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M. A. KHAN

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HYDROSTATIC FIGURE OF THE EARTH:

THEORY AND RESULTS

By

M. A. Khan†

Goddard Space Flight Center

Geodynamic Program Division

Geodynamics Branch

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† On leave of absence from the University of Hawaii

HYDROSTATIC FIGURE OF THE EARTH:

THEORY AND RESULTS

INTRODUCTION

If the earth were a fluid body, it would respond instantaneously to any stresses including those caused by its rotation until it attains a state of zero stress. The figure that the earth would assume if it were in such a state is called the hydrostatic equilibrium figure or simply hydrostatic figure or equilibrium figure. Since hydrostatic shape indicates a state of zero stress any departures from this state are particularly interesting as they show the extent of available stresses in the interior of the earth which can be invoked to explain any geophysical mechanisms which may be found or assumed to exist in the earth's interior. Apart from its traditional appeal it is because of this reason that the problem is particularly interesting to modern day geophysicists.

The mathematical theory of hydrostatic equilibrium for the earth, to the first order of small qualities, was originally developed by Clairant (1743). Radau (1885) simplified the solution of Clairant's differential equation by making an important substitution. The original purpose of the theory was that, with the then known data, the theory will provide useful information about the distribution of density in the earth. It was found however, that with the then-known data the theory led to no discrimination between widely varying laws of density. But it did yield more accurate values of flattening for the earth (assuming, of course, hydrostatic equilibrium) than were likely to be obtained by geodetic surveys. This stimulated further interest in the theory and its development was extended to the second order first by Callandreau (1889) and then by Darwin (1900). DeSitter (1938) modified the development and studied its actual application to the earth. Subsequent applications of the second order theory have been made by Bullard (1948) and Jeffreys (1962).

With the advent of artificial earth satellites it became possible to determine the actual flattening of the earth directly from the second degree harmonic coefficient of geopotential. The same coefficient, coupled with the precessional constant of the earth also yields accurate values of the earth's polar moment of inertia and hence the hydrostatic theory could now be used to yield the hydrostatic flattening as distinct from the actual flattening of the earth. Thus one could study the departures of the actual earth from its equilibrium state. Such studies were conducted by O'Keefe (1960), Henriksen (1960), Caputo (1965) and Khan (1967) in the post-artificial earth satellite era. Since there is a fundamental difference in the pre- and post-artificial earth satellite applications of the hydrostatic theory, Khan (1968, 1969) revised and extended the second order theory to suit readily the new applications and data types.

In the following pages, I recount the complete development of the theory. For the sake of brevity, however, some intermediate algebraic steps are omitted but the reader can easily reconstruct them.

THEORY OF THE EXTERNAL FIELD

The external gravitational potential V of a body symmetrical with respect to its equatorial plane and polar axis, is (see, for example, Jeffreys, 1962):

$$U = \frac{GM}{r}\left[1 - \sum\left(\frac{a_e}{r}\right)^{2n} J_{2n}P_{2n}(\sin\phi)\right] \tag{1}$$

which, accurate to the square of flattening, reduces to

$$V = \frac{GM}{r}\left[1 - \left(\frac{a_e}{r}\right)^2 J_2P_2(\sin\phi) - \left(\frac{a_e}{r}\right)^4 J_4P_4(\sin\phi) - 0(f^3)\right] \tag{2}$$

where

$P_{2n}(\sin\phi)$ = Legendre's polynomials

a_e = equatorial radius of the body

r, ϕ = geocentric coordinates

J_{2n} = constant coefficients associated with Legendre's polynomials.

$0(f^3)$ = quantities of the order of cube of flattening

U is the gravitational potential. The potential of gravity is

$$V = U + \frac{1}{3}\omega^2 r^2 [1 - P_2(\sin\phi)]$$

where ω is the rate of rotation of the body. The condition for a surface to have a constant gravity potential therefore is

$$U + \frac{1}{3}\omega^2 r^2 [1 - P_2(\sin\phi)] = V = \text{constant} \tag{3}$$

Let the equation of an equipotential surface be

$$\frac{a_e}{r} = 1 + \left(f + \frac{3}{2} f^2\right) \sin^2 \phi - \frac{1}{2} f^2 \sin^4 \phi + 0(f^3) \tag{4}$$

which in terms of the mean radius r_0 and Legendre's polynomials is expressible to the second degree, as

$$r = r_0(1 + \alpha_2 P_2 + \alpha_4 P_4)$$

$$r = r_0 \left[1 - \left(\frac{2}{3} f + \frac{23}{63} f^2\right) \cdot P_2(\sin \phi) + \frac{12}{35} f^2 P_4(\sin \phi)\right] \tag{4a}$$

Substitution of Equations 2 and 4 (or 4a) in Equation 3 yields, among other relations, the following:

$$J_2 = \frac{2}{3} f - \frac{1}{3} f^2 - \frac{1}{3} m + \frac{2}{21} mf + 0(f^3) \tag{5a}$$

$$J_4 = -\frac{4}{5} f^2 + \frac{4}{7} mf + 0(f^3) \tag{5b}$$

where

$$m = \frac{\omega^2 a^3 (1 - f)}{GM}$$

A step-by-step development of these relations is given in Khan and Woollard (1968). Equations (4a), (5a) and (5b) will be used at a later stage.

