approximation is to be consistent. For the Lagrangian analysis, a consistent expansion can be obtained if we use the rule

$$ 0\left(\left|\hat{q}^i_\eta\right|\right) = 0(\epsilon) \qquad \text{(3.45)} $$

Then the first term in $\bar{\mathcal{L}}_2$ is $0(\epsilon^2)$, and those involving $\partial\lambda_\eta/\partial\omega_\eta$ and $\partial\lambda_\eta/\partial k_\eta$ are $0(\epsilon^3)$; the same order as the lowest order approximation to $\bar{\mathcal{L}}_3$. If the expansion of $\bar{\mathcal{L}}$ is to be correct up to $0(\epsilon^3)$, the contributions from both $\bar{\mathcal{L}}_2$ and $\bar{\mathcal{L}}_3$ must be included.

3.4 Background Analysis

We now consider the equations for the background evolution, taking the behavior of the $\{\hat{a}_\eta\}$ as given. Since the $\{q_0^i\}$ are not related by a mode structure in the limit of very small perturbations, it has been necessary to include them individually in the dual expansion of $\bar{\mathcal{L}}$. The equations of motion for the background are therefore a set of Euler-Lagrange equations with $\mathcal{L}$ replaced by $\bar{\mathcal{L}}^b$. There is one equation for each variable q_0^i $(i=1,...M)$. We shall use the crudest approximation to $\bar{\mathcal{L}}^b$ which will relate the $\{q_0^i\}$ to the wave components. This is

$$
\bar{\mathcal{L}}^b = \bar{\mathcal{L}}_0\left(q_0^i,\underset{\sim}{v},\epsilon\underset{\sim}{x},\epsilon t\right) + \bar{\mathcal{L}}_2\left(q_0^i,\hat{a}_\eta(\epsilon\underset{\sim}{x},\epsilon t),\underset{\sim}{v},\epsilon\underset{\sim}{x},\epsilon t\right) \quad ,
$$

$$
= \bar{\mathcal{L}}_{0,1} + \bar{\mathcal{L}}_{0,2} + \cdots + \sum_\eta (\lambda_{\eta,1} + \lambda_{\eta,2} + \cdots)\hat{a}_\eta \hat{a}_{-\eta} \exp\left(-2\theta_{\eta i}\right). \quad (3.46)
$$

If the expansion is about an exact solution, $\bar{\mathcal{L}}_{0,1}$ is identically zero.

The procedure for the background analysis is to make a trial approximation to $\bar{\mathcal{L}}^b$ by dropping additional terms from the expansion (3.46). The corresponding equations of motion are then found from (3.5) and are used along with the ordering rule to establish the magnitude of the components $\{q_0^i\}$ in relation to the $\{|\hat{a}_\eta|\}$ and ϵ. The magnitude relations have the form

$$
q_0^i \sim \epsilon^{\mu(i)} \quad , \tag{3.47}
$$

where the constants, $\{\mu(i)\}$ are determined in the analysis.

If the approximation is self-consistent, the resulting magnitude relations between the $\{q_0^i\}$ and ϵ will show that the terms omitted from the expansion of $\bar{\mathcal{L}}^b$ are negligible compared to those retained. If not, additional trial approximations are made until self-consistency is achieved.

Because the $\{q_0^i\}$ are treated independently of one another in the background analysis, it is not possible to describe the analytic procedure in more detail unless one turns to a particular example, as in

the applications of Section 4. At this point, it is sufficient to under-
stand the general procedure and the important features of its output:
(a) a set of descriptive differential equations for the $\{q_0^i\}$, and
(b) a set of magnitude relations between the $\{q_0^i\}$ and ϵ . A flow
chart of the analysis constitutes part of Fig. 3.4 (see page   ). The
figure also illustrates the wave analyses, which will be developed in
Section 3.5. As the figure indicates, the two analyses are linked
through the dual expansion of $\overline{\mathcal{L}}$ and the ordering relations. Together,
they provide a unified description of the nonlinear behavior of the
entire medium according to (2.24).

From the description of the background analysis given above, we see
that the approximation to $\overline{\mathcal{L}}^b$ obtained from (3.46) will be adequate to
describe the quasilinear effects, i.e., the interactions between the
background and any number of monochromatic waves which do not interact
with one another directly. This is consistent with Fig. 3.3. For some
cases, such a description of the background may be quite adequate. The
example of Section 4.2 shows that the form (3.46) is sufficient to produce
the lowest order "quasilinear equation" for the evolution of a distribu-
tion function, $F(\underset{\sim}{v}, \epsilon t)$, in a warm plasma. However, one may wish to
account also for the joint effect of three or more synchronous waves on
the background; then $\overline{\mathcal{L}}^b$ must be expanded to a higher order in ϵ .
In general, the expansion used to describe the joint effects of N waves
on the background would be

$$\overline{\mathcal{L}}^b \approx \sum_{m=0}^{N} \mathcal{L}_m \quad , \tag{3.48}$$

in which the formal expressions (3.35) — (3.37), the ordering rule (3.45)
and the magnitude relations (3.47) are used to expand each term to equal
accuracy in powers of ϵ . The Euler-Lagrange equations for the $\{q_0^i\}$
then describe the background evolution to the desired accuracy. If the
number of scalar variables is large, the analytic procedure just described
can be very tedious, but in some problems there are symmetry considera-
tions which fix some of the $\{q_0^i\}$ at zero for all $\underset{\sim}{v}, \underset{\sim}{x}$ and t .

Having obtained the equations for the background evolution, we may wish to transpose the results by changing variables. This may be done in order to express the results in a form which is more familiar or more easily interpreted physically. It may also be done in order to facilitate the wave analysis, described in the next subsection. In the latter case, different reference states are employed in the wave and background analyses. Such transformations are legitimate, provided that they obey the constraint that the background state (= reference state + slow-scale perturbation components) is the same in both analyses. Suppose that we have described the background evolution by obtaining equations for the slow-scale perturbation components, $\{q_0^i\}$, from a specified reference state. For the wave analysis, it may be more convenient to employ a new "primed" reference state, in which the components $\{q_0^{i'}\}$ are identically zero, i.e., the $\{q_0^i\}$ are absorbed into the new reference state. In the new perturbation description, the averaged-Lagrangian formalism is considerably simplified, since there is no secondary expansion of $\overline{\mathcal{L}}$ in joint powers of the $\{q_0^{i'}\}$. This technique for simplifying the wave analysis is employed in the example of Section 4.2.

3.5 <u>Wave Analysis</u>

In this subsection, the background parameters are taken as given, and the wave behavior is described by way of the averaged-Lagrangian formalism. The reference state may be an approximate solution such that all slow-scale perturbations, $\{q_0^i\}$,are zero, or the $\{q_0^i\}$ may be small, in which case each λ coefficient in the expansion of $\overline{\mathcal{L}}$ represents an expansion in joint powers of the $\{q_0^i\}$. In either case, the reader should keep in mind that the background affects the waves through the λ coefficients. The effects of the waves upon one another are obvious in the following analysis. The basic steps of the wave analysis are shown as part of the complete averaged-Lagrangian method in Fig. 3.4.

Since the wave parameters $\{\hat{a}_\eta\}$ and $\{\theta_\eta\}$ are not functions of $\underset{\sim}{v}$, no information is lost by integrating $\overline{\mathcal{L}}^w$ over velocity. This integral will be denoted by $\overset{\sim}{\mathcal{L}}$, and it may be expanded in joint powers of the

$\{\hat{a}_\eta\}$ in a series

$$\tilde{\mathfrak{L}} = \tilde{\mathfrak{L}}_0 + \tilde{\mathfrak{L}}_2 + \tilde{\mathfrak{L}}_3 + \ldots \quad , \tag{3.49}$$

in which

$$\tilde{\mathfrak{L}}_m \equiv \sum_n \tilde{\mathfrak{L}}_{m,n} \quad , \qquad \tilde{\mathfrak{L}}_{m,n} \equiv \int d^3v \, \bar{\mathfrak{L}}_{m,n} \quad . \tag{3.50}$$

The formal expressions for the $\{\tilde{\mathfrak{L}}_m\}$ may be obtained through integration of Eqs. (3.35) − (3.37) over velocity. The integration leaves the equations unchanged, except that

$$\bar{\mathfrak{L}}_m \rightarrow \tilde{\mathfrak{L}}_m \quad , \qquad \lambda_{-\alpha\beta\gamma\ldots} \rightarrow \tilde{\lambda}_{-\alpha\beta\gamma\ldots} \quad , \tag{3.51}$$

where

$$\tilde{\lambda}_{-\alpha\beta\gamma\ldots} \equiv \int \lambda_{-\alpha\beta\gamma\ldots} \, d^3v \quad . \tag{3.52}$$

After making some self-consistent approximation to $\tilde{\mathfrak{L}}$, we may obtain the equations of motion by varying the mode amplitudes $\{\hat{q}_\eta\}$ in accordance with the averaged Hamilton's principle. Alternatively, we may vary the real phases $\{\theta_{\eta r}\}$ The amplitude variation differs from the phase variation in that it results in an arbitrary slow variation of all complex components $\{q_\eta^{-i} = \hat{q}_\eta^i \exp j\theta_\eta\}$. The variation of $\theta_{\eta r}$ leaves $\{|q_\eta^{-i}|\}$ unchanged. In fact, the amplitude variations alone are sufficient to generate a complete set of coupled equations for the propagating waves, plus convenient formulae for the amplitudes of the virtual waves. These are derived below. The phase variations will be discussed later in this subsection, where it is shown that they lead to a set of action-transfer relations.

3.5.1 Amplitude Variation

If we approximate $\tilde{\tilde{\mathcal{L}}}$ by $\sum_\eta \tilde{\lambda}_\eta \hat{a}_\eta \hat{a}_{-\eta} \exp -2\theta_{\eta i}$, and vary $\hat{a}_{-\eta}$, we recover the dispersion relation for the branch containing Wave η as

$$\tilde{\lambda}_\eta(\omega_\eta, \underset{\sim}{k}_\eta)\, \hat{a}_\eta = 0 \quad . \tag{3.53}$$

Unless $\hat{a}_\eta$ is zero, we must have $\tilde{\lambda}_\eta = 0$, so $\tilde{\lambda}_\eta$ may be identified with one of the factors $\{D_\eta^\ell\}$ of the dispersion relation.

The expansion of $\tilde{\tilde{\mathcal{L}}}$, up to $O(\epsilon^3)$, yields

$$\tilde{\tilde{\mathcal{L}}} = \sum_\eta \tilde{\lambda}_\eta \hat{a}_\eta \hat{a}_{-\eta} \exp\left(-2\theta_{\eta i}\right)$$

$$+ \sum_\eta \left[-\frac{j}{2}\left(\frac{\partial \tilde{\lambda}_\eta}{\partial \omega_\eta}\right)\left(\frac{\partial \hat{a}_\eta}{\partial t}\hat{a}_{-\eta} - \hat{a}_\eta \frac{\partial \hat{a}_{-\eta}}{\partial t}\right) + \frac{j}{2}\left(\frac{\partial \tilde{\lambda}_\eta}{\partial \underset{\sim}{k}_\eta}\right)\cdot\left(\frac{\partial \hat{a}_\eta}{\partial \underset{\sim}{x}}\hat{a}_\eta - \hat{a}_\eta \frac{\partial \hat{a}_{-\eta}}{\partial \underset{\sim}{x}}\right)\right] \exp\left(-2\theta_{\eta i}\right)$$

$$+ \sum_{\substack{\text{sync}\\ \alpha\beta\gamma}} \left[\tilde{\lambda}_{-\alpha,\beta,\gamma}\hat{a}_{-\alpha}\hat{a}_\beta \hat{a}_\gamma \exp j\delta\theta_{-\alpha\beta\gamma} + \tilde{\lambda}_{\alpha-\beta-\gamma}\hat{a}_\alpha \hat{a}_{-\beta}\hat{a}_{-\gamma} \exp\left(-j\delta\theta^*_{-\alpha\beta\gamma}\right)\right]$$

$$+ O(\epsilon^4) \quad . \tag{3.54}$$

The present analysis concerns propagating waves, to which (2.21) applies. For these waves, the terms in the first summation of (3.54) are therefore zero. The variation of $\hat{a}_{-\alpha}$ in the expression (3.54) yields

$$\left\{ j\left(\frac{\partial \lambda_\alpha}{\partial \omega_\alpha}\right)\left(\frac{\partial \hat{a}_\alpha}{\partial t}\right) - j\left(\frac{\partial \lambda_\alpha}{\partial \underset{\sim}{k}_\alpha}\right)\cdot\left(\frac{\partial \hat{a}_\alpha}{\partial \underset{\sim}{x}}\right) + \frac{j}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial \lambda_\alpha}{\partial \omega_\alpha}\right) - \frac{\partial}{\partial \underset{\sim}{x}}\cdot\left(\frac{\partial \lambda_\alpha}{\partial \underset{\sim}{k}_\alpha}\right)\right]\hat{a}_\alpha \right\} \exp\left(-2\theta_\alpha\right)$$

$$= \sum_{\substack{\text{sync}\\ \alpha\beta\gamma}} \lambda_{-\alpha\beta\gamma}\hat{a}_\beta \hat{a}_\gamma \exp j\delta\theta_{-\alpha\beta\gamma} + O(\epsilon^3) \quad . \tag{3.55}$$

This result, with appropriate permutations of the indices α, β, and γ , provides a complete set of coupled equations for three propagating modes.

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Equations which describe the interaction between four or more waves may be obtained in a similar manner, except that $\tilde{\mathcal{L}}$ must be expanded up to a higher order in ϵ , and virtual waves must be taken into account. Virtual waves contribute to terms of $O(\epsilon^4)$ and higher, as is shown in the four-wave interaction analysis presented later in this section.

Permutation of subscripts in (3.55), use of the symmetry properties (3.41) — (3.43), and additional algebraic manipulation produce the equations of wave evolution in the simplified standard form

$$j\left(\frac{D}{Dt}\right)_\alpha \hat{A}_\alpha = \chi \, \hat{A}_\beta \, \hat{A}_\gamma \quad ,$$

$$j\left(\frac{D}{Dt}\right)_\beta \hat{A}_\beta = \chi^* \, \hat{A}_\gamma \, \hat{A}_\alpha \quad , \qquad (3.56)$$

$$j\left(\frac{D}{Dt}\right)_\gamma \hat{A}_\gamma = \chi^* \, \hat{A}_\beta \, \hat{A}_\alpha \quad ,$$

in which the normalized amplitudes, $\{\hat{A}_\eta\}$, and the wave coupling coefficient, χ , are

$$\hat{A}_\eta = \left(\frac{\partial \lambda_\eta}{\partial \omega_\eta}\right)^{1/2} \hat{a}_\eta \exp\left(-\theta_{\eta i}\right) \ , \quad \chi \equiv \chi_{-\alpha\beta\gamma} \equiv \frac{\lambda_{-\alpha\beta\gamma} \exp j[\mathrm{Re}(\delta\theta_{-\alpha\beta\gamma})]}{\left(\frac{\partial \lambda_\alpha}{\partial \omega_\alpha}\right)^{1/2}\left(\frac{\partial \lambda_\beta}{\partial \omega_\beta}\right)^{1/2}\left(\frac{\partial \lambda_\gamma}{\partial \omega_\gamma}\right)^{1/2}} \quad . \qquad (3.57)$$

The operator $(D/Dt)_\eta$ is defined by

$$\left(\frac{D}{Dt}\right)_\eta \hat{A}_\eta \equiv \frac{\partial \hat{A}_\eta}{\partial t} + \underset{\sim}{v}_\eta \cdot \frac{\partial \hat{A}_\eta}{\partial \underset{\sim}{x}} + \left[\omega_{\eta i} + \frac{1}{2}\left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v}_\eta\right) - \underset{\sim}{k}_{\eta i} \cdot \underset{\sim}{v}_\eta\right]\hat{A}_\eta \quad , \qquad (3.58)$$

in which $\omega_{\eta i}$ and $k_{\eta i}$ are the imaginary parts of ω_η and k_η , respectively, which are determined by the small-signal dispersion relations (2.21). The vector $\underset{\sim}{v}_\eta$ is the group velocity of Wave η , i.e.,

$$\underset{\sim}{v}_\eta \equiv \frac{-(\partial \lambda_\eta / \partial \underset{\sim}{k}_\eta)}{(\partial \lambda_\eta / \partial \omega_\eta)} = \frac{d\omega_\eta}{d\underset{\sim}{k}_\eta} \quad . \qquad (3.59)$$

Equations (3.57) — (3.59) show that the $\{v_\eta\}$ and all other parameters of the equations (3.56) may be determined by evaluating the λ's and their ω and k derivatives. When the waves evolve only in one space or time dimension, are in exact synchronism, have no linear growth or damping, and propagate in a homogeneous background, the solutions to (3.56) are known in terms of Jacobian elliptic functions.[13,47]

The normalized variables, $\{\hat{A}_\eta\}$, were chosen because they lead easily to an important energy relation. To obtain this relation, multiply the equation for the η component $(\eta = \alpha, \beta, \gamma)$ by $-j\hat{A}_\eta^*$ on both sides, then add this equation to its complex conjugate. The three resulting equations imply that

$$\left(\frac{D}{Dt}\right)_\alpha^\dagger |\hat{A}_\alpha|^2 = -\left(\frac{D}{Dt}\right)_\beta^\dagger |\hat{A}_\beta|^2 = -\left(\frac{D}{Dt}\right)_\gamma^\dagger |\hat{A}_\gamma|^2 = \epsilon_{\alpha\beta\gamma} \quad , \qquad (3.60)$$

where

$$\left(\frac{D}{Dt}\right)_\eta^\dagger |\hat{A}_\eta|^2 = \left(\frac{\partial}{\partial t} + 2\omega_{\eta i}\right)|\hat{A}_\eta|^2 + \left(\frac{\partial}{\partial x} - 2k_{\sim\eta i}\right) \cdot \left(v_\eta |\hat{A}_\eta|^2\right) \qquad (3.61)$$

and

$$\epsilon_{\alpha\beta\gamma} = -j \, \chi\hat{A}_\alpha^* \, \hat{A}_\beta \, \hat{A}_\gamma + c.c. \quad . \qquad (3.62)$$

The abbreviation "c.c.," used here and later in this work, stands for the complex conjugate of the term to its left. The result (3.61) may be interpreted in terms of the generalized energy and energy flux introduced in Section 3.1.1. Consider the definition of $\mathcal{E}$ and $\mathscr{F}^s$ in (3.6) and (3.7), as they apply to a wave perturbation in the small-signal limit, where we can approximate $\mathcal{E}$ by $\mathcal{E}_2$, $\mathscr{F}^s$ by $\mathscr{F}_2^s$ and the $\{q^i\}$ by the wave components $\{q_\eta^i\}$. After bar-averaging, simplification by way of the small-signal relations, and integration over velocity, these expressions become

$$\epsilon_\eta \equiv \tilde{\mathcal{E}}_{2\eta} = \omega_\eta \left(\frac{\partial \tilde{\lambda}_\eta}{\partial \omega_\eta}\right) \hat{a}_\eta \hat{a}_\eta^* = \omega_\eta |\hat{A}_\eta|^2 \quad , \qquad (3.63)$$

and

$$\underset{\sim}{\phi}_\eta \equiv \underset{\sim}{\widetilde{\mathscr{F}}}^s_{2\eta} = \omega_\eta \left(\frac{\partial \widetilde{\lambda}_\eta}{\partial \underset{\sim}{k}_\eta} \right) \hat{a}_\eta \hat{a}^*_\eta = \underset{\sim}{v}_\eta \epsilon_\eta \quad , \tag{3.64}$$

where ϵ_η and $\underset{\sim}{\phi}_\eta$ are the averaged generalized energy and flux densities corresponding to Wave η . In obtaining these expressions, we have made use of the fact that the term $\widetilde{\mathscr{L}}_2$, which by definition appears in the expression for ϵ_η , is zero. This is true to arbitrary order in ϵ . For example, (3.53) and (3.55) may be used in (3.54) to show that $\widetilde{\mathscr{L}}_2 = 0$ up to $0(\epsilon^3)$. Equations (3.63) − (3.64) show that the relation (3.60) is equivalent to

$$\left(\frac{D}{Dt} \right)^\dagger_\alpha \left(\frac{\epsilon_\alpha}{\omega_\alpha} \right) = -\left(\frac{D}{Dt} \right)^\dagger_\beta \left(\frac{\epsilon_\beta}{\omega_\beta} \right) = -\left(\frac{D}{Dt} \right)^\dagger_\gamma \left(\frac{\epsilon_\gamma}{\omega_\gamma} \right) = \epsilon_{-\alpha\beta\gamma} \quad , \tag{3.65}$$

which we shall call the "action-transfer relations." In this equation, $\epsilon_\eta/\omega_\eta (= |\hat{A}_\eta|^2)$ is the averaged action density of Wave η , and $\epsilon_{-\alpha\beta\gamma}$ will be referred to as the three-wave coupling energy. The first two equalities of (3.65) are the three-wave Manley-Rowe relations; they are more general than the formulae (2.46), since they account for linear growth or damping and weak inhomogeneity. The action transfer relations, as a whole, contain even more information than the Manley-Rowe relations, since the latter do not specify $\epsilon_{-\alpha\beta\gamma}$.

One need not calculate the λ's and their derivatives in order to evaluate the energy components of (3.65). The definitions of ϵ_α , and $\epsilon_{-\alpha\beta\gamma}$ given in (3.62) − (3.63) show that

$$\epsilon_\eta = \int \overline{\mathscr{E}_2(\overline{q}^i_\eta + \overline{q}^i_{-\eta}, \underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t)} \, d^3v \quad , \tag{3.66}$$

and

$$\epsilon_{-\alpha\beta\gamma} = \int \overline{\mathscr{L}_3(\overline{q}^i_{-\alpha} + \overline{q}^i_\beta + \overline{q}^i_\gamma, \underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t)} \, d^3v + c.c. \quad , \tag{3.67}$$

in which it is understood that the small-signal relations are to be used to evaluate the ϵ's in terms of three scalar parameters, $\hat{a}_\alpha$, $\hat{a}_\beta$, and $\hat{a}_\gamma$. These formulas were the basis of earlier mode coupling analyses by the author.[22-24] In that work, the three-wave energy transfer relations, together with the formulae (3.66) and (3.67) were derived by a different method, based on the "arbitrary variation" approach of Section 3.2. In this earlier approach, the coupled mode equations are deduced from the action-transfer relations instead of _vice versa_. This earlier approach is not employed in the present work because it does not apply to background evolution, and is cumbersome to apply to higher order wave interactions in which virtual waves are important. The averaged-Lagrangian method of this section extends easily to account for such phenomena.

For the three-wave case, the averaged-Lagrangian approach presented here, and the alternative approach just mentioned, lead to the same equations of wave evolution, and to equivalent prescriptions for evaluating the parameters of those equations. In both cases, the parameters are obtained from the terms of the expanded Lagrangian and the small-signal relations between the $\{\bar{q}_\eta^i\}$.

3.5.2 Phase Variation

It has just been shown that the energy-transfer relations may be obtained from the coupled-mode equations by algebraic manipulation; however, it is also possible to obtain these relations from $\widetilde{\mathcal{L}}$ by means of the phase variation. The procedure is presented here for its conceptual value, even though no new equations are obtained.

