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PROPAGATION OF WAVES IN ELLIPTIC DUCTS. A THEORETICAL STUDY.

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S. Baskaran

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INTRODUCTION.

In recent years a considerable amount of work has been done on noise reduction in machines in general. In particular, attention of many workers in the field has been focussed on the noise reduction in jet engines, which radiate a high level of noise. In this context a study of sound propagation in cylindrical ducts assumes an understandable importance.

Propagation of sound in circular cylindrical ducts has been investigated extensively in connection with a jet engine compressor. It is important here to mention the work done by Tyler and Sofrin [26] which forms a basis for much work done in the field. They have shown that at very low shaft speeds a very high frequency noise is generated due to the interaction of the rotor blades and stator-vanes. Low frequency noise tends to attenuate exponentially in a duct; whereas high frequency noise propagates along the duct giving rise to spinning modes. These result in an annoying noise radiated from the open face of the duct.

From the noise control viewpoint, departures from circular symmetry of the cross-section of the duct evoke considerable interest. Elliptic duct intakes have been used in aircraft design, e.g. Boeing 727. Equations governing sound propagation in a duct of elliptic cross-section are quite well known.[2,18] Using elliptic cylindrical coordinates the wave equation separates into Mathieu equation and modified Mathieu equation. This has been discussed in the works of many authors, e.g. Chu (6), Jeffreys (13), Daymond (7) in connection with electromagnetic wave guides and oscillations in a lake with elliptic boundaries. However all work has been concentrated on the lowest order principal frequencies. These eigen frequencies decrease rapidly in the case of even solutions of the modified Mathieu equation for increasing eccentricity of the cross-section and increase in the case of odd solutions. It is, therefore, of interest to investigate the behaviour of these waves for higher order modes.

The pressure gradient normal to the duct wall vanishes under the boundary conditions that there are no reflections of the pressure fluctuations at the open end of the duct and that the duct is hard walled. This means the derivative, normal to the duct wall, of the pressure function is zero. The pressure function is a combination of Mathieu function and modified Mathieu function. Due to the continuity of pressure function in any particular cross-section of the duct, the solution are periodic. For a particular eccentricity, the eigen frequencies for various modes are obtained as the lowest parametric zeros of the derivatives of the modified Mathieu functions.

In a duct of circular section, the eigen frequencies are obtained directly as parametric zeros of the derivatives of the Bessel functions. But, in a duct of elliptic section, the eigen frequencies for the relevant eccentricities cannot be obtained in a straight forward manner. To obtain the eigen frequencies, a family of ellipses of same area, say π , are considered. For chosen positive increasing values of the parameter, the zeros of the derivative of the modified Mathieu function are found. From these the corresponding eccentricities and the eigen frequencies are computed. Then the eigen frequencies for the particular eccentricities are obtained by interpolation of the values already known.

For higher order modes, high values of the parameter have to be considered. Further the function has to be evaluated for these high values of the parameter. It is essential, therefore to check the validity and the rapidity of convergence of the various series expansions of the modified Mathieu function. The separation constant, which appears due to the separation of the wave equation, known as the characteristic value is tabulated in Blanch and Rhodes (5) for large values of the parameter. With the help of a computer program the coefficients for the series expansions for modified Mathieu functions are generated from these characteristic values. The zeros are then found and the eigen frequencies obtained as mentioned in the previous paragraph.

The eigen frequencies for ellipses of eccentricities .1 (0.1) .9 and 0.95 have been obtained for the integral orders 1-15 of the function, both even and odd. The eigen frequencies for even functions have been tabulated in Tables C and those for odd functions in Tables D.

1. PROPAGATION OF WAVES IN CIRCULAR DUCTS.

1.1 The study of noise in a jet engine compressor may be broadly classified into three main divisions:[26]

(1) generation of noise due to the movement of the rotor blades

(2) the propagation of the noise through a cylindrical duct enclosing the rotor blades

and (3) the radiation of the noise into free space from the open face of the duct.

The work described herein is mainly concerned with the propagation of noise through elliptic ducts. Propagation in circular ducts has been described extensively in the literature. A brief review of the propagation in circular ducts (which lends itself more easily for analysis) will help in a better understanding of the more general case of propagation through elliptic ducts.

1.2 The noise in a compressor engine is generated mainly due to two causes.

(1) When the rotor blades move with given angular velocity, pressure fluctuations are caused at a blade passage frequency. These pressure fluctuations rotate with rotor angular velocity giving rise to harmonically related frequencies.

(2) Due to the interaction of rotor blades and stator vanes high frequency noise is generated when

(a) the wakes of upstream stator are cut by the rotor blades

(b) the rotor blade wakes are cut by the downstream stators and (c) the rotating periodic pressure field is interrupted by reflecting objects nearby.

These give rise to very high frequency noise even at low rotor speeds. The sound field so generated is allowed to pass through a duct of rigid walls enclosing the rotor.

1.3 The sound pressure fields obey the well-known wave equation [20,21]

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \qquad (1.1)$$

x, y, z being the cartesian coordinates and c is the free space velocity of sound.

If (r, θ, z) denote cylindrical polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$t = t$$

the wave equation (1.1) takes the form

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \qquad (1.2)$$

By the separation of variables (so that the normal modes are obtained) the solution is of the form

$$p = p_r(r)\, p_\theta(\theta)\, p_z(z)\, p_t(t)$$

Substituting in equation (1.2) and dividing by p, gives

$$\frac{1}{p_r}\frac{d^2 p_r}{dr^2} + \frac{1}{r p_r}\frac{d p_r}{dr} + \frac{1}{r^2 p_\theta}\frac{d^2 p_\theta}{d\theta^2} + \frac{1}{p_z}\frac{d^2 p_z}{dz^2} - \frac{1}{c^2 p_t}\frac{d^2 p_t}{dt^2} = 0 \qquad (1.3)$$

The z and t terms can then each be equated to constants, say, $-k_z^2$ and $-\frac{\omega^2}{c^2}$ respectively, and

so $\frac{d^2 p_z}{dz^2} = -k_z^2 p_z$ and $\frac{d^2 p_t}{dt^2} = -\omega^2 p_t$

Equation (1.3) then reduces to

$$\frac{1}{p_r}\frac{d^2 p_r}{dr^2} + \frac{1}{r p_r}\frac{d p_r}{dr} + \frac{1}{r^2 p_\theta}\frac{d^2 p_\theta}{d\theta^2} + \left(-k_z^2 + \frac{\omega^2}{c^2}\right) = 0$$

If $\frac{1}{p_\theta}\frac{d^2 p_\theta}{d\theta^2} = -m^2$ (where m is a positive integer) (1.4)

the above equation reduces to

$$\frac{d^2 p_r}{dr^2} + \frac{1}{r}\frac{d p_r}{dr} + \left(-k_z^2 + \frac{\omega^2}{c^2} - \frac{m^2}{r^2}\right) p_r = 0 \qquad (1.5)$$

The equations in θ, z and t are all of the same form and their solutions are all given by imaginary exponentials.

If $k^2 = \frac{\omega^2}{c^2} - k_z^2$ (1.6)

the equation (1.5) for p_r takes the form of Bessel's equation and the solution is given by [18,26]

$$p_r = A_m J_m(kr) + B_m Y_m(kr)$$

where J_m and Y_m are Bessel's functions of the first and second kind respectively. A_m, B_m are weight constants for an arbitrary pressure distribution.

But since $Y_m \to \infty$ as $r \to \infty$ and since the pressure function is finite everywhere within the duct including the duct axis, Y_m is not included in the solution.

1.4 For a hard-walled duct, the normal pressure gradient at the duct walls is zero. The equation of the duct wall is given by $r = R$ (a constant).

Hence $$\left.\frac{\partial p}{\partial r}\right|_{r=R} = 0$$

which is the same as

$$\left.\frac{d}{dr}\left[J_m(kr)\right]\right|_{r=R} = 0$$

$$\text{i.e. } J_m'(kR) = 0 .$$

For a given value of m and R, there is an infinity $k_{m\mu}$ ($\mu = 0, 1, 2, \cdots$) of values of k which satisfies this equation, each $k_{m\mu}$ increasing in magnitude with μ. For a given μ, the pressure function has μ pressure nodes in the radial direction.

So the pressure distribution function can be written as

$$p = J_m(k_{m\mu} r) \left.\begin{matrix} \sin m\theta \\ \cos m\theta \end{matrix}\right\} e^{i(k_z z + \omega t)}$$

(At any fixed position, ω represents the circular frequency of the pressure fluctuations).

In the above equation

$$k_z^2 = \frac{\omega^2}{c^2} - k_{m\mu}^2$$

1.5 Thus for any given value of $k_{m\mu}$ only values of ω which are greater than $ck_{m\mu}$ give rise to real values of k_z. Hence the 'z' mode of the pressure function gives rise to a sinusoidal pressure distribution along the axis of the duct, causing the wave to propagate in this direction. Values of $\omega < c\,k_{m\mu}$, give imaginary values of k_z ; the pressure function becomes a negative exponential in z which results in an exponential attenuation of the pressure wave along the duct axis. This is known as the 'cut-off' phenomenon.

The values of $k_{m\mu}$ give the cut-off frequencies for the various modes m of the pressure distribution. It is interesting to note that, as μ increases in value, $k_{m\mu}$ also increase. k_{mo} gives the lowest cut-off frequency for a particular mode 'm' of the pressure function. These values may be found in any tables relating to Bessel functions - see for example Olver 22.

The decay rates for frequencies below cut-off and their significance in a practical situation has been demonstrated by several workers.[3,25,26]

2. DERIVATION OF THE WAVE EQUATION IN ELLIPTIC COORDINATE SYSTEM AND BOUNDARY CONDITIONS.

2.1 To study the behaviour of waves in an elliptic duct it becomes essential that a proper system of co-ordinates is chosen, the most convenient and relevant being the elliptic coordinate system.