THEORY OF THE INTERNAL FIELD

The condition of hydrostatic equilibrium for any point in the earth's interior is given by

$$\frac{dp}{dr} = \rho \frac{dV}{dr} \tag{6}$$

where the pressure p and the density ρ are related to the point under consideration. Thus surfaces of constant V are also surfaces of constant p and ρ i.e. the surface of equal density are equipotential surfaces. Let one such surface with a uniform density ρ' be expressed as

$$r = a'(1 + \Sigma\alpha_n P_n) \tag{7}$$

where α_n and ρ' are functions of a'. Let r_1 be the value of a for the surface of constant density through an interval point, then the potential $V(r_1)$ on this surface is the sum of potentials from (a) matter inside the shell r_1 (b) the matter outside the shell r_1 and (c) rotational potential i.e.,

$$\begin{aligned} V(r_1) = {} & \frac{4}{3}\pi G \int_0^{r_1} \rho' \frac{\partial}{\partial a'}\left(\frac{a'^3}{r} + \Sigma \frac{3}{2n+1} \frac{a'^{n+3}}{r^{n+1}} \alpha_n P_n\right) da' \\ & + \frac{4}{3}\pi G \int_{r_1}^{a} \rho' \frac{\partial}{\partial a'}\left(\frac{3}{2} a'^2 + \Sigma \frac{3}{2n+1} \frac{r^n}{a'^{n-2}} \alpha_2 P_n\right) da' \\ & + \frac{1}{2}\omega^2 r_1^2 (1 - \sin^2 \varphi') \end{aligned} \tag{8}$$

The mean density ρ_0 within the surface r_1 is

$$\rho_0 = \frac{3}{r_1^3} \int_0^{r_1} \rho' a'^2 da' \tag{9}$$

Since the second order development of the hydrostatic theory becomes somewhat complicated and thus may tend to camouflage the correct structure of the development, I will first develop the first order theory to clearly illustrate the structure of the problem and then extend it to the secord order. The development of the first order theory follows Jeffreys (1962).

Thus neglecting the difference between the geocentric and geodetic coordinates which reduces the condition that ρ and V are constant over the same surface to the condition that V varies only radially, substituting the value of r from Equation 7 in Equation 8, retaining only quantities corresponding to n = 2 and neglecting all higher order terms including those containing P_2^2, yields

$$\frac{4}{3}\pi G\left[\frac{1-\alpha_2 P_2}{r_1}\int_0^{r_1} 3\rho' a'^2 da' + \frac{3}{5}P_2\left\{\frac{1}{r_1^3}\int_0^{r_1}\rho' d(a'^5\alpha_2) + r_1^2\int_{r_1}^{a}\rho' d\alpha_2\right\}\right] + \frac{1}{3}\omega^2 r_1^2(1-P_2) \tag{10}$$

$$= F(r_1)$$

Equation (10) indicates that the function on the left hand side is constant for given r_1 i.e., V is a function of r only. Hence, coefficient of P_2 must vanish on the equipotential surface, i.e.,

$$
-\frac{\alpha_2}{r_1}\int_0^{r_1}\rho' a'^2 da' + \frac{1}{5}\left\{\frac{1}{r_1^3}\int_0^{r_1}\rho' d(a'^5\alpha_2)\right.
$$

$$
\left. + r_1^2\int_{r_1}^{a}\rho' d\alpha_2\right\} = \frac{\omega^2 r_1^2}{12\pi G} \tag{11}
$$

Multiply the above equation by r^3 and differentiate, then the distinction between r and r_1 is no longer important. This gives

$$
-\left(r^2\frac{d\alpha_2}{dr} + 2\alpha_2 r\right)\int_0^{r}\rho' a'^2 da' + r^4\int_r^{a}\rho'\frac{d\alpha_2}{da'}da' = \frac{5\omega^2 r^4}{12\pi G} \tag{12}
$$

Divide by r^4, differentiate again.

$$
\left(\frac{d^2\alpha_2}{dr^2} - \frac{6\alpha_2}{r^2}\right)\int_0^{r}\rho' a'^2 da' + 2\rho r^2\left(\frac{d\alpha_2}{dr} + \frac{\alpha_2}{r}\right) = 0 \tag{13}
$$

Now substitute from Equation 9:

$$
\rho_0\left(\frac{d^2\alpha_2}{dr^2} - \frac{6\alpha_2}{r^2}\right) + \frac{6\rho}{r}\left(\frac{d\alpha_2}{dr} + \frac{\alpha_2}{r}\right) = 0 \tag{14}
$$

This is Clairaut's (1743) differential equation. Equation (12) is particularly interesting. For $r = a$, it becomes

$$
-\left(a^2\frac{d\alpha_2}{da} + 2\alpha_2 a\right)\int_0^{a}\rho' a'^2 da' = \frac{5\omega^2 a^4}{12\pi G} \tag{15}
$$

The mass of the whole body is

$$M = 4\pi \int_0^a \rho' a'^2 da' = \frac{4}{3}\pi a^3 \rho_0(a)$$

where $\rho_0(a)$ is the mean density of the mass bounded by a. Equation (15) then simplify

$$a\frac{d\alpha_2}{da} + 2\alpha_2 = -\frac{5\omega^2 a^3}{3GM} = -\frac{5}{3}m_e$$

where

$$m_e = \frac{\omega^2 a^3}{GM} \tag{16}$$

But to the first order

$$m_e \simeq m = \frac{\omega^2 a^3(1 - f)}{GM}$$

Thus

$$\frac{a}{\alpha_2(a)}\left(\frac{d\alpha_2}{da}\right)_a = -\frac{5m}{3\alpha_2(a)} - 2 \tag{17}$$

where a as a subscript or in the parenthesis denotes the value of the quantity at r = a.