The quantities to be varied are the $\{\theta_{\eta r}\}$, which are the real parts of the phases $\{\theta_\eta\}$. It is important to remember that this implies a simultaneous variation in $\theta_{-\eta r}$ since θ_η and $\theta_{-\eta}$ are related by our original assumption (2.15). Consider the variation of $\theta_{\alpha r}$ in the expression for $\widetilde{\mathcal{L}}$ given by (3.54), keeping in mind the fact that $\underset{\sim}{k}_\alpha$ and ω_α are spatial and temporal gradients of θ_α . The

corresponding Euler-Lagrange equation is

$$\left[\frac{d}{dt}\left(\frac{1}{2}\frac{\partial\lambda_\alpha}{\partial\omega_\alpha}\hat{a}_\alpha\hat{a}_{-\alpha}\right) - \frac{d}{d\underset{\sim}{x}}\cdot\left(\frac{1}{2}\frac{\partial\lambda_\alpha}{\partial\underset{\sim}{k}_\alpha}\hat{a}_\alpha\hat{a}_{-\alpha}\right)\right] = -j3\lambda_{-\alpha\beta\gamma}\hat{a}_{-\alpha}\hat{a}_\beta\hat{a}_\gamma + \text{c.c.} \quad (3.68)$$

which will be recognized as one of the action-transfer relations, (3.65). The other two relations follow from the variations of $\theta_{\beta r}$ and $\theta_{\gamma r}$.

The results just obtained are in keeping with the rule that a more restrictive variation yields less information: the action-transfer relations do not tell us as much as do the equations of wave evolution (3.56), which were obtained by the amplitude variation. The phase variation used above is more restrictive, because it changes only the imaginary part of $\ln(\bar{a}_\eta)$. The amplitude variation changes both this and $|\bar{a}_\eta|$. In principle, the phase variation could do something that the amplitude variation cannot: vary the frequency and wavevector of Wave η. However, the expansion procedure used here for $\widetilde{\mathcal{L}}$ neglects all but the first order derivatives of θ_η. Therefore, in this analysis, the variation $\delta\theta_\eta$ in the averaged-Hamilton's principle may not include changes in ω_η or $\underset{\sim}{k}_\eta$. Both Dougherty[20] and Dysthe[21] have noted in earlier work that the phase variation leads to the action-transfer relation for three waves, with background perturbations neglected.

3.5.3 Higher Order Nonlinear Effects

It was shown in Section 2 that if the interaction involves four or more waves, the virtual waves affect the equations of motion for the propagating modes. If the effects of virtual waves are to be accounted for, it will be necessary to calculate their amplitudes, and the averaged-Lagrangian approach provides a convenient means of doing this. Consider the expansion of $\widetilde{\mathcal{L}}$ up to $O(\epsilon^3)$ in (3.54). The quantity to be varied is $\hat{a}_{-\alpha}$. In the preceding analysis, α has been assumed to be a propagating mode, and the term $\lambda_\alpha\hat{a}_\alpha\hat{a}_{-\alpha}$ in $\widetilde{\mathcal{L}}$ made no contribution because of the linear dispersion relation. Now, however, α is a virtual wave, denoted by $(\beta + \gamma)$, and $\lambda_{(\beta + \gamma)}$ is not zero. In this case,

the amplitude variation in (3.54) gives

$$\lambda_{(\beta+\gamma)}\hat{a}_{(\beta+\gamma)} = -\lambda_{-(\beta+\gamma)\beta\gamma}\hat{a}_{\beta}\hat{a}_{\gamma} \exp j\delta\theta_{-\alpha\beta\gamma} + O(\epsilon^3) \qquad . \qquad (3.69)$$

Equation (3.69) implies that $|\hat{a}_{(\beta+\gamma)}|$ is of $O(\epsilon^2)$. This does not contradict the ordering rule (3.45) however, since ϵ is associated with $|\hat{a}_{\eta}|$ only when η is a propagating mode.

For the interaction of four propagating modes, the ordering rule shows that $\tilde{\mathcal{L}}$ must be expanded through $O(\epsilon^4)$, in which case

$$\tilde{\mathcal{L}} = \sum_{\alpha} \left[\lambda_{\alpha}|\hat{a}_{\alpha}|^2 - \frac{j}{2}\left(\frac{\partial\lambda_{\alpha}}{\partial\omega_{\alpha}}\right)\left(\frac{\partial\hat{a}_{\alpha}}{\partial t}\hat{a}_{\alpha}^* - \hat{a}_{\alpha}\frac{\partial\hat{a}_{\alpha}^*}{\partial t}\right) \right.$$

$$+ \frac{j}{2}\left(\frac{\partial\lambda_{\alpha}}{\partial\underset{\sim}{k}_{\alpha}}\right)\cdot\left(\frac{\partial\hat{a}_{\alpha}}{\partial\underset{\sim}{x}}\hat{a}_{\alpha}^* - \hat{a}_{\alpha}\frac{\partial\hat{a}_{\alpha}^*}{\partial\underset{\sim}{x}}\right) + \frac{1}{2}\left(\frac{\partial^2\lambda_{\alpha}}{\partial\omega_{\alpha}^2}\right)\left|\frac{\partial\hat{a}_{\alpha}}{\partial t}\right|^2$$

$$\left. - \frac{1}{2}\frac{\partial^2\lambda_{\alpha}}{\partial\omega_{\alpha}\partial\underset{\sim}{k}_{\alpha}}\cdot\left(\frac{\partial\hat{a}}{\partial t}\frac{\partial\hat{a}^*}{\partial\underset{\sim}{x}} + \frac{\partial\hat{a}}{\partial\underset{\sim}{x}}\frac{\partial\hat{a}^*}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2\lambda_{\alpha}}{\partial\underset{\sim}{k}_{\alpha}^2}\right) : \left|\frac{\partial\hat{a}_{\alpha}}{\partial\underset{\sim}{x}}\right|^2 \right]\exp\left(-2\theta_{\alpha i}\right)$$

$$+ \sum_{\substack{\text{sync}\\\alpha\beta\gamma\delta}} \lambda_{\alpha-\beta(-\gamma-\delta)}\hat{a}_{\alpha}\hat{a}_{\beta}^*\hat{a}_{(\gamma+\delta)}^* \exp j\delta\theta_{-\alpha\beta\gamma\delta} + \text{c.c.}$$

$$+ \sum_{\substack{\text{sync}\\\alpha\beta\gamma\delta}} \lambda_{\alpha-\beta-\gamma-\delta}\hat{a}_{\alpha}\hat{a}_{\beta}^*\hat{a}_{\gamma}^*\hat{a}_{\delta}^* \exp j\delta\theta_{-\alpha\beta\gamma\delta} + \text{c.c.} + O(\epsilon^5) \qquad . \qquad (3.70)$$

For brevity, we have used the fact that for real solutions $\hat{a}_{-\eta} = \hat{a}_{\eta}^*$ $(\eta=\alpha,\beta,\gamma,\delta)$. The symbol ":" is introduced here to denote the inner product of the two matrices to either side of it. In this case, the matrix to its right is a dyad.

As in the three-wave analysis, we shall be concerned here with propagating waves, for which the terms $\{\lambda_{\eta}|\hat{a}_{\eta}|^2\}$ are identically zero. They will henceforth be dropped from the above expansion. The expansion implicitly contains the assumption that no three propagating modes have

$(\omega,\underset{\sim}{k})$ values which satisfy the three-wave synchronism conditions. That assumption is often valid, since for some systems the three-wave synchronism cannot be satisfied at all,[11,30] and in cases where three-wave synchronism is possible, it may prove to be possible only over limited sections of the dispersion curves. Thus, the Lagrangian (3.70) may well describe the lowest order nonlinear interaction between waves. Dysthe[21] has discussed the special case of the decay of a wave into sidebands

$$(\omega_\alpha, \underset{\sim}{k}_\alpha)_r = (\omega_\alpha, \underset{\sim}{k}_\alpha)_r - (\omega_\alpha + \delta\omega, \underset{\sim}{k}_\alpha + \underset{\sim}{\delta k})_r - (\omega_\alpha - \delta\omega, \underset{\sim}{k}_\alpha - \underset{\sim}{\delta k})_r \quad , \quad (3.71)$$

where the frequency shift $\delta\omega \; (=\underset{\sim}{\delta k}/\underset{\sim}{v}_\alpha)$ is small, and has also commented on self-action effects:

$$(\omega_\alpha, \underset{\sim}{k}_\alpha)_r = (\omega_\alpha, \underset{\sim}{k}_\alpha)_r + (\omega_\alpha, \underset{\sim}{k}_\alpha)_r + (\omega_{-\alpha}, \underset{\sim}{k}_{-\alpha})_r \quad . \quad\quad (3.72)$$

These interactions are generally possible, regardless of the shape of the dispersion curves. Another combination which automatically satisfies synchronism is

$$(\omega_\alpha, \underset{\sim}{k}_\alpha)_r = (\omega_\alpha, \underset{\sim}{k}_\alpha)_r + (\omega_\beta, \underset{\sim}{k}_\beta)_r + (\omega_{-\beta}, \underset{\sim}{k}_{-\beta})_r \quad . \quad\quad (3.73)$$

This interaction is the only one which has any effect if the "random phase approximation," mentioned earlier, is used to go from the interaction equations for monochromatic waves to an equation for the evolution of a wave spectrum; the point is illustrated by the example of Section 4.2. Here, however, we shall be concerned with the general equations for four distinct, propagating waves

$$(\omega_\alpha, \underset{\sim}{k}_\alpha)_r = (\omega_\beta, \underset{\sim}{k}_\beta)_r + (\omega_\gamma, \underset{\sim}{k}_\gamma)_r + (\omega_\delta, \underset{\sim}{k}_\delta)_r \quad . \quad\quad (3.74)$$

At this point, for illustrative purposes, the analysis will be restricted to interactions in which the vectors $\{(\omega_\eta, \underset{\sim}{k}_\eta)\}$ are real.

Variation of $\hat{a}_\alpha^*$ $(=\hat{a}_{-\alpha})$ in the expression of (3.70) gives

$$-j\left(\frac{\partial\lambda_\alpha}{\partial\omega_\alpha}\right)\left(\frac{\partial\hat{a}_\alpha}{\partial t} + \underset{\sim}{v}_\alpha \cdot \frac{\partial\hat{a}_\alpha}{\partial\underset{\sim}{x}}\right) - \frac{j}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial\lambda_\alpha}{\partial\omega_\alpha}\right) + \frac{\partial}{\partial\underset{\sim}{x}}\cdot\left(\underset{\sim}{v}_\alpha\frac{\partial\lambda_\alpha}{\partial\omega_\alpha}\right)\right]\hat{a}_\alpha$$

$$-\frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial^2\lambda_\alpha}{\partial\omega_\alpha^2}\right) - \frac{\partial}{\partial\underset{\sim}{x}}\cdot\left(\frac{\partial^2\lambda_\alpha}{\partial\omega_\alpha\partial\underset{\sim}{k}_\alpha}\right)\right]\frac{\partial\hat{a}_\alpha}{\partial t} + \frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial^2\lambda_\alpha}{\partial\omega_\alpha\partial\underset{\sim}{k}_\alpha}\right) - \frac{\partial}{\partial\underset{\sim}{x}}\cdot\left(\frac{\partial^2\lambda_\alpha}{\partial\underset{\sim}{k}_\alpha^2}\right)\right]\cdot\frac{\partial\hat{a}_\alpha}{\partial\underset{\sim}{x}}$$

$$-\frac{1}{2}\left(\frac{\partial^2\lambda_\alpha}{\partial\omega_\alpha^2}\right)\frac{\partial^2\hat{a}_\alpha}{\partial t^2} + \left(\frac{\partial^2\lambda_\alpha}{\partial\omega_\alpha\partial\underset{\sim}{k}_\alpha}\right)\cdot\frac{\partial^2\hat{a}_\alpha}{\partial t\partial\underset{\sim}{x}} - \frac{1}{2}\left(\frac{\partial^2\lambda_\alpha}{\partial\underset{\sim}{k}_\alpha^2}\right):\frac{\partial^2\hat{a}_\alpha}{\partial\underset{\sim}{x}^2}$$

$$= -\sum_{\substack{\text{sync}\\\alpha\beta\gamma\delta}} \lambda_{-\alpha\beta(\gamma+\delta)}\hat{a}_\beta\hat{a}_{(\gamma+\delta)}\exp j\delta\theta_{-\alpha\beta(\gamma+\delta)} + \lambda_{-\alpha\beta\gamma\delta}\hat{a}_\beta\hat{a}_\gamma\hat{a}_\delta\exp j\delta\theta_{-\alpha\beta\gamma\delta}$$

$$= \sum_{\substack{\text{sync}\\\alpha\beta\gamma\delta}}\left\{\frac{\lambda_{-\alpha\beta(\gamma+\delta)}\lambda_{(\gamma+\delta)\gamma\delta}}{\lambda_{(\gamma+\delta)}} - \lambda_{-\alpha\beta\gamma\delta}\right\}\hat{a}_\beta\hat{a}_\gamma\hat{a}_\delta\exp j\delta\theta_{-\alpha\beta\gamma\delta} \qquad . \qquad (3.75)$$

The second equality follows from (3.69) for $\hat{a}_{(\gamma+\delta)}$. The above result is rather complicated, but it covers a general problem which includes the self-action, sideband decay, and phase-invariant interactions characterized by the synchronism conditions (3.71) − (3.74). The result (3.75) describes the weak nonlinear interaction of four discrete modes, propagating in directions which are constrained only by the synchronism requirement, in a medium which may be spatially anisotropic and weakly inhomogeneous in both space and time.

In special cases (3.75) can be simplified considerably, as we shall demonstrate for the case in which the background state is homogeneous. The effects of inhomogeneity are contained in the bracketed terms on the left hand side of (3.75). These are now dropped. We may also use the fact that in a homogeneous medium, in the absence of three-wave synchronism,

$$\frac{d\hat{a}_\alpha}{dt} + \underset{\sim}{v}_\alpha \cdot \frac{d\hat{a}_\alpha}{d\underset{\sim}{x}} + 0(\epsilon^3) = 0 \qquad . \qquad (3.76)$$

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This relation is just a special case of (3.55). It may be used to express $d\hat{a}_\alpha/dt$ as $-\underset{\sim}{v}_\alpha \cdot (d\hat{a}_\alpha/d\underset{\sim}{x})$ in the highest order terms of (3.75), with negligible error. The result may be cast into the form

$$\frac{1}{\omega_\alpha}\left(\frac{d\hat{a}_\alpha}{dt} + \underset{\sim}{v} \cdot \frac{d\hat{a}_\alpha}{d\underset{\sim}{x}}\right) + \frac{j}{2\omega_\alpha(\partial\lambda_\alpha/\partial\omega_\alpha)}\left(\frac{\partial\underset{\sim}{v}_\alpha}{\partial\underset{\sim}{k}_\alpha}\right):\left(\frac{\partial^2\hat{a}_\alpha}{\partial\underset{\sim}{x}^2}\right)$$

$$= j\sum_{\substack{\text{sync}\\ \alpha\beta\gamma\delta}}\frac{\left[\lambda_{-\alpha\beta(\gamma+\delta)}\lambda_{(\gamma+\delta)\gamma\delta}/\lambda_{(\gamma+\delta)}\right] - \lambda_{-\alpha\beta\gamma\delta}}{\omega_\alpha(\partial\lambda_\alpha/\partial\omega_\alpha)}\hat{a}_\beta\hat{a}_\gamma\hat{a}_\delta \exp j\delta\theta_{-\alpha\beta\gamma\delta} \qquad .$$

$$(3.77)$$

Dysthe[21] has presented a special case of this result in which he has pointed out that the terms involving $\partial\underset{\sim}{v}_\alpha/\partial\underset{\sim}{k}$ represent the effects of dispersion in the medium. The right hand side of (3.77) accounts for self-action effects and the coupling of α with other modes.

3.6 Outline of the Averaged-Lagrangian Method

At this point, all of the basic concepts and procedures in the averaged-Lagrangian analysis have been introduced. They have already been applied to obtain standard forms for the equations of wave evolution, in certain cases of fundamental interest. It is now possible to systematize the approach. The steps of the method are stated below, where they are ordered in such a way as to ensure both efficiency and broad applicability. A flow chart of the entire analysis is shown in Fig. 3.4.

The given information is a Lagrangian density, $\mathcal{L}$, in phase space or positional space only; the absence of velocity coordinates does not change the analysis, except to eliminate the averaging over $\underset{\sim}{v}$.

1. Dual Expansion of $\overline{\mathcal{L}}$

(1.1) Choose a convenient reference state, defined by parameters which change only on the slow scale in position and time, and which differ from the true parameters of the medium only by small perturbation components. In some cases, the choice of reference state is a matter of taste. In others, it is dictated by the problem.

(1.2) Expand $\mathcal{L}$ in joint powers of the perturbations, $\{q^i\}$, from the reference state. Through the substitution $q^i = q_w^i + q_0^i$, evaluate, in algebraic form, the terms $\{\mathcal{L}_{m,n}\}$ of the dual expansion, where subscripts m and n indicate the order of each term in joint powers of the wave components, $\{q_w^i\}$, and background components, $\{q_0^i\}$, respectively. In any tractable problem, it will be sufficient to evaluate the expansion terms only for the lower values of m and n .

(1.3) From $\mathcal{L}_2$ $(\equiv \Sigma_n \mathcal{L}_{2,n})$, obtain the Euler-Lagrange equations for the $\{q_w^i\}$, and solve to determine the dispersion relations and the small-signal relations between wave parameters:

$$D_\eta(\omega_\eta, \underset{\sim}{k}_\eta, q_0^i, \epsilon \underset{\sim}{x}, \epsilon t) = 0 , \qquad \hat{q}_\eta^i = c_\eta^i(\omega_\eta, \underset{\sim}{k}_\eta, q_0^i, \underset{\sim}{v}, \epsilon \underset{\sim}{x}, \epsilon t) \, \hat{a}_\eta ,$$
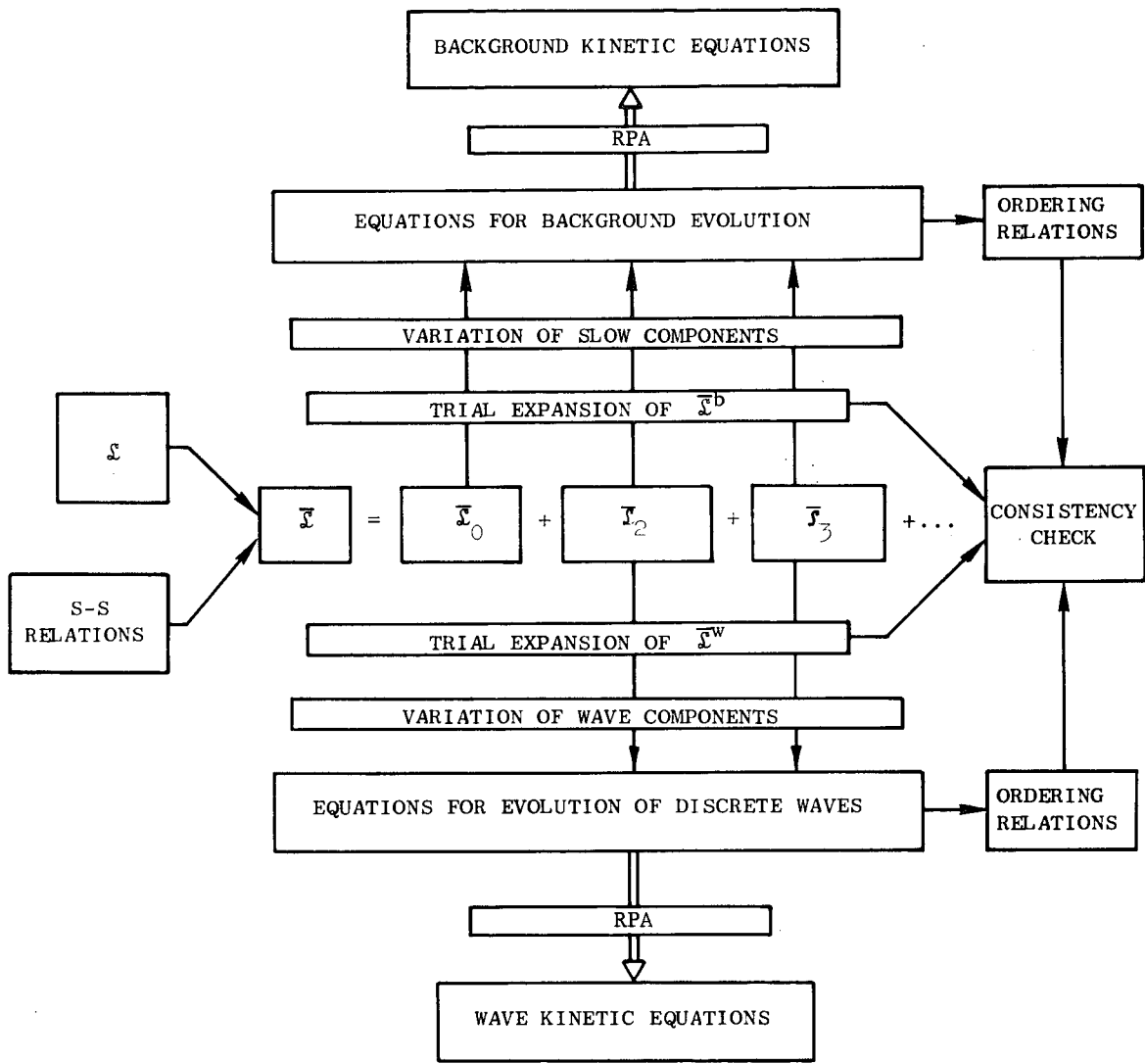
Fig. 3.4   Flow chart for the averaged-Lagrangian analysis.   From a
           trial approximation to $\overline{\mathcal{L}}$, the wave and background equations
           are obtained separately by the variation of the corresponding
           parameters, in keeping with the averaged Hamilton's principle.
           The equations establish the order of the variables in terms
           of the characteristic small quantity, ϵ, and the ordering
           relations determine whether or not the trial approximation is
           self-consistent.   The process is repeated until a self-
           consistent approximation is found.   Application of the RPA,
           to the equations for monochromatic waves, yields the kinetic
           equations, which describe a weakly turbulent medium.

where $\hat{a}_\eta$ is a convenient amplitude parameter for Wave η , and the $\{c_\eta^i\}$ are the mode structure coefficients.

(1.4) Employ the small-signal relations, together with a limited space-time (bar) averaging, to find the explicit expressions for the terms $\{\overline{\mathcal{L}}_m \equiv \Sigma_n \overline{\mathcal{L}}_{m,n}\}$, where

$$\overline{\mathcal{L}}_0 = \mathcal{L}_0 \quad , \qquad \overline{\mathcal{L}}_1 = 0 \quad ,$$

$$\overline{\mathcal{L}}_2 = \sum_\eta \lambda_\eta \hat{a}_\eta \hat{a}_{-\eta} \exp{-2\theta_\eta i} \quad ,$$

and

$$\overline{\mathcal{L}}_m = \sum_{\substack{sync \\ \alpha\beta\dots m}} \left(\lambda_{-\alpha\beta\dots m} \hat{a}_{-\alpha} \hat{a}_\beta \dots \hat{a}_m \exp{j\delta\theta_{\alpha\beta\dots m}} + c.c \right) \quad ,$$

for $m > 2$. The coefficients $\{\lambda_{\alpha\beta\dots}(\omega_\alpha, \underset{\sim}{k}_\alpha, \omega_\beta, \underset{\sim}{k}_\beta, \dots, \underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t)\}$, are determined in the averaging process.

2. Background Analysis.

(2.1) Make an approximation to $\overline{\mathcal{L}}$, which describes the lowest-order wave-background interaction effects. To do this, use

$$\overline{\mathcal{L}}^b \approx \sum_n (\overline{\mathcal{L}}_{0,n} + \overline{\mathcal{L}}_{2,n}) \quad ,$$

and simplify this approximation further, by dropping all but a few of the terms in the secondary expansion. Those retained must correspond to low values of n . From the trial approximation, obtain the Euler-Lagrange equations for the $\{q_0^i(\underset{\sim}{v}, \epsilon\underset{\sim}{x}, \epsilon t)\}$, and from these and the ordering rule

$$\hat{a}_\eta \sim \epsilon \quad ,$$

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find the positive integers, $\{\mu(i)\}$, such that

$$q_0^i \sim \epsilon^{\mu(i)} \quad .$$

If the approximation is consistent, these ordering relations will show that the terms neglected in the trial approximation to $\overline{\mathcal{L}}^b$ make a negligible contribution to the equations. If this is not the case, make a new trial approximation, and repeat the process until a self-consistent approximation is found. The result will be the lowest order background equations, plus a consistent set of ordering relations.

