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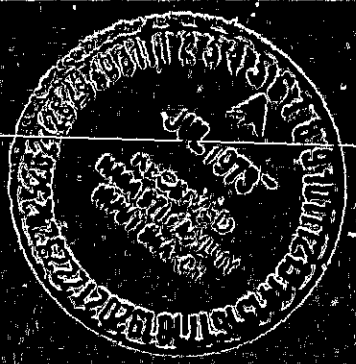
APPLICATION OF THE METHOD OF LOCAL POTENTIAL TO THE ANALYSIS OF TURBULENT SHEAR FLOWS

By T. D. Reed



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ABSTRACT

It has been found that, in general, the local potential cannot be employed to obtain approximate solutions for the various correlations of turbulent properties which appear in the time averaged form of the conservation equations.

Although the method of local potential is equivalent to the Galerkin method when the self-consistent condition is applied, the local potential can also be applied as an iterative algorithm in place of using the self-consistent condition. This procedure offers an alternative to the Galerkin method and may be useful in obtaining approximate solutions for the total turbulent velocity. In addition, for certain simple turbulent shear flows the iterative algorithm may permit approximate, but non-empirical, solutions by modeling only the mean velocity and the Reynolds stress.

FOREWORD

The work reported herein was performed under Contract No. NAS1-11666 from June 1972 through March 1973. Technical monitoring was provided by NASA Langley Research Center.

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NOMENCLATURE

C_n	nth coefficient in a series solution
D	rate at which turbulent kinetic energy is dissipated into heat, $\mu\overline{(u_{i,j} + u_{j,i})\, u_{i,j}}$
e	internal energy per unit mass
$\overset{+}{F}$	denotes steady state, Navier-Stokes operator
F_n	a generalized flow normal to the boundaries, eqn. (3)
G	an integral over space and time of the residual squared, eqn. (28)
J_α	generalized flux or rate of the αth transport process
K_1	Prandtl-von Karman constant appearing in the law of the wall
K_2	Constant appearing in the law of the wall, eqn. (A-8)
ℓ	Prandtl's turbulent mixing length
P	total, time dependent pressure
$\overline{P}$	mean or time averaged pressure
p'	fluctuating, time dependent pressure
R	residual obtained by substituting an approximate solution into a differential operator
s	bounding surface
t	time
U_j	component of mean velocity
u_j	component of turbulent velocity fluctuations
u_τ	friction velocity, $(\tau_w/\rho)^{1/2}$
V_j	jth component of total, time dependent velocity
v	volume of system

w	width of flow
X_α	generalized force of the αth transport process
X_1, X_2, X_3	Cartesian coordinates

Greek letters

ε	eddy viscosity
μ	molecular viscosity
ν	kinematic viscosity, μ/ρ
ρ	density
τ_{ij}	fluid shear stress tensor, $\mu(V_{i,j} + V_{j,i})$
τ_w	viscous shear stress at the boundary
ϕ	the local potential
Ψ_n	nth specified function in a series solution

Subscript

0	denotes stationary state values

APPLICATIONS OF THE LOCAL POTENTIAL TO THE ANALYSIS OF TURBULENT SHEAR FLOWS

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I. INTRODUCTION

As is well known, when the conservation equations are applied to a turbulent flow, with properties separated into a mean and a fluctuating component, and the equations are time averaged, the number of unknowns exceed the number of equations. The nonlinearity of the equations is the basic reason why such an approach leads to difficulties. The general problem of reducing the number of unknowns to equal the number of equations is known as the closure problem in the classical theory of turbulence. However, if the total, time dependent, velocity of a turbulent field were calculated, using the time dependent conservation equations, there would be no closure problem since in this case the number of unknowns equal the number of equations.

In an effort to attain closure, Prandtl, von Karman, and other early pioneers assumed that the Reynolds stress was proportional to the slope of the mean velocity profile and determined the constant of proportionally empirically. This type of closure is usually referred to as an eddy viscosity or a mixing length theory. Unfortunately, the constants of proportionally usually vary with changing flow conditions and across a given flow. This is especially true for flows with large pressure gradients and/or abrupt changes in the boundary conditions, e.g., reference 1. In addition, it is generally recognized that the mixing length theories of turbulent transport have no rigorous basis, e.g., Champagne, et. al. [ref. 2].

During the past decade more complex mathematical models of turbulent shear flows have been developed based on higher moments of the Navier-Stokes equations, e.g., Reynolds [ref. 3]. However, these procedures require more numerous empirical constants which must be chosen by curve fitting the theory to selected experimental data. Although the goal of this approach is to attain "universal constants" which would characterize all types of turbulent flows, the constants determined in this manner are inevitably found to vary with flow conditions. In the case of nonequilibrium turbulent flows*, it appears that no amount of empiricism can equal the required calculation of all dependent variables as they vary along a rapidly changing flow.

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* A nonequilibrium turbulent flow may be defined as one for which the boundary conditions change more rapidly than the characteristic relaxation times of the turbulence.

The objective of this work was to eliminate the need for empirical constants and solve the closure problem for the case of two-dimensional, incompressible, turbulent boundary layers by applying the theory of the local potential, Glansdorff and Prigogine [refs. 4 & 5]. The method of local potential casts the differential equations of conservation in terms of a variational integral and is based on the following idea. If a thermodynamic system is not in equilibrium with given boundary conditions, the system always evolves toward the equilibrium state in such a way that the magnitude of all gradients decrease with time. This is the basic idea underlying the local potential which attempts to characterize transport processes in terms of an integral which decreases and attains a minimum at the equilibrium state. Although the local potential provides no new physical information, it does reformulate the conservation equations into the form of a variational principle and thereby permit the calculation of approximate solutions.