If (x, y, z) are the cartesian coordinates and (ξ, η, z) the elliptic cylindrical coordinates, the following relation holds between them:

$$x = h \cosh \xi \cos \eta$$

$$y = h \sinh \xi \sin \eta$$

$$z = z$$

and when the wave equation (1.1) is considered, the time variable is kept unchanged.

In any z-constant plane, the η-constant curves are hyperbolae; the ξ-constant curves are ellipses with major axis $2h \cosh \xi$ and minor axis $2h \sinh \xi$. The foci are at $(+h, 0)$ and $(-h, 0)$. (Fig. 2)

With this transformation the wave equation takes the form

$$\frac{2}{h^2(\cosh 2\xi - \cos 2\eta)} \left\{ \frac{\partial^2 p}{\partial \xi^2} + \frac{\partial^2 p}{\partial \eta^2} \right\} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \qquad (2.1)$$

To obtain a separable solution, as before, can be written as

$$p = p_\xi(\xi)\, p_\eta(\eta)\, p_z(z)\, p_t(t)$$

so that equation (2.1) reduces to, on substitution of the value of p and division by p,

$$\frac{2}{h^2(\cosh 2\xi - \cos 2\eta)}\left\{\frac{1}{p_\xi}\frac{d^2p_\xi}{d\xi^2} + \frac{1}{p_\eta}\frac{d^2p_\eta}{d\eta^2}\right\} + \frac{1}{p_z}\frac{d^2p_z}{dz^2} - \frac{1}{c^2p_t}\frac{d^2p_t}{dt^2} = 0$$

Equating z and t terms to constants $-k_z^2$ and $\frac{\omega^2}{c^2}$ respectively,

$$\frac{d^2p_z}{dz^2} = -k_z^2 p_z \quad \text{and} \quad \frac{d^2p_t}{dt^2} = -\frac{\omega^2}{c^2}p_t$$

and

$$\frac{1}{p_\xi}\frac{d^2p_\xi}{d\xi^2} + \frac{1}{p_\eta}\frac{d^2p_\eta}{d\eta^2} + \left(\frac{\omega^2}{c^2} - k_z^2\right)\frac{h^2}{2}(\cosh 2\xi - \cos 2\eta) = 0$$

Let $$\frac{1}{p_\xi}\frac{d^2p_\xi}{d\xi^2} + \left(\frac{\omega^2}{c^2} - k_z^2\right)\frac{h^2}{2}\cosh 2\xi = a$$

and $$\frac{1}{p_\eta}\frac{d^2p_\eta}{d\eta^2} - \frac{h^2}{2}\left(\frac{\omega^2}{c^2} - k_z^2\right)\cos 2\eta = -a$$

where 'a' is a separation constant, known as the characteristic number.

The above equations can now be written out separately as follows:

$$\frac{d^2p_\eta}{d\eta^2} + \left[a - \frac{h^2}{2}\left(\frac{\omega^2}{c^2} - k_z^2\right)\cos 2\eta\right]p_\eta = 0 \qquad (2.2)$$

$$\frac{d^2p_\xi}{d\xi^2} - \left[a - \frac{h^2}{2}\left(\frac{\omega^2}{c^2} - k_z^2\right)\cosh 2\xi\right]p_\xi = 0 \qquad (2.3)$$

If $$q = \frac{h^2}{4}\left(\frac{\omega^2}{c^2} - k_z^2\right). \qquad (2.4)$$

the equations (2.2) and (2.3) reduce to the form

$$\frac{d^2 p_\eta}{d\eta^2} + (a - 2q\cos 2\eta) p_\eta = 0 \tag{2.5}$$

$$\frac{d^2 p_\xi}{d\xi^2} - (a - 2q\cosh 2\xi) p_\xi = 0 \tag{2.6}$$

which are the canonical forms of Mathieu's equation and the modified Mathieu equation.[2,19]

It can be seen here that if $\eta = i\xi$ equation (2.5) transforms to equation (2.6) and conversely if $\xi = i\eta$, equation (2.6) transforms to equation (2.5).

The separation constant 'a' here is such that as $q \to 0$, $a \to m^2$, where m is a positive integer.

2.2 The solutions of equation (2.5) are the even and odd Mathieu functions, ce_m and se_m of order m, when $a \to m^2$ as $q \to 0$. These functions ce_m, se_m are functions of both η and q and are such that as $q \to 0$, $ce_m \to \cos m\eta$ and $se_m \to \sin m\eta$. The even 'cosine-elliptic' functions correspond to the cosine functions in the circular case and the odd 'sine-elliptic' functions correspond to the sine functions. In a circular duct the periodic sinusoidal waves of order m around its periphery can be of either sinusoidal or cosinusoidal form depending on their values at any particular vectorial axis. In an elliptic duct, these wave forms are dependent on the axis of symmetry. For the same value of a and q, ce_m and se_m form a fundamental system of solution for the Mathieu equation (2.5). The solution, therefore, can be written as

$$p_\eta = A_c ce_m(\eta, q) + A_s se_m(\eta, q)$$

which gives us the pressure distribution in the circumferential mode. The ce_m and se_m are periodic with period π or 2π depending on m being even or odd.

The solutions of the modified Mathieu equation (2.6) are the modified Mathieu functions, also called radial Mathieu Functions as they represent the radial pressure distribution in the duct. It can be shown that as $q \to 0$, the equation (2.6) reduces to the Bessel's equation (1.5).[19] The radial Mathieu functions Ce_m and Se_m of the first kind are the even 'cosh-elliptic' and odd 'sinh-elliptic' functions respectively and they are periodic with period πi or $2\pi i$. As a matter of fact

$$Ce_m(\xi, q) = ce_m(i\eta, q)$$
$$Se_m(\xi, q) = -i\, se_m(i\eta, q)$$

There exist radial Mathieu functions of the second, third and fourth kind, but these functions do not satisfy (as shall be shown) the necessary boundary conditions and hence are not included in the pressure distribution function.

2.3 The boundary conditions to be satisfied by the pressure function p in a reference plane $z = 0$ are as follows:

(a) Since p is single-valued, it is periodic in η with a maximum period of 2π.

(b) $p(\xi, \eta)$ is continuous in the duct and in particular, it is continuous across the interfocal line,

i.e. $p(0, \eta) = p(0, -\eta)$

(c) On crossing the interfocal line, there is continuity of the pressure gradient, so that

$$\frac{\partial p}{\partial \xi}(0, \eta) = -\frac{\partial p}{\partial \xi}(0, -\eta)$$

and (d) the component of the pressure gradient normal to the wall of the duct (it being hard walled) is zero at the walls. On this boundary $\xi = \xi_0$.

Hence

$$\left.\frac{\partial p}{\partial \xi}\right|_{\xi = \xi_0} = 0$$

The functions ce_m and se_m are periodic in η (and with period π or 2π, so long as m is a positive integer). Hence they satisfy condition (a).

If $Ce_m(\xi, q)$ is a solution of equation (2.6), $Ce_m(0, q)$ is a constant. $ce_m(\eta) = ce_m(-\eta)$, as ce_m are even functions.

So $Ce_m(\xi)\, ce_m(\eta)$ satisfies condition (b).

Since $Se_m(0, q)$, $Se_m(0)\, se_m(\eta) = Se_m(0)\, se_m(-\eta)$. Hence $Se_m(\xi)\, se_m(\eta)$ satisfies condition (b)

Consider now $Ce_m(\xi)\, se_m(\eta)$. Since $se_m(\eta) = -se_m(-\eta)$, se_m being odd, $Ce_m(0)\, se_m(\eta) \neq Ce_m(0) \times se_m(-\eta)$. So $Ce_m(\xi)\, se_m(\eta)$ does not satisfy condition (b). On the same lines it can be proved that $Se_m(\xi)\, ce_m(\eta)$ does not satisfy condition (b).

It can be proved that the combinations $Ce_m(\xi) \times ce_m(\eta)$ and $Se_m(\xi)\, se_m(\eta)$ satisfy the conditions (a), (b) and (c) and are in fact the only possible combinations of Mathieu and modified Mathieu functions satisfying the boundary conditions (a), (b) and (c). The solutions of the second, third and fourth kinds fail to satisfy the condition (c) and hence are not included in the pressure function.

Hence the only acceptable solutions of the wave equation for the pressure field are given by

$$p = \left.\begin{array}{l} C_m\, Ce_m(\xi, q)\, ce_m(\eta, q) \\ S_m\, Se_m(\xi, q)\, se_m(\eta, q) \end{array}\right\} e^{ik_z z} \cos(\omega t + \alpha)$$

which satisfy the conditions (a), (b) and (c).

Now the condition (d) requires that when $\xi = \xi_0$ (i.e. on the boundary of the elliptic duct) $\frac{\partial p}{\partial \xi} = 0$ which is

$$\frac{d}{d\xi} Ce_m(\xi, q)\Bigg|_{\xi=\xi_0} = 0 \quad \text{or} \quad \frac{d}{d\xi} Se_m(\xi, q)\Bigg|_{\xi=\xi_0} = 0 \qquad (2.7,\ 2.8)$$

corresponding to each value of m, these equations, for a given value ξ_0 of ξ, give an infinity of positive values of q, which satisfy the above equations. The first equation gives a set of values $q_{m\mu}$ ($\mu = 0, 1, 2, \cdots$) which satisfy it. The second equation gives a set of values $\bar{q}_{m\mu}$ (say) which satisfy it. These are known as the parametric zeros of the functions. For each value of μ, both

$q_{m\mu}$ and $\bar{q}_{m\mu}$ give the nodal (zero pressure) ellipses in the reference plane. Since Ce_m and Se_m from a fundamental system of solution for equation (2.6), it is to be noted that none of the $q_{m\mu}$ and $\bar{q}_{m\mu}$ are the same for the same value of m and μ and a given characteristic value a.[11]

3. PREVIOUS WORK ON ELLIPTIC WAVE PROPAGATION.

3.1 From equation (2.4),

$$q = \frac{h^2}{4}\left(\frac{\omega^2}{c^2} - k_z^2\right)$$

the important relationship

$$k_z = \left(\frac{\omega^2}{c^2} - \frac{4q}{h^2}\right)^{1/2} \qquad (3.1)$$

is obtained. $4q/h^2$ here corresponds to the 'k' mentioned in the circular case. As $q \to 0$, equation (2.6) reduces to equation (1.4) as mentioned earlier. Indeed, this corresponds to $h \to 0$ with the ellipses becoming circles.