(2.2) Obtain higher order background equations from more accurate approximations to $\overline{\mathcal{L}}^b$. The joint interactions of N waves with the background are accounted for by the approximation

$$\overline{\mathcal{L}}^b \approx \overline{\mathcal{L}}_0 + \sum_{m=2}^{N} \overline{\mathcal{L}}_m \quad ,$$

in which each term is expressed to the same order in ϵ , by means of a truncated secondary expansion

$$\mathcal{L}_m \approx \mathcal{L}_{m,0} + \mathcal{L}_{m,1} + \cdots \quad ,$$

and the operator expansion,

$$\omega_\eta \rightarrow \omega_\eta - j(\partial/\partial t)_\eta \quad , \qquad \underset{\sim}{k}_\eta \rightarrow \underset{\sim}{k}_\eta + j(\partial/\partial \underset{\sim}{x})_\eta \quad ,$$

which accounts for the changing wave amplitudes. The subscript, η , on an operator indicates that it operates only on $\hat{a}_\eta$. The ordering relations of Step (2.1) are sufficient to establish the length to which the expansion processes must be carried. From the higher order approximation to $\overline{\mathcal{L}}^b$, the background equations are obtained by the Euler-Lagrange formula, applied separately to each variable, q_0^i $(i=1,\ldots M)$.

3. Wave Analysis

(3.1) Make the formal expansion of $\overline{\mathcal{L}}^W$, which does not depend on the explicit form of $\overline{\mathcal{L}}$. To account for nonlinear interactions of N , and fewer, synchronous waves, use the approximation

$$\tilde{\overline{\mathcal{L}}}^W \approx \sum_{m=2}^{N} \tilde{\overline{\mathcal{L}}}_m \quad ,$$

in which the terms are expressed in the general forms

$$\tilde{\overline{\mathcal{L}}}_m = \sum_{\substack{\text{sync} \\ \alpha\beta\ldots m}} \left\{ \tilde{\lambda}_{-\alpha\beta\ldots m} \hat{a}_\alpha \hat{a}_\beta \ldots \hat{a}_m \exp j\delta\theta_{-\alpha\beta\ldots m} + c.c. \right\} \quad ,$$

where

$$\tilde{\lambda}_{-\alpha\beta\ldots}(\omega_\alpha, \underset{\sim}{k}_\alpha, \omega_\beta, \underset{\sim}{k}_\beta, \ldots) \equiv \int \lambda_{-\alpha\beta\ldots} \, d^3v \quad .$$

All terms of $\overline{\mathcal{L}}^W$ are to be expressed to the same order in ϵ , by means of the operator expansion used previously in Step (2.2).

(3.2) Conduct the formal wave analysis. Using the expanded formal expression for $\tilde{\overline{\mathcal{L}}}^W$, apply the averaged Hamilton's principle to the variation of a single amplitude component, say $\hat{a}_{-\alpha}$, to obtain an equation for the evolution of $\hat{a}_\alpha$. This equation yields expressions for the amplitudes of virtual waves, such as $\hat{a}_{(\mu+\nu+\ldots)}$, and these, in turn, are to be used to express the equation for $\hat{a}_\alpha$ exclusively in terms of the other propagating waves $\beta, \gamma, \delta, \ldots$. This done, use the appropriate permutations of subscripts (indicated by alternative forms of the synchronism condition) to obtain a complete set of coupled-mode equations. The coefficients of these equations are expressed in general form, in terms of the $\tilde{\lambda}$'s and their derivatives. Note that although this step may be quite involved, it need be followed only once for a particular type of interaction, as

characterized by the synchronism conditions. From the general solution, one may obtain many specific solutions by evaluating the $\tilde{\lambda}$'s .

(3.3) Choose a convenient reference state. If the state chosen is that of Step 1, proceed with the subsequent steps. If the reference state is to be changed (for instance to one in which all $\{q_0^i\}$ are zero), then $\bar{\mathcal{L}}$, and the λ's must be re-evaluated.

(3.4) Integrate $\bar{\mathcal{L}}$ over the velocity coordinate, if it is used in the description of the medium, to obtain the expansion terms $\{\tilde{\bar{\mathcal{L}}}_{m,n}\}$ and the corresponding $\tilde{\lambda}$'s in explicit form.

(3.5) Determine the wave coupling parameters by substituting the specific form for the $\tilde{\lambda}$'s into the general form of the equations found in Step (3.2). In doing so, express each term in the equations to the same order in ϵ , by employing the ordering relations, and the dual expansion

$$\tilde{\lambda}_{-\alpha\beta\ldots} = \sum_n \tilde{\lambda}_{-\alpha\beta\ldots,n} \quad ,$$

to account for the effects of the background perturbations, $\{q_0^i\}$.

(3.6) Describe the evolution of the frequencies and wavenumbers. The equations obtained in Step (3.2) describe the evolution of the $\{\hat{a}_\eta(\epsilon\underset{\sim}{x},\epsilon t)\}$. The wave analysis must be completed by equations which describe the evolution of the $\{\omega_\eta\}$ and $\{\underset{\sim}{k}_\eta\}$. The needed equations are simply

$$D_\eta(\omega_\eta,\underset{\sim}{k}_\eta, q_0^i , \epsilon\underset{\sim}{x}, \epsilon t) = 0, \qquad \underset{\sim}{v}_\eta = (\partial\omega_\eta/\partial\underset{\sim}{k}_\eta) \quad ,$$

where $\underset{\sim}{v}_\eta$, the group velocity, is the velocity of a nearly monochromatic wavepacket. The evolution of ω_η and $\underset{\sim}{k}_\eta$ is determined by tracing the wavepacket, in accordance with the above equations.

When it is applicable, the averaged-Lagrangian method is sufficient to yield a complete set of equations for nonlinear wave and background evolution in the weak-coupling limit. The method, as it is outlined

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above, pertains only to monochromatic waves. However, the equations may be modified to account for a continuous wave spectrum, by means of the RPA (random-phase approximation), to be described in the next section. For this reason, the application of the RPA has been shown in Fig. 3.4. The limits on the applicability of the averaged-Lagrangian method have already been established through the theoretical development of this section. They will be reviewed at the beginning of Section 5, which is primarily concerned with the applicability and capabilities of the averaged-Lagrangian approach.

4. APPLICATIONS

In this section, the Lagrangian method will be applied to two non-linear plasma models. In the first, the plasma is treated in the hydrodynamic approximation. In the second, it is described statistically by a distribution function $f(\underset{\sim}{v},\underset{\sim}{x},t)$. In these applications, both wave and background perturbations will be taken into account.

For other applications, the reader may refer to earlier work by the author, in which more restrictive, but similar, methods have been applied to wave-wave interactions only.[22-24] That work was based on the three-wave action-transfer relations introduced in Section 3.5.1, and the applications were restricted accordingly. In one of these early applications,[23] the general formula for the wave coupling coefficients was derived for three electromagnetic waves propagating at arbitrary angles in a cold, homogeneous magnetoplasma with negligible ion motion. Because of their complexity, these coefficients had been previously found only in special cases.

4.1 Ion-acoustic Waves in a Plasma

The first application of the averaged-Lagrangian method is to a plasma in which ion-acoustic waves are propagating. These waves are analogous to acoustic waves in a neutral gas. In the ion-acoustic mode, the wave periods are long enough to ensure that the electron and ion charge densities are equalized by electrostatic forces. Here we shall assume that the particles are singly ionized, so that the electron and ion densities are equal. We shall also assume that $T_i \ll T_e$, where T_i and T_e are the ion and electron temperatures, respectively. In this case, Landau damping of the waves is negligible, and in the hydrodynamic model, the pressure effects are contributed primarily by the electrons, and the inertial effects are contributed primarily by the ions. We shall assume that three discrete, synchronous, ion-acoustic waves are propagating collinearly in the plasma. First, we shall apply the averaged-Lagrangian method to obtain the interaction equations;

then we shall further simplify the equations by means of additional physical assumptions. Finally, we shall state an analytic solution for that special case.

4.1.1 Formal Lagrangian Description

In this analysis, it is convenient to choose an equilibrium state as the reference state for the perturbation expansion of $\mathcal{L}$. The state chosen is homogeneous; it has a charged particle density n_0 , electron temperature T_0 , and electron pressure P_0 , related by

$$P_0 = n_0 K T_0 \qquad\qquad (4.1)$$

where K is Boltzmann's constant. The symbols n , and P will denote the perturbed particle density and electron pressure, respectively, and v will denote the perturbed velocity. The treatment is one dimensional, so that these parameters are scalar functions of time, t , and position, z measured in the direction of propagation. The Lagrangian density for the system may be obtained through the classical identification of $\mathcal{L}$ with $\mathcal{T}-\mathcal{U}$ where $\mathcal{U}$ is the potential energy density gained from work done against electron pressure. The total Lagrangian has been obtained from this prescription by Dysthe.[48] It is

$$\int \mathcal{L} d\tau = \int \left[\frac{1}{2} Mnv^2 + n_0 \int_{n_0}^{n} P \, d\!\left(\frac{1}{n}\right) \right] d\tau \qquad , \qquad (4.2)$$

where M is the ion mass and the integration is over the volume of the system. We must now express this integral in terms of one or more independent generalized variables. To this end, let z_p denote the particle position, and let $z_p = z + \zeta(z,t)$, so that $v = \partial\zeta/\partial t$. The total Lagrangian is then

$$\int \mathcal{L} d\tau = \int \left[\frac{1}{2} Mn_0 \left(\frac{\partial\zeta}{\partial t}\right)^2 - U(n) \right] d\tau \qquad , \qquad (4.3)$$

where

$$U(n) \equiv -n_0 \int_{n_0}^{n} P \, d(1/n) \quad . \tag{4.4}$$

One may now make use of the fact that $nd\tau = n_0 d\tau_0 = nJd\tau_0$, where the Jacobian, J, is $(1 + \partial\zeta/\partial z)$. This yields

$$\mathcal{L} = \frac{1}{2} Mn_0 \left(\frac{\partial\zeta}{\partial t}\right)^2 - U\left(\frac{n_0}{1 + \partial\zeta/\partial z}\right) \quad . \tag{4.5}$$

The adiabatic gas law is used for perturbations about equilibrium. Thus

$$Pn^{-\gamma} = P_0 n_0^{-\gamma} \quad . \tag{4.6}$$

Neglecting terms with no dependence on n, one obtains

$$U(n) = \frac{n_0 KT}{(\gamma-1)} \left\{ \left(\frac{n}{n_0}\right)^{\gamma-1} - 1 \right\} \quad , \tag{4.7}$$

and an expansion in powers of $\partial\zeta/\partial z$ yields

$$U\left(\frac{n_0}{1+\partial\zeta/\partial z}\right) = n_0 KT\left[-\frac{\partial\zeta}{\partial z} + \frac{\gamma}{2}\left(\frac{\partial\zeta}{\partial z}\right)^2 - \gamma\left(\frac{\gamma+1}{6}\right)\left(\frac{\partial\zeta}{\partial z}\right)^3 + \cdots \right] \quad . \tag{4.8}$$

Constant terms which depend only on the reference state parameters have been omitted in this expansion. The Lagrangian may now be expanded in powers of the displacement, ζ, to yield the series $\mathcal{L} = \mathcal{L}_{(0)} + \mathcal{L}_{(1)} + \mathcal{L}_{(3)} + \cdots$.
The term $\mathcal{L}_{(0)}$ is irrelevant to the Lagrangian analysis, and so is $\mathcal{L}_{(1)}$ in this case, since the reference state is an exact solution. The first two relevant terms of the series are consequently

$$\mathcal{L}_{(2)} = \frac{Mn_0}{2}\left(\frac{\partial\zeta}{\partial t}\right)^2 - \frac{n_0 KT\gamma}{2}\left(\frac{\partial\zeta}{\partial z}\right)^2 \quad ,$$

and

$$\mathcal{L}_{(3)} = \frac{n_0 KT\gamma(\gamma+1)}{6}\left(\frac{\partial\zeta}{\partial z}\right)^3 \quad . \tag{4.9}$$

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From these, we may construct the dual expansion by substituting $\zeta_0 + \zeta_w$ for ζ. We make a trial approximation to $\overline{\mathcal{L}}$ by retaining only a few of the lowest order terms in this expansion. The first trial approximation is

$$\overline{\mathcal{L}} \approx \overline{\mathcal{L}}_0 + \overline{\mathcal{L}}_2 + \overline{\mathcal{L}}_3 \qquad , \qquad (4.10)$$

in which

$$\overline{\mathcal{L}}_0 \approx \overline{\mathcal{L}}_{0,2} = \frac{Mn_0}{2}\left(\frac{\partial \zeta_0}{\partial t}\right)^2 - \frac{n_0 KT\gamma}{2}\left(\frac{\partial \zeta_0}{\partial z}\right)^2 \qquad ,$$

$$\overline{\mathcal{L}}_2 \approx \overline{\mathcal{L}}_{2,0} + \overline{\mathcal{L}}_{2,1} = \frac{Mn_0}{2}\overline{\left(\frac{\partial \zeta_w}{\partial t}\right)^2} - \frac{n_0 KT\gamma}{2}\overline{\left(\frac{\partial \zeta_w}{\partial z}\right)^2} + \frac{n_0 KT\gamma(\gamma+1)}{2}\overline{\left(\frac{\partial \zeta_0}{\partial z}\right)\left(\frac{\partial \zeta_w}{\partial z}\right)^2} \qquad ,$$

$$\overline{\mathcal{L}}_3 \approx \overline{\mathcal{L}}_{3,0} = \frac{n_0 KT\gamma(\gamma+1)}{6}\overline{\left(\frac{\partial \zeta_w}{\partial z}\right)^3} \qquad . \qquad (4.11)$$

The approximation (4.10) — (4.11) is reasonable because $\overline{\mathcal{L}}_{2,0}$ and $\overline{\mathcal{L}}_{0,2}$ are the terms which govern the independent wave and background behavior, $\overline{\mathcal{L}}_{3,0}$ is the first term in the dual expansion which gives rise to coupling between wave components, and $\overline{\mathcal{L}}_{2,1}$ is the first term which gives rise to coupling between the wave and background components. In order to see whether this is a valid approximation, we must proceed with the analysis up to the derivation of the background equations of motion. This will indicate the order of ζ_0 with respect to ϵ, and we may check to see if the approximation is self-consistent up to the desired order in ϵ.

We shall now simplify $\overline{\mathcal{L}}$ by means of the small-signal relations. To obtain these relations, we use the assumed form of ζ_w:

$$\zeta_w = \sum_\eta \left(\hat{\zeta}_\eta \exp j\theta_\eta + \hat{\zeta}_{-\eta} \exp -j\theta_\eta\right) \qquad , \qquad \theta = \text{real} \qquad . \qquad (4.12)$$

When this is substituted into the trial approximation for $\bar{\mathcal{L}}_2$, the result is

$$\bar{\mathcal{L}}_2 \approx \sum_\eta \lambda_\eta \, \hat{\zeta}_\eta \, \hat{\zeta}_{-\eta} \quad , \tag{4.13}$$

where

$$\lambda_\eta \equiv Mn_0 \left\{ \omega_\eta^2 - c_s^2 \left[1 - \left(\gamma+1\right)\left(\frac{\partial \zeta_0}{\partial z}\right) \right] k_\eta^2 \right\} \quad , \tag{4.14}$$

and

$$c_s \equiv \left(\frac{\gamma KT}{M}\right)^{1/2} \quad . \tag{4.15}$$

When this expression for $\bar{\mathcal{L}}_2$ is used in the averaged-Hamilton's principle, and $\zeta_{-\eta}$ is varied, the corresponding Euler-Lagrange equation is $\lambda_\eta \hat{\zeta}_\eta = 0$, and this may be solved for the small-signal relations. For this problem the only such relation is the dispersion relation $\lambda_\eta = 0$, or

$$\omega_\eta^2 = c_s^2 \left[1 - \left(\gamma+1\right)\left(\frac{\partial \zeta_0}{\partial z}\right) \right] k_\eta^2 \quad . \tag{4.16}$$

If there had been more than one total perturbation variable in this problem, λ_η would have been a tensor, and the small-signal relations would have included relations between the mode variables, $\bar{q}_\eta^i = c_\eta^i \, \bar{a}_\eta$. These would be used to express $\bar{\mathcal{L}}_2$ in terms of the $\{q_0^i\}$ and one scalar parameter for each wave component. In this instance, however, the only total perturbation variable is the scalar, ζ . We express $\bar{\mathcal{L}}$ in the desired form merely by using the form for ζ_w given by (4.12), and requiring that, for each component, ω_η and k_η satisfy (4.16). We find that for the first trial approximation

$$\bar{\mathcal{L}}_{trial} = \frac{Mn_0}{2}\left(\frac{\partial \zeta_0}{\partial t}\right)^2 - \frac{n_0 KT\gamma}{2}\left(\frac{\partial \zeta_0}{\partial z}\right)^2 + \sum_\eta \lambda_\eta \hat{\zeta}_\eta \hat{\zeta}_{-\eta}$$
$$+ \sum_{\substack{sync \\ \alpha\beta\gamma}} \left(\lambda^{-\alpha\beta\gamma} \, \hat{\zeta}_{-\alpha} \hat{\zeta}_\beta \hat{\zeta}_\gamma \exp j\delta\theta_{-\alpha\beta\gamma} + c.c. \right) \tag{4.17}$$

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where λ_η is defined in (4.14), and

$$\lambda^{-\alpha\beta\gamma} \equiv j\left[n_0 KT\gamma(\gamma+1)\right] k_{-\alpha} k_\beta k_\gamma \qquad . \tag{4.18}$$

This completes Step 1 of the Lagrangian method, for the first trial approximation.

4.1.2 Equations of Background Evolution

In this analysis, we wish to describe the wave-background interaction only to lowest order in ϵ . For this, it suffices to use the trial approximation

$$\overline{\mathcal{L}}^b \approx \overline{\mathcal{L}}_{0,2} + \overline{\mathcal{L}}_{2,1} \qquad ,$$

that is

$$\overline{\mathcal{L}}^b \approx \left\{\frac{Mn_0}{2}\right\}\left\{\left(\frac{\partial\zeta_0}{\partial t}\right)^2 - c_s^2\left[\left(\frac{\partial\zeta_0}{\partial z}\right)^2 - 2\left(\gamma+1\right)\left(\frac{\partial\zeta_0}{\partial z}\right)\sum_\eta k_\eta^2 |\hat{\zeta}_\eta|^2\right]\right\} \qquad . \tag{4.19}$$

This expression is the same as that for $\overline{\mathcal{L}}$, except that terms which do not involve ζ_0 have been dropped. In this approximation, the Euler-Lagrange equation for ζ_0 is

$$\left(\frac{\partial^2\zeta_0}{\partial t^2}\right) - c_s^2\left(\frac{\partial^2\zeta_0}{\partial z^2}\right) + 2(\gamma+1)c_s^2\sum_\eta k_\eta^2 \frac{\partial}{\partial z}|\hat{\zeta}_\eta|^2 = 0 \qquad . \tag{4.20}$$

ORDERING RELATIONS: from Section 3 we have the ordering rules

$$\zeta_\eta \sim \epsilon \qquad (\eta = \alpha,\beta,\gamma) \qquad ,$$

$$(\partial\zeta_\mu/\partial z) \sim (\partial\zeta_\mu/\partial t) \sim \epsilon\zeta_\mu \qquad (\mu = 0,\alpha,\beta,\gamma) \qquad . \tag{4.21}$$

When these are used to order the terms of (4.20) with respect to ϵ , the result is

$$\zeta_0 \sim \epsilon \quad , \qquad (\partial\zeta_0/\partial z) \sim (\partial\zeta_0/\partial t) \sim \epsilon^2 \qquad . \tag{4.22}$$

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CONSISTENCY CHECK: the ordering relations (4.21) — (4.22) show that the original trial approximation is inconsistent; the terms $\bar{\Sigma}_{0,2}$ and $\bar{\Sigma}_{2,1}$ are of $O(\epsilon^4)$, and, if the approximation is to be consistent to that order, the term $\bar{\Sigma}_{4,0}$ must be included in the approximation also, i.e., four-wave effects must be considered. However, in this example, only three-wave effects will be considered, and $\bar{\Sigma}$ will be expanded only through order ϵ^3 . For this example, then, the valid approximation for $\bar{\Sigma}$ through $O(\epsilon^3)$ is

$$\bar{\Sigma} \approx \bar{\Sigma}_{2,0} + \bar{\Sigma}_{3,0} = \sum_{\eta} \lambda_{\eta} \hat{\zeta}_{\eta} \hat{\zeta}_{-\eta} + \sum_{\substack{\text{sync} \\ \alpha\beta\gamma}} \left(\lambda_{-\alpha\beta\gamma} \hat{\zeta}_{-\alpha} \hat{\zeta}_{\beta} \hat{\zeta}_{\gamma} + \text{c.c.} \right) \quad , \quad (4.23)$$

where

$$\lambda_{\eta} \approx M n_0 \left[\omega_{\eta}^2 - c_s^2 k_{\eta}^2 \right] \quad ,$$

$$(4.24)$$

$$\lambda_{-\alpha\beta\gamma} \approx j n_0 KT\gamma(\gamma+1) k_{-\alpha} k_{\beta} k_{\gamma} \quad .$$

In this approximation, the wave components interact only with one another. They are independent of any background fluctuations, which are negligible. For this example, then, we need to analyze only the wave behavior.