Unfortunately, the local potential is not a true potential or a variational principle in the classical sense. By this we mean that the integral may <u>not</u> be required to be stationary with respect to each of the dependent variables which appear in the integrand of the defining integral. The reason for this is that Glansdorff and Prigogine constructed the local potential so that when it is required to be stationary, with respect to certain dependent variables, the differential equations for conservation of mass, momentum, and energy are restored. In order to accomplish this it is necessary to treat the dependent variables as known, fixed quantities at certain places within the variational integral and at other places allow them to vary. This aliasing of variables is discussed in section II and is the reason why the local potential is called a restricted variational principle.

Section III of this report discusses why the local potential cannot be used to calculate approximate solutions for the various turbulent correlations. With the exception of the simple case discussed in the appendix, the discussion in section III establishes the fact that the local potential cannot be used to resolve the classical closure problem.

On March 8, 1973 the author discussed the local potential and the closure problem with Profs. I. Prigogine and R. S. Schechter at the University of Texas in Austin. It was their opinion that, in general, the total turbulent velocity should be modeled in the search for approximate, analytical solutions. Thus, section IV briefly discusses the recent work of Orszag, which models the total turbulent velocity, and its relation to the theory of the local potential.

II. DISCUSSION OF THE EVOLUTION CRITERION AND THE LOCAL POTENTIAL

In the original paper by Glansdorff and Prigogine [ref. 4] a general evolution criterion is developed which has the following form for time independent boundary conditions.

$$\frac{\partial \Phi}{\partial t} = \int \sum_{\alpha} J'_{\alpha} \frac{\partial X'_{\alpha}}{\partial t} \, dv \leq 0 \tag{1}$$

Φ represents the local potential and J'_{α} and X'_{α} represent, respectively, generalized flows (or fluxes) and generalized forces (or gradients) which are defined so as to describe irreversible processes that include <u>both</u> dissipative phenomena and mechanical convection. Thus the J'_{α} and X'_{α} are generalizations of the flows and forces usually discussed in the classical thermodynamic theory of irreversible processes.*

The derivation of eqn. (1) begins with multiplying the conservation of mass, momentum and energy equations by certain multipliers.

$$\left(\frac{\partial \rho}{\partial t} = \text{(steady form of continuity eqn.)} \right) \frac{\partial M_1}{\partial t} \tag{2a}$$

$$\left(\rho \frac{\partial V_i}{\partial t} = \text{(steady form of momentum eqn.)} \right) \frac{\partial M_2}{\partial t} \tag{2b}$$

$$\left(\frac{\partial (\rho e)}{\partial t} = \text{(steady form of energy eqn.)} \right) \frac{\partial M_3}{\partial t} \tag{2c}$$

Summing the left hand sides, it can be shown by using results from classical thermodynamics that the sum is always nonpositive. Once the inequality is established, the right hand sides of eqns. (2) are summed and integrated over the volume of the system to obtain

$$\int \sum_{n} F_n \, ds + \int \sum_{\alpha} J'_{\alpha} \frac{\partial X'_{\alpha}}{\partial t} \, dv \leq 0 \, . \tag{3}$$

The surface integral arises from application of the divergence theorem to part of the volume integral. The surface integral permits the introduction of boundary conditions and may be thought of as a flow term associated with the exchange of mass and energy across the boundary s. For time independent boundary conditions it can be shown that the surface integral is zero and inequality (3) reduces to (1). The remaining volume integral is a source term associated with the irreversible processes occurring within the system.

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*In the classical theory it is shown that the internal rate of entropy production for purely dissipative processes can be written in the form $\sum J_{\alpha} X_{\alpha}$ - entropy per unit volume per unit time.

The evolution criterion is an expression of the following fact. When thermodynamic systems are not in equilibrium with given, time independent, boundary conditions, they always evolve toward the equilibrium state in such a way that the magnitude of all gradients decrease with time. At equilibrium, the gradients cease to change. Therefore, the local potential Φ can only decrease with time and assume a minimum value at the equilibrium or stationary state.*

For the purpose of obtaining an expression for Φ alone, rather than its time derivative, a zero subscript is introduced to denote quantities evaluated at the stationary state and as such are:

(1) treated as if they were known, fixed functions of the spatial coordinates and have zero derivatives with respect to non-subscripted, dependent variables, i.e., they are not subject to variation; however, they do have non-zero derivatives with respect to the independent variables (spatial coordinates), and

(2) time independent for time independent boundary conditions.+

This allows the fluxes and forces to be expanded in a Taylor series in the neighborhood of the stationary state.

$$J = J_o + \delta J + \text{h.o.t.} \tag{4a}$$

$$X = X_o + \delta X + \text{h.o.t.} \tag{4b}$$

$$\frac{\partial X}{\partial t} = \frac{\partial (\delta X)}{\partial t} + \text{h.o.t.} \tag{4c}$$

Glansdorff and Prigogine [ref. 4] chose to linearize the product occurring in eqn. (1) by neglecting all the terms in eqn. (4a) except the first.

$$\sum_\alpha J'_\alpha \frac{\partial X'_\alpha}{\partial t} = \sum_\alpha (J'_\alpha)_o \frac{\partial X'_\alpha}{\partial t} = \frac{\partial}{\partial t} \left[\sum_\alpha (J'_\alpha)_o X'_\alpha \right] \tag{5}$$

This allows the elimination of the time derivative, which appears in the evolution criterion, eqn. (1), and thereby obtain an expression for Φ alone.