Let a new dependent variable η be

$$\eta = \frac{d \log \alpha_2}{d \log r} = \frac{r d\alpha_2}{\alpha_2 dr} \tag{18}$$

Then

$$\frac{d\alpha_2}{dr} = \frac{\eta \alpha_2}{r}$$

$$\frac{d^2 \alpha_2}{dr^2} = \frac{\alpha}{r} \cdot \frac{d\eta}{dr} + \frac{\eta}{r} \cdot \frac{r}{\alpha_2} \cdot \frac{d\alpha_2}{dr} \cdot \frac{\alpha_2}{r} - \frac{\eta \alpha_2}{r^2} \tag{19}$$

$$= \alpha \left(\frac{1}{r} \frac{d\eta}{dr} + \frac{\eta^2 - \eta}{r^2} \right)$$

Substitute the above in Equation (18) and multiply the result by $r^2/\alpha_2 \rho_0$ to get

$$r \frac{d\eta}{dr} + \eta^2 - \eta - 6 + \frac{6\rho}{\rho_0} (\eta + 1) = 0 \tag{20}$$

Differentiating both sides of equation (9), we get

$$\rho r^2 = \frac{1}{3} \frac{d}{dr} (\rho_0 r^3) \tag{21}$$

which on simplification gives

$$\frac{\rho}{\rho_0} = 1 + \frac{1}{3} \frac{r}{\rho_0} \frac{d\rho_0}{dr} \tag{21a}$$

Hence Equation (20) becomes

$$r \frac{d\eta}{dr} + \eta^2 + 5\eta + 2 \frac{r}{\rho_0} \frac{d\rho_0}{dr} (\eta + 1) = 0 \tag{22}$$

Now consider

$$\frac{\frac{d}{dr} (\rho_0 r^5 \sqrt{1 + \eta})}{\rho_0 r^5 \sqrt{1 + \eta}} = \frac{1}{\rho_0} \frac{d\rho_0}{dr} + \frac{5}{r} + \frac{1}{2(1 + \eta)} \frac{d\eta}{dr} \tag{23}$$

which gives

$$\frac{d\eta}{dr} = \frac{2\sqrt{1 + \eta}}{\rho_0 r^5} \frac{d}{dr} (\rho_0 r^5 \sqrt{1 + \eta}) - \frac{2}{\rho_0} \cdot \frac{d\rho_0}{dr} (1 + \eta) - \frac{10(1 + \eta)}{r} \tag{24}$$

Substitution of this in Equation (22) yields

$$\frac{d}{dr} (\rho_0 r^5 \sqrt{1 + \eta}) = 5\rho_0 r^4 \psi(\eta) \tag{25}$$

where

$$\psi(\eta) = \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1 + \eta}} \tag{26}$$

This is Radau's equation as the substitution in Equation (18) was conceived by Radau (1885). The importance of the substitution lies in the fact that, for any reasonable law of density variation, the function $\psi(\eta)$ has the remarkable property of never departing from 1 by more than 8 parts in 10^4. Then to an accuracy of this order

$$\frac{d}{dr}\left(\rho_0 r^5 \sqrt{1+\eta}\right) = 5\rho_0 r^4 \tag{27}$$

To the same degree of accuracy, the polar moment of inertia C is

$$C = \frac{8}{3}\pi \int_0^a \rho(r) r^4 dr \tag{28}$$

which, on integration by parts, yields

$$C = \frac{8}{9} \int_0^a \left(3r^4 \rho_0(r) + r^5 \frac{d\rho_0}{dr}\right) dr$$

or

$$C = \frac{2}{3} Ma^2 - \frac{16}{9}\pi \int_0^a \rho_0 r^4 dr \tag{29}$$

But the integral in the above equation is

$$\int_0^a \rho_0 r^4 dr = \frac{3}{20} \frac{Ma^2}{\pi} \sqrt{1+\eta(a)} \tag{30}$$

Thus

$$C = \frac{2}{3} Ma^2 - \frac{4}{15} Ma^2 \sqrt{1+\eta(a)} \tag{31}$$

or

$$\eta(a) = \left[\frac{5}{2} - \frac{15}{4} \cdot \frac{C}{Ma^2}\right]^2 - 1 \tag{32}$$

But from Equation (18), the value of η at r = a in

$$\eta(a) = \frac{a}{\alpha_2(a)} \cdot \left(\frac{d\alpha_2}{dr}\right)_a$$

which, from Equation 17, is

$$\eta(a) = \frac{5}{3} \frac{m}{\alpha_2(a)} - 2 \tag{33}$$

It is in the choice of this coefficient $\alpha_2(a)$ which links the external potential theory to the hydrostatic theory. The Equations (4) or (4a), (5) and (6) of the external potential theory are derived without any assumption about the density distribution inside the earth. Hence, these are valid whether or not hydrostatic equilibrium exists in the earth. Thus, if the earth were in hydrostatic equilibrium, the exterior surface defined by Equation (7) must match that defined by Equation (4) or (4a). This should be intuitively clear: if the hydrostatic theory were true for the earth the surfaces defined by the theories of external and internal fields must be coincident at the outer boundary. To the first order, Equation (4a) becomes

$$r = r_0 \left[1 - \frac{2}{3} f_h P_2(\sin \phi)\right] + O(f^2) \tag{34}$$

and

$$\alpha_2 = -\frac{2}{3} f_h$$

where the subscript h refers to hydrostatic flattening. Consequently from Equations (32) and (33)

$$\frac{C}{Ma^2} = \frac{2}{3} - \frac{4}{15} \sqrt{\frac{5}{2} \frac{m}{f_h} - 1} \tag{35}$$

or

$$f_h = \frac{5}{2} m \bigg/ \left[1 + \left(\frac{5}{2} - \frac{15}{4} \frac{C}{Ma^2} \right)^2 \right] \tag{36}$$

The first order development illustrates clearly the structure of the hydrostatic equilibrium problem. The development of the second order theory is somewhat complicated but follows exactly the procedure outlined above. Now all terms $\leq 0(f^2)$ must be retained and the simplifying assumptions modified accordingly. The treatment given below follows deSitter (1938) and Khan (1968, 1969).