4.1.3 Equations of Wave Evolution

In the approximation (4.23) it is understood that all terms are to be expanded through $O(\epsilon^3)$ by means of the formal expansion procedure of (2.30). The result is just a special case of (3.44) yielding

$$\bar{\Sigma}^w = -\sum_{\eta} \left[\frac{j}{2} \left(\frac{\partial \lambda_{\eta}}{\partial \omega_{\eta}} \right) \left(\frac{\partial \hat{\zeta}_{\eta}}{\partial t} \hat{\zeta}_{-\eta} - \zeta_{\eta} \frac{\partial \hat{\zeta}_{-\eta}}{\partial t} \right) - \frac{j}{2} \left(\frac{\partial \lambda_{\eta}}{\partial k_{\eta}} \right) \left(\frac{\partial \hat{\zeta}_{\eta}}{\partial z} \zeta_{-\eta} - \zeta_{\eta} \frac{\partial \hat{\zeta}_{-\eta}}{\partial z} \right) \right]$$

$$+ \sum_{\substack{\text{sync} \\ \alpha\beta\gamma}} \left(\lambda_{-\alpha\beta\gamma} \zeta_{-\alpha} \zeta_{\beta} \zeta_{\gamma} \exp j\delta\theta_{-\alpha\beta\gamma} + \text{c.c.} \right) \quad . \quad (4.25)$$

The term $\lambda_\eta \hat{\xi}_\eta \hat{\xi}_{-\eta}$ does not appear in the expansion because we choose ω_η and k_η to be solutions of the dispersion relation, i.e., they are chosen such that λ_η is zero. For the form of $\bar{\mathcal{L}}^W$ given above, the formal analysis has already been carried out in Section 3.5, and the equations of motion are the one-dimensional limit of (3.46), with the normalized variables and coupling coefficients given by (3.47) and the expressions for the λ's in (4.24). Specifically, we have

$$j\left(\frac{\partial}{\partial t} + v_\alpha \frac{\partial}{\partial z}\right)\hat{A}_\alpha = \chi\,\hat{A}_\beta\hat{A}_\gamma \quad ,$$

$$j\left(\frac{\partial}{\partial t} + v_\beta \frac{\partial}{\partial z}\right)\hat{A}_\beta = \chi^*\hat{A}_\gamma^*\hat{A}_\alpha \quad ,$$

$$j\left(\frac{\partial}{\partial t} + v_\gamma \frac{\partial}{\partial z}\right)\hat{A}_\gamma = \chi^*\hat{A}_\beta^*\hat{A}_\alpha \quad , \tag{4.26}$$

in which, for $\eta = \alpha,\beta,\gamma$,

$$\hat{A}_\eta = (2Mn_0\omega_\eta)^{1/2}\,\hat{\xi}_\eta \quad , \qquad v_\eta = \pm\,c_s \quad , \qquad \omega_\eta^2 = c_s^2 k_\eta^2 \quad ,$$

$$\chi = -j\,\frac{n_0 KT\gamma(\gamma+1)k_\alpha k_\beta k_\gamma \exp\,\delta\theta_{-\alpha\beta\gamma}}{(2Mn_0)^{3/2}(\omega_\alpha\omega_\beta\omega_\gamma)^{1/2}} \quad . \tag{4.27}$$

This completes the derivation of the interaction equations by the averaged-Lagrangian method. For the case $\gamma = 1$, the equations have been obtained previously by Ohnuma and Hatta, who employed an iterative analysis.[49]

4.1.4 Solution Procedure

By means of additional physical and mathematical assumptions, it is possible to simplify the interaction equations to a form for which analytic solutions are known. First, we shall represent the complex wave amplitudes in terms of real quantities $\{\mathcal{A}_\eta\}$, and $\{\phi_\eta\}$, such that

$$\hat{A}_\eta \equiv \mathcal{A}_\eta \exp\,j\phi_\eta \quad , \qquad \mathcal{A}_\eta \equiv |\hat{A}_\eta| \quad (\eta=\alpha,\beta,\gamma) \quad . \tag{4.28}$$

The real, normalized wave parameters, $\{A_\eta\}$, are thus given by

$$A_\eta \equiv \hat{A}_\eta \exp j\theta_\eta + \text{c.c.} \equiv \mathscr{A}_\eta \left[\exp j(\theta_\eta + \phi_\eta) + \text{c.c.} \right] \qquad . \qquad (4.29)$$

All information about the phases of the waves may thus be accounted for by the sums $\{\theta_\eta + \phi_\eta\}$. Because the background inhomogeneity is negligible for this problem, the $\{\theta_\eta\}$ have the form

$$\theta_\eta = \left[\omega_\eta t - k_\eta z + \phi'_\eta(\epsilon z, \epsilon t) \right] \qquad (4.30)$$

in which the $\{\phi'_\eta\}$ are real quantities. We shall stipulate that

$$\phi'_\eta \equiv 0 \qquad (\eta = \alpha, \beta, \gamma) \qquad , \qquad (4.31)$$

for the rest of this analysis. This causes no loss of generality in the application of the interaction equations (4.26), since any slow-scale changes in the relative phases of the waves may still be accounted for by the $\{\phi_\eta\}$ of (4.28) — (4.29). Now we shall restrict consideration to a relatively simple problem, in which the waves are excited by a continuous oscillation of the displacement, ζ, at $z = 0$. The boundary conditions are assumed to be

$$A(0,t) = \sum_\eta \mathscr{A}_{\eta 0} \exp j(\omega_\eta t + \phi_{\eta 0}) + \text{c.c.} \qquad , \qquad (4.32)$$

$$\omega_\alpha = \omega_\beta + \omega_\gamma \qquad , \qquad (4.33)$$

where the $\{\mathscr{A}_{\eta 0}\}$, $\{\phi_{\eta 0}\}$, and $\{\omega_\eta\}$ are specified constants, and

$$0 \leq \phi_{\eta 0} < 2\pi \qquad . \qquad (4.34)$$

The boundary conditions (4.32) — (4.34) imply that the wave parameters

change only according to the distance z from the source of excitation, i.e.,

$$\mathscr{A}_\eta = \mathscr{A}_\eta(\epsilon z) \quad , \qquad \phi_\eta = \phi_\eta(\epsilon z) \qquad (\eta = \alpha, \beta, \gamma) \quad . \qquad (4.35)$$

We are considering waves which propagate away from the source, so that $\omega_\eta = c_s k_\eta \ (\eta = \alpha, \beta, \gamma)$. Thus, (4.33) is sufficient to ensure that the ω's and k's are in synchronism for all z . This fact, together with (4.31) and (4.27), implies that

$$\delta\theta_{-\alpha\beta\gamma} = 0 \quad , \qquad X = -j|X| \quad , \qquad (4.36)$$

where $|X|$ is a constant.

When the interaction equations (4.26) are expressed in the notation of (4.28) − (4.29), they may be simplified for the problem at hand by means of (4.35) − (4.36). The real and imaginary parts of the equations may then be separated, to produce the four equations

$$\frac{\partial}{\partial z}\mathscr{A}_\alpha = -\frac{|X|}{c_s}\mathscr{A}_\beta \mathscr{A}_\gamma \sin\phi \quad , \qquad (4.37)$$

$$\frac{\partial}{\partial z}\mathscr{A}_\beta = \frac{|X|}{c_s}\mathscr{A}_\gamma \mathscr{A}_\alpha \sin\phi \quad , \qquad (4.38)$$

$$\frac{\partial}{\partial z}\mathscr{A}_\gamma = \frac{|X|}{c_s}\mathscr{A}_\beta \mathscr{A}_\alpha \sin\phi \quad , \qquad (4.39)$$

$$\frac{\partial}{\partial z}\phi = \left(\frac{|X|}{c_s}\right)\left(\frac{\mathscr{A}_\alpha \mathscr{A}_\beta}{\mathscr{A}_\gamma} + \frac{\mathscr{A}_\alpha \mathscr{A}_\gamma}{\mathscr{A}_\beta} - \frac{\mathscr{A}_\beta \mathscr{A}_\gamma}{\mathscr{A}_\alpha}\right)\cos\phi \quad , \qquad (4.40)$$

where

$$\phi \equiv \phi_\beta + \phi_\gamma - \phi_\alpha + \frac{\pi}{2} \quad . \qquad (4.41)$$

Equation (4.40) may be integrated to show that

$$\mathscr{A}_\alpha \mathscr{A}_\beta \mathscr{A}_\gamma \cos\phi = \Gamma \quad , \qquad (4.42)$$

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where the constant, Γ , is determined by the boundary conditions. Another set of constants is obtained by integrating the Manley-Rowe relations, which we obtain from (4.37) — (4.39) in the manner of Section 3.5.1. The result is

$$N_\alpha + N_\beta = M_1 \quad ,$$

$$N_\alpha + N_\gamma = M_2 \quad , \tag{4.43}$$

$$N_\beta - N_\gamma = M_3 \quad ,$$

where

$$N_\eta \equiv \mathscr{A}_\eta^2 = \frac{\epsilon_\eta}{\omega_\eta} = \left\{\begin{matrix} \text{action density} \\ \text{of Wave } \eta \end{matrix}\right\} \quad , \tag{4.44}$$

and M_1 , M_2 , and M_3 are constants. The conservation relations may be used together with (4.37) — (4.40) to obtain an equation which involves only N_α as a variable. This is

$$\frac{\partial}{\partial z} N_\alpha = -2 \frac{|\chi|}{c_s} \left[N_\alpha (M_1 - N_\alpha)(M_2 - N_\alpha) - \Gamma^2 \right]^{1/2} \quad , \tag{4.45}$$

the solution to which is

$$N_\alpha = N_1 + (N_2 - N_1) \, \text{sn}^2 \left\{ \frac{|\chi|}{c_s} (N_3 - N_1)^{1/2} (z - z_\alpha) \, \Big| \, m \right\} \quad , \tag{4.46}$$

where $\text{sn}\{\mu | m\}$ is a Jacobian Elliptic function, with argument μ and parameter m . The constants, N_1 , N_2 , and N_3, are the three solutions of

$$N(M_1 - N)(M_2 - N) - \Gamma^2 = 0 \quad , \tag{4.47}$$

ordered such that

$$0 < N_1 < N_2 < N_3 \quad . \tag{4.48}$$

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β

The parameter m is given by

$$m \equiv (N_2 - N_1)/(N_3 - N_1) \quad , \qquad\qquad (4.49)$$

and the constant z_0 is chosen so that N_α has the correct value at $z = 0$:

$$\mathrm{sn}\left\{ - \frac{x}{c_s}(N_3 - N_1)^{1/2} z_0 \mid m \right\} = \left(\frac{N_\alpha(0) - N_1}{N_2 - N_1} \right)^{1/2} \quad . \qquad\qquad (4.50)$$

4.1.5 Interpretation of the Solution

For the steps leading from (4.45) to the solution, (4.46), the reader is referred to an earlier exposition by Sagdeev and Galeev.[13] Here, we shall merely discuss some characteristics of the solution. It appears that a solution of the form to be described was first obtained for coupled modes of oscillation by Armstrong et al in 1962.[47]

The function $\mathrm{sn}\{\mu \mid m\}$ may be defined mathematically in terms of an elliptic integral:

$$\mu \equiv \int_0^\psi \frac{d\theta}{(1 - m \sin^2\theta)^{1/2}} \quad . \qquad\qquad (4.51)$$

With this definition of μ , $\mathrm{sn}\{\mu \mid m\}$ is defined as $\sin \psi$, and for real μ it oscillates between 1 and -1 with a quarter period

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{(1 - m \sin^2\theta)^{1/2}} \quad . \qquad\qquad (4.52)$$

We have, for any m , the identities

$$\mathrm{sn}\{2nK(m) \mid m\} = 0 \quad , \qquad \mathrm{sn}\{(2n+1)K(m) \mid m\} = (-1)^n \quad , \qquad (4.53)$$

where n is any integer. The parameters of the solution (4.46) depend on the boundary conditions for the waves, which define Γ , N_1 , N_2 , N_3 , m , and z_0 . Let us consider the solutions in the special case $\Gamma = 0$, in the manner of Sagdeev and Galeev.[13] Suppose that at $z = 0$

we have

$$\Gamma^2 = N_\alpha(0)N_\beta(0)N_\gamma(0) \cos^2 \phi = 0 \quad . \qquad (4.54)$$

This can occur if the amplitude of one of the waves is zero at the source, or if $\phi(0) = (n+1/2)\pi$, where n is an integer. Since Γ is a constant of the motion, the problem is the same in either case. Suppose that $N_\gamma(0) = 0$, $N_\alpha(0) \neq 0$ and $N_\beta(0) \neq 0$. As soon as nonlinear interaction makes $N_\gamma(z)$ finite $\phi(z)$ must be frozen at $(n+1/2)\pi$ in order to keep Γ equal to zero. Now it may be seen from the definition (3.62), that the coupling energy of the energy transfer relation, (3.65), is a maximum for any given wave amplitudes if $\phi = (n+1/2)\pi$; so the case we are considering is the one which leads to the maximum rate of action transfer between the modes. The condition $\Gamma = 0$ (for all z) shows that Wave γ , starting with zero amplitude, initially grows in space with a phase ϕ_γ such that $\phi_\gamma + \phi_\beta - \phi_\alpha + \frac{\pi}{2} = (n+1/2)\pi$: it adjusts its relative phase to ensure the maximum growth rate. Now we have not yet specified any relation between N_β and N_α . It turns out that the solutions divide into two classes, depending on whether N_α , the action density of the higher frequency wave, is greater or less than N_β . We shall examine a particular case of each type:

CASE A: consider the problem specified by the boundary conditions

$$N_\beta(0) \gg N_\alpha(0) \quad , \qquad N_\gamma(0) = 0 \quad . \qquad (4.55)$$

The condition that one wave is much larger than the other two waves is called the parametric limit. In this problem, the parametric limit obviously obtains near $z = 0$, and we shall see that, for this case, it obtains for all z . The large amplitude wave β (the 'pump') exchanges energy with wave α (the 'signal') and wave γ (the 'idler'). For the boundary conditions (4.55) the parameters of the solution are
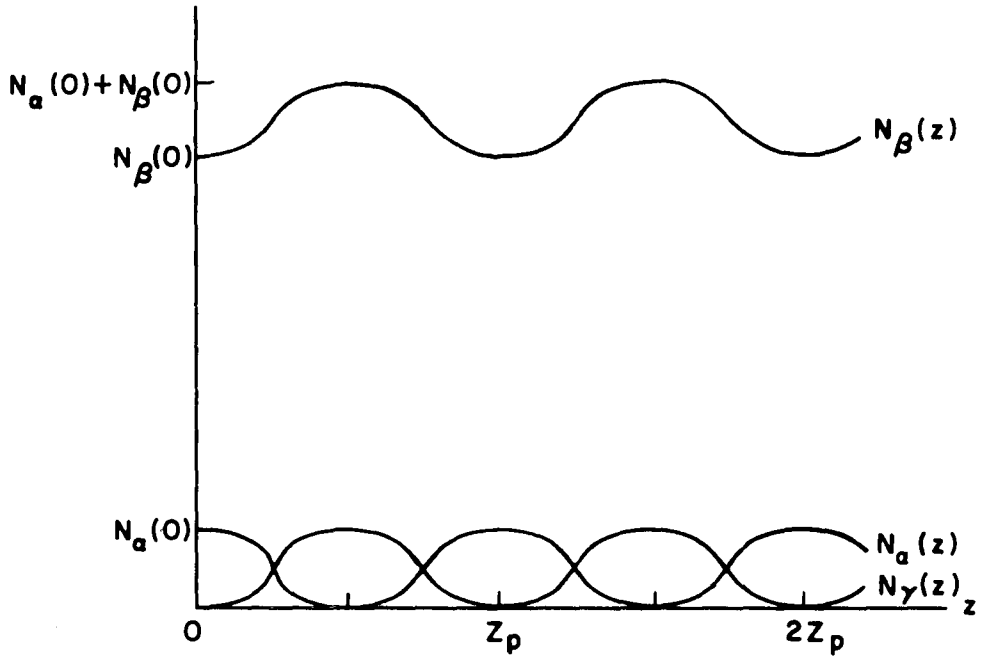
Fig. 4.1   Solution, in Case A, for three coupled ion-acoustic waves,
propagating away from a constant, planar, source.
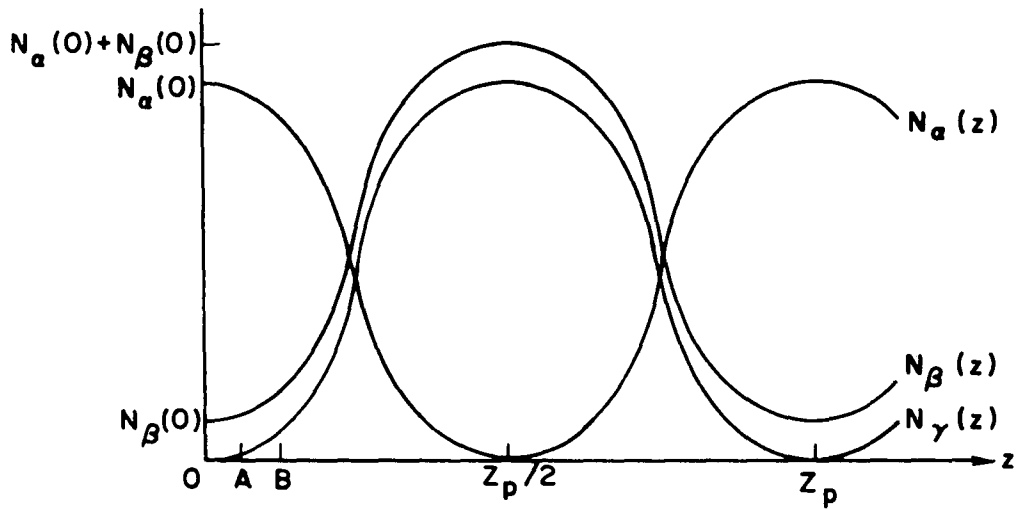
Fig. 4.2   Solution, in Case B, for three coupled ion-acoustic waves,
propagating away from a constant, planar, source.

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given by (4.47) through (4.50)

$$\Gamma = 0 \quad , \qquad N_1 = 0 \quad , \qquad N_2 = N_\alpha(0) \quad , \qquad N_3 = N_\alpha(0) + N_\beta(0)$$

$$m = \frac{N_\alpha(0)}{N_\beta(0) + N_\alpha(0)} \approx 0 \quad , \qquad K \approx \frac{\pi}{2} \quad , \qquad z_0 \approx 0 \quad . \qquad (4.56)$$

The solution for $N_\alpha(z)$ is sketched in Fig. 4.1, as are the curves for $N_\beta(z)$ and $N_\gamma(z)$. These are obtained from the boundary conditions and N_α by means of the Manley-Rowe relations (4.43). They are

$$N_\beta(z) = N_\beta(0) + N_\alpha(0) - N_\alpha(z) \quad , \qquad (4.57)$$

and

$$N_\gamma(z) = N_\alpha(0) - N_\alpha(z) \quad . \qquad (4.58)$$

Note that the solutions for N_α , N_β , and N_γ are all periodic. Each has the same period, Z_p , given by

$$Z_p = \frac{c_s \pi}{|X| \left[N_\alpha(0) + N_\beta(0) \right]^{1/2}} \quad . \qquad (4.59)$$

The magnitude of the wave interaction in Case A is limited; the pump never loses an appreciable fraction of its initial action density. This is guaranteed by the boundary conditions (4.55) and the Manley-Rowe relations (4.43), and is made obvious by Fig. 4.1. In Case A, therefore, the parametric limit holds for all z .

CASE B: consider the problem specified by the boundary conditions

$$N_\alpha(0) \gg N_\beta(0) \quad , \quad N_\gamma(0) = 0 \quad . \qquad (4.60)$$

Again, the parametric approximation holds near $z = 0$, but the Manley-Rowe relations no longer prevent N_α from becoming significantly smaller

than its initial value, even to the point of becoming zero. The parameters of the solution for N_α are

$$\Gamma = 0 \quad , \qquad N_1 = 0 \quad , \qquad N_2 = N_\alpha(0) \quad ,$$

(4.61)

$$N_3 = N_\alpha(0) + N_\beta(0) \quad , \qquad m = \frac{N_\alpha(0)}{N_\alpha(0) + N_\beta(0)} \approx 1 \quad , \qquad z_0 \approx K(m) \quad .$$

The values of N_β and N_γ are determined from N_α by (4.57) – (4.58). The solutions for the three waves are sketched in Fig. 4.2. Note that they are periodic in z, with a period, z_p, given by

$$z_p \approx \frac{2c_s K(m)}{|\chi| \left[N_\alpha(0) \right]^{1/2}} \quad .$$

(4.62)

In this case, Wave α, which is initially the pump, decreases in amplitude with z until all its energy has been transferred to Waves β and γ. At some point, say B on the figure, N_α has decreased so much that the parametric approximation is no longer valid: the pump amplitude may no longer be considered a constant in z. The boundary conditions of this case are well suited to the purpose of amplification. We see that if the high frequency wave is externally excited to a high amplitude, it can transfer a significant fraction of its energy to a signal wave, which is externally excited at a low level.

The interval [A,B] in Fig. 4.2 is the one in which a direct parametric analysis is valid. Throughout this interval, $N_\alpha \gg N_\beta, N_\gamma$. The pump, Wave α, is essentially constant in amplitude and phase. Consequently, we may solve the two equations

$$j \frac{\partial \hat{A}_\beta}{\partial z} = \frac{\chi}{c_s} \hat{A}_\gamma^* \hat{A}_\alpha \quad ,$$

(4.63)

$$j \frac{\partial \hat{A}_\gamma}{\partial z} = \frac{\chi}{c_s} \hat{A}_\beta^* \hat{A}_\alpha \quad ,$$

(4.64)

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in which $\hat{A}_\alpha$ is a constant. These equations follow from the boundary
conditions and (3.56). From (4.63) — (4.64) we obtain

$$\frac{\partial^2 \hat{A}_\beta}{\partial z^2} = \frac{|x|^2}{c_s^2} \hat{A}_\beta \, |\hat{A}_\alpha|^2 \quad , \tag{4.65}$$

and

$$\frac{\partial^2 \hat{A}_\gamma}{\partial z^2} = \frac{|x|^2}{c_s^2} \hat{A}_\gamma \, |\hat{A}_\alpha|^2 \quad . \tag{4.66}$$

Equations (4.65) — (4.66) obviously have solutions for $\hat{A}_\beta$ and $\hat{A}_\gamma$
which vary exponentially $\exp \pm \gamma z$, where γ , the parametric growth
rate, is $|x\hat{A}_\alpha|/c_s$. The more complete solution to this subsection indi-
cates that the exponential solution is an approximation to the behavior
of the curves in the region AB of Fig. 4.2. Parametric amplifiers
operate within that region. The signal and pump are externally excited;
the idler is generally allowed to grow from noise.

4.1.6 Discussion

The problem solved in this subsection is a rather restrictive one:
a one-dimensional boundary value problem in which only three discrete
waves are excited and the background perturbations are negligible. The
small-signal propagation parameters $\{\omega_\eta\}$ and $\{k_\eta\}$ are real, and the
medium is homogeneous, so that the coupling coefficient, X , is a real
constant. In this case it has been possible to obtain an analytic solu-
tion, but the analysis was by no means simple. The general three-wave
interaction equations (3.46) are even more complicated, and to this
author's knowledge, no general analytic solution to them is known, even
when background perturbations are negligible.

The averaged-Lagrangian method was not intended to eliminate all
these complexities. It is an efficient, unified method for obtaining
the interaction equations and their parameters, and it led to the con-
venient general form for the equations of wave evolution, that was used

in this example. The averaged-Lagrangian method generates the inter-
action equations; solution of these equations is a process which has
been outlined here in order to illustrate the other aspects of nonlinear
analysis.

4.2 Electrostatic Waves in a Vlasov Plasma

Our second application of the averaged-Lagrangian method is to a
plasma which is described by the Vlasov equation and Maxwell's equations.
In this subsection, an appropriate Lagrangian is stated and interpreted,
then used in conjunction with the averaged-Lagrangian method to obtain
equations for the evolution of four discrete electrostatic waves and the
plasma background. The analysis allows for fluctuations in one spatial
dimension only; it is quasistatic in the sense that perturbations in
the magnetic field are neglected.

After the interaction equations are obtained, they are transformed
to describe the evolution of a spectrum of interacting waves and the
plasma background. The equations are simplified by means of additional
physical assumptions, including the random-phase approximation, commonly
used in weak turbulence theory. The resulting equations are shown to be
equivalent, in the appropriate order of approximation, to the wave and
particle kinetic equations, as they appear in the literature of weak
plasma turbulence.

4.2.1 The Plasma

In the Vlasov description of a plasma, particle collisions are neg-
lected, and the state of the charged particles of species s is defined
by a distribution function in phase space, $F_s(\underset{\sim}{v},\underset{\sim}{x},t)$. The definition
is such that

$$n_s = \int d^2v \, F_s \tag{4.67}$$

where the integration is over all velocity space, and n_s is the particle
density for the species. The electromagnetic state of the system is given

by the vector fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$, or the corresponding potentials φ and $\underset{\sim}{A}$.