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*It is noted on P. 130 of the book by Glansdorff and Prigogine [ref. 5] that $\partial\Phi/\partial t$ is less than zero only "around stable solutions." If the system is unstable, fluctuations amplify and Φ increases.

+Formulation of the local potential for time dependent processes is discussed on P. 146 of reference 5.

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t} \int \sum_{\alpha} (J'_\alpha)_o X'_\alpha \, dv \leq 0$$

$$\therefore \Phi = \int \sum_{\alpha} (J'_\alpha)_o X'_\alpha \, dv \qquad (6)$$

The derivation of this general expression for the local potential is concluded by showing that when Φ is required to be stationary with respect to each of the multipliers M_1, M_2, M_3 appearing in eqns. (2), the steady form of the conservation equations are obtained and, in addition, $\delta^2\Phi > 0$. Since stationarity is a necessary condition that must be satisfied in order for a minimum to exist, it is concluded that minimizing Φ with respect to the reference state (denoted by the zero subscript) is consistent with:

(1) the evolution criterion, eqn. (1), and

(2) the conservation equations for steady state processes.

The advantage of the local potential theory is that it permits one to obtain approximate solutions for nonlinear, transport processes. In practice, this is usually done by:

(1) assuming a series representation* for the unknown variables, e.g., a power series,

(2) substituting these into the integrand of the local potential,

(3) requiring Φ to be stationary with respect to each coefficient,

(4) dropping the zero subscripts (this is called the "self-consistent condition" in the local potential theory, and in effect requires the solutions to the variational equations to equal the values corresponding to the steady state), and

(5) finally, solving the resulting set of nonlinear (algebraic or ordinary differential) equations to obtain numerical values for the various coefficients.

This basic procedure is known as the Ritz method. Numerous applications to laminar flow problems can be found in the monograph by Schechter [ref. 6].

However, there is an essential point that is not appropriately emphasized in reference 4. Specifically, the number of variables which can be approximated by a series representation can not exceed the number of conservation equations. This point is discussed further in the next section and is related to the problem of modeling turbulent flows.

*The admissible series are required to satisfy the boundary conditions for all values of the coefficients and, theoretically, should form a complete set.

III. APPLICATION OF THE LOCAL POTENTIAL TO TURBULENT SHEAR FLOWS

Since eqn. (1) is not an exact differential, the minimum which Φ assumes is of a special nature. On page 152 of reference 6, an example of one-dimensional heat conduction through a slab of unit thickness is used to demonstrate that indeed the local potential assumes a true minimum only if the exact steady state values of the generalized fluxes, $(J'_a)_o$, are substituted into eqn. (6). If some other values for the J'_a are used, it is possible to find smaller values of Φ. However, use of the Ritz method in conjunction with the self-consistent condition (i.e., dropping the zero subscript after variational derivatives are taken), permits the calculation of approximate values of the fluxes and forces which bring the values of Φ arbitrarily close to the desired minimum. One may consult P. 136 of reference 5 for additional discussion.

The desired minimum is relative to the generalized fluxes which satisfy the steady form of the conservation equations. Therefore, one must be sure to impose only those stationary conditions on Φ which produce equations identical to the conservation equations. This requirement excludes any additional equations and, in effect, restricts the number of variables with respect to which Φ may be made stationary. Thus, one can not use the idea of minimizing Φ to solve for a number of variables greater than the number of conservation equations. If eqn. (1) were an exact differential (or Φ were a true potential), it would be possible, in general, to solve for any number of variables which might appear in the integrand by simply requiring the potential to be stationary with respect to each variable. In fact, Gage, et.al. [ref. 7] have shown that such a potential does not exist for general transport processes which obey the conservation of mass, momentum and energy.

The importance of the above discussion to the modeling of turbulent flows is as follows. Reed [ref. 8] has shown that the local potential for an isothermal, incompressible, steady flow can be written in the following form.

$$T\Phi = \int (V_i P - \tau_{ij} V_j) N_i \, ds + \int \Big(-[V_i]_o \frac{\partial P}{\partial X_i} + [V_i]_o \mu \frac{\partial^2 V_i}{\partial X_j^2} + \mu\Big(\frac{\partial V_i}{\partial X_j} + \frac{\partial V_j}{\partial X_i}\Big) \frac{\partial V_i}{\partial X_j} + \rho [V_j \frac{\partial V_i}{\partial X_j}]_o V_i\Big) \, dv \tag{7}$$

If $T\Phi$ is required to be stationary with respect to P and V_i, the following forms of the conservation of mass and momentum equations result.*

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* The surface integral does not contribute to the Euler-Lagrange equations for fixed boundary values of V_i and P.

$$\frac{\partial V_i}{\partial X_i} = 0 \qquad (8)$$

$$\rho V_j \frac{\partial V_i}{\partial X_j} = \mu \frac{\partial^2 V_i}{\partial X_j^2} \qquad (9)$$

These equations are the appropriate form of the conservation equations for steady, incompressible flow without pressure gradients, body forces, etc.