Development of the second order theory becomes somewhat simpler if we choose as independent variable the mean radius of a surface of equal density deSitter, 1938). Further simplification is possible by expressing the mean radius in terms of the mean radius of the outer surface as unity and the density in terms of mean density of the body as unity. With these modifications, the potential V at any point (r, ϕ) within the earth (Equation 8), correct to the second order, is

$$\begin{aligned} V = \frac{4}{3}\pi G \Bigg[3 \int_0^\beta \frac{\rho}{r} \beta^2 d\beta - \frac{2}{5} \Bigg\{ \Bigg[\int_0^\beta \frac{\rho}{r^3} \frac{d}{d\beta} \left(\alpha' + \frac{2}{7} \alpha^2 \right) \beta^5 \Bigg] d\beta \\ + \int_\beta^1 r^2 \rho \frac{d}{d\beta} \left[\alpha' + \frac{16}{21} \alpha^2 \right] \Bigg\} P_2(\sin\phi') \\ + \left\{ \frac{12}{35} \int_0^\beta \frac{\rho}{r^5} \frac{d}{d\beta} [\alpha^2 \cdot \beta^7] \cdot d\beta + \frac{32}{105} \int_\beta^1 r^4 \cdot \rho \frac{d}{d\beta} (\beta^{-2}) \, d\beta \right\} P_4(\sin\phi') \\ + \frac{1}{3} \omega^2 r^2 [1 - P_2(\sin\phi')] \end{aligned} \tag{37}$$

where β is the mean radius of an arbitrary surface of equal density expressed in terms of the mean radius of the outer surface as unity and ρ the density, expressed in terms of the mean density as unity. Units of ρ are different from the previously used ρ'. In this notation, $\rho(\beta)$ the mean density within the surface β (corresponding to Equation (9)) is

$$\rho(\beta) = \frac{3}{\beta^3} \int_0^\beta \rho\beta^2 d\beta \tag{38}$$

so that for the surface r = a, i.e., $\beta = 1$,

$$\rho(r) = \rho(1) = 1 \tag{39}$$

The equivalent of Equation (7), which defines a surface of equal density, also in this case an equipotential surface (Equation 4), becomes

$$r = \beta \left[1 - \left(\frac{2}{3}\alpha' + \frac{4}{9}\alpha^2\right) P_2(\sin\phi') + \frac{12}{35}\alpha^2 P_4(\sin\phi')\right] \tag{40}$$

where for convenience of algebraic manipulation, the quantity α' is introduced:

$$\alpha' = \alpha - \frac{5}{42}\alpha^2$$

In the second order theory $\alpha'^2 = \alpha^2$. Substitution of

$$\frac{1}{r},\ \frac{1}{r^3},\ \frac{1}{r^5},\ r^2 \text{ and } r^4$$

from Equation (40) in Equation (37) and the condition that the resulting equation must define an equipotential surface, yields

$$\left[\alpha' + \frac{2}{7}\alpha^2 - \frac{1}{2}m(\beta)\right] \cdot \frac{3}{\beta^3}\int_0^\beta \rho\beta^2 d\beta$$

$$- \frac{3}{5\beta^5}\int_0^\beta \rho \frac{d}{d\beta}\left[\left(\alpha' + \frac{2}{7}\alpha^2\right)\beta^5\right] d\beta$$

$$+\left(\frac{4}{7}\alpha - \frac{3}{5}\right)\int_\beta^1 \rho \frac{d}{d\beta}\left(\alpha' + \frac{16}{21}\alpha^2\right) d\beta - \frac{4}{21}\alpha m(r_1) = 0 \qquad (40)$$

where

$$m(\beta) = \frac{\omega^2\beta^3}{GM}$$

and

$$m(r_1) = \frac{\omega^2 r_1^3}{GM}$$

with β and r_1, denoting the radius of the same surface in different units. Also

$$9\frac{\alpha^2}{\beta^3}\int_0^\beta \rho\beta^2 d\beta + \frac{3}{\beta^7}\int_0^\beta \rho\frac{d}{d\beta}\left[\alpha^2 \cdot \beta^7\right] d\beta + \frac{8}{3}\beta^2\int_\beta^1 \rho \frac{d}{d\beta}(\beta^{-2})\, d\beta$$

$$- \frac{6\alpha}{\beta^5}\int_0^\beta \rho \frac{d}{d\beta}\left[\left(\alpha' + \frac{2}{7}\alpha^2\right)\beta^5\right] d\beta = 0 \qquad (41)$$