It is convenient to describe the plasma with the following dimensionless parameters, since the normalization is with respect to standard plasma parameters, and no physical constants then appear in the equations of motion

$$\underset{\sim}{E}' \equiv \frac{q_0 \underset{\sim}{E}}{cm_0\omega_0} \quad , \quad \underset{\sim}{H}' \equiv \frac{q_0\underset{\sim}{B}}{m_0\omega_0} \quad , \quad \underset{\sim}{V} \equiv \frac{\underset{\sim}{v}}{c} \quad , \quad \underset{\sim}{X} \equiv \frac{\omega_0 \underset{\sim}{x}}{c} \quad , \quad T = \omega_0 t$$

(4.68)

$$\Phi' \equiv \frac{q_0\varphi}{m_0 c^2} \quad , \quad \underset{\sim}{A}' \equiv \frac{q_0\underset{\sim}{A}}{m_0 c} \quad , \quad M_s \equiv \frac{m_s}{m_0} \quad , \quad Q_s \equiv \frac{q_s}{q_0} \quad , \quad N_s \equiv \frac{n_s}{n_0} \quad , \quad F_s \equiv \frac{cf_s}{n_0} \quad .$$

The quantities on the right hand sides of these equations are in rationalized MKS units, and the normalization is with respect to a density n_0 , of reference particles of charge q_0 , and mass m_0 . The quantity ω_0 is the reference plasma frequency $(n_0 q_0^2/\epsilon_0 m_0)^{1/2}$, and c is the speed of light in vacuum. Henceforth, only the dimensionless variables will be used in the analysis. Therefore, the primes on the normalized electromagnetic field quantities may be dropped without causing confusion. We may obtain the electric and magnetic fields from the vector and scalar potentials according to the equations

$$\underset{\sim}{E} = -\nabla\Phi - \frac{\partial \underset{\sim}{A}}{\partial t} \quad , \qquad \underset{\sim}{H} = \nabla \times \underset{\sim}{A} \quad . \qquad (4.69)$$

Use of these potentials guarantees that two of the four Maxwell equations in $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are satisfied. The remaining Maxwell equations and the Vlasov equation are

$$\nabla \times \underset{\sim}{H} = \sum_s Q_s \int F_s(\underset{\sim}{V},\underset{\sim}{X}) \, \underset{\sim}{V} \, d\underset{\sim}{V} + \frac{\partial \underset{\sim}{E}}{\partial T} \quad , \quad \nabla \cdot \underset{\sim}{E} = \sum_s Q_s N_s \quad ,$$

(4.70)

$$\frac{\partial F_s}{\partial T} + \underset{\sim}{V} \cdot \nabla F_s + \frac{Q_s}{M_s}(\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{H}) \cdot \nabla_V F_s = 0 \quad ,$$

where the summations are over particle species.

4.2.2 The Plasma Lagrangian

To apply the averaged-Lagrangian method, we must have a Lagrangian density which yields equations of motion which are equivalent, but not necessarily identical, to those of (4.70). A suitable expression for the plasma Lagrangian was established independently in papers by Low[27] and Sturrock[50] in 1958. In both analyses, the desired Lagrangian was obtained by combining the well-known Lagrangians for (a) the electromagnetic fields, and (b) a charged particle moving in specified electromagnetic fields. Also, both analyses describe the state of the plasma particles by means of perturbation variables which correspond to the displacements of particle cells in phase space. Strictly speaking, Sturrock's analysis was limited to a single plasma stream; however, through integration over a continuum of streams, his expressions lead directly to the correct Lagrangian for a Vlasov plasma. Sturrock's Lagrangian analysis was carried out in relativistically covariant form, and he went on to apply his Lagrangian to derive energy-momentum stress relations of the type discussed previously — in the classical limit — in Section 3. Low, using a classical analysis, carried out the integration over particle velocity distributions in such a manner that all field and particle components are referred to a common reference frame $(\underset{\sim}{V}, \underset{\sim}{X}, T)$. However, in doing so, Low imposed certain constraints on his variables which were unnecessarily restrictive and which led to difficulties in physical interpretation. In 1971, this author and H. Kim[21] presented an alternative derivation of the Low Lagrangian which avoided the difficulties associated with the original analysis, and it is upon this basis that the plasma Lagrangian will be employed here. The derivation will not be repeated; the result will simply be stated and interpreted.

The reference state is described by distribution functions $\{F_{sr}\}$, one for each species, s, and fields Φ_r and $\underset{\sim}{A}_r$. We shall require that these correspond to some exact solution to the Maxwell-Vlasov equations, which fluctuates with $\underset{\sim}{X}$ and T only on the slow scale. The variables in $\mathcal{L}$ are the field perturbations, $\underset{\sim}{\Phi}$ and $\underset{\sim}{A}$, and 'displacement vectors', $\{\underset{\sim}{\Delta}_s\}$, which represent displacements of cells in phase space, as illustrated in Fig. 4.3 for a single particle species.

The figure shows a cell, σ_p , in phase space at $(\underset{\sim}{V}', \underset{\sim}{X}')$ at time T , following a trajectory in phase space in the presence of a perturbed field. If the perturbation were absent, the particles would be in the reference cell σ_r , at position $(\underset{\sim}{V}, \underset{\sim}{X})$ at the same instant of time. The perturbed position of the cell is obtained from the displacement $\underset{\sim}{\Delta}(\underset{\sim}{V}, \underset{\sim}{X}, T)$ according to

$$\underset{\sim}{X}' = \underset{\sim}{X} + \underset{\sim}{\Delta} \ , \qquad \underset{\sim}{V}' = \underset{\sim}{V} + \dot{\underset{\sim}{\Delta}} \ . \qquad (4.71)$$
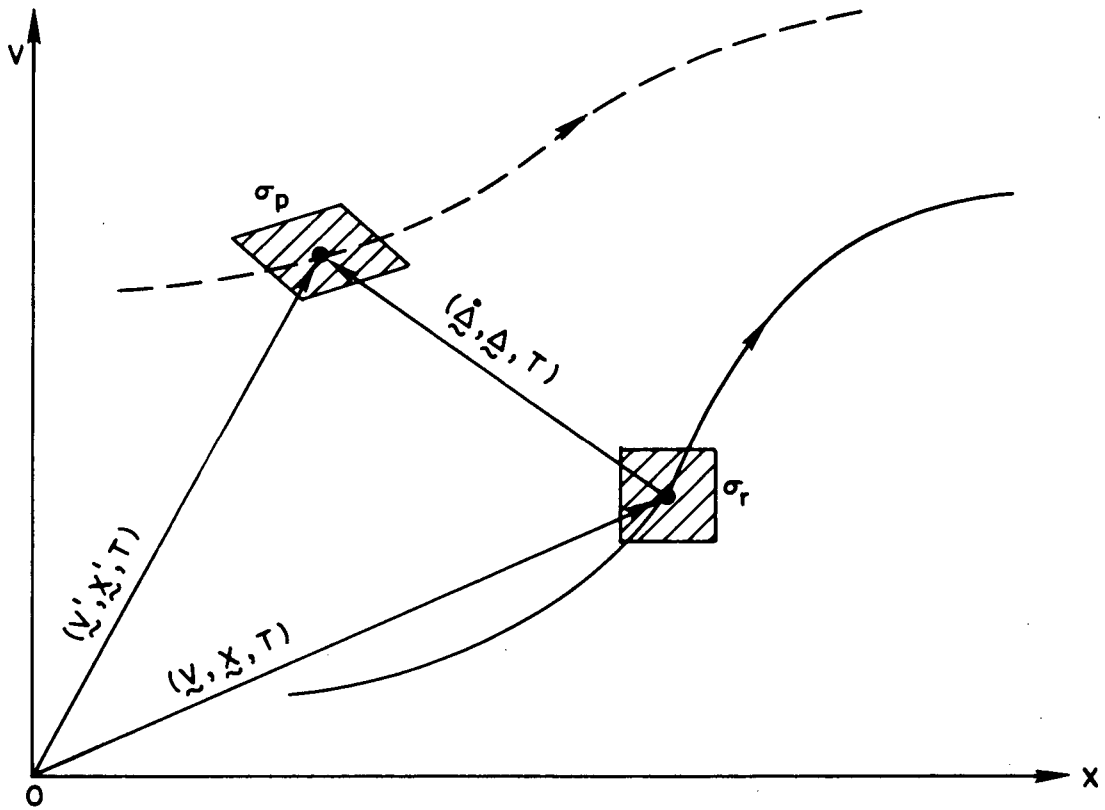
Fig. 4.3   Definition of the displacement variable, $\underset{\sim}{\Delta}(\underset{\sim}{V}, \underset{\sim}{X}, T)$.   In the presence of perturbations, a particle cell, σ_p, follows the trajectory in phase space indicated by the broken lines. If the perturbations were not present, the particles of would be in the reference cell, σ_r, at the same instant of time.   The positional displacement of σ_p from the reference cell is $\underset{\sim}{\Delta}$.   The velocity displacement is $\dot{\underset{\sim}{\Delta}}$.

The dot operator in this equation is defined by

$$\overset{\cdot}{\underset{\sim}{\Delta}} \equiv \left(\frac{\partial}{\partial T}\right)\underset{\sim}{\Delta} + \left(\underset{\sim}{V}\cdot\frac{\partial}{\partial X}\right)\underset{\sim}{\Delta} + \left(\underset{\sim}{a}_r\cdot\frac{\partial}{\partial V}\right)\underset{\sim}{\Delta} \quad , \qquad (4.72)$$

in which $\underset{\sim}{a}_r$ is the acceleration of a particle at $(\underset{\sim}{V},\underset{\sim}{X})$ in the reference state.

The boundary of the perturbed cell is defined by a one-to-one matching of particles with their virtual positions in the unperturbed cell, σ_r . Since there are no particle collisions, both cells contain the same constant number of particles as they travel through phase space, and this number is equal to both

$$\int_{\sigma_r} F_r d^3V\, d^3X \,, \quad \text{and} \quad \int_{\sigma_p} F_p d^3V' d^3X' \,,$$

where F_r and F_p are the unperturbed and perturbed distribution functions, respectively. Over sufficiently small cells, F_r and F_p have negligible fluctuations, so particle conservation and the definition of the perturbed cell imply that

$$F_p(\underset{\sim}{V}',\underset{\sim}{X}',T)d^3V' d^3X' = F_r(\underset{\sim}{V},\underset{\sim}{X},T)d^3V\, d^3X \quad . \qquad (4.73)$$

Therefore

$$F_p(\underset{\sim}{V}',\underset{\sim}{X}',T) = F_r(\underset{\sim}{V},\underset{\sim}{X},T)/\mathscr{J} \quad , \qquad (4.74)$$

where $\mathscr{J}$ is the Jacobian,

$$\mathscr{J} \equiv \frac{\partial(\underset{\sim}{V}',\underset{\sim}{X}')}{\partial(\underset{\sim}{V},\underset{\sim}{X})} \quad . \qquad (4.75)$$

Using these definitions, one may write the Lagrangian for a multi-species Vlasov plasma in the form

$$L = \int d^3X\, d^3V\, \left\{ \psi(\underset{\sim}{V})\mathscr{L}^f + \sum_s F_{sr}(\underset{\sim}{V},\underset{\sim}{X},T)\mathscr{L}_s^p \right\} \quad , \qquad (4.76)$$

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where F_{sr} is the reference distribution function for particle species s , the terms $\{\mathcal{L}_s^p\}$ and $\mathcal{L}^f$ are Lagrangian densities associated with the particle and field behavior, respectively, and $\psi(\underset{\sim}{V})$ is any well-behaved function of $\underset{\sim}{V}$ which approaches zero as $|\underset{\sim}{V}| \to \infty$ and which has the property that $\int \psi(\underset{\sim}{V})d^3V = 1$. The functional forms of $\mathcal{L}_s^p$ and $\mathcal{L}^f$ are

$$\mathcal{L}_s^p \equiv \frac{M}{2} (\underset{\sim}{V}+\dot{\underset{\sim}{\Delta}}_s)^2 - Q\left[\Phi_r + \Phi - (\underset{\sim}{V}+\dot{\underset{\sim}{\Delta}}_s)\cdot(\underset{\sim}{A}_r+\underset{\sim}{A})\right] \quad , \tag{4.77}$$

$$\mathcal{L}^f \equiv \frac{1}{2} \left(\nabla\Phi_r + \frac{\partial\underset{\sim}{A}_r}{\partial T} + \nabla\Phi + \frac{\partial\underset{\sim}{A}}{\partial T}\right)^2 - \frac{1}{2} (\nabla\times\underset{\sim}{A}_r + \nabla\times\underset{\sim}{A})^2 \quad , \tag{4.78}$$

in which the subscript r denotes reference state quantities. Field quantities in $\mathcal{L}_s^p$ are to be evaluated at $(\underset{\sim}{V}+\dot{\underset{\sim}{\Delta}}_s,\underset{\sim}{X}+\underset{\sim}{\Delta}_s,T)$; all other quantities indicated above are evaluated at $(\underset{\sim}{V},\underset{\sim}{X},T)$. For the derivation of (4.76) — (4.78) the reader is referred to reference 24.

From (4.76) it is clear that the Lagrangian density for the plasma is

$$\mathcal{L} = \psi\mathcal{L}^f(\Phi,\underset{\sim}{A}) + \sum_s F_{sr}\mathcal{L}_s^p(\Phi,\underset{\sim}{A},\underset{\sim}{\Delta}_s) \quad , \tag{4.79}$$

in which the dependent variables are the perturbations ϕ, $\underset{\sim}{A}$ and $\{\underset{\sim}{\Delta}_s\}$. The independent variables, $\underset{\sim}{V}$, $\underset{\sim}{X}$, and T , may be treated as Eulerian coordinates. The Lagrangian density (4.79) generates equations for ϕ , $\underset{\sim}{A}$ and $\underset{\sim}{\Delta}$ which are equivalent to the Maxwell-Vlasov equations (4.70), provided that the reference state parameters are consistent with the assumptions used in the derivation of L: i.e., the $\{F_{sr}\}$, ϕ_r , and $\underset{\sim}{A}_r$ must satisfy the equations (4.70) exactly. In reference 24, this assertion was verified directly in the linear limit. The two sets of equations were linearized in the perturbation variables, and the resulting equations were shown to be equivalent through a transformation of variables. There has been no attempt to verify the equivalence of the equations for each higher (nonlinear) order of approximation. The assertion that they are equivalent in their physical implications is based on the arguments used to obtain the Lagrangian.

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The expression for L in (4.79) is just that obtained by Low. However, Low was in error in his interpretation of the displacement coordinate $\underset{\sim}{\Delta}$, and this affected his transformation from the displacement description of the plasma particles to the Vlasov description. In that description, the state of a particle species may be given by $F_r + F_\delta$, where $F_\delta(\underset{\sim}{V}, \underset{\sim}{X}, T)$ is the perturbation of the distribution function, defined by

$$F_p(V,X,T) \equiv F_\delta(V,X,T) + F_r(V,X,T) \quad . \quad (4.80)$$

The transformation of coordinates will be denoted symbolically by $[\underset{\sim}{\Delta} \rightarrow F_\delta]$. Low's interpretation led to an unnecessary "consistency condition" on $\underset{\sim}{\Delta}$ and to a formula for $[\underset{\sim}{\Delta} \rightarrow F_\delta]$ which is not correct in general. In our example, the perturbation of the plasma background by the waves will be studied, so the correct transformation will be needed in order to calculate $\bar{F}_p$. The necessary equation is developed below after the manner of reference 24. Certain authors, using a quantum mechanical approach, have employed the Low results without qualification in the analysis of nonlinear plasma waves.[30,51-53] However, it appears that in this work Low's "consistency condition" has not been imposed, and the interaction equations have been expressed in the electromagnetic field quantities only. The perturbation of the plasma distribution function was not studied, so Low's version of $[\underset{\sim}{\Delta} \rightarrow F_\delta]$ was not used, and no errors were introduced.

4.2.3 The Perturbed Distribution Function

Let us now consider the equations for the transformation $[\underset{\sim}{\Delta} \rightarrow F_\delta]$. The analysis is restricted to one spatial dimension because this simplifies the presentation, and because the results adequately describe the example considered later in this section. In this case, the coordinates, V, X, and T , and the displacement variable, $\Delta(V,X,T)$ may be treated as scalars. To find F , we employ the definition of the perturbed cell, which implies

$$F_r(V,X,T)dXdV = F_p(V',X',T)dX'dV' = F_p(V',X',T) \mathscr{J} dXdV \quad , \quad (4.81)$$

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$$X' = X + \Delta \quad , \qquad V' = V + \dot{\Delta} \quad , \qquad (4.82)$$

and

$$\mathcal{J} \equiv \frac{\partial(V',X')}{\partial(V,X)} = \begin{vmatrix} (1 + \partial\Delta/\partial X) & (\partial\Delta/\partial V) \\ & \\ (\partial\dot{\Delta}/\partial X) & (1 + \partial\dot{\Delta}/\partial V) \end{vmatrix} \quad . \qquad (4.83)$$

Equations (4.81) and (4.83) imply that

$$F_p(V',X',T) = F\left[1 - \frac{\partial\Delta}{\partial X} - \frac{\partial\dot{\Delta}}{\partial V} + \left(\frac{\partial\Delta}{\partial X}\right)^2 + \frac{\partial\Delta}{\partial X}\frac{\partial\dot{\Delta}}{\partial V} + \frac{\partial\dot{\Delta}}{\partial X}\frac{\partial\Delta}{\partial V} + \left(\frac{\partial\dot{\Delta}}{\partial V}\right)^2\right]$$

$$(4.84)$$

$$+ \text{ (higher order terms) } .$$

In this and the subsequent equations, all quantities and their derivatives are evaluated at (V,X,T) unless otherwise indicated. For comparison with the above expression, we have the result of a direct Taylor expansion of F

$$F_r(V',X',T) = F_r + \Delta\frac{\partial}{\partial X}F_r + \dot{\Delta}\frac{\partial}{\partial V}F_r$$

$$+ \frac{1}{2}\Delta^2\frac{\partial^2}{\partial X^2}F_r + \Delta\dot{\Delta}\frac{\partial^2}{\partial X\partial V}F_r + \dot{\Delta}^2\frac{\partial^2}{\partial V^2}F_r \quad .$$

$$(4.85)$$

$$+ \text{ (higher order terms)}$$

An expression for $F_\delta(V',X',T)$ may now be obtained through the use of (4.84) and (4.85) in the definition (4.80). However, the function of interest is $F_\delta(V'-\dot{\Delta}, X'-\Delta, T)$ [$=F_\delta(V,X,T)$] . This is obtained from $F_\delta(V',X',T)$ through an additional Taylor expansion, and the resulting

formula is

$$
F_\delta(V,X,T) \approx -\frac{\partial}{\partial X}(\Delta F_r) + \frac{\partial}{\partial X}\left(\Delta \frac{\partial \Delta}{\partial X} F_r\right) + \frac{1}{2}\frac{\partial}{\partial X}\left(\Delta^2 \frac{\partial F_r}{\partial X}\right)
$$

$$
-\frac{\partial}{\partial V}(\Delta F_r) + \frac{\partial}{\partial V}\left(\dot\Delta \frac{\partial \dot\Delta}{\partial V} F_r\right) + \frac{1}{2}\frac{\partial}{\partial V}\left(\dot\Delta^2 \frac{\partial F_r}{\partial V}\right) \quad . \quad (4.86)
$$

$$
+ \frac{\partial^2}{\partial X \partial V}(\Delta\dot\Delta F_r) \quad .
$$

The terms neglected in this approximation are of the third power, and higher, in Δ and its derivatives.

4.2.4   The Problem

For clarity of illustration, we shall apply the averaged-Lagrangian method to a problem which is defined by much more restrictive assumptions than those used in the derivation of L . Consequently, the warm plasma Lagrangian will be employed in a simpler, more specialized form. In this example, we shall assume that (a) the plasma is homogeneous except in one spatial direction, so there are only three independent scalar coordinates; V, X, and T ; (b) on the slow scale, the plasma is homogeneous in X, so that the wave parameters are functions of ϵT only, and the averaged particle distribution functions $\{\overline{F}_{sp}\}$ are functions only of V and ϵT ; (c) the only significant nonlinear wave interaction process is the four-wave interaction, characterized by the synchronism conditions (2.37); and (d) the plasma behavior is adequately described by the Maxwell-Vlasov equations in the quasistatic limit

$$
\frac{\partial^2 \Phi_p}{\partial x^2} = -\sum_s Q_s N_{sp} = -\sum_s Q_s \int dV\, F_{sp} \quad ,
$$

$$
\frac{\partial F_{sp}}{\partial T} + V\frac{\partial F_{sp}}{\partial X} - \frac{Q_s}{M_s}\frac{\partial \Phi_p}{\partial X}\frac{\partial F_{sp}}{\partial V} = 0 \quad . \quad\quad (4.87)
$$

In keeping with these assumptions, the electrostatic Lagrangian density for this system is obtained from (4.76) by dropping $\underset{\sim}{A}$ as a variable, and specializing the result to the case of one spatial dimension. The process yields

$$\mathcal{L} = \frac{1}{2} \psi(V)[\nabla\Phi_r(X,T) + \nabla\Phi(X,T)]^2$$

$$+ \Sigma_s F_{sr}(V,X,T)\left\{\frac{1}{2}\,[V+\dot{\Delta}]^2 - \Phi_r(X+\Delta,\,T) - \Phi(X+\Delta,\,T)\right\} \quad , \tag{4.88}$$

where

$$\dot{\Delta} = \left(\frac{\partial}{\partial T} + V\frac{\partial}{\partial X} - \frac{\partial\Phi_r}{\partial X}\frac{\partial}{\partial V}\right)\Delta \quad . \tag{4.89}$$

For this Lagrangian, the reference state is defined in general by a potential field, Φ , and a set of distribution functions $\{F_{sr}\}$; however, because of Assumption (b) there can be no static electric field in the reference state, i.e. Φ_r is constant in X , and will be dropped from the analysis.

This analysis will be concerned only with the lower order equations for wave and background evolution, and interactions of five or more waves will be neglected. The only terms which will be needed from the expansion of $\mathcal{L}$ are

$$\mathcal{L}_{(2)} = \frac{1}{2}\psi(V)\left(\frac{\partial\Phi}{\partial X}\right)^2 + \sum_s F_{sr}\left[\frac{1}{2}M_s\dot{\Delta}_s^2 - Q_s\Delta_s\frac{\partial\Phi}{\partial X}\right] \quad ,$$

$$\mathcal{L}_{(3)} = -\sum_s \frac{1}{2}Q_s F_{sr}\Delta_s^2\frac{\partial^2\Phi}{\partial X^2} \quad , \qquad \mathcal{L}_{(4)} = -\sum_s \frac{1}{6}Q_s F_{sr}\Delta_s^3\frac{\partial^3\Phi}{\partial X^3} \quad . \tag{4.90}$$

At this point, the formal Lagrangian description differs from that of the first example in a matter of technique: different reference states are adopted for the wave and background analyses, respectively. For the wave analysis, the reference state is defined by the averaged distribution

functions $\{F_{sr}^{w} \equiv \overline{F}_{sp}\}$, while for the background analysis, the reference state is chosen to be an exact solution defined by functions $\{F_{sr}^{b}(V)\}$, which are essentially constant in time.

In the wave Lagrangian density, $\mathcal{L}^{w}$, all slow background fluctuations are absorbed into the reference state, so there are no slow perturbation components $\{\Delta_{s0}\}$. The choice of reference state thus eliminates the dual character of the expansion for $\mathcal{L}^{w}$ and $\overline{\mathcal{L}}^{w}$. There is only a single expansion, in joint powers of the wave components $\{\Delta_{s\eta}\}$ and $\{\Phi_{\eta}\}$. The subsequent wave analysis is simplified accordingly.

In $\mathcal{L}^{b}$, however, a dual expansion is still needed. The small-signal wave relations are used to express $\overline{\mathcal{L}}^{b}$ to the necessary order in ϵ . Later in the analysis the Euler-Lagrange equations for the $\{\Delta_{s0}\}$ are used in conjunction with the expression (4.86) for the $F_{s\delta}$, in order to obtain equations for the $\{\overline{F}_{sp}\}$. We then have a complete set of interaction equations for the wave and background evolution.

Because the Lagrangian description is different for the wave and background analyses, the rest of the formal Lagrangian description is presented in the following two subsections.