However, for turbulent flow it is standard practice to introduce a Reynolds decomposition of the total velocity into a time-averaged or mean component and a fluctuating component. For statistically steady, turbulent flow we write

$$V_i (X_j,t) = U_i (X_j) + u_i (X_j,t) \qquad (10)$$

where $$\overline{u_i(X_j,t)} = \lim_{T\to\infty} \frac{\int_o^T u_i(X_j,t)}{T} dt = 0, \quad i = 1, 2, 3 .$$

If eqn. (10) and a similar expression for the pressure are substituted into the volume integral of eqn. (7) , it is shown in reference 8 that the following equation results after taking a time average.

$$\int_v \Big(-[U_i]_o \frac{\partial \overline{P}}{\partial X_i} + \mu [U_i]_o \frac{\partial^2 U_i}{\partial X_j^2} + \mu \Big(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\Big) \frac{\partial U_i}{\partial X_j} + \mu \overline{\Big(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\Big) \frac{\partial u_i}{\partial X_j}} + \rho \Big(U_j \frac{\partial U_i}{\partial X_j} + \overline{u_j \frac{\partial u_i}{\partial X_j}} \Big)_o U_i \Big) dv \qquad (11)$$

Requiring this integral to be stationary with respect to the mean velocity U leads to the following well-known form of the momentum equation.

$$\rho U_j \frac{\partial U_i}{\partial X_j} = \frac{\partial}{\partial X_j} \Big(\mu \frac{\partial U_i}{\partial X_j} - \rho \overline{u_i u_j} \Big) \qquad (12)$$

The primary objective of the local potential theory is to provide us with a tool for obtaining approximate solutions to the conservations equations. In applications of the theory we substitute some series approximations for the unknown dependent variables and then require the local potential to be stationary with respect to the unknown coefficients, e.g.,

$$\tilde{U}(X) \equiv \sum_{n=1}^{N} C_n \Psi_n(X) \tag{13}$$

$$\frac{\delta\Phi}{\delta\tilde{U}} = \frac{\delta\Phi}{\delta C_1}\frac{\partial C_1}{\partial\tilde{U}} + \frac{\delta\Phi}{\delta C_2}\frac{\partial C_2}{\partial\tilde{U}} + \cdots + \frac{\delta\Phi}{\delta C_N}\frac{\partial C_N}{\partial\tilde{U}} = 0 \tag{14}$$

= conservation of momentum equation [eqn. (12)]

In general, $\partial C_n/\partial\tilde{U} = \Psi_n^{-1}$ is non-zero and the $\Psi_n(X)$ are linearly independent for an admissible series. Therefore, if Φ is to be stationary with respect to the approximate mean velocity distribution $\tilde{U}(X)$, we must require

$$\begin{aligned} &\delta\Phi/\delta C_1 = 0 \\ &\delta\Phi/\delta C_2 = 0 \\ &\quad\cdot \\ &\quad\cdot \\ &\quad\cdot \\ &\delta\Phi/\delta C_N = 0 \end{aligned} \tag{15}$$

Equations of this type are usually referred to as the Ritz equations.

However, eqns. (15) can not be solved to obtain numerical values for the C_n because of the unknown Reynolds stresses $\overline{u_i u_j}$. A series representation for the Reynolds stresses will not be fruitful because requiring the local potential to be stationary with respect to $\overline{u_i u_j}$ does not lead to a valid form of the conservation equations.* In addition, the momentum equation, eqn. (12), can not be used to express the Reynolds stress in terms of the mean velocity since this causes all the terms in $\delta\Phi/\delta U = 0$ to cancel with one another and forces the equality to be identically satisfied for all values of U, i.e., eqn. (11) becomes:

$$\int_v \Big(-[U_i]_o \frac{\partial\overline{P}}{\partial X_i} + \mu[U_i]_o \frac{\partial^2 U_i}{\partial X_j^2} + \mu\Big(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\Big)\frac{\partial U_i}{\partial X_j} + \mu\Big(\overline{\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}}\Big)\overline{\frac{\partial u_i}{\partial X_j}} + \mu\Big(\frac{\partial^2 U_i}{\partial X_j^2}\Big)_o U_i\Big)\, dv, \tag{16}$$

*Owing to the zero subscript on the Reynolds stresses which appear in eqn. (11), $\delta\Phi/\delta(\overline{u_i u_j})$ is identically zero. Even though it is possible to rearrange the local potential so as to make $\overline{u_i u_j}$ appear without a zero subscript, there is no way to obtain a valid differential equation of conservation from

and requiring this integral to be stationary with respect to U_i leads to

$$\mu \left(\frac{\partial^2 U_i}{\partial X_j^2}\right)_o - 2\mu \frac{\partial^2 U_i}{\partial X_j^2} + \mu \frac{\partial^2 [U_i]_c}{\partial X_j^2} = 0. \qquad (17)$$

Upon applying the self-consistent condition, i.e., dropping the zero subscript after variational derivatives are taken, we obtain an identity which can not provide any information on the U_i. This is the basic reason why only the dissipation term, $D \equiv \mu \left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right] \frac{\partial u_i}{\partial X_j}$, remained in the analysis of reference 8, where eqn. (11) was simplified for the case of plane Couette flow.