The manipulation of Equation (40) in the manner of Equation (11), and the introduction of the variable η which is now defined as

$$\eta = \frac{d \log \alpha'}{d \log \beta} = \frac{\beta}{\alpha'} \frac{d\alpha'}{d\beta} \tag{42}$$

yields

$$\left[\eta\alpha' \left(1 + \frac{4}{7} \alpha\right) - 3\alpha' \left(1 + \frac{2}{7} \alpha\right)\right] \cdot \frac{3}{\beta^3} \int_0^\beta \rho\beta^2 d\beta$$

$$- \frac{4}{21} \eta\alpha' m(r_1) + \frac{4}{7} \eta\alpha' \int_\beta^1 \rho \frac{d}{d\beta} \left(\alpha' - \frac{16}{21} \alpha^2\right) d\beta$$

$$+ \frac{3}{\beta^5} \int_0^\beta \rho \frac{d}{d\beta} \left[\left(\alpha' + \frac{2}{7} \alpha^2\right) \beta^5\right] d\beta = 0 \tag{43}$$

which, on redifferentiation and manipulation in the manner of Equations (12) through (21) yields the second order counterpart of Equation (22) as

$$\beta \frac{d\eta}{dr} + \eta^2 + 5\eta - 2\zeta(1 + \eta) - \frac{4}{21} \zeta\xi = 0 \tag{44}$$

where

$$\zeta = - \frac{\beta}{\rho(\beta)} \frac{d\rho(\beta)}{d\beta} = 3 \left(1 - \frac{\rho}{\rho(\beta)}\right) \tag{45}$$

as yielded by the differentiation of Equation (38), in a manner analogous to Equations (21) and (21a). Also

$$\xi = 7m(\beta) (1 + \eta) - 3\alpha(1 + \eta)^2 - 4\alpha \tag{46}$$

Like Equation (12), Equation (43) is particularly interesting. For r = a i.e., $\beta = 1$, it yields after simplification

$$\eta(a)\ f_h' = 3f_h' - \frac{6}{7} f_h^2 + \frac{4}{7} m f_h - J_h \left(5 + \frac{10}{21} f_h + \frac{20}{21} m\right) \tag{47}$$

where η(a) denotes the value of η for the outer surface on which α and α' also become α' (a) = f_h' and α(a) = f_h i.e. f_h denotes the value of hydrostatic flattening for the outer surface and J_h is the hydrostatic counterpart of $J = 3/2\ J_2$.

Following the procedure outlined in Equations (23) and (24), Equation (44) yields the second order counterparts of Equations (25) and (26) i.e.,

$$\frac{d}{d\beta}\left[\rho(\beta)\ \beta^5 \sqrt{1+\eta}\right] = 5\rho(\beta)\ \beta^4 \psi(\eta) \tag{48}$$

where

$$\psi(\eta) = \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2 + \frac{2}{105}\zeta\xi}{\sqrt{1+\eta}} \tag{49}$$

Integration of Equation (48) yields

$$\int_0^\beta \rho(\beta)\ \beta^4 \psi(\eta)\ d\beta = \frac{1}{5}\rho(\beta)\ \beta^5 \sqrt{1+\eta}$$

or

$$\int_0^\beta \rho(\beta)\ \beta^4 d\beta = \frac{1}{5}\rho(\beta)\ \beta^5 \frac{\sqrt{1+\eta}}{1+\lambda} = \frac{3}{20}\ \frac{M\beta^2}{\pi} \cdot \frac{\sqrt{1+\eta}}{1+\lambda}$$

which for the outer surface yields

$$\int_0^1 \rho(\beta)\ \beta^4 d\beta = \frac{3}{2}\frac{M}{\pi}\ \frac{\sqrt{1+\eta\ (a)}}{1+\lambda\ (a)} \tag{50}$$

where $1 + \lambda$ is the average value of the function $\psi(\eta)$ over the range of integration and λ (a) denotes the value of λ for the outer surface.

The polar moment of inertia C is given by the second order counterpart of Equation (29) i.e.,

$$C = \frac{8}{3}\pi \int_0^\beta \rho\beta^4 d\beta + \frac{2}{3}(C - A) \tag{51}$$

where A is the moment of inertia of the body around its equatorial diameter. Integrating the above equation by parts

$$C = \frac{2}{3}M - \frac{16}{9}\pi \int_0^1 \rho(\beta)\, \beta^4 d\beta + \frac{2}{3}(C - A) \tag{52}$$

and by substitution of Equation (50) in (52)

$$C = \frac{2}{3}Ma^2 \left(1 - \frac{2}{3} f_h\right) - \frac{4}{15} Ma^2 \left(1 - \frac{2}{3} f_h\right) \frac{\sqrt{1 + \eta(a)}}{1 + \lambda(a)}$$

$$+ \frac{2}{3}(C - A) \tag{53}$$

where a is now the equatorial radius. Consequently

$$\frac{3}{2}\, \frac{C}{Ma^2} = 1 - \frac{2}{3} f_h - \frac{2}{5}\left(1 - \frac{2}{3} f_h\right) \frac{\sqrt{1 + \eta(a)}}{1 + \lambda(a)} + J_{2h} \tag{54}$$

or

$$\eta(a) = [1 + \lambda(a)]^2 \left[\frac{5}{2}\left(1 - \frac{2}{3} f_h + J_{2h} - \frac{3}{2}\frac{C}{Ma^2}\right)\left(1 - \frac{2}{3} f_h\right)^{-1}\right]^2 - 1 \tag{55}$$

and from Equation (47)

$$[1 + \lambda(a)]^2 \left[\frac{5}{2} \left(1 - \frac{2}{3} f_h + J_{2h} - \frac{3}{2} \frac{C}{Ma^2}\right) \left(1 - \frac{2}{3} f_h\right)^{-1} \right]^2 - 1$$

$$= 3 - \frac{6}{7} f_h'^{-1} f_h^2 + \frac{4}{7} m f_h f_h'^{-1} - \frac{3}{2} J_{2h} f_h'^{-1} \left(5 + \frac{10}{21} f_h + \frac{20}{21} m\right) \tag{56}$$