If ω/c is such that $\omega/c < 2\sqrt{q_{m\mu}}/h$, where $q_{m\mu}$ is a parametric zero obtained from equation (2.7), the value of k_z from (3.1) becomes imaginary; the pressure function becomes a negative exponential resulting in an exponential decay of waves along the axial direction of the duct.

If $\omega/c \geq 2\sqrt{q_{m\mu}}/h$, real values of k_z are obtained and this mode is propagated along the duct without attenuation.

For a particular m, q_{mo} gives the parametric zero such that $\omega/c \geq 2\sqrt{q_{mo}}/h$, which gives the cut-off frequency for the particular mode m. The higher order radial modes associated with $q_{m\mu}$ are of less significance and will not be considered further.

Though the theory of propagation and attenuation of waves with elliptic boundary conditions have been considered by many authors [6,7,10,13], actual determination of the 'non-dimensionalised' cut-off frequencies, though only for the lowest order modes, 0 and 1, have been found only by Chu and by Daymond. In what follows work done by Jeffreys, Chu and Daymond is described as a basis for further development.

3.2 [13] In an elliptical lake of uniform depth d, let ζ represent the vertical displacement from its equilibrium position of the water surface. Let the lake be stationary in space and let ζ be an imaginary exponential function of the time variable t. If ζ is so small that its second and higher powers could be neglected, it can be proved that ζ satisfies the reduced wave equation

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\omega^2}{c^2}\zeta = 0 \qquad (3.2)$$

$c^2 = gd$, c being the free wave velocity in an unrestricted expanse of water of uniform depth d and g being the acceleration due to gravity. x, y coordinates are chosen with axes along the major and minor axes respectively of the elliptic boundary, origin being at its centre (Fig. 2).

Transforming to elliptic coordinates (see chapter II) the equation (3.2) reduces to

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} + 2k^2(\cosh 2\xi - \cos 2\eta)\zeta = 0, \quad 2k = \frac{\omega h}{c} \qquad (3.3)$$

The form of ζ is then given by

$$\zeta(\xi,\eta,t) = \left.\begin{matrix} C_m\, Ce_m(\xi,q)\, ce_m(\eta,q) \\ S_m\, Se_m(\xi,q)\, se_m(\eta,q) \end{matrix}\right\} e^{i(\omega_m t+\epsilon_m)} \qquad (3.4)$$

ω_m is the angular frequency of vibration of the mth mode and ϵ_m is the relative phase angle.

The velocity normal to an ellemental arcual length ds of an ellipse, that is, in the direction dn, is given by

$$u_n = \frac{ig}{\omega}\frac{\partial \zeta}{\partial n} = \frac{ig}{\omega l_1}\frac{\partial \zeta}{\partial \xi} \quad \text{where } dn = l_1 d\xi,\ l_1 = \frac{h}{\sqrt{2}}(\cosh 2\xi - \cos 2\eta)^{1/2}$$

On the boundary $\xi = \xi_0$, the normal velocity is zero and hence

$$\left[\frac{\partial \zeta}{\partial \xi}\right]_{\xi=\xi_0} = 0$$

which reduces to $Ce_m'(\xi_0,q) = 0$ and $Se_m'(\xi_0,q) = 0$ from equation (3.4).

The values $q_{m\mu}$ and $\bar{q}_{m\mu}$ satisfying the above respective equations have now to be found.

Consider the lowest modes $m=1, \mu=1$ when ζ is proportional to $Ce_1(\xi,q_{11})\,ce_1(\eta,q_{11})$. Let us consider an ellipse of given eccentricity.

Suppose the boundary of the elliptic lake has eccentricity $e = 0.8$. Then $\cosh \xi_0 = 1/e = 1.25$, $\exp(\xi_0) = 2$, $\exp(-\xi_0) = 0.5$ and $\xi_0 = .6931$

From the expansions of $Ce_1(\xi_0, q)$ (see Chapter IV)*

* Though the function is normalised differently.

Therefore

$$Ce_1(\xi_0, q) \simeq -\alpha^3\left(\frac{1}{18}\cosh 7\xi_0 - \frac{4}{9}\cosh 5\xi_0 + \frac{1}{3}\cosh 3\xi_0\right) + \alpha^2\left(\frac{1}{3}\cosh 5\xi_0 - \cosh 3\xi_0\right) - \alpha\cosh 3\xi_0 + \cosh\xi_0$$

where $\alpha = \frac{1}{8}q$

Therefore

$$Ce_1'(\xi_0, q) \simeq -\alpha^3\left(\frac{7}{18}\sinh 7\xi_0 - \frac{20}{9}\sinh 5\xi_0 + \sinh 3\xi_0\right) - \alpha^2\left(\frac{5}{3}\sinh 5\xi_0 - 3\sinh 3\xi_0\right) - 3\alpha\sinh 3\xi_0 + \sinh\xi_0$$

Since $Ce_1'(\xi_0, q) = 0$, substituting in terms of $\exp(\xi_0)$ and $\exp(-\xi_0)$ for $\sinh\xi_0$, an equation in α is obtained. The smallest positive root of this equation gives $\alpha \simeq 0{\cdot}07$, i.e. $q \simeq 0{\cdot}56$, and since $4q = \omega^2 h^2/c^2$ and $h = ae$

$$\omega_1^2 = \frac{4q_{11}c^2}{a^2e^2}$$

Therefore $\omega_1 \simeq 1{\cdot}87\, c/a$, which gives the pulsatance of the water waves mainly parallel to the direction of the major axis. From Bessel function tables, the lowest root for a circular lake is given by $\omega_1 a/c \simeq 1{\cdot}84$ or $\omega_1 = 1{\cdot}84\, c/a$

Thus we see that the pulsatance of this mode for a lake of eccentricity $e = 0{\cdot}8$ is only slightly different from the same for a circular lake.

Let $\zeta \propto Se_1(\xi, \bar{q}_{11})\, se_1(\eta, \bar{q}_{11})$, in which case the oscillations are across the major axis.

$$Se_1'(\xi_0; \bar{q}) \simeq \bar{\alpha}^3\left(\frac{7}{18}\cosh 7\xi_0 + \frac{20}{9}\cosh 5\xi_0 + \cosh 3\xi_0\right) + \bar{\alpha}^2\left(\frac{5}{3}\cosh 5\xi_0 + 3\cosh 3\xi_0\right) - 3\bar{\alpha}\,(\cosh 3\xi_0) + \cosh \xi_0$$

where $\bar{\alpha} = \frac{1}{8}\bar{q}$

Considering an elliptic boundary of eccentricity $e = 0{\cdot}8$ the equation $Se_1'(\xi_0, \bar{q}) = 0$ reduces to a cubic in $\bar{\alpha}$, whose smallest positive root $\bar{\alpha} \simeq .175$, so that $\bar{q}_{11} \simeq 1{\cdot}4$.

Hence, since $\bar{\omega}_1^{\,2} = \frac{4\bar{q}_{11}c^2}{a^2e^2}$, $\bar{\omega}_1 = 2{\cdot}96\frac{c}{a} = 1{\cdot}78\frac{c}{b}$

Comparing $\bar{\omega}_1$, and ω_1,

$$\frac{\bar{\omega}_1}{\omega_1} = \frac{2{\cdot}96}{1{\cdot}87} = 1{\cdot}58$$

which shows that the pulsatance across the major axis is about 40 per cent more than that parallel to the major axis.

When the eccentricity of the lake boundary tends to unity, the lake is a long narrow ellipse. Now, since $\cosh \xi_0 = 1/e$, ξ_0 has values very near zero and $\sinh \xi_0 \simeq \xi_0$.

If $\zeta \simeq Ce_1(\xi, q_{11})\, ce_1(\eta, q_{11})$, the equation $Ce_1'(\xi_0, q) = 0$ reduces to $5{\cdot}39\alpha^3 - 0{\cdot}667\alpha^2 - 9\alpha + 1 = 0$, where $\alpha = 1/8q$.

The smallest positive root of this equation is

$\alpha \simeq 0 \cdot 111$, so $q_{11} = 0 \cdot 888$.

Hence as $e \to 1$, we get $\omega_1 = 1 \cdot 886\, c/a$ which shows that the effect of elongating an ellipse on the frequency for the lowest mode is quite negligible.

3.3 In the paper by Chu [6], electro-magnetic waves in a perfectly conducting elliptic metal pipe have been discussed. The values of the critical cut-off wavelengths have been calculated for the electrical field E waves and the magnetic field H-waves.

For an electro-magnetic wave, the Maxwell's equations have to be solved, assuming a boundary condition that the tangential component of the electrical field in air must vanish. It is further assumed that the metal pipe has perfect conductivity. With the help of these equations, the wave equations for E_z and H_z components of the electric and magnetic field respectively are obtained. As the wave is propagating in an elliptic guide, both E_z and H_z satisfy the Mathieu equation (2.5) and the modified Mathieu equation (2.6).