Furthermore, it can be shown that requiring eqn. (11) to be stationary with respect to the correlation representing the dissipation is impossible since the dissipation appears linearly in the integrand. The variational derivative of eqn. (11) with respect to D is

$$\frac{\delta \text{ eqn. (11)}}{\delta D} = \int (1)\, dv \neq 0. \qquad (18)$$

Therefore, the only way that this integral can be stationary with respect to D is for D to be identically zero, which can only be true for laminar flow. A great deal of effort was expended during this contract to introduce a series representation for D and apply one or more constraints having a form similar to the integral of the turbulent kinetic energy equation. The objective was to obtain a formulation which would permit the local potential to be stationary with respect to D and simultaneously restrain D to be non-zero, as it must be for turbulent flow. However, as shown by eqn. (18), this procedure cannot lead to any form of the conservation equations. Thus, it is fruitless to attempt to substitute a series representation for D and apply the Ritz technique to determine the coefficients.

The general conclusion is that the various correlations, which appear in the time-averaged form of the conservation equations for turbulent flow, can not be individually modeled by a separate series representation. The reason being that such a procedure does not lead to equations equivalent to the conservation equations.

It appears that Nihoul [ref.9] encountered this same basic problem in a somewhat different manner. Nihoul attempted to apply the local potential to the analysis of turbulent channel flow. He also introduced a Reynolds decomposition and was able to show that requiring his form* of the local

*The form of the local potential is not unique owing to a range of possible choices of the M_1, M_2 and M_3 which appear in eqns. (2), Glansdorff and Prigogine [ref. 5], P. 125.

potential to be stationary with respect to the mean velocity did indeed restore the correct form of the time-averaged, conservation of momentum equation. However, he observed that this produced no information on the Reynolds stress. In an attempt to gain information on the turbulence, Nihoul chose to represent the velocity fluctuations in terms of a Fourier series having unknown coefficients which were assumed to vary across the flow. Unfortunately, requiring the local potential to be stationary with respect to the velocity fluctuations u_i does not restore any form of the differential equations of conservation. This then invalidates any attempt to calculate the coefficients appearing in the Fourier series via the Ritz technique. This is true even though Nihoul is able to show that the integral of the turbulent kinetic energy equation can be constructed by multiplying the Euler-Lagrange equations, corresponding to $\delta(T\Phi)/\delta u_i = 0$, with u_i and integrating over the flow volume.

A valid local potential, as described in reference 5, P. 129, must restore one of the differential conservation equations when Φ is required to be stationary with respect to each unknown, dependent variable.+ This means that if we introduce a Reynolds decomposition of the flow into a mean and a fluctuating component, then the following must hold in order to have a valid variational principle.

$$\frac{\delta\Phi}{\delta V_i} = \frac{\delta\Phi}{\delta U_i} + \frac{\delta\Phi}{\delta u_i} \tag{19}$$

$$= \text{Navier-Stokes equation.}$$

If the flow is statistical steady, eqn. (10) relates the total velocity to the mean and fluctuating velocities. In this case the following equations must result for incompressible flow.

$$\frac{\delta\Phi}{\delta U_i} = 0 \Rightarrow \rho \frac{\partial}{\partial X_j} [U_j U_i + \overline{u_j u_i}] + \frac{\partial \bar{P}}{\partial X_i} - \frac{\partial}{\partial X_j} \left(\mu \frac{\partial U_i}{\partial X_j} \right) = 0 \tag{20}$$

$$\frac{\delta\Phi}{\delta u_i} = 0 \Rightarrow \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial X_j} [U_j u_i + u_j U_i + u_j u_i - \overline{u_j u_i}] + \frac{\partial p'}{\partial X_i}$$

$$- \frac{\partial}{\partial X_j} \left(\mu \frac{\partial u_i}{\partial X_j} \right) = 0 \tag{21}$$

+When this is true, the variable may be approximated by a series with as many terms as is desired. The coefficients may then be calculated via the Ritz technique.

As of this time, no one has attempted to construct a local potential which produces these results. Since this approach requires a modeling of the time dependent fluctuations, a large number of terms are necessary to adequately describe the known spectrum of turbulence. This usually leads to equations that become unwieldy and computationally impractical. However, a new approach to the general problem of obtaining analytical solutions for turbulent flows is discussed in the following section.

IV. GENERAL APPROACH TO MODELING TURBULENT FLOWS

Rather than attempt to model the velocity fluctuations separately from the mean velocity as Nihoul [ref. 9] suggested, Orszag [ref. 10] has shown that it is feasible to model the total velocity $\vec{V}$ with a Fourier series. Orszag [ref. 11] has successfully computed the decay with time of an isotropic box of turbulence using

$$\vec{V}(\vec{X},\ t) = \sum_{k} \vec{u}\ (2\ \vec{k},\ t)\ e^{2\ i\ \vec{k}\ \cdot\ \vec{X}}, \tag{22}$$

where $\vec{k}$, the vector wave number, was restricted to integer components within the range $|k_j| < (242)^{1/2}$ for $j = 1, 2, 3$. Thus, the number of distinct wave vectors retained in eqn. (19) was $\{2(242)^{1/2} - 1\}^3 = 27{,}305!$ Orszag used the Galerkin method to calculate the Fourier coefficients $\vec{u}(2\ \vec{k},\ t)$ which gave an approximate solution to the complete, time-dependent Navier-Stokes equations. In order to reduce the computational effort to a practical size, Orszag [ref. 10] developed a fast Fourier transform algorithm which helped to reduce the computing time from several hours to a few seconds per time step.* Orszag [ref. 11] claims that his solutions "are at least as accurate as finite-difference simulations using $(64)^3$ grid points (786,432 velocity values)" within his model box of isotropic turbulence.