whereas the polar moment of inertia is determined by

$$\frac{C}{Ma^2} = \frac{J_2}{H} \tag{57}$$

where

$$J_2 = \frac{C - A}{Ma^2}$$

and

$$H = \frac{C - A}{C}$$

The quality J_2 is the coefficient of the second harmonic in the spherical harmonic expansion of geopotential and is directly determinable from the orbital motion of an artificial earth satellite to a high degree of accuracy. The quantity H is determined also highly accurately from the precession of the earth's axis due to the torque exerted by the moon and the sun. Thus, in Equation (56), if λ (a) can be assigned an appropriate value, a knowledge of J_{2h} should yield f_h as the polar moment of inertia has already been determined by Equation (57). But $J_{2h} \neq J_2$ in case of the earth and thus is not known. We have reasoned earlier, however, that the solutions of external potential and internal potential theories must match at the outer boundary i.e., Equations (5a) and (56) must be simultaneously satisfied for the outer boundary. To avoid any confusion, rewrite Equation (5a) as

$$J_{2h} = \frac{2}{3} f_h - \frac{1}{3} f_h^2 - \frac{1}{3} m + \frac{2}{21} m f_h. \tag{58}$$

Equation (56) can then be solved immediately with the help of Equation (58).

However, Equations (47) and (55), or Equation (56), can be recast to obtain a more convenient expression which gives f_h explicitly in terms of other parameters. To do this, write Equation (55) as

$$\eta(a) = \frac{25}{4} F^2 q'^2 \left(\frac{1 - \Delta_1}{1 - \Delta_2}\right)^2 - 1 \tag{59}$$

where

$$q = \frac{3}{2} \frac{C}{Ma^2}$$

$$\left.\begin{aligned} q' &= 1 - q \\ \Delta_1 &= \frac{\frac{2}{3}\left(f_h - J_h - \frac{2}{3} f_h^2\right)}{q'} \\ \Delta_2 &= \frac{2}{3}\left(f_h - \frac{2}{3} f_h^2\right) \\ F &= 1 + \lambda(a) \end{aligned}\right\} \tag{60}$$

It is instructive to note that Δ_1 and Δ_2 are both of the order of flattening. Simplifying Equation (59), one obtains

$$\eta(a) = \frac{25}{4} F^2 q'^2 \, [1 + 2(\Delta_2 - \Delta_1) \tag{61}$$

$$+ (\Delta_1{}^2 + 3\Delta_2{}^2 - 4\Delta_1\Delta_2)] - 1 = \eta_0 + \eta_1 + \eta_2$$

where

$$\eta_0 = \frac{25}{4} F^2 q'^2 - 1$$

$$\eta_1 = \frac{25}{2} F^2 q'^2 (\Delta_2 - \Delta_1) = \frac{25}{3} F^2 q' \left(J_h - q f_h + \frac{2}{3} q f_h{}^2 \right) \tag{62}$$

and

$$\eta_2 = \frac{25}{4} F^2 q'^2 (\Delta_1{}^2 + 3\Delta_2{}^2 - 4\Delta_1\Delta_2)$$

Note that the quantity η_1 is of the order of f_h, whereas η_2 is of the order of $f_h{}^2$.

Using this value of $\eta(a)$, Equation (47) can be written as

$$A f_h{}^2 + (\eta_0 - 3 + \delta_1) f_h + 5J_h + \delta_2 = 0 \tag{63}$$

where

$$\left.\begin{aligned}
A &= \frac{17}{14} - \frac{5}{42}\,\eta_0 - \frac{25}{3}\,F^2\,q\,q' \\
\delta_1 &= \frac{25}{3}F^2\,q'\,J_h - \frac{4}{7}\,m + \frac{10}{21}\,J_h \\
\delta_2 &= \frac{20}{21}\,m\,J_h
\end{aligned}\right\} \tag{64}$$

and

Note that δ_1 is approximately of the order of f_h, whereas δ_2 is of the order of $f_h^{\,2}$.

Equation (63) gives the required expression for f_h which, correct to the second order of small quantities, is

$$f_h = \frac{1}{3 - \eta_0}\left[(5J_h + \delta_2) + \frac{(5J_h + \delta_2)\,\delta_1}{3 - \eta_0} + \frac{25\,A\,J_h^{\,2}}{(3 - \eta_0)^2}\right] \tag{65}$$

Also, from Equation (58)

$$f_h = J_h\left(1 + \frac{5}{14}\,m + \frac{1}{2}\,J_h\right) + \frac{1}{2}\,m + \frac{3}{56}\,m^2 \tag{66}$$

Simultaneous solution of Equation (65) and (66) would then yield the correct value of f_h.

RESULTS

Shape of the Hydrostatic Earth:

The following data are adopted in the computation of f_h:

$$J_2 = 1082.646 \times 10^{-6}$$

$$H = 3273.64 \times 10^{-6}$$

and therefore

$$\frac{C}{Ma^2} = 0.33071598$$

$$m = 3449.80 \times 10^{-6}$$

$$\lambda(a) = 0$$

The resulting value of the hydrostatic flattening is

$$f_h = \frac{1}{299.75 \pm 0.05}$$

The corresponding value of hydrostatic J_2 is

$$J_{2h} = 1071.66 \times 10^{-6}$$

The value of actual flattening f corresponding to $J_2 = 1082.696 \times 10^{-6}$ is

$$f = \frac{1}{298.25 \pm 0.05}$$

and therefore

$$f - f_h = 16.722 \times 10^{-6}$$

The choice of $\lambda(a) = 0$ needs some justification. The value of $\lambda(a)$ lies between 1.3×10^{-6} (Jeffreys, 1963) and 1.6×10^{-6} (Bullard, 1948) as calculated from the known density distribution of the earth. Thus it may appear at first sight that choice of λ is not compatible with the second order theory. However, selection of $\lambda(a) = 0$, instead of $\lambda(a) = 1.3 \times 10^{-4}$, results in f_h being greater by 6×10^{-7} only whereas the difference $f - f_h = 16.722 \times 10^{-6}$. Hence our choice of $\lambda(a) = 0$ and consequently $\psi(\eta) = 1$ does not affect the value of f_h to any significant degree. This is demonstrated in Figure 1.