4.2.5 Wave Analysis

For the waves, the dual expansion is exactly the same as the simple perturbation expansion $[\Phi = \Phi_{w}, \Delta = \Delta_{w}, \mathcal{L}_{(m)} = \mathcal{L}_{m}]$. The small-signal equations of motion are the Euler-Lagrange equations of $\mathcal{L}_{2}$, given by

$$-M_{s}\Delta_{s} - Q_{s}\frac{\partial\Phi}{\partial X} = 0 \quad , \qquad -\frac{\partial^{2}\Phi}{\partial X^{2}} + \sum_{s}Q_{s}\int dV \frac{\partial(\Delta_{s}\overline{F}_{sp})}{\partial X} = 0 \quad . \quad (4.91)$$

In keeping with (2.15), we adopt the complex notation

$$\Delta_{s} = \Sigma_{\eta}(\hat{\Delta}_{s\eta} \exp j\theta_{\eta} + \hat{\Delta}_{s-\eta} \exp -j\theta_{\eta}^{*}) \quad , \qquad (4.92)$$

and

$$\Phi = \Sigma_{\eta}(\hat{\Phi}_{\eta} \exp j\theta_{\eta} + \hat{\Phi}_{\eta} \exp -j\theta_{\eta}^{*}) \quad . \qquad (4.93)$$

Substitution of (4.92) – (4.93) into (4.91) produces the small-signal relations

$$M_s(\Omega - KV)^2 \hat{\Delta}_s(V) + jKQ_s\hat{\Phi} = 0 \quad , \qquad -jK\hat{\Phi} - \sum_s Q_s \int \hat{\Delta}_s(V)\overline{F}_{sp}(V)dV = 0 \quad ,$$

(4.94)

in which Ω denotes $\partial\theta/\partial T$, and K denotes $\partial\theta/\partial X$. The above equations combine to give the well-known small-signal dispersion relation for electrostatic waves,[54]

$$1 - \sum_s \frac{Q_s^2}{M_s} \int \frac{\overline{F}_{sp}(V)}{(\Omega - KV)^2} dV = 0 \quad .$$

(4.95)

Assumption (c) implies that $\tilde{\overline{\mathcal{I}}}^w$ may be well approximated by $\tilde{\overline{\mathcal{I}}}_2 + \tilde{\overline{\mathcal{I}}}_3 + \tilde{\overline{\mathcal{I}}}_4$. Use of the small-signal relations in (4.90) allows us to express $\tilde{\overline{\mathcal{I}}}^w$ in the form

$$\tilde{\overline{\mathcal{I}}}^w = \sum_\eta \tilde{\lambda}_\eta \hat{\Phi}_\eta \hat{\Phi}_{-\eta} \exp -2\theta_{\eta i} + \sum_{\substack{sync \\ \alpha\beta\gamma}} \left[\tilde{\lambda}_{-\alpha\beta\gamma} \hat{\Phi}_{-\alpha}\hat{\Phi}_\beta\hat{\Phi}_\gamma \exp j\delta\theta_{-\alpha\beta\gamma} + c.c. \right]$$

$$+ \sum_{\substack{sync \\ \alpha\beta\gamma\delta}} \left[\tilde{\lambda}_{-\alpha\beta\gamma\delta} \hat{\Phi}_{-\alpha}\hat{\Phi}_\beta\hat{\Phi}_\gamma\hat{\Phi}_\delta \exp j\delta\theta_{-\alpha\beta\gamma\delta} + c.c. \right] \quad ,$$

(4.96)

where

$$\tilde{\lambda}_\eta = K_\eta^2 - \sum_s \int \overline{F}_{sp} \frac{Q_s^2 K_\eta^2}{M_s(\Omega_\eta - K_\eta V)^2} dV \quad ,$$

(4.97)

$$\lambda_{-\alpha\beta\gamma} = -K_\alpha K_\beta K_\gamma \sum_s \frac{Q_s^3}{M_s^2} \int \overline{F}_{sp} \frac{K_\alpha(\Omega_\alpha - K_\alpha V)^2 - K_\beta(\Omega_\beta - K_\beta V)^2 - K_\gamma(\Omega_\gamma - K_\gamma V)^2}{(\Omega_\alpha - K_\alpha V)^2(\Omega_\beta - K_\beta V)^2(\Omega_\gamma - K_\gamma V)^2} dV \quad ,$$

(4.98)

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and

$$\lambda_{-\alpha\beta\gamma\delta} = -K_\alpha K_\beta K_\gamma K_\delta \sum_s \left\{ \frac{Q_s^4}{M_s^3} \right.$$

$$\left. \times \int \bar{F}_{sp} \frac{K_\alpha^2(\Omega_\alpha - K_\alpha V)^2 + K_\beta^2(\Omega_\beta - K_\beta V)^2 + K_\gamma^2(\Omega_\gamma - K_\gamma V)^2 + K_\delta^2(\Omega_\delta - K_\delta V)^2}{M_s^3(\Omega_\alpha - K_\alpha V)^2(\Omega_\beta - K_\beta V)^2(\Omega_\gamma - K_\gamma V)^2(\Omega_\delta - K_\delta V)^2} \, dV \right\} \quad .$$

(4.99)

The wave-wave interaction, corresponding to the problem defined in Assumptions (a) — (d) of Section 4.2.4, is a special case of the four-wave interaction considered in Section 3.5.3. Therefore, the equation for the evolution of any wave component, α , may be obtained by appropriately specializing the formal result (3.77). This process yields

$$\frac{\partial}{\partial T} \hat{A}_\alpha + [\Omega_{\alpha i} - j(\delta\Omega)_\alpha] \hat{A}_\alpha = j \sum_{\substack{\text{sync} \\ \alpha\beta\gamma\delta}} X_{-\alpha\beta\gamma\delta} \hat{A}_\beta \hat{A}_\gamma \hat{A}_\delta$$

(4.100)

in which

$$\hat{A}_\eta \equiv (\partial\lambda_\eta/\partial\Omega_\eta)^{1/2} \hat{\Phi}_\eta \exp -\theta_{\eta i} \quad , $$

(4.101)

$$(\delta\Omega)_\eta \equiv \frac{1}{2(\partial\lambda_\eta/\partial\Omega_\eta)^{1/2}} \frac{d}{dT} \left[\frac{\partial^2\lambda_\eta}{\partial\Omega_\eta^2} \frac{d}{dT} \left(\frac{1}{\partial\lambda_\eta/\partial\Omega_\eta} \right)^{1/2} \right] \quad , $$

(4.102)

$$X_{-\alpha\beta\gamma\delta} \equiv \frac{\left\{ \left[\lambda_{-\alpha\beta(\gamma+\delta)} \lambda_{(\gamma+\delta)\gamma\delta} / \lambda_{(\gamma+\delta)} \right] - \lambda_{-\alpha\beta\gamma\delta} \right\} \exp j\delta\theta_{-\alpha\beta\gamma\delta}}{(\partial\lambda_\alpha/\partial\Omega_\alpha)^{1/2}(\partial\lambda_\beta/\partial\Omega_\beta)^{1/2}(\partial\lambda_\gamma/\partial\Omega_\gamma)^{1/2}(\partial\lambda_\delta/\partial\Omega_\delta)^{1/2}} \quad , $$

(4.103)

and $\Omega_{\eta i}$ is the imaginary part of $\Omega_\eta(K_\eta)$ defined by (4.94). The λ's are given by (4.97) — (4.99).

4.2.6 Background Analysis

For this analysis, we expand $\bar{\Sigma}^b$ about a static reference state in which the particle distribution functions are the set $\{F_{sr}^b(V)\}$. For now, we shall use the approximation,

$$\dot{\bar{\Sigma}}^b \approx \bar{\Sigma}_{0,2} + \bar{\Sigma}_{2,1} + \bar{\Sigma}_{2,2}$$

$$\approx \sum_s F_{sr}^b \left[\frac{M_s}{2} \left(\frac{\partial \Delta_{s0}}{\partial T} \right)^2 - Q\Delta_{s0} \overline{\left(\Delta_{sw} \frac{\partial^2 \Phi_w}{\partial x^2} \right)} - \frac{Q_s}{2} \Delta_{s0}^2 \overline{\left(\Delta_{sw} \frac{\partial^3 \Phi_w}{\partial x^3} \right)} \right] . \quad (4.104)$$

This approximation follows from the expansion terms of (4.83), and the assumption that the slow perturbation components $\{\Delta_{s0}\}$ are functions of V and T only. The small-signal relations for the wave components are now used to simplify $\bar{\Sigma}^b$ further. Use of (4.92) − (4.94) in the above expression transforms it to

$$\bar{\Sigma}^b = \sum_s F_{sr}^b \left\{ \frac{M_s}{2} \left(\frac{\partial \Delta_{s0}}{\partial T} \right)^2 - \frac{Q_s^2}{M_s} \sum_\eta \left[\Delta_{s0} \left(\frac{jK_\eta^3 |\hat{\Phi}_\eta|^2}{[\Omega_\eta - K_\eta V]^2} + c.c. \right) + \Delta_{s0}^2 \frac{2K_\eta^4 |\hat{\Phi}_\eta|^2}{[\Omega_\eta - K_\eta V]^2} \right] \right\} .$$

$$(4.105)$$

The corresponding Euler-Lagrange equation for Δ_{s0} is

$$\frac{\partial^2}{\partial T^2} \Delta_{s0} + 4 \frac{Q_s^2}{M_s^2} \sum_\eta \frac{K_\eta^4 |\hat{\Phi}_\eta|^2}{(\Omega_\eta - K_\eta V)^2} \Delta_{s0} + 4 \frac{Q_s^2}{M_s^2} \sum_\eta \frac{K_\eta^3 \Omega_\eta i |\hat{\Phi}_\eta|^2}{(\Omega_\eta - K_\eta V)^3} = 0, \quad (4.106)$$

and in our ordering scheme this implies

$$\epsilon^2 \Delta_{s0} \sim \epsilon |\hat{\Phi}_\eta|^2, \quad \text{or} \quad \Delta_{s0} \sim \epsilon . \quad (4.107)$$

For compatibility with the terminology of the preceding section, one may express the wave parameters of (4.106) in terms of the normalized parameters $\{A_\eta\}$ by means of the definition (4.101).

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Since the system is spatially homogeneous for all T , the averaged potential, Φ_0 , is zero. Thus, if the $\{F_{sr}\}$, the initial state, and the wave parameters are given, (4.106) fully describes the evolution of the plasma background. However, this description of the background evolution is rather unsatisfactory in two respects. First, for a given choice of the $\{F_{sr}(V)\}$, the equations are valid only for small $\{\Delta_{s0}\}$. These may be made small at $T = 0$ by the choice of reference state, but study of (4.106) shows that its solutions may, in general, become quite large before some steady state is reached. Second, even in the limit of small $\{\Delta_{s0}\}$, the background equations, taken together with the equations for the waves (4.100), do not constitute a complete description of the system: the set of equations is not closed. This is due to the use of different reference states in the two analyses.

For a complete and more versatile description of the wave-plasma system, a different form is needed for the equations of background evolution. This may be obtained from (4.106) by using the basic properties of the displacement variables $\{\Delta_s\}$ in the plasma Lagrangian. The $\{\Delta_s\}$ are related to the perturbations $\{F_s\}$ by (4.86), which closes the set of equations. We shall see in the next section that through these relations the $\{\Delta_{s0}\}$ may be eliminated from the new set of equations entirely, so that the limitations just mentioned no longer apply.

4.2.7 The Interaction Equations

Let us now consider how the wave and background equations may be written in a closed form. To close the set of equations, it is necessary to use the background equations (4.106), together with the properties of the $\{\Delta_s\}$, in order to obtain an equation for the evolution of the $\{\bar{F}_{sp}\}$. These distribution functions completely describe the state of the plasma background, and once their behavior is specified, the wave evolution is specified also by (4.100).

The wave analysis was based on an expansion of $\bar{\mathcal{L}}^w$ through $O(\epsilon^4)$, and an inspection of (4.96) — (4.99) in the light of the ordering rule $(\Phi_\eta \sim \epsilon)$ reveals that for this level of accuracy the $\{\bar{F}_{sp}\}$ must be described through $O(\epsilon^2)$. Equation (4.86) provides a suitable expression

for $\overline{F}_{sp}(=F_{sr} + \overline{F}_{s\delta})$, which in this spatially homogeneous system is

$$\overline{F}_{sp} = F_{sr}^b - \frac{\partial}{\partial V}\left(\overline{\frac{\partial \Delta_{s0}}{\partial T} F_{sr}^b}\right) + \frac{\partial}{\partial V}\left(\overline{\dot{\Delta}_{sw}\frac{\partial \Delta_{sw}}{\partial V} F_{sr}^b}\right) + \frac{1}{2}\frac{\partial}{\partial V}\left(\overline{\dot{\Delta}_{sw}^2 \frac{\partial F_{sr}^b}{\partial V}}\right) + 0(\epsilon^3) \quad .$$

(4.108)

In this expression, the component Δ_w is to be expressed in terms of the wave parameters $\{\hat{\Phi}_\eta\}$ by means of the small-signal relations. The accuracy of this approximation may be verified by means of the ordering relations (4.107). Use of the small-signal relations in (4.108), followed by a differentiation of the equation with respect to time yields

$$\frac{\partial \overline{F}_{sp}}{\partial T} = -\frac{\partial}{\partial V}\left(\frac{\partial^2 \Delta_{s0}}{\partial T^2} F_{sr}^b\right) + \frac{\partial}{\partial V}\left\{\frac{\partial}{\partial T}\left(\frac{Q_s^2}{M_s^2}\sum_\eta \frac{2K_\eta^3|\overline{\Phi}_\eta|^2}{(\Omega_\eta - K_\eta V)^3}\right) F_{sr}^b\right\}$$

$$+ \frac{\partial}{\partial V}\left\{\frac{1}{2}\frac{\partial}{\partial T}\sum_\eta \frac{Q_s^2 K_\eta^2|\hat{\Phi}_\eta|^2}{M_s^2(\Omega_\eta - K_\eta V)^2}\frac{\partial F_{sr}^b}{\partial V}\right\} \quad .$$

(4.109)

The quantity $\partial^2\Delta_{s0}/\partial T^2$ may be eliminated from this equation by means of (4.106). When this is done, it is clear that the second term on the right hand side of (4.109) partially cancels the first. The equation for $\overline{F}_s$ becomes

$$\frac{\partial \overline{F}_{sp}}{\partial T} = \frac{\partial}{\partial V}\left(D_s \frac{\partial F_{sr}^b}{\partial V}\right) + \frac{\partial}{\partial V}\left(\sum_\eta \frac{4Q_s^2 K_\eta^4|\Phi_\eta|^2}{M_s^2(\Omega_\eta - K_\eta V)^2}\Delta_{s0}F_{sr}^b\right)$$

(4.110)

where

$$D_s = \sum_\eta \frac{Q_s^2 K_\eta^2 \Omega_\eta i|\hat{\Phi}_\eta|^2}{M_s^2(\Omega_\eta - K_\eta V)^2} \quad .$$

(4.111)

Even in this form, the background equation is not entirely satisfactory; it still involves an unspecified reference distribution function, F_{sr}^b, and the displacement component, Δ_{s0}, which is not a measurable quantity.

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This difficulty may be overcome by means of a simple device, however. We shall allow F^b_{sr} to be slowly time-varying, and consider the particular limit $F^b_{sr} \rightarrow \overline{F}_{sp}$, in which $\Delta_{s0} \rightarrow 0$. Then

$$\frac{\partial \overline{F}_{sp}}{\partial T} = \frac{\partial}{\partial V} \left(D_s \frac{\partial \overline{F}_{sp}}{\partial V} \right) \quad , \tag{4.112}$$

which has the form of a diffusion equation in velocity space. In the derivation of (4.103) it was assumed that F^b_{sr} was essentially constant in time, in the sense that its time derivatives made a negligible contribution to the equations of motion. Now the choice $F^b_{sr} = \overline{F}_{sp}$ is consistent with that assumption, as one may verify from the ordering relations

$$\frac{\partial}{\partial T} \overline{F}_{sp} \sim \epsilon^3 \quad , \tag{4.113}$$

which follow from (4.107) and (4.109). Use of this relation in the preceding analysis shows that the terms involving $\partial F^b_{sr}/\partial T$ do indeed have a negligible effect on (4.103) when $[F^b_{sr} \rightarrow \overline{F}_s]$. The expression (4.112) for $(\partial \overline{F}_{sp}/\partial T)$ is therefore a self-consistent approximation through order ϵ^3 .

4.2.8 Weak Turbulence

The equations (4.100) and (4.113) comprise a closed set which describe the evolution of a number of monochromatic waves, and the distribution functions $\{\overline{F}_{sp}\}$, in a self-consistent manner through $0(\epsilon^3)$. Let us now consider how these equations may be applied in the limit of weak turbulence, where the modes of oscillation are so closely spaced in wavenumber that they may be regarded as a continuum. In this limit, the evolution of the $\{F_{sp}\}$ is still adequately described by (4.113). This is, in fact, the well-known quasilinear diffusion equation, and (4.111) is the usual expression for the quasilinear diffusion coefficient, $D(V)$.[55]

Conventionally, however, the diffusion equation is derived by way of the Vlasov equation.

In their present form, the equations for wave evolution are difficult to apply in the weak turbulence limit. This is because of the sum over synchronous mode combinations which appears on the right hand side of (4.100). Given any one such combination $(\alpha, \beta, \gamma, \delta)$ the continuous spectral distribution assumes that there are other nearly synchronous combinations $(\alpha, \beta', \gamma', \delta')$ at nearby values of Ω and K. These other combinations have various phase mismatches $\{\delta\theta_{-\alpha\beta'\gamma'\delta'}\}$ and they collectively influence the evolution of Wave α in a manner which is not immediately clear. However, using an argument due to R. K. Fisher and J. L. Hirshfield[56] we shall infer how these effects may be interpreted and how the interaction equations may be simplified. The argument follows.

We have already considered the case of interacting wavepackets which have negligible width, and for that case (4.100) may be applied without difficulty. Consider now the wavepacket β, of finite frequency width $\delta\Omega_\beta$. Its phase, θ_β, is constant only over time intervals much less than δT_β, where $\delta T_\beta \equiv 1/\Omega_\beta$. Over time intervals much larger than δT_β, there will be negligible nonlinear coupling between Waves β and α, because $\delta\theta_{-\alpha\beta\gamma\delta}$ is not slowly varying on that time scale and $\exp j\delta\theta_{-\alpha\beta\gamma\delta}$ vanishes under the time average. In the turbulent limit, the width of the wavepacket is indistinguishable from the spectral width, $\delta\Omega_s$. Then δT_s ($= 1/\delta\Omega_s$) may become very small for all waves in the spectrum, and it may be much smaller than the time scale of observation. In that case, no synchronous interaction which is affected by the phase relationships between modes $\alpha, \beta\gamma, \ldots$etc. can contribute to the interaction equations; the corresponding interaction terms must therefore be dropped from (4.100).

The assumption that the interaction equations may be simplified in this manner is known in weak turbulence theory as the 'random-phase approximation' (RPA). Various other arguments have been offered to support it,[14,57] of which that of Davidson appears to be the most rigorous. The theoretical basis will not be explored further in this work.

The RPA, which is a standard analytic technique, will simply be applied to the problem at hand. To do this we multiply both sides of (4.100) by $\hat{A}_{-\alpha}$, then add the resulting equation to its complex conjugate to obtain

$$\left(\frac{\partial}{\partial T} + 2\Omega_{\alpha i}\right)|\hat{A}_\alpha|^2 = \sum_{\substack{\text{sync} \\ \alpha\beta\gamma\delta}} \left[jX_{-\alpha\beta\gamma\delta}\hat{A}^*_\alpha\hat{A}_\beta\hat{A}_\gamma\hat{A}_\delta + \text{c.c.}\right] \quad , \qquad (4.114)$$

but under the RPA, the only combinations we allow under the sum are those for which $\delta\theta_{-\alpha\beta\gamma\delta}$ is identically zero. This is true only for combinations which are automatically synchronous, such as that produced in the limit

$$(-\alpha,\beta,\gamma,\delta) \rightarrow (-\alpha,\alpha,-\beta,\beta) \quad . \qquad (4.115)$$

Consequently, we have

$$\left(\frac{\partial}{\partial T} + 2\Omega_{\alpha i}\right)|\hat{A}_\alpha|^2 = \sum_{\alpha,\beta} 3\left[jX_{-\alpha,\alpha,-\beta,\beta} + \text{c.c.}\right]|\hat{A}_\alpha|^2|\hat{A}_\beta|^2 \quad . \qquad (4.116)$$

This equation shows that under the RPA the equations of wave evolution relate the action densities $\{|A_\eta|^2\}$, or quantities proportional to them. They do not relate the complex amplitudes, as in the analysis of monochromatic waves, because under the RPA the right hand side of (4.93) is replaced by zero. In keeping with the interpretation given earlier, one may say that the coupling term, $j|X_{-\alpha\beta\gamma\delta}A_\beta A_\gamma A_\delta| \exp\delta\theta_{-\alpha\beta\gamma\delta}$, makes a contribution to $\partial\hat{A}/\partial T$ which has a rapidly varying complex phase angle $(\delta\theta_{-\alpha\beta\gamma\delta})$. Consequently, this contribution vanishes under the bar average. On the other hand, the magnitude of the coupling term fluctuates only on the slow scale. The coupling term can therefore make an averaged contribution to $|\partial\hat{A}/\partial T|$; this is the effect described by (4.114).

The description of the plasma system in the limit of weak turbulence is now complete. The interaction equations for the plasma background and spectral evolution are (4.112) and (4.116) respectively. These equations are part of the hierarchy of wave and particle kinetic equations, which have been developed by other methods in the plasma literature.[58] A brief interpretation of the kinetic equations is given here. For more detailed discussion the reader may refer to the works just cited. The processes described by the interaction equations are illustrated schematically in Fig. 4.4, where the quasiparticle picture for the waves is employed as a conceptual aid. The number of quasiparticles associated with frequency $\Omega_\eta(K_\eta)$ is $|\hat{A}_\eta|^2/\hbar$.

The quasilinear diffusion equation (4.112), describes the effects of quasiparticle emission and absorption shown in Fig. 4.4(a). There, a plasma particle emits or absorbs a quasiparticle of frequency Ω_η and wavenumber K_η . It undergoes a corresponding change in energy, $\hbar\Omega$, and in momentum, $\hbar K$. The probability of quasiparticle emission or absorption is determined by the linear growth or damping parameter, $\Omega_{\eta i}$. Since energy and momentum must be conserved, the only particles which affect the emission/absorption processes are those of velocity V such that $\Omega_\eta = K_\eta V$, and this is the origin of the resonance in the diffusion coefficient. The diffusion equation describes the collective effects of emission/absorption on the particles, assuming that the spectral action densities (quasiparticle populations) are given.