Hence their forms are given by

$$\left.\begin{array}{l} E_z \\ \\ H_z \end{array}\right\} = \left.\begin{array}{l} C_m \, Ce_m(\xi, q)\, ce_m(\eta, q) \\ \\ S_m \, Se_m(\xi, q)\, se_m(\eta, q) \end{array}\right\} e^{i(\omega t - k_z z)} \qquad \begin{array}{r} (3.5) \\ \\ (3.6) \end{array}$$

m th order of the functions being considered.

If the longitudinal component of the electric field is zero then $E_z = 0$ and the propagating wave is called an H-wave. So the components of the field in this case are given by

H_z = as in equation (3.6)

$H_\xi = \frac{\beta}{\mu\omega} E_\eta$ (μ being the permeability of the medium)

$H_\eta = \frac{\beta}{\mu\omega} E_\xi$

$E_z = 0$

$$E_\xi = \frac{\mu\omega}{k_1^2 h (\cosh 2\xi - \cos 2\eta)^{1/2}} \begin{Bmatrix} C_m \, Ce_m(\xi, q)\, ce_m'(\eta, q) \\ S_m \, Se_m(\xi, q)\, se_m'(\eta, q) \end{Bmatrix} \times e^{i(\omega t - k_z z)}$$

$(k_1 h = 2\sqrt{q})$

$$E_\eta = \frac{\mu\omega}{k_1^2 h (\cosh 2\xi - \cos 2\eta)^{1/2}} \begin{Bmatrix} C_m \, Ce_m'(\xi, q)\, ce_m(\eta, q) \\ S_m \, Se_m'(\xi, q)\, se_m(\eta, q) \end{Bmatrix} \times e^{i(\omega t - k_z z)}$$

Since the tangential component of the electric field E_η must vanish on the boundary $\xi = \xi_0$,

$$Ce_m'(\xi_0, q) = 0$$

$$Se_m'(\xi_0, q) = 0$$

These equations determine values q_{m0}, $\bar{q}_{m0}$ of q which give the critical cut-off frequencies of the m th order wave. It is particularly interesting to note that the sound waves in a duct behave very similarly to these H-waves.

If the longitudinal component of the magnetic field vanishes while both the transverse and the longitudinal components of the electric field exist, this wave is called an E-wave. In this case $H_z = 0$, and E_z is given by equation (3.5). These waves satisfy the boundary conditions $Ce_m(\xi_0, q) = 0$ and $Se_m(\xi_0, q) = 0$.

The wavelengths λ_{mo} for cut-off condition are given by the relation $k_{mo} = 2\pi/\lambda_{mo}$, where $k_{mo} = 2 \times \sqrt{q_{mo}}/h$ or $2\sqrt{\bar{q}_{mo}}/h$ depending on whether even or odd function is considered.

Fig. 3 represents the ratios λ_0/p as functions of the eccentricity e of the ellipses; λ_0 are cut-off wavelengths for orders 0 and 1, p is the periphery of the pipe given by

$$p = h\cosh\xi_0 \int_0^{2\pi} \left\{1 - (\cos\eta/\cosh\xi_0)^2\right\}^{1/2} d\eta$$

3.4 The above gave an idea of the behaviour of the waves when a circular pipe was deformed into an elliptic one. The values of the cut-off frequencies were found not independent of the periphery of the ellipse. In what follows [7], the ellipse is so deformed that the area of the cross-section remains constant, while the eccentricity changes. Let this constant area be π , which is also the area of a circle of unit radius.

An important property [14] connected with the reduced wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + k^2 p = 0 \qquad (3.7)$$

for a region R is given below. The cut-off frequencies for ellipses of different sizes but same eccentricity can be scaled from this property.

If the equation (3.7) is valid under the boundary condition $\partial p / \partial n = 0$, on the boundary of R, then there is an infinity of values of k, k_i $(i = 0, 1, 2, \cdots)$ such that

$$0 = k_0 < k_1 \leq k_2 \leq \cdots$$

which satisfy the equation (3.7). These k_i are known as the eigen values of equation (3.7).

If A is the area of the region R, $\bar{A}$ is the area of another region $\bar{R}$ which is similar to R and if $\bar{k}_i$ are the values of k satisfying equation (3.7) for the region $\bar{R}$, then

$$\frac{\bar{k}_i^2}{k_i^2} = \frac{A}{\bar{A}} \qquad (3.8)$$

Hence, the eigen values $\bar{k}_i$ corresponding to any other elliptical boundary with the same eccentricity e can be found with the help of equation (3.8), once the eigen values corresponding to an ellipse, with area π and eccentricity e, have been found.

Consider an ellipse of eccentricity e and area π. The reduced wave equation (3.7) when transformed to elliptic coordinates gives

$$\frac{d^2 p_\eta}{d\eta^2} + (a - 2q \cos 2\eta) p_\eta = 0$$

$$\frac{d^2 p_\xi}{d^2 \xi^2} - (a - 2q \cosh 2\xi) p_\xi = 0$$

where

$$p = p_\xi(\xi)\, p_\eta(\eta) \quad \text{and} \quad k^2 = 4q/h^2$$

Since the area of the ellipse is π , $h^2 = e^2(1-e^2)^{-1/2}$

Let $s = 4q = k^2 h^2 = k^2 e^2 (1-e^2)^{-1/2}$ (3.9)

If $\xi = \xi_0$ on the boundary, then $\cosh \xi_0 = 1/e$

If $u = \sqrt{q}\, \exp(\xi_0)$ and $v = \sqrt{q}\, \exp(-\xi_0)$

$$\omega_0 = u + v = 2\sqrt{q} \cosh \xi_0 = \sqrt{s} \cosh \xi_0$$

Hence $\omega_0^2 = s \cosh^2 \xi_0 = s/e^2$ (3.10)

and since $s = k^2 e^2 (1-e^2)^{-1/2}$ (from (3.9))

$$\omega_0^2 = k^2 (1-e^2)^{-1/2} \qquad (3.11)$$

The boundary conditions reduce to:

$$Ce_m'(\xi_0, q) = 0 \quad \text{and} \quad Se_{2m+1}'(\xi_0, q) = 0, \; m = 0, 1, 2, \ldots$$

Determining the parametric zeros by direct evaluation is not possible in practice. The method used to obtain the values of $k\,(= 2\sqrt{q}/h)$ is given in detail in Chapter V. In Daymond's work [7] the principal frequencies have been found. It shall be discussed how the cut-off frequencies have been found for higher orders. The values calculated for the lowest order agree with those found in Daymond.

4. MATHIEU FUNCTIONS AND EXPANSIONS IN SERIES.

4.1

$$\frac{d^2 z}{d\eta^2} + (a - 2q\cos 2\eta) z = 0 \qquad (4.1)$$

$$\frac{d^2 z}{d\xi^2} - (a - 2q\cosh 2\xi) z = 0 \qquad (4.2)$$

are Mathieu equation and modified Mathieu equation respectively.

The solutions of equation (4.1) can be classified into four categories as follows:

(i) $ce_{2n}(\eta, q)$, even functions with period π

(ii) $ce_{2n+1}(\eta, q)$, even functions with period 2π

(iii) $se_{2n+2}(\eta, q)$, odd functions with period π

(iv) $se_{2n+1}(\eta, q)$, odd functions with period 2π

(Only periodic functions occur in the pressure function and hence they are considered here).

Functions Ce_{2n}, Ce_{2n+1}, Se_{2n+2}, Se_{2n+1} respectively correspond to the above functions and are solutions of equation (4.2). These functions have imaginary periods.

The functions ce_m and se_m can be expanded in Fourier series

$$ce_{2n}(\eta, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2\eta r \qquad (a_{2n}) \quad (4.3)\ \text{(i)}$$

$$ce_{2n+1}(\eta, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)\eta \qquad (a_{2n+1}) \qquad (4.3)\ \text{(ii)}$$

$$se_{2n+2}(\eta, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)\eta \qquad (b_{2n+2}) \qquad (4.3)\ \text{(iii)}$$

$$se_{2n+1}(\eta, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)\eta \qquad (b_{2n+1}) \qquad (4.3)\ \text{(iv)}$$

where the coefficients A,B are functions of q , and so are the characteristic numbers a, b. The functions in (4.3) are normalised such that

$$\frac{1}{\pi}\int_0^{2\pi} ce_m^2(\eta, q)\, d\eta = \frac{1}{\pi}\int_0^{2\pi} se_m^2(\eta, q)\, d\eta = 1 \qquad (4.4)$$

for all q . From (4.3) and the orthogonal property of the circular functions, (4.4) gives

$$2\left[A_0^{(2n)}\right]^2 + \sum_{r=1}^{\infty}\left[A_{2r}^{(2n)}\right]^2 = 1 = \sum_{r=0}^{\infty}\left[A_{2r+1}^{(2n+1)}\right]^2$$
$$= \sum_{r=0}^{\infty}\left[B_{2r+2}^{(2n+2)}\right]^2 = \sum_{r=0}^{\infty}\left[B_{2r+1}^{(2n+1)}\right]^2 \qquad (4.5)$$

The constant term in the expansion of $ce_0(\eta, q)$ is $2^{-1/2}$, for if $q = 0$, the equation (4.4) would become

$$\frac{1}{\pi}\int_0^{2\pi} d\eta = 2$$

If $\xi = i\eta$, then the solutions of (4.2) can be written as

$$Ce_{2n}(\xi, q) = ce_{2n}(i\eta, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2r\xi \qquad (a_{2n})\ \text{(i)}$$

$$Ce_{2n+1}(\xi,q) = ce_{2n+1}(i\eta,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh(2r+1)\xi \quad (a_{2n+1}) \quad \text{(ii)}$$

$$Se_{2n+2}(\xi,q) = se_{2n+2}(i\eta,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh(2r+2)\xi \quad (b_{2n+2}) \quad \text{(iii)}$$

$$Se_{2n+1}(\xi,q) = se_{2n+1}(i\eta,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh(2r+1)\xi \quad (b_{2n+1}) \quad \text{(iv)} \qquad (4.6)$$

The A, B, a and b being the same as in (4.3) for the same q.