<u>Size of the Hydrostatic Earth:</u>

The size of the hydrostatic earth ellipsoid can be computed by assuming that it is volumetrically equivalent to the actual earth. Thus if a and b are the equatorial and polar radii of the actual earth and a_h and b_h their hydrostatic counterparts, we have

$$a_h = a\left(\frac{1 - f}{1 - f_h}\right)^{1/3}$$

For a = 6378, 140 meters and C = 6356,755 meters (corresponding to f = 1/298.255), we obtain

$$a_h = 6378{,}104 \text{ meters}$$

$$b_h = 6356{,}826 \text{ meters}$$

resulting in

$$a - a_h = 36 \text{ meters}$$

$$b - b_h = -71 \text{ meters}$$

<u>Gravity Field Referred to the Hydrostatic Figure:</u>

As discussed earlier, since the hydrostatic figure is a figure of zero stress, it constitutes a geophysically meaningful reference figure for gravity anomalies for use in studying the internal state of the earth. Since these gravity anomaly maps are generally available with reference to either the International Gravity Formula or Gravity Formula, 1967, Table 1 lists the conversion factors to be applied to the above-noted gravity anomalies in order to refer them to the hydrostatic reference figure. Note that these are the conversion factors for the gravity anomalies. For converting absolute gravity values from the international reference ellipsoid to the hydrostatic references ellipsoid the Potsdam datum correction must also be taken into account.

BIBLIOGRAPHY

1. Bullard, E. C., The figure of the earth, M.N.R.A.S., Geophys. Suppl. 5, 186-192, 1948.

2. Caputo, M., The minimum strength of the earth, J. Geophys. Res., 70, 955-963, 1965.

3. Clairaut, A. C., Theorie de la Figure de la Terre, Paris, 1743.

4. deSitter, W., On the system of astronomical constants, Bull. Astr. Insts. Netherlands, 8, 307, 213-229, 1938.

5. Henriksen, S. W., The hydrostatic flattening of the earth, Ann. Int. Geophys. Year, 12, 197-198, 1960.

6. Jeffreys, H., The Earth, Cambridge University Press, Cambridge, 1962.

7. Jeffreys, H., On the hydrostatic theory of the figure of the earth, Geophys. J., 8, 196-202, 1963.

8. Khan, M. A., Some parameters of a hydrostatic earth, Trans. Am. Geophys. Union, 48, 1, 56, 1967.

9. Khan, M. A., On the Equilibrium figure of the earth, Hawaii Inst. of Geophys. Ref. No. HIG-68-10, 1968.

10. Khan, M. A., General solution of the problem of hydrostatic equilibrium of the earth, Geophys. J. R. Astro. Soc. 18, 177-188, 1969.

11. Khan, M. A. and G. P. Wollard, A review of the perturbation theory as applied to the determination of geopotential, Hawaii Inst. of Geophysics Rep. No. HIG-68-1, 1968.

12. Kazai, Y., New determination of zonal harmonic coefficients of the earth's gravitational potential, Smith. Astrophys. Obsv. Sp. Rept. No. 165, 1964.

13. O'Keefe, J. A., Determination of the earth's gravitational field, In H. Kallmann (ed.), Space Research I, 448-457, 1960.

14. Radau, R., Compte Rendus De Science, C. R. Acad. Sci. Paris, 100, 972-974, 1885.

15. Tisserand, F., Mecanique´ Celeste, 4 Vols., Paris, 1889-1896.

Table 1: Corrections in Milligals for the Interconversion of the International, 1967 and Equilibrium Reference Ellipsoids

Latitude (Degrees)	(1)	(2)	(3)
90.00	9.10	-9.00	-18.11
89.00	9.10	-9.00	-18.10
88.00	9.09	-8.99	-18.07
87.00	9.07	-8.97	-18.03
86.00	9.04	-8.94	-17.97
85.00	9.00	-8.90	-17.90
84.00	8.95	-8.86	-17.81
83.00	8.90	-8.80	-17.71
82.00	8.84	-8.74	-17.59
81.00	8.77	-8.68	-17.45
80.00	8.70	-8.60	-17.30
79.00	8.61	-8.52	-17.13
78.00	8.52	-8.43	-16.94
77.00	8.42	-8.33	-16.75
76.00	8.31	-8.22	-16.53
75.00	8.20	-8.11	-16.31
74.00	8.08	-7.99	-16.06
73.00	7.95	-7.86	-15.81
72.00	7.81	-7.73	-15.54

Table 1–(continued)

Latitude (Degrees)	(1)	(2)	(3)
71.00	7.67	-7.58	-15.26
70.00	7.52	-7.44	-14.96
69.00	7.37	-7.28	-14.65
68.00	7.21	-7.12	-14.33
67.00	7.04	-6.96	-14.00
66.00	6.87	-6.79	-13.65
65.00	6.69	-6.61	-13.30
64.00	6.50	-6.43	-12.93
63.00	6.31	-6.24	-12.56
62.00	6.12	-6.05	-12.17
61.00	5.92	-5.85	-11.77
60.00	5.72	-5.65	-11.37
59.00	5.51	-5.45	-10.96
58.00	5.30	-5.24	-10.54
57.00	5.08	-5.02	-10.11
56.00	4.87	-4.81	- 9.67
55.00	4.64	-4.59	- 9.23
54.00	4.42	-4.37	- 8.79
53.00	4.19	-4.14	- 8.33
52.00	3.96	-3.91	- 7.87