Along this same line, Orszag [ref. 10] has pointed out a number of advantages of integral methods, which use series expansions for the dependent variables, as opposed to finite-difference schemes. Some of these advantages are:

(1) For a given number of independent degrees of freedom (i.e., number of space grid-points or number of terms in the series expansion) integral methods are likely to give considerably more accurate results.

(2) Certain important integral constraints, that are necessary to maintain numerical stability, are automatically satisfied.

(3) Appropriate choice of the functions appearing in the series allows the boundary conditions to be satisfied exactly and the utilization of any symmetries and invariances of the flow being modeled.

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*The interested reader should consult Orszag's papers [refs. 10 & 11] for additional discussion.

In the past the disadvantage was that integral methods, which required a large number of terms for a satisfactory model, were extremely inefficient compared to finite-difference simulations. However, the transform algorithms developed by Orszag have reversed this situation. Indeed, it is generally concluded by most workers in turbulence that it is not feasible to model turbulence with finite-difference schemes and existing digital computers, Emmons [ref. 12]. This no longer appears to be the case using Orszag's approach.

In the case of turbulent shear flows with no-slip boundary conditions, Orszag [ref. 10] recommends that Chebyshev polynomials be used in place of a Fourier series. Preliminary work on this type of application is reported by Orszag in reference 13. Apparently, he is actively pursuing applications of the method of weighted residuals to turbulent shear flows of practical interest [ref. 14].

At this point, it is relevant to ask: what is the relation between the Galerkin method and the method of local potential? In order to answer this question, first let us review the mechanics of the Galerkin procedure and its interpretation. Basically, the Galerkin method is a mathematical tool for obtaining approximate solutions to either ordinary or partial differential equations. The unknown exact solution to the differential equation is approximated by a series with each term consisting of the product of a specified function of one or more of the independent variables and an unknown coefficient. In the case of ordinary differential equations the coefficients are simply constants, but for partial differential equations the coefficients are functions of one of the independent variables. Numerical values for the coefficients are calculated by solving the so-called Galerkin equations.

The Galerkin equations are derived by requiring the residual, obtained by substituting the assumed series solution into the differential equation, to be orthogonal to each of the specified functions which appear in the series. For example, the Navier-Stokes equations have the following form of a mixed initial-boundary value problem.

$$\partial \vec{V}(\vec{X}, t)/\partial t = \vec{F}[\vec{V}(\vec{X}, t), \vec{X}, t] \tag{23}$$

where $\vec{F}(\vec{V}, \vec{X}, t)$ involves partial derivatives of $\vec{V}$ with respect to the three spatial components of $\vec{X}$. If the Galerkin method is employed to obtain an approximate solution, a series of the form

$$\tilde{V} = \sum_{n=1}^{N} C_n(t)\, \vec{\Psi}_n(\vec{X}) \tag{24}$$

is substituted into eqn. (23).* Since the series is at best only an approximate solution to the differential equation, the residual is defined as

$$\vec{R}(\tilde{V}, \vec{X}, t) \equiv \partial \tilde{V}/\partial t - \vec{F}(\tilde{V}, \vec{X}, t). \tag{25}$$

*The $\vec{\Psi}_n(\vec{X})$ are chosen so as to (1) satisfy the boundary conditions on $\vec{V}$ regardless of the values of the $C_n(t)$ and (2) be linearly independent, i.e., orthogonal.

Using this nomenclature, the Galerkin orthogonally conditions have the following form.

$$\int \vec{\Psi}_n \cdot \vec{R}\, dv = \int \vec{\Psi}_n \cdot \left(\partial \tilde{V}/\partial t - \vec{F} \right) dv = 0 , \quad n = 1, 2, \cdots N \tag{26}$$

By substituting the series defining $\tilde{V}$ into eqn. (26) we obtain

$$\frac{\partial C_n}{\partial t} \int \left(\vec{\Psi}_n \cdot \sum_{m=1}^{N} \vec{\Psi}_m \right) dv = \int \left(\vec{\Psi}_n \cdot \vec{F} \{ \sum_{m=1}^{N} C_m \vec{\Psi}_m, \vec{X}, t \} \right) dv \quad . \tag{27}$$

These N first order, ordinary differential equations may then be solved for the coefficients $C_n(t)^*$. Completion of this task provides the approximate solution $\tilde{V}$, eqn. (24).

Orszag [ref.10] has noted that the Galerkin method has an interesting interpretation which is analogous to a restricted variational principle. By assuming the $C_n(t)$ are known for arbitrary time $t > 0$, which is equivalent to zero subscripting in the local potential theory, the Galerkin equations are obtained by requiring the following integral to be stationary with respect to the derivatives $C_{n,t}$.

$$G \equiv \int \int \left| \vec{R}_o \right|^2 dv\, dt = \int \int \left| \partial \tilde{V}/\partial t - \vec{F}_o \right|^2 dv\, dt$$

$$= \int \int \left| \sum_n C_{n,t} \vec{\Psi}_n - \vec{F} \left(\sum_n (C_n)_o \vec{\Psi}_n, \vec{X}, t \right) \right|^2 dv\, dt \tag{28}$$

Now if we require G to be stationary with respect to $C_{n,t}$, we obtain

$$\frac{\delta G}{\delta C_{n,t}} = 0 = \frac{\partial \left(\int \left| \vec{R}_o \right|^2 dv \right)}{\partial C_{n,t}} = 2 \int \left(\vec{R}_o \cdot \partial \vec{R}_o / \partial C_{n,t} \right) dv \tag{29}$$

which is equivalent to eqn. (27) after the zero subscript is dropped. Thus, the Galerkin equations give the best approximation, in the least squares sense, to the derivatives $C_{n,t}$ at each instant of time t if the $C_n(t)$ are assumed to be known.