4.2 At this point, another method of normalisation of the solutions of equation (4.1) is given. [22] (This has been used by most American authors). The solutions in this case are Se_r, So_r (not to be confused with Se_r, solutions of (4.2)) normalised by the conditions

$$Se_r(s,0) = 0 \quad \text{and} \quad \frac{d}{d\eta}\left[So_r(s,\eta)\right] = 1$$

where $s = 4q$, q is the same as in equation (4.1).

The expansions of Se_r and So_r are given respectively by

$$Se_{2n+p}(s,\eta) = \sum_{r=0}^{\infty} De_{2r+p}^{(2n+p)} \cos(2r+p)\eta \qquad (be)_{2n+p}$$

and

$$So_{2n+p}(s,\eta) = \sum_{r=0}^{\infty} Do_{2r+p}^{(2n+p)} \sin(2r+p)\eta \qquad (bo)_{2n+p}$$

where $p = 0$ or 1 according as the function under consideration is of period π or 2π and $be_r = a_r + 2q$, $bo_r = b_r + 2q$,

a_r, b_r being the even and odd characteristic numbers in (4.3) respectively.

The normalisation condition gives

$$\sum_{r=0}^{\infty} De_{2r+p} = 1 \ , \quad \sum_{r=0}^{\infty} (2r+p) Do_{2r+p} = 1 \ .$$

The relationship between the coefficients here and the coefficients in (4.3) is given by

$$De^{(m)}_{2r+p} = A^{(m)}_{2r+p} / A$$

and

$$Do^{(m)}_{2r+p} = B^{(m)}_{2r+p} / B$$

where

$$\frac{1}{A^2} = \left[2\left(De^{(m)}_0\right)^2 + \left(De^{(m)}_2\right)^2 + \cdots \right] \text{ or } \left[\left(De^{(m)}_1\right)^2 + \left(De^{(m)}_3\right)^2 + \cdots \right]$$

and

$$\frac{1}{B^2} = \sum_{r=0}^{\infty} \left(Do^{(m)}_{2r+p}\right)^2$$

4.3 On substituting the value of ce_{2n} from (4.3) in the Mathieu equation (4.1) and equating to zero the coefficients of $\cos 2r\eta$, the following recurrence relation is obtained[19]

$$\begin{aligned} &a A_0 - q A_2 = 0 \\ &-2q A_0 + (a-4) A_2 - q A_4 = 0 \\ &-q A_{2r-2} + (a - 4r^2) A_{2r} - q A_{2r+2} = 0 \end{aligned} \qquad (4.7)$$

Let A_{2r} decrease to zero as r becomes very large compared to the order of the function.

If $v_{2r} = A_{2r+2} / A_{2r}$, the third of the equations (4.7) gives

$$-\frac{q}{v_{2r-2}} + (a - 4r^2) - q v_{2r} = 0 \qquad \left(\text{As } \frac{A_{2r-2}}{A_{2r}} = \frac{1}{v_{2r-2}}\right)$$

Therefore

$$(a - 4r^2) v_{2r-2} - q v_{2r} v_{2r-2} - q = 0$$

Hence

$$v_{2r} + \frac{1}{v_{2r-2}} = \frac{a - 4r^2}{q}$$

If $(r+1)$ is written for r,

$$v_{2r+2} + \frac{1}{v_{2r}} = \frac{a - 4(r+1)^2}{q}$$

So, when $r \to \infty$, $v_{2r+2} + \dfrac{1}{v_{2r}} \to -\infty$

Thus v_{2r} cannot have a unique finite limit other than zero.

Hence

$$v_{2r} \to 0 \quad \text{or} \quad |v_{2r}| \to \infty \quad \text{as } r \to \infty$$

As the coefficients A_{2r} are so found that $A_{2r} \to 0$ as $r \to \infty$, v_{2r} too must approach zero. Hence one of the solutions of the third of the equations (4.7) tends to zero when $r \to \infty$ while the other tends to ∞. But we cannot have a solution tending to ∞. Hence when r is very large $v_{2r+2}\, v_{2r} \ll 1$ and $a \ll 4(r+1)^2$. With $a,\ q$ finite

$$|v_{2r}| \sim \frac{q}{4(r+1)^2} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

So for a real η as $|\cos 2r\eta| \leq 1$, $ce_{2n}(\eta, q)$ is absolutely convergent. Since for any η in a given closed interval, it is possible to find an M independent of η such that

$$\left| \frac{A_{2r} \cos 2r\eta}{A_{2r-2} \cos (2r-2)\eta} \right| \sim \frac{q}{4(r+1)^2} < M$$

the series is uniformly convergent. Thus the expansion (4.3) for ce_{2n} represents a continuous function for all real η.

When η is not real (i.e. when $\eta = i\xi$)

$$\left| \frac{u_{r+1}}{u_r} \right| = \left| \frac{A_{2(r+1)}}{A_{2r}} \quad \frac{\cos 2(r+1)\eta}{\cos 2r\eta} \right| = |v_{2r}| \left| \frac{\cosh 2(r+1)\xi}{\cosh 2r\xi} \right|$$

$$\simeq \frac{q}{4(r+1)^2}\, e^{2|\xi|} \rightarrow 0 \text{ as } r \rightarrow \infty$$

so long as $|\xi|$ remains finite, i.e. ξ is finite. From the above, it follows that the fourier-series expansion for Ce_{2n} is absolutely and uniformly convergent for any finite ξ. On the same lines of proof as above, taking the relevant difference equations, it can be proved that the series expansions (4.6) for Ce_{2n+1}, Se_{2n+2}, Se_{2n+1} are all absolutely and uniformly convergent. Since the expansions are uniformly convergent for a given q and for any finite ξ, the series can be differentiated term by term and the series expansions so obtained for the derivations are absolutely and uniformly convergent for any finite ξ.

4.4 It will now be shown that the solutions of equation (4.2) can be expressed as a rapidly converging Bessel function series.[9, 19]

If $\varsigma = 2k\cosh\xi$, where $k^2 = q > 0$, then the equation (4.2) becomes

$$(\varsigma^2 - 4k^2)\frac{d^2 z}{d\varsigma^2} + \varsigma\frac{dz}{d\varsigma} + \left\{\varsigma^2 - (a+2q)\right\} z = 0 \qquad (4.8)$$

Suppose

$$z = \sum_{m=0}^{\infty} (-1)^m C_{2m} J_{2m}(\varsigma) \qquad (4.9)$$

is a solution of the above equation (4.8). Substituting the value of z from (4.9) into (4.8), we obtain

$$\sum_{m=0}^{\infty} (-1)^m C_{2m}\left[(\varsigma^2 - 4k^2) J_{2m}'' + \varsigma J_{2m}' + (\varsigma^2 - p^2) J_{2m}\right] = 0 \qquad (4.10)$$

where

$$p^2 = a + 2q = a + 2k^2$$

Since J_{2m} are solutions of Bessel's equation,

$$\rho^2 \frac{d^2 z}{d\rho^2} + \rho \frac{dz}{d\rho} - (4m^2 - \rho^2) z = 0,$$

$$\rho^2 J_{2m}'' + \rho J_{2m}' + \rho^2 J_{2m} = 4m^2 J_{2m} \qquad (4.11)$$

Also

$$4 J_{2m}'' = J_{2m-2} - 2 J_{2m} + J_{2m+2} \qquad (4.12)$$

from the recurrence relationship between Bessel function and its derivative.

So, from (4.11), (4.12), the equation (4.10) becomes

$$\sum_{m=0}^{\infty} (-1)^m c_{2m} \left[(4m^2 - p^2) J_{2m} - k^2 (J_{2m-2} - 2J_{2m} + J_{2m+2}) \right] = 0 \qquad (4.13)$$

or $$\sum_{m=0}^{\infty} (-1)^m c_{2m} \left[(4m^2 - a) J_{2m} - k^2 (J_{2m-2} + J_{2m+2}) \right] = 0 \qquad (4.14)$$

Equating the coefficients of J_{2m} to zero (m = 0,1,2, ...)

$$\left. \begin{array}{l} a c_0 - k^2 c_2 = 0 \\ (a-4) c_2 - k^2 (c_4 + 2c_0) = 0 \\ (a - 4m^2) c_{2m} - k^2 (c_{2m+2} + c_{2m-2}) = 0, \quad m \geq 2 \end{array} \right\} \qquad (4.15)$$

Comparing equations (4.15) and equations (4.7), they are found to be the same, with A_m in (4.7) replaced by c_m.