The equation for the spectral evolution (4.116) describes the effects of two processes. The first is that of Fig. 4.4(a); the effect on the waves is accounted for by the presence of $\Omega_{\alpha i}$ in the equation. The second process is shown in Fig. 4.4(b); it is the scattering of a quasi-particle by a particle. In this process the quasiparticle exchanges energy and momentum, hence it changes its Ω and K values, but the number of quasiparticles is conserved in the process. The effects of the scattering appear on the right hand side of the equation. The rate at which a quasiparticle is scattered from state α into β and vice versa is shown to be proportional to the product of the populations, which is in agreement with the quantum picture.[59] The scattering process
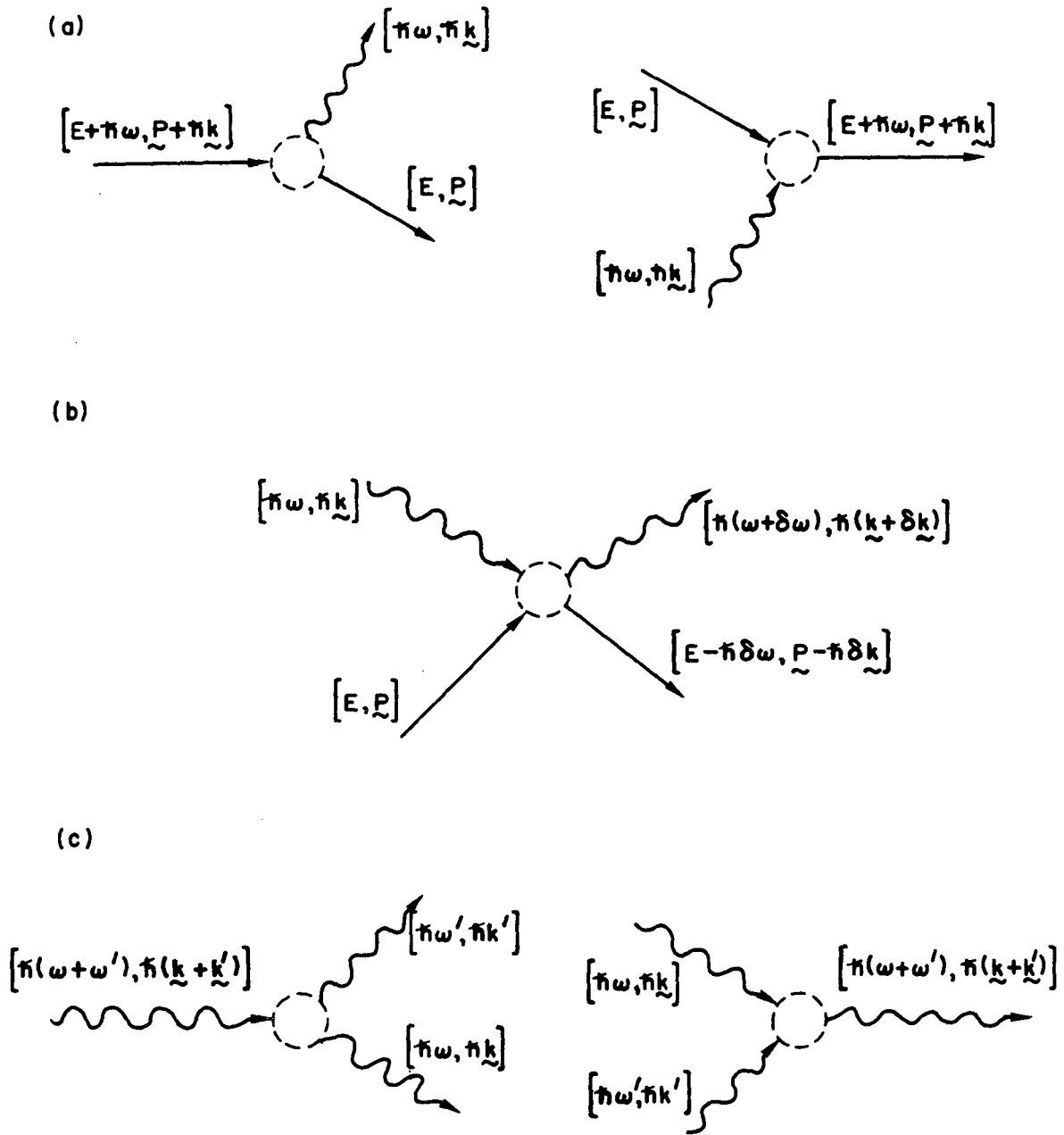
Fig. 4.4 The quantum interpretation of three plasma processes described by the wave and background kinetic equations. Part (a) shows the quasilinear interactions, the emission or absorption of a quasi-particle by a particle. Part (b) shows a nonlinear wave-particle interaction, the scattering of a quasiparticle by a particle. Part (c) shows the nonlinear three-wave interactions, the decay of a quasiparticle into two others, and the inverse process. Energy and momentum are denoted in the figure by brackets, as in [energy, momentum].

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of (4.116) involves the interaction of a particle with two waves. It is
thus a nonlinear wave-particle interaction.

The quasiparticle decay process illustrated in Fig. 4.4(c) is the
quantum equivalent of the three-wave interaction. It has not been
accounted for in our analysis, because we assumed that three-wave syn-
chronism cannot be satisfied, except perhaps over regions of K which
are so small as to be negligible. This is a valid approximation for
many possible spectral distributions, as an examination of the electro-
static dispersion relations will reveal; and it is a completely valid
assumption for interactions of Langmuir waves in a Maxwellian electron
plasma. In that case, the shape of the dispersion curve makes three-
wave synchronism impossible. When synchronous interactions of
three waves are important in the plasma, they may be accounted for in
the analysis by application of the RPA to (3.56).

4.2.9 Discussion

The analysis of this example, like that of the first, has gone
beyond the essential application of the averaged-Lagrangian method in
order to show how that method fits into the general scheme of nonlinear
analysis. Again, the reader should bear in mind that the method itself
is designed to generate the interaction equations in an efficient and
illuminating manner. It is not intended to solve them. In fact, as
this example illustrates, the interaction equations produced by the
method may involve an unfamiliar set of variables. In such cases, addi-
tional manipulation is necessary if the equations are to be expressed in
conventional terms. The example also illustrates, that even in a highly
symmetric system, and even after the application of the RPA, the inter-
action equations may be complicated to solve. Extensive discussion of
dynamic and asymptotic solutions of the quasilinear equations may be
found in the plasma literature.[12,13,18]

Because the plasma has been described statistically in this example,
rather than hydrodynamically, the Lagrangian formalism is employed more
fully than in the first application. Also, because of the distribution
of the plasma in velocity space, the linear theory may predict wave

growth or damping. This causes a perturbation of the background which is of sufficiently low order that it must be taken into account in the wave analysis. As the $\{\bar{F}_{sp}\}$ change, the dispersion relation changes too (on the slow scale), and this changes the rate of growth or damping. Beyond this, the effects of the evolving distribution function have been neglected in the wave analysis.

An interesting feature of the warm plasma considered here is that the waves may have negative energy $(\epsilon_\eta < 0)$. This does not affect the quasilinear or scattering processes described by (4.116), but if the system allows for three-wave (decay-type) interactions between waves of positive and negative energy, there will be an explosive instability of the type mentioned in Section 2.4.4. The parameters of the instability are the coefficients for the three-wave interaction. For this electro-static plasma system, these have been obtained by way of the averaged-Lagrangian method in reference 24. A detailed discussion of their application to explosive instabilities has been given by Dysthe,[60] who obtained the equations by conventional means.

This example was intended to illustrate how the four-wave interactions may be dealt with in the case of discrete waves, and how the RPA may be applied to the equations. However, the additional manipulations involved in the application of the RPA may lead one to suspect that in turbulence studies the averaged-Lagrangian method may be clumsy in comparison with the quantum mechanical approach, which is more directly oriented toward the turbulent case. Actually, the averaged-Lagrangian approach is quite efficient, and has definite advantages when the physics of the medium does not really require a quantum description. Section 3 has shown that the averaged-Lagrangian method produces equations for the wave evolution in standard forms. The appropriate form is determined by the symmetry of the medium, and the types of interaction processes under consideration, where each type is characterized by its synchronism conditions. The standard form equations are independent of the particular algebraic expression for $\mathcal{L}$. The RPA may be applied to these equations, thereby producing standard forms for the wave kinetic equations and their coef-ficients. Equation (4.116) is one such form. Once this is done the

process need not be repeated. For a given medium, the particular
expression for $\mathcal{L}$ is simply used in conjunction with the standard
expressions to calculate the coefficients.

5. APPLICABILITY AND CAPABILITIES OF THE

AVERAGED-LAGRANGIAN APPROACH

In this section, we shall explore certain topics in physics and mathematics in regard to the applicability, capabilities, and possible extensions of the averaged-Lagrangian method. We shall first review the basic assumptions of the theory. Then, in order to interpret them, we shall consider: the effects of boundaries between nonlinear media; the problem of formulating a Lagrangian for a given nonlinear medium or set of nonlinear equations; dissipative media, and the related topic of variational principles other than Hamilton's. Finally, we shall survey the various types of nonlinear effects which are within the domain of the method.

5.1 Basic Assumptions of the Averaged-Lagrangian Theory

The averaged-Lagrangian method of Section 3 will yield a complete set of equations for the wave and background evolution under the conditions that (a) the medium is continuous over a region much larger than the wave periods, and it has a suitable Lagrangian description as defined in (3.1) and (3.2), (b) the Lagrangian density may be expanded in powers of the perturbations from a reference state which is either an exact solution to the full nonlinear equations or differs only slightly from one, (c) the perturbations $\{q^i = q_0^i + \Sigma_\eta q_\eta^i\}$ are small enough that the weak-coupling approximation is valid, and (d) the inhomogeneity in the background state, defined by the reference state plus the $\{q_0^i\}$, is a slow fluctuation, i.e. it is on a scale $\epsilon\underline{x}$, ϵt much slower than the periods of any waves present in the medium.

The method is based on a perturbation expansion, and if it is to be useful, the exact solution of condition (b) must be known. For some problems, there is a series or continuum of such solutions. The exact solution need not be an equilibrium state, although the latter is always an exact solution. An exact solution is simply a complete set of parameters which satisfy the equations of motion. The state need not be stable with respect to a perturbation of these parameters. An example of such a solution is a Maxwellian Plasma through which a beam passes, with no waves present. If there is no perturbation, the beam and plasma remain unchanged, but even a very small amplitude wave perturbation may lead, eventually, to large changes in the beam and plasma distribution functions through the well-known beam-plasma instability.[61]

The Lagrangian analysis has relied on a clear separation in scale between the periods of the waves in the medium and other slowly varying quantities, such as the $\{q_0^i\}$. This means that the method does not apply if the wave components have ω and $\underset{\sim}{k}$ values which extend down to zero.

5.2 The Effects of Boundaries

If a region of a medium is described by a Lagrangian density which has only a slow scale dependence on position within the region, we shall call this region "unified." If two unified regions are separated by a transition region which is not described by some $\mathcal{L}$, or in which $\mathcal{L}$ changes rapidly, we shall refer to the transition region as a "boundary" between the unified regions. If the width of the boundary region is much less than the wavelengths under consideration, it will be referred to as a "sharp boundary."

Since the averaged-Lagrangian method assumes slowly varying wave parameters, it does not, as it stands, describe the nonlinear interaction of waves as they propagate across a boundary region. However, it does apply when the boundaries are parallel to the direction of propagation.

The effect of such parallel boundaries is simply to add a transverse mode structure to the small-signal waves. If the propagation is in the z direction, the spatial dependence of the small-signal modes is

$$\overline{a}_\eta = \hat{a}_\eta(\epsilon z, \epsilon t) M_\eta(x, y, \epsilon z, \epsilon t) \exp j\theta_\eta(z, t) \quad . \qquad (5.1)$$

The variables in the averaged Hamilton's principle are $\hat{a}_\eta$ and θ_η. Since there is no variation with respect to the x and y coordinates, the Euler-Lagrange equations involve only the integral of $\overline{\mathcal{L}}$ over these coordinates. It is through these integrals that M_η affects the mode coupling parameters. The same formulas developed in Section 3 apply to this case, except that $\tilde{\lambda}$ now is an integral of $\overline{\lambda}$ over x and y as well as $\underset{\sim}{v}$.

For some problems, it would be desirable to have an extended averaged-Lagrangian method which accounts for the nonlinear effects of boundary regions. We shall not actually develop such a method here, but we shall consider the first theoretical step: the inclusion of the boundary terms in the action integral.

Let us consider a unified region, R , bounded by a surface S; and let us assume that within and on S , the medium is described by a Lagrangian density, $\mathcal{L}$, which depends only on $\underset{\sim}{x}$, t, the $\{q^i\}$, and their first derivatives. When $\mathcal{L}$, or the variations $\{\delta q^i\}$, vanish on S , Hamilton's principle yields the Euler-Lagrange equations (3.3). However, when boundary terms are included, an integration of the action integral (3.1) by parts gives

$$0 = \int_{t_1}^{t_2} dt \int_R d^3x \, \mathcal{L}(q^i + \delta q^i, \underset{\sim}{x}, t)$$

$$= \int_{t_1}^{t_2} dt \int_R d^3x \, [E-L]_i \delta q^i + \int_{t_1}^{t_2} dt \int_S ds \, [E-L-S]_i \delta q^i \quad , \qquad (5.2)$$

where

$$[\text{E-L}]_i \equiv \left\{ \frac{\partial \mathcal{L}}{\partial q^i} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial q^i/\partial t)} - \sum_k \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial (\partial q^i/\partial x_k)} \right\} \quad , \qquad (5.3)$$

and

$$[\text{E-L-S}]_i \equiv \left\{ \frac{\partial \mathcal{L}}{\partial q^i}\underset{\sim}{n} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial q^i/\partial t)}\underset{\sim}{n} + \sum_k \frac{\partial \mathcal{L}}{\partial (\partial q^i/\partial x_k)}\underset{\sim}{e}_K \right\} \cdot \underset{\sim}{n} \qquad . \quad (5.4)$$

Here, $\underset{\sim}{n}$ is a unit vector, pointing normally outward from the differential surface element ds, and the $\{\underset{\sim}{e}_k\}$ are cartesian unit vectors, corresponding to the coordinates $\{x_k\}$. Because δq^i may be of arbitrary form within and on S, (5.2) implies that for (i=1,...,M)

$$[\text{E-L}]_i = 0 \quad (\text{in R}), \quad [\text{E-L-S}]_i = 0 \quad (\text{on S}) \quad . \qquad (5.5)$$

The first of these equations is simply the Euler-Lagrange Equation (3.3). The second is the boundary condition for the variable q^i at S . Thus, the boundary conditions, as well as the equations of motion are given by Hamilton's principle.

For two unified regions, separated by a single boundary, the relations between the surface terms of the regions tend to simplify considerably in the limit of zero boundary width. It therefore seems probable, that by including surface terms in the Lagrangian analysis, we may usefully extend the averaged-Lagrangian method to account for nonlinear interactions at sharp boundaries. Such interactions are important, for instance, in problems of coherent optics, which have been treated theoretically by Bloembergen and Pershan.[62] When the boundary is sharp, the wave parameters on either side may be treated as independent variables in the Lagrangian analysis. They are to be related by nonlinear boundary conditions which derive from the contribution of the boundary to the total Lagrangian. By introducing different variables for waves on each side of the boundary, we can maintain

consistency with the averaged-Lagrangian concept that changes in the wave amplitudes are on the slow scale in x and t .

The averaged-Lagrangian method cannot be extended to describe wave interactions in boundaries of finite width, because, in that case, it is necessary to trace the changes in the wave parameters continuously through the region. The changes are too rapid for description by the averaging approach developed in the present work.

5.3 Formulation of the Lagrangian

The application of the averaged-Lagrangian method is possible only when there is a known functional $\mathcal{L}$, of a set of dependent variables $\{q^i\}$, such that the set of equations $\{E-L\}$, generated by Hamilton's principle, completely describe the model for the medium. In this discussion, the formulation of $\mathcal{L}$ will be referred to as the "general inverse problem." The term "general" is used in order to distinguish the problem from a more restrictive mathematical problem to be discussed later in this subsection. There is no necessity for the medium to be real in the physical sense; the model used for it may be either physical or mathematical. In either case, however, the variable parameters of the model must be associated with a set of descriptive equations, $\{M\}$, which describe their behavior. We shall say that a Lagrangian description of the medium is "complete" if and only if the equations generated from $\mathcal{L}$ through Hamilton's principle correctly describe the behavior of all variable scalar parameters, $\{p^i\}$, of the model. In this sense, they are equivalent to the equations $\{M\}$. In equivalent sets of equations, the scalar variables may be different in definition and in number. Any complete description, however, defines transformations which specify each parameter, p^j , in terms of the $\{q^i\}$. This was illustrated in Section 4.2, where the Lagrangian analysis of a warm plasma described the same phenomena as the Maxwell-Vlasov equations, by using different variables for the particle perturbations. An important aspect of the general inverse problem is that its solution, $\mathcal{L}$, may involve some

variables, $\{q_s^i\}$, which are not involved in the transformation relation. These will be referred to as "supplementary variables," and the remainder of the $\{q^i\}$ will be referred to as the "primary variables", $\{q_p^i\}$. Consider the Lagrangian description of a medium in which $\mathcal{L}$ involves both types of variables. The equations $\{E-L\}$ are equivalent to $\{M\}$, i.e. all solutions $\{q^i\}$ to $\{E-L\}$ specify all solutions $\{p^j\}$ to $\{M\}$. The Lagrangian description is associated with a more general mathematical model, however, in that $\{E-L\}$ has solutions for the $\{q_s^i\}$ as well. In general, these solutions are not of interest in themselves, but we shall see in the course of this discussion that they sometimes may be given a physical interpretation.

Section 3.1.1 has shown that a Lagrangian description implies that, for any independent variable, t , the equations $\{E-L\}$ imply a conservation relation. Equation (3.8) is the form of that relation when $\mathcal{L}$ is a function of the independent variables, the $\{q^i\}$ and their first derivatives only. The existence of such relations provides information about the Lagrangian description of dissipative physical media. Consider a dissipative medium, described by a set of primary variables, $\{p^j\}$, and equations $\{M\}$. For such a medium, these equations and variables cannot be used to construct an energy conservation relation of the form (3.8), where t is the time. This implies that if a Lagrangian description of a dissipative medium does exist, it can be formulated only through the introduction of supplementary variables, $\{q_s^i\}$.

5.3.1 The Physical Approach to the Variational Formulation

The basic physical approach to the general inverse problem follows the line of reasoning used in the first example of Section 4. Kinetic and potential energy states are associated with the variables, then a trial expression for $\mathcal{L}$ is constructed according to the classical prescription (3.12). Except for purely mechanical systems it is generally not clear how to divide the energy into "kinetic" and "potential" parts, so the choice for $\mathcal{L}$ must be ultimately justified by the correctness of the resulting equations. Although this approach is essentially heuristic, it has been used with considerable success. A variant of this approach

is the analogy with a mechanical system discussed earlier in Section 3.1, and another is that used by Low[27] in his formulation of a warm plasma Lagrangian. Low constructed the needed Lagrangian by combining other known Lagrangians.

If the Lagrangian, obtained by any of the methods mentioned above, is not expressed in a convenient set of variables, one may make a transformation to a new set of variables without invalidating the variational principle. The principle itself involves small changes in the physical state of the medium, and is not dependent on the parameters used to characterize that state. The variations must be consistent with any constraints on the system, however, and the variables must be independent of one another.

An attempt to clarify the energy-state approach to the Lagrangian formulation has been made by Penfield and Haus.[45] They have categorized the variables of a physical system as either "force like" or "geometric" in order to aid in the identification of energy terms with $\mathscr{T}$ or $\mathscr{U}$. The result of this effort appears to be a more clearly expressed version of the classical rule (3.12), but a rigorous justification of the rule from physical principles is still lacking.

For a dissipative system, the physical approach is inapplicable because there is no conserved energy density, $\mathscr{T} + \mathscr{U}$, associated with the dissipative model. However, a physical approach to overcoming this difficulty is to consider a more complete model than that which describes the system alone. The new model combines the dissipative system with its conjugate, i.e. the system to which the energy is lost through random processes. The combined system is conservative and may thus be described by a Lagrangian. The variables which describe the conjugate system are supplementary variables with respect to the original, dissipative model. With respect to the combined models, however, all of the generalized variables are primary. The introduction of additional variables to describe the conjugate system is one disadvantage of this approach. Another is that the physics may become much more complex. Consider, for purposes of illustration, a plasma in which inelastic collisions occur between the charged and neutral particles. The collisions may be described

as conservative processes only if the excitation energies of the neutral particles are accounted for. If this were done through a quantum mechanical description of the particle behavior, and the classical theory we have developed in this investigation would not apply. The nonlinear behavior of a quantum model may be treated by the second-quantization approach mentioned in Section 1. The reader should keep in mind that, although the classical Lagrangian formalism cannot account for true dissipation, such as collisional loss, it is still capable of accounting for non-random exchange of energy between a medium and external sources. Therefore, the system need not be thermodynamically closed, merely non-dissipative. Furthermore, the Lagrangian description can account for Landau damping of an electromagnetic wave in a plasma. This damping is implicit in the results of Section 4.2. The Landau damping process involves no loss of energy, since it occurs even in a collisionless plasma. The process simply results in a transfer of energy from the fields of the waves into fine scale fluctuations in the distribution functions of the plasma particles.[63]

5.3.2 The Mathematical Approach to the Variational Formulation

Much of the mathematical literature on variational formulations concerns itself with the following restricted version of the general inverse problem. Take, as given, a set $\{M\}$, of M coupled partial differential equations in M variables, $\{p^i\}$ $\dot{.}$ The equations are defined in an $(\ell + 1)$ - dimensional volume V of the coordinates $\{x_k : k=0,\ldots\ell\}$. The problem is of two parts: (a) determine whether the system $\{M\}$ is equivalent to the set generated by

$$\delta J = 0 \quad , \qquad J \equiv \int_V d^{\ell+1}x \; \mathcal{L}(p^i,\underset{\sim}{x}) \quad , \qquad (5.6)$$

and (b) if this is the case, prescribe how the functional $\mathcal{L}$ may be constructed. This problem differs from the general inverse problem, in that the variables of $\mathcal{L}$ are required to be identical to those of $\{M\}$. The use of supplementary variables is not allowed. This problem will be

referred to henceforth as the "strict inverse problem." A set of equations for which the variational formulation described in (a) is possible will be labeled "self sufficient." The label has been used previously by Becker.[64]

The strict inverse problem proves to be quite complicated, and it has not been solved in its full generality. In the literature, the problem was first defined and solved for special cases by Darboux,[65] in a geometrical treatise, and by Hirsch,[66] in a mathematical paper devoted entirely to the subject. Both of these treatments dealt only with ordinary differential equations. Subsequently, the problem has been treated in increased generality by others. Significant work has been published by Kurschak,[67] Davis,[68] La Paz[69] and Douglas.[70] A text by Funk[71] provides a survey and an introduction to the work on this subject through 1970.

All published solutions to the special inverse problem have been obtained for differential equations in which the number of independent and dependent variables, and the order of the derivatives, are restricted. The most general solutions published to date appear to be those of La Paz and Douglas. Both allow only for $\mathcal{L}$ to be a function of the $\{p^i\}$ and their first derivatives. La Paz treated the system of second order differential equations having many independent variables and only one dependent variable. Douglas also considered second order equations, but allowed for up to three dependent variables and two independent variables, requiring that the equations be linear in the second derivatives of the independent variables.

In unpublished work, Peng[72] has solved a problem in which the numbers of dependent and independent variables are allowed to be arbitrary. Otherwise, his assumptions are identical to those of Douglas. The problem as formulated by Peng is the most general case of a special inverse problem for which a solution is known to this author. The need for more general solutions is illustrated by the Euler-Lagrange equations of the warm plasma Lagrangian (4.79). These are not of a general form for which the special inverse problem has been solved.

Certain sets of partial differential equations are not self-sufficient. Some of these sets have been identified in the course of purely mathematical investigations[68] and others appear in the mathematical description of physical processes. Gage et al[73] have shown that the general thermokinetic equations are not self-sufficient, and other examples are to be found among the equations used to describe dissipative media. A simple one is the damped wave equation

$$\frac{\partial^2 q}{\partial t^2} + \sigma \frac{\partial q}{\partial t} + \frac{1}{c_p^2} \frac{\partial^2 q}{\partial x^2} = 0 \quad . \tag{5.7}$$

In this equation, q is the independent variable, and the independent variables are position, x, and time, t. The constants σ and c_p are the damping coefficient and the characteristic speed of propagation, respectively. The problem is self-sufficient only if σ is zero.

The use of supplementary variables leads to the solution of the general inverse problem for many non-self-sufficient sets of equations, including (5.7) and others which describe dissipative media. Two mathematical techniques for Lagrangian formulations through the use of supplementary variables are discussed in the remainder of this subsection.