At this point one may begin to suspect that the Galerkin method and the method of local potential are closely related. In fact, for the case of non-linear heat conduction it can be shown that the Ritz equations derived from the local potential are identical to the Galerkin equations, e.g., Glansdorff and Prigogine [ref. 5], P. 134. According to Finlayson and Scriven [ref. 15], Finlayson demonstrated in his Ph.D. thesis the equivalence of the self-consistent approximation scheme of the local potential and the Galerkin method for the case of steady states involving the coupled Navier-Stokes and energy equations. However, since here the primary interest is in applications to time dependent turbulent flows, it is relevant to demonstrate that an equation equivalent to eqn. (27) can be obtained from the local potential.

The general form of the local potential for time dependent motion of a multicomponent fluid is presented on P. 146 of reference 5. For the case of time dependent motion of an isothermal, incompressible, single component fluid with known, time independent boundary conditions the local potential reduces to

$$T\Phi = -\int_s (\tau_{ij})_o \, V_j N_i \, ds + \int_v \left(V_i (P_{,i})_o + \mu (V_{i,j} + V_{j,i})_o \, V_{i,j} + \rho (V_{i,t} + V_j \, V_{i,j})_o \, V_i \right) dv \quad . \tag{30}$$

Now assuming that the boundary conditions specify V_i on the bounding surface s, the Euler-Lagrange equations, which apply when eqn. (30) is required to be stationary with respect to V_i, are

$$\frac{\delta (T\Phi)}{\delta V_i} = 0 = \frac{\partial L}{\partial V_i} - \frac{\partial}{\partial X_j} \left(\frac{\partial L}{\partial V_{i,j}} \right) \quad . \tag{31}$$

Where L denotes the integrand of the volume integral in eqn. (30). Substituting for L, taking the indicated derivatives, and using the continuity equation, the Navier-Stokes equations for an isothermal, incompressible flow with no external body forces is obtained.

$$0 = \rho (V_{i,t} + V_j \, V_{i,j}) + P_{,i} - \mu V_{i,jj} \tag{32}$$

At this point, we are ready to assume a series solution for the total velocity in the form of eqn. (24) and require the local potential, eqn. (30), to be stationary with respect to each of the coefficients $C_n(t)$. Using the vector notation employed in eqn. (23), the Ritz equations are found to be identical to eqn. (27), i.e.,

$$\frac{\partial L}{\partial C_n} = \int \left(\rho \, \vec{\Psi}_n \cdot \sum_m \left(\frac{\partial C_m}{\partial t} \, \vec{\Psi}_m \right)_o - \vec{\Psi}_n \cdot \vec{F}_o \left[\sum_m C_m \, \vec{\Psi}_m, \, \vec{X}, \, t \right] \right) dv = 0$$

or since the $\vec{\Psi}_n$ are orthogonal,

$$\frac{\partial C_n}{\partial t} \int \left(|\vec{\Psi}_n|^2 \right) dv = \int \left(\vec{\Psi}_n \cdot \vec{F} \right) dv \quad . \tag{33}$$

It should be noted that the zero subscripts have been dropped from eqn. (33) in accordance with the self-consistent condition. Thus, the Galerkin and local potential methods lead to the same set of equations which must be solved to obtain an approximate solution.

The only significant difference between the two methods appears to be the algorithm for successive approximations which the local potential permits if the self-consistent condition is replaced by the following iteration procedure. If a completely specified trial solution, which satisfies the

boundary conditions, is substituted for all subscripted variables and a trial solution with unknown coefficients (as before) is calculated by making the local potential stationary, then one can obtain a set of successive approximations by taking each solution and substituting it for the corresponding zero subscripted ariables. The iteration process is continued until the solutions agree with the subscripted variables within some small tolerance, Glansdorff and Prigogine [ref. 5], P. 141.

A possible application of this iterative process to a simple turbulent shear flow is discussed in the appendix. Also, it is here noted that the efficient methods developed by Orszag to model the complete turbulent velocity could be used in conjunction with the iterative version of the local potential. Such a procedure should lead to a solution with higher order accuracy for a given series representation and thereby offer an improvement over the Galerkin method. However, the practical utility and economy of such an approach needs to be studied by comparing the refined answers obtained by the iterated local potential with Galerkin results corresponding to one or more additional terms in a given series approximation.

V. CONCLUSIONS

Based on the analyses conducted for this contract, the following conclusions were reached.

(1) The individual correlations of turbulent properties, which appear in the time-averaged form of the conservations equations, cannot be modeled by substituting a series representation with unknown coefficients into the method of local potential and calculating the coefficients via the Ritz technique.*

(2) The Galerkin and local potential methods are identical when the self-consistent condition of the local potential theory is applied.

(3) The local potential can be applied as an iterative algorithm in place of using the self-consistent condition. This procedure offers an alternative to the Galerkin method. For certain simple turbulent shear flows the iterative process may permit approximate, but non-empirical, solutions; an example is discussed in the appendix.

(4) The general approach of Orszag, to the modeling of turbulence, can be utilized in a local potential analysis. For a given series solution, the iterative algorithm may lead to an improvement over the Galerkin method.