Hence C_{2m} are proportional to A_{2m}.

$$Ce_{2n}(\xi, q) = \sum_{m=0}^{\infty} A_{2m}^{(2n)} \cosh 2m\xi$$

C_{2m} are proportional to A_{2m} and it can be noticed that $\cosh 2m\xi$ and $J_{2m}(\cosh 2m\xi)$ are both periodic with period πi

Hence $K \sum_{m=0}^{\infty} (-1)^m A_{2m} J_{2m} \; (2k \cosh \xi)$ are solutions of the equation (4.2)

If $\xi = \frac{\pi i}{2}$, $J_{2m}(0) = 0$ for all m, except $J_0(0)$ which is = 1

and

$$Ce_{2n}\left(\frac{\pi i}{2}, q\right) = ce_{2n}\left(\frac{\pi}{2}, q\right) / A_0^{(2n)}$$

Therefore

$$Ce_{2n}(\xi, q) = \frac{Ce_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{m=0}^{\infty} (-1)^m A_{2m}^{(2n)} J_{2m}(2k\cosh\xi) \quad (4.16)$$

Following arguments similar to above, we obtain the series solutions for Ce_{2n+1}, Se_{2n+2}, Se_{2n+1} as follows:

$$Ce_{2n+1}(\xi, q) = \frac{ce'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k\, A_1^{(2n+1)}} \sum_{m=0}^{\infty} (-1)^m A_{2m+1}^{(2n+1)} J_{2m+1}(2k\cosh\xi) \quad (4.17)$$

$$Se_{2n+2}(\xi, q) = -\frac{se'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \tanh\xi \times \sum_{m=0}^{\infty} (-1)^m (2m+2) B_{2m+2}^{(2n+2)} J_{2m+2}(2k\cosh\xi) \quad (4.18)$$

and

$$Se_{2n+1}(\xi, q) = \frac{se_{2n+1}\left(\frac{\pi}{2}, q\right)}{k\, B_1^{(2n+1)}} \tanh\xi \times \sum_{m=0}^{\infty} (-1)^m (2m+1) B_{2m+1}^{(2n+1)} J_{2m+1}(2k\cosh\xi) \quad (4.19)$$

4.5 Here a new set of functions are introduced, which are respectively proportional to functions (4.16) - (4.19), but are normalised differently.

$$Je_{2n+p}(\xi,\eta) = \sqrt{\frac{\pi}{2}}\,(-1)^n \sum_{m=0}^{\infty} (-1)^m De_{2m+p}\, J_{2m+p}(2k\cosh\xi),$$

$$p = 0 \text{ or } 1. \tag{4.20}$$

$$Jo_{2n+p}(\xi,\eta) = \sqrt{\frac{\pi}{2}}\,(-1)^n \tanh\xi \sum_{m=0}^{\infty} (-1)^m (2m+p) \times$$

$$Do_{2m+p}\, J_{2m+p}(2k\cosh\xi) \tag{4.21}$$

As in Article 4.4, it can be proved, by direct substitution in equation (4.2), that expansion (4.20) and (4.21) are solutions of the equation (4.2) and again as these are periodic functions with period πi or $2\pi i$ corresponding to the functions with the same period, it follows that Je_{2n} are proportional to Ce_{2n}, Jo_{2n+1} to Se_{2n+1} Je_{2n+1} to Ce_{2n+1} and Jo_{2n+2} to So_{2n+2}.

The Je and Jo have been used in the computations in this work, as they are more convenient than the Ce and Se .

The convergence of the above series is discussed at the end of this chapter.

4.6 Consider the reduced wave equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + k_1^2 G = 0 \tag{4.22}$$

From Chapter II, it is known that this equation takes the form

$$\frac{\partial^2 G}{\partial \chi^2} + \frac{\partial^2 G}{\partial \varphi^2} + 2q(\cosh 2\chi - \cos 2\varphi)G = 0 \tag{4.23}$$

where $k, h = 2\sqrt{q}$, when the cartesian coordinates (x,y) are transformed to elliptic coordinates (χ, ς).

If in the above, $\chi = i\xi$, so that $x = h\cos\xi\cos\varsigma$, $y = ih\sin\xi\sin\varsigma$ (these are the modified elliptic coordinates) equation (4.23) takes the form

$$\frac{\partial^2 G}{\partial \xi^2} - \frac{\partial^2 G}{\partial \varsigma^2} + 2q(\cos 2\xi - \cos 2\varsigma)G = 0 \tag{4.24}$$

It shall now be proved that any solution of the above equation (4.24) forms a nucleus for an integral equation involving a solution of the Mathieu's equation.[2,19] The main aim of this theorem is to obtain a series expansion for the modified Mathieu function in a product of Bessel functions.

4.7 If (i) $\phi(\xi)$ is a function satisfying the Mathieu's equation for a particular value of a and q,

(ii) $G(\xi, \varsigma)$ is an analytic function of ξ, ς (ξ belongs to a region X in the complex ξ-plane, ς belongs to a region Z in the complex ς-plane) satisfying equation (4.24)

and (iii) (a) the values of the expression $\left[G(\xi,\varsigma) \times \frac{d}{d\varsigma}\phi(\varsigma) - \frac{\partial G(\xi,\varsigma)}{\partial \varsigma}\phi(\varsigma)\right]$ at the ends of a path C of integration, lying wholly within the Z-region of the ς-plane, are the same

and also (b) $\int_C G(\xi, \varsigma)\phi(\varsigma)\,d\varsigma$ exists for all ξ in X, and the integral converges uniformly in ξ for all ξ in X, if it is singular;

then
$$x(\xi) = \int_C G(\xi,\zeta)\phi(\zeta)d\zeta$$
satisfies the Mathieu's equation for all ξ in X for the same a and q for which $\phi(\xi)$ satisfies Mathieu's equation.

Now the proof.

Because of the hypothesis (iii) (b), the integral $\int_C G(\xi,\zeta)\phi(\zeta)d\zeta$ can be differentiated under the integral sign.

Hence
$$\frac{d^2x}{d\xi^2} = \int \frac{\partial^2 G}{\partial \xi^2}(\xi,\zeta)\,\phi(\zeta)d\zeta$$

Therefore
$$\frac{d^2x}{d\xi^2} + (a-2q\cos 2\xi)x = \int_C \left\{\frac{\partial^2 G}{\partial \xi^2} + (a-2q\cos 2\xi)G\right\} \times \phi(\zeta)d\zeta,$$
$$= \int \left\{\frac{\partial^2 G}{\partial \zeta^2} + (a-2q\cos 2\zeta)G\right\}\phi(\zeta)d\zeta \qquad (4.25)$$

(from hypothesis (ii))

Integrating $\int_C \frac{\partial^2 G}{\partial \zeta^2}\phi(\zeta)d\zeta$ by parts, we obtain
$$\int_C \frac{\partial^2 G}{\partial \zeta^2}\phi(\zeta)d\zeta = \left[\frac{\partial G}{\partial \zeta}\phi(\zeta)\right]_C - \int_C \frac{\partial G}{\partial \zeta}\phi'(\zeta)d\zeta$$
$$= \left[\frac{\partial G}{\partial \zeta}\phi(\zeta)\right]_C - \left[G\phi'(\zeta)\right]_C + \int_C G\phi''(\zeta)d\zeta$$
$$= \left[\phi(\zeta)\frac{\partial G}{\partial \zeta} - G\frac{d\phi}{d\zeta}\right]_C + \int_C G\phi''(\zeta)d\zeta$$
$$= \int_C \frac{d^2\phi}{d\zeta^2}G\,d\zeta \qquad (4.26)$$

(due to hypothesis (iii) (a))

So
$$\frac{d^2x}{d\xi^2} + (a-2q\cos 2\xi)x = \int_C \left[\frac{d^2\phi}{d\zeta^2} + (a-2q\cos 2\zeta)\phi\right]G\,d\zeta$$
(from (4.25) and (4.26))

By hypothesis (i), ϕ satisfies Mathieu's equation hence the integral on the right vanishes and

$$\frac{d^2 x}{d\xi^2} + (a - 2q \cos 2\xi) x = 0$$

which shows that x satisfies the Mathieu's equation.
From above it follows that:

If $\phi(\xi)$ is a basically periodic solution of Mathieu's equation, $G(\xi,\varsigma)$ is symmetric in ξ, ς, with the same period as $\phi(\xi)$, and the integration is over any finite period of $\phi(\varsigma)$, then $\phi(\xi)$ satisfies the integral equation

$$\phi(\xi) = \lambda \int_C G(\xi, \varsigma) \phi(\varsigma) d\varsigma \qquad (4.27)$$

where the integral is not improper.
As the integrand is basically periodic in ξ, the integral itself is basically periodic in ξ. By the theorem above, it satisfies the same Mathieu equation as $\phi(\xi)$. Since there is only one basically periodic solution of this equation, the integral itself is a multiple of $\phi(\xi)$. Hence the equation (4.27). It may happen that the integral itself is zero, in which case λ can be taken to be infinite and the result holds.

4.8 Thus we know that any basically periodic solution of equation (4.24) in modified elliptic coordinates can be the nucleus of the integral equation (4.27).

Consider a solution of the reduced wave equation (4.22) in cylindrical polar coordinates.

Now $x = \rho\cos\theta$ and $y = \rho\sin\theta$,

which reduces the equation (4.22) to two second order linear equations

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k_1^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

and $\frac{d^2T}{d\theta^2} + \nu^2 T = 0$ (ν a positive integer)

(see Chapter I)

where $G = R(\rho)\,T(\theta)$

Hence the solution of equation (4.22) can be given as

$$\left.\begin{matrix} J_\nu(k_1\rho) \\ Y_\nu(k_1\rho) \end{matrix}\right\} \begin{matrix} \cos(\nu\theta) \\ \sin(\nu\theta) \end{matrix}$$

Since $x = h\cos\xi\cos\varsigma$, $y = i\,h\sin\xi\sin\varsigma$

$$\rho = h\left[\tfrac{1}{2}(\cos 2\xi + \cos 2\varsigma)\right]^{1/2} \tag{4.28}$$

and since $k_1 h = 2\sqrt{q}$,

$$k_1\rho = \left[2q\,(\cos 2\xi + \cos 2\varsigma)\right]^{1/2}$$

The expressions for $\cos\theta$ and $\sin\theta$ are given as follows:

$$\cos\theta = \frac{\cos\xi\cos\varsigma}{\left[\tfrac{1}{2}(\cos 2\xi + \cos 2\varsigma)\right]^{1/2}} \quad \left(= \frac{x}{\rho}\right)$$

$$\sin\theta = \frac{i\sin\xi\sin\varsigma}{\left[\tfrac{1}{2}(\cos 2\xi + \cos 2\varsigma)\right]^{1/2}} \quad \left(= \frac{y}{\rho}\right)$$

DeMoivre's therem gives $\cos\nu\theta + i\sin\nu\theta = (\cos\theta + i\sin\theta)^{\nu}$

Hence equating the real and imaginary parts and substituting the values of cos θ and sin θ , the expressions for cos $\nu\theta$ and sin $\nu\theta$ are obtained.