DUAL VARIATIONAL FORMULATION: One approach to the general inverse problem involves a "dual" Lagrangian formulation, in which each primary variable, q^i, has associated with it a supplementary variable, $q^{i\dagger}$. For simplicity of exposition, we shall consider a single nonlinear descriptive equation which has only one independent variable, x, and one dependent variable, q. The discussion follows that given in a monograph by Becker.[51] The equation is

$$A_{Op} q + g = 0 \quad , \tag{5.8}$$

in which A_{Op} is a nonlinear differential operator, and g is a function of x only. Whether or not this equation is self-sufficient depends on the nature of A_{Op}. A special self-sufficient case is that in which A_{Op}

is a linear operator such that

$$\int_a^b dx \; q_2 (A_{Op} q_1) = \int_a^b dx (A_{Op} q_2) q_1 + \left\{ \begin{array}{c} \text{boundary} \\ \text{terms} \end{array} \right\} \quad , \qquad (5.9)$$

for any well-behaved functions $q_1(x)$ and $q_2(x)$. An operator with the property (5.9) is called "self-adjoint," and when A is such an operator, the variational principle which corresponds to (5.4) is

$$\delta J = 0 \quad , \qquad J \equiv \int_a^b dx \left[\frac{1}{2} q A_{Op} q - qg \right] \quad . \qquad (5.10)$$

Even when A is not a self-adjoint linear operator, (5.8) admits of a variational description through the use of a supplementary variable. The variational principle is

$$\delta J = \int_a^b dx \left[A_{Op} q - g \right] \delta q^\dagger = 0 \quad , \qquad (5.11)$$

in which $q^\dagger$ is a second independent variable, called the "adjoint" of q. An integration of (5.10) with respect to $q^\dagger$ yields

$$J = \int_a^b dx \; \mathcal{L} \quad , \qquad \mathcal{L} \equiv q^\dagger A_{Op} q - gq^\dagger + h(q,z) \quad , \qquad (5.12)$$

where h is an arbitrary function of x and q on the interval a-b. Clearly the variation of $q^\dagger$, under the assumption that δJ is zero, gives the desired equation, (5.8). The variation of q, on the other hand, gives

$$\delta J = \int_a^b dx \left[A_{Op}^+ q^\dagger + \frac{\partial h}{\partial q} - \frac{\partial}{\partial x} \frac{\partial h}{\partial (\partial q / \partial x)} + \ldots \right] \delta q = 0 \quad , \qquad (5.13)$$

in which A is the "adjoint operator," defined to be such that

$$\int_a^b dx\, q^\dagger A_{Op} q = \int_a^b dx (A_{Op}^\dagger q^\dagger) q \; .$$ (5.14)

Since δq is of arbitrary shape in a-b , (5.13) implies the "adjoint equation"

$$A_{Op}^\dagger q^\dagger + \frac{\partial h}{\partial q} - \frac{\partial}{\partial x} \frac{\partial h}{\partial(\partial q/\partial x)} + \ldots = 0 \; .$$ (5.15)

The definition for $A_{Op}^\dagger$ is meaningful if and only if the variables and the variations of the problem are such that (5.14) is the result of an integration by parts. To insure that this is the case we impose boundary conditions of the form

$$B(q^\dagger, q, \delta q^\dagger, \delta q, x)_{x=b} - B(q^\dagger, q, \delta q^\dagger, \delta q, x)_{x=a} = 0 \; ,$$ (5.16)

where B is determined by (5.14) and the form of A_{Op}. The equation (5.16) defines the boundary conditions needed for the complete specification of the adjoint problem.

It is known that dual variational descriptions may be given for certain dissipative media other than those described by an equation of the form (5.8); but we make no claim here that a dual formulation is possible for each member of the entire class of dissipative systems. The mathematical limitations of the dual variational formulation do not appear to be established precisely enough to prove or disprove such a claim.

For any set of M linear inhomogeneous partial differential equations, in M dependent variables and any number of independent variables, the dual variational description is straightforward. The approach to solving such an inverse problem is the same as that which led to (5.11).

The dual variational formulation for nonlinear equations is a more complicated problem which has been studied, for example, by Becker.[64]

The analysis of a set of dual equations is generally more complicated than the analysis of the primary set of equations, because the number of independent variables and boundary conditions is greater for the dual set. The dual analysis may also be more complicated because of the form of the adjoint equations and boundary conditions. These forms may be unfamiliar, even when the primary problem is not. Furthermore, when the primary problem describes a physical system, information about the solution may sometimes come from physical insight. The adjoint variables, on the other hand, may have no physical interpretation. Even in obtaining approximate solutions to the adjoint problem, considerable analytic effort may be involved.

The averaged-Lagrangian method may be applied to problems of nonlinear wave interaction even when the variational formulation is of the dual type. In the analysis, however, the small-signal relations must be found for the set of adjoint variables $\{q^{i\dagger}\}$ as well as for the set of primary variables $\{q^i\}$. Each set will have its own closed set of small-signal relations because in the linear limit, the primary and adjoint variables are related only by the adjoint boundary conditions. Since the primary equations describe wave behavior, the solution of the adjoint equations is straightforward in the small-signal limit: they must have wave solutions also.

LAGRANGE MULTIPLIERS: Another use of supplementary variables, in the solution of the general inverse problem, is in the form of Lagrange multipliers. These are used when a functional $\mathcal{L}(q_p^i)$ is known to generate a given set of equations through a variational principle which is more restrictive than (5.6). The additional restriction is that the $\{q_p^i\}$ must be varied in keeping with a set of constraint equations $\{M'\}$. Through the introduction of supplementary variables $\{q_s^i\}$ it is possible to use the constraints and $\mathcal{L}$ to construct a functional $\mathcal{L}''$ which has the following property: A Hamilton-type variational principle, with $\mathcal{L}''$

the functional kernel, generates a set of equations M'' such that

$$\{M''\} = \{M\} \cup \{M'\} \qquad (5.17)$$

That is, $\{M''\}$ is equivalent to the conjunction of the sets of equations $\{M\}$ and $\{M'\}$. The method of Lagrange multipliers is a familiar one in the literature of physics and variational calculus. A description of the analytic steps is given in the standard texts on the subject, such as that of Goldstein,[74] or Courant and Hilbert.[75]

5.4 Other Variational Principles

There exists a classical technique for accounting for dissipative effects in a modified Lagrangian description. The label "modified" is used here to indicate that the description does not employ a Hamilton-type variational principle. It is, however, associated with the variational principle of virtual work, known in classical mechanics as D'Alembert's principle.[76] Use of the principle of virtual work leads to the formulation of a dissipation function, $\Re$, which is sometimes called the Rayleigh dissipation function. It accounts for the effect of dissipative forces in the medium. The functional $\Re$ is added to $\mathcal{L}$, the Lagrangian density which describes the behavior of the medium in the absence of dissipative forces, and the descriptive differential equations are obtained from $\mathcal{L} + \Re$ by means of a modified Euler-Lagrange formula. Examples of the application of the principle of virtual work in non-mechanical systems have been given by Biot,[77] who developed a modified Lagrangian description for heat transfer processes. The averaged-Lagrangian method, as it has been developed here, cannot be used in conjunction with variational principles which are not of the Hamilton type.

5.5 Plasma Lagrangians

This subsection is devoted to a survey of the known plasma Lagrangians, which may be usefully characterized and discussed in relation to two archetypes. The first of these is the Low Lagrangian for a warm plasma.

This Lagrangian, which was employed in Section 4.2, generates equations which are equivalent to the Maxwell-Vlasov equations. The second archetype is an ideal hydrodynamic plasma Lagrangian which, by definition, generates the equivalent of the hydrodynamic plasma equations, i.e. the electromagnetic equations in conjunction with the velocity moments of the Vlasov equation. The moment equations comprise a hierarchy of hydrodynamic equations which is useful when the effects of wave-particle resonant interactions, such as Landau damping, are not significant, provided also that the series of equations may be truncated at some velocity moment without much loss of accuracy. Henceforth in this discussion, the term "hydrodynamic" will refer to the description of a single beam. The hydrodynamic equations generally used in plasma theory are the first three moment equations: the continuity equation, the momentum transfer equation, and the heat transfer equation.

The Low Lagrangian appears to be the only useful Lagrangian of its type which is presently known. It has the convenient property that it may be expanded about a given reference state in a set of field-like independent variables. The reference frame is a set of Eulerian independent variables. A different formulation of this Lagrangian was used as a starting point by Low, but in that form the Lagrangian is divided into two terms, in which the independent variables are of the Eulerian and Lagrangian types, respectively. In that form, the Lagrangian is unsuitable for use with the averaged-Lagrangian method.

Hydrodynamic Lagrangians have been the subject of numerous publications, and a variety of forms have been proposed. These differ in the choice of variables, the constraints imposed on them, the physical assumptions about the medium, and the accuracy to which the hydrodynamic approximation has been carried. The formulation of a hydrodynamic Lagrangian with respect to a system of Lagrangian coordinates was the subject of a paper by Eckart[78] in 1960. However, the main interest in the literature has been in solving the more difficult problem of formulating the Lagrangian in an Eulerian coordinate system. Eulerian formulations are, in general, the most convenient analytically, and they are of the greatest relevance to the present investigation. Henceforth, this

discussion refers only to those variational principles which involve Eulerian coordinate systems.

Early progress in the variational description of hydrodynamic media was reported by Taub,[79] in 1947, and by Davydov,[80] in 1949. Taub formulated a Lagrangian for an inviscid, perfectly compressible fluid with no heat conductivity; Davydov concerned himself with finding the appropriate canonical variables for the fluid Lagrangian, and he gave a Hamiltonian description of the medium. Neither of these authors included the electromagnetic fields in the variational description. In 1961, Katz[81] considered the inclusion of the field behavior in a Lagrangian description of an inviscid, perfect, compressible plasma. Subsequent papers by Su,[82] Lundgren,[83] Calkin,[84] and Merches,[85] discuss and solve various forms of a related but different problem: the formulation of a Lagrangian which yields the magnetohydrodynamic equations. This problem is different, in that an additional constitutive relation, infinite conductivity, is incorporated into the descriptive equations. In the aforementioned magnetohydrodynamic Lagrangians, the pressure is assumed to be isotropic, but in a relatively well-known paper, in 1962, Newcomb[86] formulated Lagrangian expressions for cases in which the pressure tensor is required only to be symmetric about the axis of the externally applied magnetic field. Problems of boundaries were considered by Taub, and by Wenger,[87] who extended the magnetohydrodynamic Lagrangian description to include surface Lagrangian densities, for a particular model of flow in a channel with fixed boundaries. Application of Hamilton's principle to the Lagrangian surface terms gives the correct boundary conditions on the variables for his model. A Hamiltonian formulation for magnetohydrodynamics of an ideal liquid was contributed by Zakharov and Kuznetsov[88] in 1971.

Recent work on the Lagrangian description of non-magnetohydrodynamic fluid and plasma models has been published by Penfield and Haus,[89] in 1966, and in 1971, Zakharov[90] authored a paper on the Hamiltonian description for such a model. The published work up to the present time appears to provide Lagrangian descriptions for inviscid, compressible fluids which have current transport through conduction. Ohm's law has not been taken

into account except in the magnetohydrodynamic case in which the conductivity is assumed to be infinite. No viscosity or ohmic loss resulting from particle collisions has been taken into account in any of these descriptions, and the equations generated by the Lagrangians do not describe moment equations of higher order than the second, i.e. the momentum transfer equation. In recent unpublished work, Peng[91] has shown how one may include the heat transfer equation, as an additional constraint, and account for elastic collisions between particles, by the addition of an interaction energy term to the Lagrangian.

In the case of a perfectly cold plasma (no thermal motions) the Vlasov and hydrodynamic descriptions are equivalent. For this simple case, the most useful formulations for the Lagrangian seem to be of two types. These differ in their choice of variables. One type, formulated by Sturrock[43] uses a set of displacement variables to describe the particle perturbation. This representation most closely resembles that used later by Low in formulating for the Vlasov plasma. The second type of formulation employs the commonly used Clebsch variables of hydrodynamics, and is thereby the closest to the hydrodynamic formulations mentioned above.

The Lagrangian for a cold plasma stream, expressed in the Clebsch variables, was used by Buneman[63] to obtain a warm plasma Lagrangian in a form quite different from those already mentioned. Buneman considered the approximation of a warm plasma by a finite number of non-vortical cold streams, and formulated a stream-model Lagrangian in terms of the Clebsch variables for each stream. In his formulation, $\mathcal{L}$ consists of $\mathcal{L}_{(2)} + \mathcal{L}_{(3)}$; there are no higher order terms. However, the number of independent variables is much larger than in the Low description if the number of streams is very large. The stream-model tends to a Vlasov description of the plasma, in the limit of a large number of streams. The stream-model Lagrangian appears to be well suited to computer simulation of plasma phenomena, and is an alternative to microscopic simulation techniques, which involve the tracking of a large number of particles, or groups of particles, in time and space. The averaged-Lagrangian method could be applied to the stream-model

Lagrangian because the stream description uses Eulerian position co-ordinates, t and $\underset{\sim}{x}$. However, for many streams, the number of independent variables is quite large, and the analysis would be extremely tedious.

5.6 Capabilities of the Averaged-Lagrangian Method

The preceding sections have shown how the assumptions made in the development of the averaged-Lagrangian method set the limits on its applicability. These limits define the domain in which the method is valid, but they do not, in themselves, convey a picture of what problems lie within these limits. For that purpose, we shall now consider how the problems within the domain of the method may differ in their various characteristics. Hereafter in this subsection it will be assumed that the basic assumptions of the method, summarized in Section 5.1, are valid for all of the problems under consideration.

The media to which the averaged-Lagrangian method may be applied constitute a broad class of physical and mathematical systems, of which three-dimensional physical media are only a subclass. The analysis does not require that the dependent variables $\{q^i\}$ be given any parti-cular physical interpretation, nor does it require that independent variables, $\underset{\sim}{v}, \underset{\sim}{x}$ and t actually represent velocity, position and time coordinates. This notation has been chosen simply because it is generally convenient in the description of non-relativistic physical media, in-cluding plasmas. The Lagrangian analysis merely requires that the sets $\{q^i\}$ and $\{x_k,t\}$ each have at least one element, and that the small signal solutions for the $\{q^i\}$ must be oscillatory with respect to the t and x coordinates. The $\{q^i\}$ need not be oscillatory with respect to the v coordinates, if they are used. The description of a Vlasov plasma required seven independent scalar variables, which repre-sent velocity, position, and time. On the other hand, the properties of a fluid or a solid are functions of position and time only, and a collec-tion of coupled oscillators constitutes a medium, for our purposes, in which there is only a single independent variable, the time. Of course, the oscillators may not be fixed in space, so the position of an oscil-lator, s , may be denoted by a positional vector, $\underset{\sim}{r}^s$. That vector,

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however, is a dependent variable which has the argument t . It does not play the role of an x-coordinate.

The averaged-Lagrangian method may be applied to a medium which is weakly inhomogeneous in the x and t coordinates. The medium may be strongly anisotropic, and although its state must be close to an exact solution of the descriptive equations, it need not necessarily be near a state of equilibrium. The method applies equally well to waves in unbounded media, to waves which propagate parallel to fixed boundaries, and to oscillations which evolve in time only, such as standing waves in a cavity. If the waves propagate across boundaries, the method applies only within the region of continuous propagation.

For monochromatic waves, the method yields equations which describe the effects of linear growth or damping, weak inhomogeneity, and nonlinear wave-wave and wave-background interactions to all orders in the small parameter ϵ . In the case of plane waves in a large continuous region, the method allows for propagation of the waves at arbitrary angles with respect to one another and any axes of symmetry in the reference state. We have seen that the wave-wave interaction effects accounted for by the method include the interesting special cases of sideband decay and the self-action of a single, large amplitude, monochromatic wave. The effect of synchronism mismatch in wave-wave interactions is also accounted for.

The method is effective for the nonlinear analysis of monochromatic waves, because it yields equations for the wave evolution in a standard form. That is, the structure of the equations and the formulas for their coefficients do not depend on the physics or mathematics of the medium, except through the assumption that a Lagrangian description is possible. The standard form depends only on the synchronism conditions which characterize the interaction processes. For a particular medium, the interaction equations are obtained explicitly by substitution of $\mathcal{L}$ and the corresponding small-signal relations into the standard form equations.

The method is also effective for the analysis of continuous wave spectra in the weak turbulence limit. This is done through application

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of the random-phase approximation to the standard form equations for the monochromatic waves. The resulting wave-kinetic equations will also have a standard form, for a particular interaction process, and once these are obtained, the coefficients may be calculated for a particular medium by inserting the expression for $\mathcal{L}$ into these formulas. This technique, and its counterpart for monochromatic waves, are quite similar in calculational efficiency. However, the standard formulas obtained through the RPA tend to be somewhat simpler algebraically than those for the monochromatic wave analysis.

The background evolution differs in many respects from the wave evolution, and its description by the averaged-Lagrangian method differs accordingly. For a medium in which waves propagate in the linear régime without growth or damping, there is no wave-background interaction. We have seen from the example of Section 4.1 that in this case, the method yields background equations which are uncoupled from those of the waves. When there is linear growth or damping, the method yields the quasi-linear equations of background evolution, as in the example of Section 4.2, and, when the analysis is carried to higher orders of ϵ, the equations include the effects of nonlinear wave-background interactions - such as nonlinear wave-particle interactions in a warm plasma. The approximations, to the various orders in ϵ, form a hierarchy of background equations.

Application of the RPA to the hierarchy for the monochromatic wave case will generate another hierarchy, the background kinetic equations. The term "kinetic" indicates that the wave parameters which appear in these equations are the wave energies, as opposed to the complex wave amplitudes which appear in the other background equations. We have seen in Section 4.2 how the background kinetic equations may be obtained for a plasma example. In plasma nomenclature, these are called the "particle kinetic equations" because the background parameters which change are the particle distribution functions.

The utility of the averaged-Lagrangian approach to the background analysis is in its compatibility with the wave analysis. Use of an averaged-Lagrangian formalism in both analyses is aesthetically pleasing,

and it makes possible the efficient ordering procedure described in Section 3. Without a unified formalism it would be more difficult to make self-consistent approximations for the wave and background equations.

6. CONCLUSIONS

This dissertation essentially completes a program of investigation which has resulted in earlier publications and co-publications by this author.[22-24] The goal of the program has been to provide a conceptually useful and analytically efficient method for the analysis of nonlinear wave effects, in continuous media which have a classical Lagrangian description. The insight and elegance provided by Hamilton's principle have been utilized as (a) as means of establishing general properties of the interaction equations, such as the Manley-Rowe relations and the necessary conditions for explosive instability, and (b) a means by which standard expressions for the wave interaction parameters may be obtained. We have already shown that the resulting averaged-Lagrangian method has characteristics of elegance, convenience, and versatility which are desirable in the analysis of complicated nonlinear wave phenomena, such as those which occur in plasmas. Here we shall review the original contributions of the work, and discuss future applications of the averaged-Lagrangian approach to plasma and non-plasma problems.

6.1 Original Contributions of the Work

The original contributions of this work are contained in (a) the averaged-Lagrangian method itself, (b) the derivation of standard formulas for nonlinear wave coupling coefficients, and (c) the application of the method to specific nonlinear wave problems in plasma physics. Here we shall review these contributions in more detail.

The averaged-Lagrangian method presented here is closely related to previous work by others, notably that of Dougherty[20,25] and Dysthe.[21] The new feature of this method is the manner in which it incorporates the background analysis. This is done here by means of a dual expansion of $\bar{\mathcal{L}}$ in terms of the wave and background variables. An ordering procedure, based on this expansion, is used as a means of approximating the wave and background equations in a mutually consistent manner. No other Lagrangian method with this feature has come to the attention of the author. In fact, with the exception of the papers by Dougherty, it appears that no

other Lagrangian analyses describe the background evolution at all. Dougherty, however, has noted the possibility of describing the background evolution through the Lagrangian formalism, and he has shown how this may be done for certain simple examples. He has not, however, developed the background analysis in a manner which is well suited to the more complicated plasma problems.

In Section 3 it was shown how the wave interaction equations and their coefficients may be derived in standard form, in terms of $\bar{\mathcal{L}}$. These are significant results, as we have already seen; in fact, they are the reason why the averaged-Lagrangian method is so attractive in nonlinear wave analysis. Once the standard forms are established for the wave-wave interaction processes, we need only to insert the specific form of $\bar{\mathcal{L}}$ in order to completely specify the wave evolution for a particular medium. The functional $\bar{\mathcal{L}}$ supplies all the detailed information about the medium's physical or mathematical properties. For the three-wave case, a complete statement of the standard formulas was given by this author and F.W. Crawford in 1970.[23] Previous to this, in 1966, Siegman[33] had inferred the general form of the interaction equations from classical Hamiltonian theory. He did not, however, derive useful formulas for the interaction coefficients.

In this dissertation, the plasma applications have been chosen with a view toward illustrating the power and elegance of the averaged-Lagrangian method. They were not intended to generate new results, but they do contain original conceptual features, which provide additional insight into the problems. An example is the derivation in Section 4.2 of the quasilinear diffusion equation, a result which is already well-known in the plasma literature. The new contribution of the present analysis is the derivation of this relation by means of the displacement variables of the Low Lagrangian.

In earlier publications by this author the averaged-Lagrangian method has been applied to wave-wave interaction problems only. The phenomena considered were three-wave interactions in a nonlinear transmission line, a cold plasma, and a Vlasov plasma. Of these, only the cold plasma application was intended to produce new results, i.e. the

interaction equations and coupling coefficients for any three Appelton-Hartree waves, propagating at arbitrary angles with respect to one another and the static magnetic field in a cold, homogeneous magneto-plasma. The results are of interest because of their generality. Previously, the three-wave interactions in a cold plasma had been analyzed only for more specialized cases.

6.2 Future Applications of the Method

The preceding sections have shown that the limitations of the averaged-Lagrangian method stem primarily from two of its features. First, it requires a Lagrangian description of the medium, and second, it is a perturbation theory. In plasma applications of the method, the first limitation is not very severe. We have shown in Section 5.5 that the known plasma Lagrangians describe most of the important plasma models. The major exceptions are the dissipative collisional models. However, the discussion of mathematical techniques, in Sections 5.3 - 5.4, suggests that through the introduction of supplementary variables it may be possible to describe any dissipative physical medium by a variational principle of the Hamilton type. We have neither proved nor disproved that proposition here, because of the mathematical complexities involved.

The limitations which stem from the perturbation nature of the method are much more severe than those discussed above, because the perturbation assumptions firmly limit the method to the description of weak nonlinear effects in weakly inhomogeneous plasmas. The domain of the method is broad enough, however, to encompass almost all of the non-linear effects which are describable by any plasma theories available at this time. Therefore, the prospects for application of the method in the immediate future are bright. In the long run, however, it may be that perturbation analyses will be overshadowed in importance as development proceeds on techniques for the description of strong turbulence.

Much important work remains to be done in the analysis of nonlinear plasma effects. For example, except in the case of an unbounded, spatially homogeneous, Vlasov plasma, the background evolution in plasmas has still not been thoroughly investigated. In nonlinear plasma theory,

the greatest progress has been made in the study of wave-wave interactions. There are, however, a great many different problems to choose from, i.e. problems which differ in the choice of plasma model, the types of waves and order of wave interaction considered, boundary conditions, and symmetry assumptions. Because of the large number of problems, only a fraction of them have been studied to date. These have generally been chosen because of their relevance to an experiment, their relative simplicity, or their general theoretical interest.

A popular class of problems is that of three- or four-wave interaction processes in an infinite homogeneous plasma. The problems of this class vary in the model chosen for the plasma and in the types of waves considered. For monochromatic waves in a Vlasov plasma, a rather comprehensive analysis of three- and four-wave interactions has been undertaken by Suramlishvili.[52-53] He employed the Low Lagrangian in formulas obtained from a semi-quantum mechanical approach. The resulting expressions for the interaction coefficients are extremely general. Unfortunately, they are also extremely complicated, and they appear to contain some significant errors which were noted by this author in reference 24. For the more specialized wave interaction problems, it is often more convenient and instructive to simplify the analysis from the beginning, rather than to specialize complicated general results like those of Suramlishvili. For such specialized analyses, and for the study of the unsolved problems of nonlinear plasma theory, the averaged-Lagrangian method promises to be an effective tool.

At the present time, the most promising plasma applications of the averaged-Lagrangian method appear to be in the study of weak turbulence as it related to fusion devices, and in the study of wave scattering by density fluctuations in the ionosphere.[92] Possible applications of the method outside of plasma physics are particularly promising in the fields of coherent optics and microwave acoustics. In both of these fields, the nonlinear behavior of monochromatic waves is under active study, and the wave-wave interaction effects appear to have practical device applications.[5,10]

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