Table 1–(continued)

Latitude (Degrees)	(1)	(2)	(3)
51.00	3.73	-3.68	- 7.41
50.00	3.50	-3.45	- 6.95
49.00	3.26	-3.22	- 6.48
48.00	3.02	-2.99	- 6.01
47.00	2.79	-2.75	- 5.54
46.00	2.55	-2.51	- 5.06
45.00	2.31	-2.28	- 4.59
44.00	2.07	-2.04	- 4.11
43.00	1.83	-1.81	- 3.64
42.00	1.59	-1.57	- 3.16
41.00	1.35	-1.33	- 2.69
40.00	1.12	-1.10	- 2.22
39.00	0.88	-0.87	- 1.75
38.00	0.65	-0.64	- 1.29
37.00	0.42	-0.41	- 0.82
36.00	0.09	-0.18	- 0.37
35.00	-0.04	0.04	0.08
34.00	-0.27	0.26	0.53
33.00	-0.49	0.48	0.97
32.00	-0.70	0.70	1.40

Table 1–(continued)

Latitude (Degrees)	(1)	(2)	(3)
31.00	-0.92	0.91	1.83
30.00	-1.13	1.12	2.25
29.00	-1.34	1.32	2.66
28.00	-1.54	1.52	3.06
27.00	-1.74	1.72	3.45
26.00	-1.93	1.91	3.84
25.00	-2.12	2.09	4.21
24.00	-2.30	2.27	4.57
23.00	-2.48	2.45	4.92
22.00	-2.65	2.62	5.26
21.00	-2.81	2.78	5.59
20.00	-2.97	2.94	5.90
19.00	-3.12	3.09	6.21
18.00	-3.27	3.23	6.50
17.00	-3.41	3.37	6.77
16.00	-3.54	3.50	7.03
15.00	-3.66	3.62	7.28
14.00	-3.78	3.74	7.52
13.00	-3.89	3.84	7.73
12.00	-3.99	3.95	7.94

Table 1–(continued)

Latitude (Degrees)	(1)	(2)	(3)
11.00	-4.09	4.04	8.13
10.00	-4.18	4.12	8.30
9.00	-4.25	4.20	8.46
8.00	-4.32	4.27	8.60
7.00	-4.39	4.33	8.72
6.00	-4.44	4.39	8.83
5.00	-4.49	4.43	8.92
4.00	-4.53	4.47	9.00
3.00	-4.56	4.50	9.06
2.00	-4.58	4.52	9.10
1.00	-4.59	4.53	9.12
0.0	-4.59	4.54	9.13

Column (1): For change from the "Reference Ellipsoid 1967" to the International Reference Ellipsoid, add the correction to the gravity anomaly. For reverse operation subtract the correction.

Column (2): For change from the "Reference Ellipsoid 1967" to the Equilibrium Reference Ellipsoid, add the correction to the gravity anomaly. For reverse operation subtract the correction.

Column (3): For change from the International Reference Ellipsoid to the Equilibrium Reference Ellipsoid, add the correction to the gravity anomaly. For reverse operation subtract the correction.

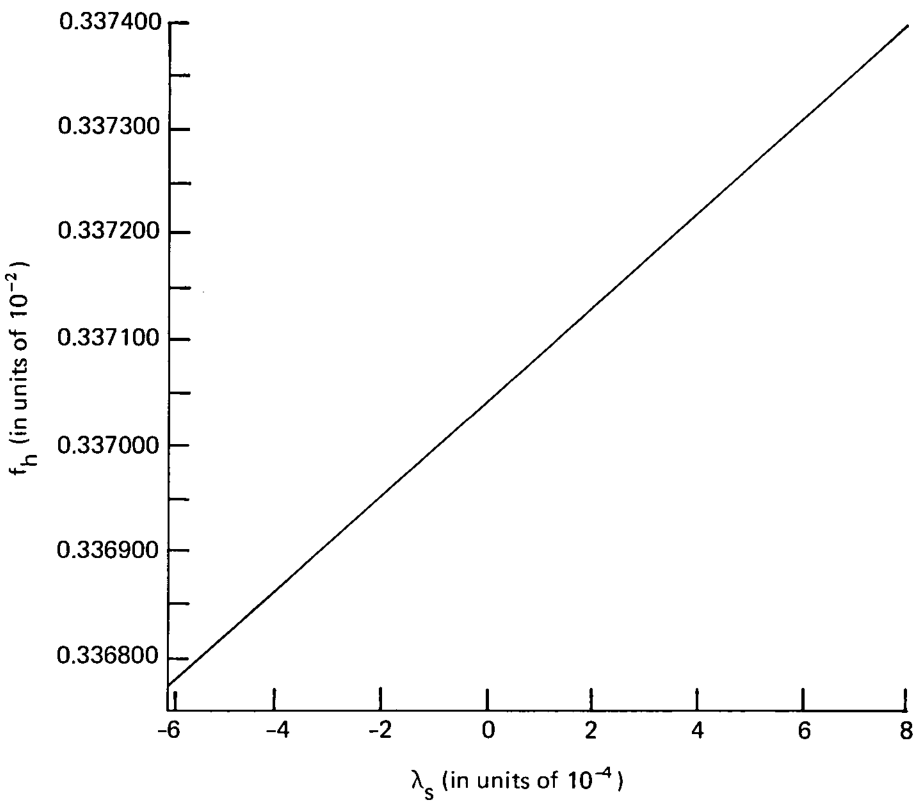

Figure 1. Influence of $\lambda(a)$ on the value of f_h (after Khan, 1968).