*The same conclusion applies to any form of the method of weighted residuals.

APPENDIX

APPLICATION OF THE ITERATIVE VERSION OF THE LOCAL POTENTIAL TO TURBULENT SHEAR FLOWS

The purpose of this appendix is to suggest a way that the subject iterative algorithm may be used to obtain a non-empirical solution for simple turbulent shear flows, e.g., plane Couette, plane channel, and straight pipe flows.

The following equation can be obtained from eqn. (30) by:

(1) introducing a Reynolds decomposition of the velocity and pressure,

(2) interpreting the zero-subscript to denote a time average,

(3) employing the assumption of ergodic turbulence on surfaces parallel to the boundaries,* Reed [ref. 8], in order to eliminate terms linear in a fluctuating quantity, and

(4) assuming steady flow.

$$T\Phi = -\int_s (\tau_{ij})_o \, U_j \, N_i \, ds + \int_v \left(U_i (\overline{P}_{,i})_o + \mu (U_{i,j} + U_{j,i})_o \, U_{i,j} + \rho \left(U_j \, U_{i,j} + \overline{u_j u_{i,j}}\right)_o \, U_i\right) dv \qquad (A\text{-}1)$$

If the mean velocity is fixed at the boundaries and $T\Phi$ is required to be stationary with respect to U_i, we obtain a well-known form of the conservation of momentum equation.

$$\frac{\delta (T\Phi)}{\delta U_i} = 0 = \rho \left(U_j U_{i,j} + \overline{u_j u_{i,j}}\right)_o + (\overline{P}_{,i})_o - \mu \left(U_{i,jj}\right)_o \qquad (A\text{-}2)$$

Now if we introduce an eddy viscosity to relate the Reynolds stress to the gradient of the mean velocity, the volume integral appearing in eqn. (A-1) becomes

$$\int_v \left(U_i \{\rho (U_j \, U_{i,j} - \varepsilon \, U_{i,j}) + \overline{P}_{,i}\}_o + \mu \left(U_{i,j} + U_{j,i}\right)_o U_{i,j}\right) dv \, . \qquad (A\text{-}3)$$

For two-dimensional flows with a uni-directional mean velocity, $\overset{+}{U} \equiv U_1(X_2)$, this equation reduces to

$$\int_o^w \left(U\{-\rho \, \varepsilon \, U_{,2} + \overline{P}_{,1}\}_o + \mu \left(U_{,2}\right)_o \, U_{,2}\right) dX_2 \, . \qquad (A\text{-}4)$$

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* This assumes averages over surfaces parallel to the boundaries are equiva-

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The integrals over X_1 and X_3 are ignored because they do not contribute to the Euler-Lagrange equations.

At this point, we can introduce Prandtl's assumption of a mixing length which is related to the eddy viscosity by

$$\epsilon = \ell^2 U_{,2} . \tag{A-5}$$

Prandtl assumed that the mixing length is proportional to distance from a solid boundary, i.e. ,

$$\ell = K_1 X_2 . \tag{A-6}$$

This assumption leads to the well-known logarithmic velocity distribution, Hinze [ref. 16], P. 467.

$$\frac{U}{u_\tau} = \frac{1}{K_1} \ln \frac{X_2 u_\tau}{\nu} + \text{const.} \tag{A-7}$$

Unfortunately, this expression has a singularity at $X_2 = 0$. In order to bypass this difficulty and provide the correct linear relation near the wall, Rasmussen and Karamcheti [ref. 17] derived an implicit expression for mean velocity distribution.

$$\frac{X_2 u_\tau}{\nu} = \frac{U}{u_\tau} + K_2 \{ \cosh (K_1 U/u_\tau) - .5 (K_1 U/u_\tau)^2 - 1 \} \tag{A-8}$$

One of the primary conclusions of this report is that the local potential can not be stationary with respect to correlations like the Reynolds stress. Therefore, the Reynolds stress must be treated as if it were known when calculating variations of the local potential. However, it is legitimate to require eqn. (A-4) to be stationary with respect to U.

Thus, if the Rasmussen and Karamcheti representation for the velocity is used in conjunction with the Ritz technique, the local potential can be required to be stationary with respect to K_1 and K_2.

An iteration algorithm can now be formulated for obtaining simultaneous solutions for the mean velocity and the Reynolds stress, without the aid of any empirical constants. The iteration begins by assuming an initial guess for K_1 which is used to express the initial distribution of the Reynolds stress as a function of the slope of the mean velocity profile, i.e.,

$$- \rho \overline{u_1 u_2} = \rho \ell^2 (U_{,2})^2$$

$$= \rho (K_1 X_2 U_{,2})^2 . \tag{A-9}$$

Also, this same K_1 is used, together with an initial guess for K_2, to calculate the zero-subscripted values of the velocity and its derivatives. Thus, all the zero-subscripted quantities appearing in eqn. (A-4) are known quantities and, therefore, cannot vary.

Next, the integral is required to be stationary with respect to K_1 and K_2. The two resulting Ritz equations may be solved to obtain new values for K_1 and K_2. These new values are then used to calculate a new estimate of the Reynolds stress and the zero-subscripted quantities. This changes the Ritz equations which must then be solved again, etc. The iteration would continue until the values of K_1 and K_2 agreed with the corresponding previous iterate to within some small tolerance. Thus, this procedure would result in the calculation of a turbulent shear flow without the aid of empirical constants.

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