Consider the expression

$$\int_0^{\pi} J_{2\nu}(k_1\rho)\cos 2\nu\theta \, ce_{2\nu}(\varsigma,q)\,d\varsigma$$

Obviously $ce_{2\nu}(\varsigma,q)$ satisfies condition (i) Article 4.7. $J_{2\nu}(k_1\rho)$ is symmetric in ξ and ς (since ρ is symmetric in ξ and ς) and since cos $\nu\theta$ = $\cos\left\{\nu\cos^{-1}\left(\frac{\sqrt{2}\cos\xi\cos\varsigma}{(\cos 2\xi+\cos 2\varsigma)^{1/2}}\right)\right\}$, so is cos $\nu\theta$ and for that matter, so is sin $\nu\theta$.

Hence $J_{2\nu}(k_1\rho)\cos 2\nu\theta$ satisfies condition (ii) Article 4.7.

Since ρ, θ are periodic in ξ with period πi , so is $J_{2\nu}(k_1\rho)\cos\nu\theta$.

Hence condition (iii) (a) of article 4.7 is satisfied and so is condition (iii) (b), due to the nature of the function $J_{2\nu}(k_1\rho)\cos 2\nu\theta$. As a result of Article 4.7, ce_{2n} , therefore satisfies the integral equation

$$ce_{2n}(\xi,q) = \lambda\int_0^{\pi} J_{2\nu}(k_1\rho)\cos 2\nu\theta \, ce_{2n}(\varsigma,q)\,d\varsigma \qquad (4.29)$$

4.9 In the equation (4.29) let $\nu = 0$.

Then that equation can be written as

$$ce_{2n}(\xi,q) = \lambda\int_0^{\pi} J_0\left[\left\{2q(\cos 2\xi+\cos 2\varsigma)\right\}^{1/2}\right] ce_{2n}(\varsigma,q)\,d\varsigma \qquad (4.30)$$

Consider the addition formula for Bessel function of the first kind and order zero.

$$J_0\left[(x^2+y^2+2xy\cos\theta)^{1/2}\right] = J_0(x)J_0(y)+2\sum_{r=1}^{\infty}(-1)^r J_r(x)J_r(y)\cos r\theta \qquad (4.31)$$

Let $x = ke^{i\xi}$, $y = ke^{-i\xi}$, then

$$J_0\left[(x^2+y^2+2xy\cos\theta)^{1/2}\right] = J_0\left[(k^2e^{2i\xi}+k^2e^{-2i\xi}+2k^2\cos\theta)^{1/2}\right]$$

$$= J_0\left[\left\{2k^2(\cos 2\xi+\cos\theta)\right\}^{1/2}\right]$$

If $\theta = 2\varsigma$ and $k^2 = q$

$$J_0\left[(x^2+y^2+2xy\cos\theta)^{1/2}\right] = J_0\left[\left\{2q(\cos 2\xi+\cos 2\varsigma)\right\}^{1/2}\right]$$

Therefore applying formula (4.31),

$$J_0\left[\left\{2q(\cos 2\xi+\cos 2\varsigma)\right\}^{1/2}\right] = J_0(ke^{i\xi})J_0(ke^{-i\xi})+2\sum_{r=1}^{\infty}(-1)^r J_r(ke^{i\xi})J_r(ke^{-i\xi})\cos 2r\varsigma$$

If $\xi = 0$,

$$J_0(2k\cos\varsigma) = J_0^2(k)+2\sum_{r=1}^{\infty}(-1)^r J_r^2(k)\cos 2r\varsigma$$

Therefore, from the integral equation (4.29)

$$ce_{2n}(0,q) = \lambda_{2n}\int_0^{\pi}\left[J_0^2(k)+2\sum_{r=1}^{\infty}(-1)^r J_r^2(k)\cos 2r\varsigma\right]\times\sum_{m=0}^{\infty}A_{2m}^{(2n)}\cos 2m\varsigma\, d\varsigma$$

Orthogonality of the circular functions gives

$$\int_0^{\pi}\cos 2r\varsigma\cos 2m\varsigma\, d\varsigma = \begin{cases} 0, & r \neq m \\ \pi/2, & r = m \end{cases} \qquad (4.32)$$

and

$$\int_0^{\pi} \cos 2m\xi \, d\xi = 0 \tag{4.33}$$

Hence

$$\int_0^{\pi}\left[J_0^2(k) + 2\sum_{r=1}^{\infty}(-1)^r J_r^2(k)\cos 2r\xi\right]\cos 2m\xi \, d\xi = (-1)^m \pi J_m^2(k)$$

Therefore

$$ce_{2n}(0,q) = \lambda_{2n}\pi \sum_{m=0}^{\infty}(-1)^m A_{2m}^{(2n)} J_m^2(k)$$

and so $\lambda_{2n} = \dfrac{ce_{2n}(0,q)}{\pi \sum_{m=0}^{\infty}(-1)^m A_{2m}^{(2n)} J_m^2(k)}$ where $q=k^2$ (4.34)

A simpler expression for λ_{2n} can be obtained as follows: If in the expansion (4.16) for ce_{2n}, we write 2ξ for ξ, it becomes

$$ce_{2n}(\xi,q) = \frac{ce_{2n}(\frac{\pi}{2},q)}{A_0^{(2n)}} \sum_{m=0}^{\infty}(-1)^m A_{2m}^{(2n)} J_{2m}(2k\cos\xi)$$

$$ce_{2n}(\xi,q) = \sum_{m=0}^{\infty} A_{2m}^{(2n)} \cos 2m\xi$$

Therefore

$$\sum_{m=0}^{\infty} A_{2m}^{(2n)} \cos 2m\xi = \frac{ce_{2n}(\frac{\pi}{2},q)}{A_0^{(2n)}} \sum_{m=0}^{\infty}(-1)^m A_{2m}^{(2n)} J_{2m}(2k\cos\xi)$$

As the series on both sides of the above equation are uniformly convergent (see articles 4.3, 4.12), term by term integration is permissible. Integrating both sides with respect to ξ from 0 to π, we get

$$\int_0^{\pi}\sum_{m=0}^{\infty} A_{2m}^{(2n)} \cos 2m\xi \, d\xi = \frac{ce_{2n}(\frac{\pi}{2},q)}{A_0^{(2n)}} \int_0^{\pi}\sum_{m=0}^{\infty}(-1)^m A_{2m}^{(2n)} J_{2m}(2k\cos\xi)\, d\xi$$

or

$$\pi A_0^{(2n)} = \frac{ce_{2n}(\frac{\pi}{2},q)}{A_0^{(2n)}} \pi \sum_{m=0}^{\infty}(-1)^m A_{2m}^{(2n)} J_m^2(k)$$

Therefore

$$\sum_{m=0}^{\infty} (-1)^m A_{2m}^{(2n)} J_m^2(k) = \frac{\left[A_0^{(2n)}\right]^2}{ce_{2n}(\frac{\pi}{2},q)}$$

Hence

$$\lambda_{2n} = \frac{ce_{2n}(0,q)\, ce_{2n}(\frac{\pi}{2},q)}{\pi\left[A_0^{(2n)}\right]^2} \tag{4.35}$$

Hence, from equation (4.30),

$$ce_{2n}(\xi,q) = \frac{ce_{2n}(0,q)\, ce_{2n}(\frac{\pi}{2},q)}{\pi\left[A_0^{(2n)}\right]^2} \times$$

$$\int_0^{\pi} J_0\left[k\left\{2(\cos 2\xi + \cos 2\varsigma)\right\}^{1/2}\right] ce_{2n}(\varsigma,q)\,d\varsigma$$

Since $ce_{2n}(\varsigma,q) = \sum_{m=0}^{\infty} A_{2m}^{(2n)} \cos 2m\varsigma$

and

$$J_0\left[k\left\{2(\cos 2\xi+\cos 2\varsigma)\right\}^{1/2}\right] = J_0(ke^{i\xi})J_0(ke^{-i\xi}) + 2\sum_{r=1}^{\infty}(-1)^r J_r(ke^{i\xi})J_r(ke^{-i\xi})\cos 2r\varsigma$$

$$ce_{2n}(\xi,q) = \frac{ce_{2n}(0,q)\, ce_{2n}(\frac{\pi}{2},q)}{\pi\left[A_0^{(2n)}\right]^2} \times \int_0^{\pi}\Big[J_0(ke^{i\xi})J_0(ke^{-i\xi}) + 2\sum_{r=1}^{\infty}(-1)^r J_r(ke^{-i\xi})J_r(ke^{i\xi})\cos 2r\varsigma\Big]\sum_{m=0}^{\infty}A_{2m}^{(2n)}\cos 2m\varsigma\,d\varsigma$$

$$= \lambda_{2n}\int_0^{\pi} J_0(ke^{i\xi})\,J_0(ke^{-i\xi})\sum_{m=0}^{\infty}A_{2m}^{(2n)}\cos 2m\varsigma\,d\varsigma$$

$$+ 2\lambda_{2n}\int_0^{\pi}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}(-1)^r J_r(ke^{i\xi})J_r(ke^{-i\xi})A_{2m}^{(2n)}\cos 2r\varsigma\cos 2m\varsigma\,d\varsigma$$

In the above equation, the first term gives

$\lambda_{2n} J_0(ke^{i\xi})J_0(ke^{-i\xi})\,\pi A_0^{(2n)}$, from equation (4.33) and the second term gives

$$\lambda_{2n}\pi\sum_{r=1}^{\infty}(-1)^r J_r(ke^{i\xi})J_r(ke^{-i\xi})A_{2r}^{(2n)}$$

Hence

$$ce_{2n}(\xi,q)=\lambda_{2n}\pi\sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)}J_r(ke^{i\xi})J_r(ke^{-i\xi}) \tag{4.36}$$

If in equation (4.36) we write $i\xi$ for ξ, the equation becomes

$$Ce_{2n}(\xi,q)=\pi\lambda_{2n}\sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)}J_r(ke^{\xi})J_r(ke^{-\xi}) \tag{4.37}$$

where λ_{2n} has the value given by equations (4.34) or (4.35).

This is the expansion for $Ce_{2n}(\xi,q)$ in Bessel function product series.

4.10 Obtaining a series of the form (4.37) for the other functions Ce_{2n+1}, Se_{2n+2}, Se_{2n+1} is not as straight forward as above.

We obtain, for Ce_{2n+1}, (say), a series of the form

$$\text{constant} \times \cosh\xi\sum_{r=0}^{\infty}\bar{A}_{2r+1}^{(2n+1)}J_{r+1}(ke^{i\xi})J_{r+1}(ke^{-i\xi})$$

where $\bar{A}_{2r+1}^{(2n+1)}$ are not the same as $A_{2r+1}^{(2n+1)}$. This can be obtained using the method described in articles 4.7, 4.8 and 4.9.

We can also have expansions with the same coefficients, but in this case the expansion functions are more complicated. In what follows, a series expansion for $Ce_{2n+1}(\xi,q)$ is obtained.

As in articles 4.7 and 4.8, it can be proved that

$$ce_{2n+1}(\xi,q) = \phi_{2n+1}\int_0^{2\pi} J_{2\nu+1}(k_1\rho)\cos(2\nu+1)\theta \sum_{s=0}^{\infty} A_{2s+1}^{(2n+1)} \cos(2s+1)\eta\, d\eta$$

If we write $i\xi$ for ξ

$$Ce_{2n+1}(\xi,q) = \phi_{2n+1}\int_0^{2\pi} J_{2\nu+1}(k_1\rho)\cos(2\nu+1)\theta \sum_{s=0}^{\infty} A_{2s+1}^{(2n+1)} \cos(2s+1)\eta\, d\eta$$

where

$$k_1\rho = k\left[2(\cos 2\eta + \cosh 2\xi)\right]^{1/2} = k(e^{2\xi} + e^{-2\xi} + 2\cos 2\eta)^{1/2}$$

For any cylinder function Z,

$$\frac{Z_\nu(\omega)}{\omega^\nu} = 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} \frac{(-1)^m (m+\nu) J_{m+\nu}(v_1) Z_{m+\nu}(v_2)}{v_1^{\nu} v_2^{\nu}} C_m^\nu(\cos 2\eta) \qquad (4.38)$$

where $\omega^2 = v_1^2 + v_2^2 + 2v_1 v_2 \cos 2\eta$, $|v_1| < |v_2|$ and C_m^ν are the Gegenbauer's polynomials, (coefficients of x^m in the expansion of $(1-2x\cos 2\eta + x^2)^{-\nu}$ in the ascending powers of x), so that

$$C_m^\nu(\cos 2\eta) = \sum_{s=0}^{\le m/2} \frac{(-1)^s 2^{m-2s} \Gamma(m+\nu-s) \cos^{m-2s} 2\eta}{(m-2s)!\, s!\, \Gamma(\nu)}$$

In particular if $\nu = 1$,

$$C_m^1(\cos 2\eta) = \sum_{s=0}^{\le m/2} \frac{(-1)^s 2^{m-2s} \Gamma(m+1-s) \cos^{m-2s} 2\eta}{(m-2s)!\, s!}$$

From the above we obtain

$$\cos\eta\, C_m^1(\cos 2\eta) = \frac{\sin(2m+2)\eta}{2\sin\eta} = \sum_{p=0}^{m} \cos(2p+1)\eta$$

(As $\sum_{p=0}^{m} 2\sin\eta\cos(2p+1)\eta = \sin 2\eta + (\sin 4\eta - \sin 2\eta) + \cdots$

$$\cdots + [\sin(2m+2)\eta - \sin 2m\eta]$$

$$= \sin(2m+2)\eta\,)$$

$$\cos\theta = \frac{x}{\rho} = \frac{h}{\rho}\cosh\xi\cos\eta = \frac{2k\cosh\xi\cos\eta}{k_1\rho}$$

If in equation (4.38) $Z_\nu = J_\nu$, $v_1 = ke^{-\xi}$, $v_2 = ke^{\xi}$ and $\nu = 1$

$$J_1(k_1\rho)\cos\theta = \frac{J_1(k_1\rho)}{k_1\rho}\cdot 2k\cosh\xi\cos\eta$$

$$= 2k\cosh\xi \cdot 2\sum_{m=0}^{\infty} \frac{(-1)^m (m+1) J_{m+1}(v_1) J_{m+1}(v_2)}{v_1 v_2} \times$$

$$C_m^1(\cos 2\eta)\cos\eta.$$

$$= \frac{4}{k}\cosh\xi \sum_{m=0}^{\infty} (-1)^m (m+1)\,\omega_{m+1} \sum_{p=0}^{m} \cos(2p+1)\eta$$

$$(\text{as } v_1 v_2 = k^2)$$

where $\omega_{m+1} = J_{m+1}(v_1)\,J_{m+1}(v_2)$

Hence

$$Ce_{2n+1}(\xi,\eta) = \phi_{2n+1}\,\frac{4}{k}\cosh\xi \sum_{m=0}^{\infty}(-1)^m (m+1)\,\omega_{m+1} \times$$

$$\int_0^{2\pi} \sum_{p=0}^{m} \cos(2p+1)\eta \sum_{s=0}^{\infty} A_{2s+1}^{(2n+1)} \cos(2s+1)\eta\, d\eta$$

As

$$\int_0^{2\pi} \cos(2p+1)\eta\,\cos(2s+1)\eta\, d\eta = \begin{cases} 0, & p \neq s \\ \pi, & p = s \end{cases}$$

Now

$$(m+1)\int_0^{2\pi} \sum_{p=0}^{m} \cos(2p+1)\eta \sum_{s=0}^{\infty} A_{2s+1}^{(2n+1)} \cos(2s+1)\eta\, d\eta$$

$$= \pi(m+1)\,(A_1 + A_3 + A_5 + \cdots + A_{2m+1})$$

$$= \pi\,\bar{A}_{2m+1}^{(2n+1)} \quad \text{(say)}$$

Then

$$Ce_{2n+1}(\xi,q) = \phi_{2n+1}\frac{4\pi\cosh\xi}{k}\sum_{m=0}^{\infty}(-1)^m \bar{A}_{2m+1}^{(2n+1)} J_{m+1}(ke^{\xi}) J_{m+1}(ke^{-\xi})$$

where $\bar{A}_{2m+1}^{(2n+1)} = (m+1)\left(A_1^{(2n+1)} + A_3^{(2n+1)} + A_5^{(2n+1)} + \cdots + A_{2m+1}^{(2n+1)}\right).$

To obtain the series with A_{2r+1}^{2n+1} as coefficients,[19]

consider $W_r = J_r(v_1) J_{r+1}(v_2) + J_{r+1}(v_1) J_r(v_2)$

Using the recurrence relation

$$J_n(x) = \frac{2(n+1)}{x} J_{n+1}(x) - J_{n+2}(x),$$

$$W_r = J_{r+1}(v_2)\left[\frac{2(r+1)}{v_1} J_{r+1}(v_1) - J_{r+2}(v_1)\right]$$

$$+ J_{r+1}(v_1)\left[\frac{2(r+1)}{v_2} J_{r+1}(v_2) - J_{r+2}(v_2)\right]$$

$$= 2(r+1) J_{r+1}(v_1) J_{r+1}(v_2)\left[\frac{1}{v_1} + \frac{1}{v_2}\right]$$

$$- \left[J_{r+1}(v_2) J_{r+2}(v_1) + J_{r+1}(v_1) J_{r+2}(v_2)\right]$$

$$= \frac{4}{k}(r+1)\cosh\xi\, \omega_{r+1} - W_{r+1}$$

Using this recurrence relation,

$$W_r = \frac{4}{k}\cosh\xi\left[(r+1)\omega_{r+1} - (r+2)\omega_{r+2} + (r+3)\omega_{r+3} - \cdots\right]$$

So $\sum_{m=0}^{\infty} (-1)^m A_{2m+1}^{(2n+1)} W_m$

$$= \frac{4}{k}\cosh\xi\Big[A_1^{(2n+1)}(\omega_1 - 2\omega_2 + 3\omega_3 - \cdots) - A_3^{(2n+1)}(2\omega_2 - 3\omega_3 + 4\omega_4 - \cdots)$$

$$+ A_5^{(2n+1)}(3\omega_3 - 4\omega_4 + \cdots) - \cdots\Big]$$

$$= \frac{4}{k}\cosh\xi\sum_{m=0}^{\infty}(-1)^m(m+1)\left(A_1^{(2n+1)} + A_3^{(2n+1)} + \cdots + A_{2m+1}^{(2n+1)}\right)\omega_{m+1}$$

$$= \frac{4}{k}\cosh\xi\sum_{m=0}^{\infty}(-1)^m \bar{A}_{2m+1}^{(2n+1)} J_{m+1}(ke^{\xi}) J_{m+1}(ke^{-\xi})$$

Hence

$$Ce_{2n+1}(\xi,q) = \pi\,\phi_{2n+1}\sum_{m=0}^{\infty}(-1)^m A_{2m+1}^{(2n+1)} W_m$$

where

$$W_m = J_m(ke^{-\xi})J_{m+1}(ke^{\xi}) + J_m(ke^{\xi})J_{m+1}(ke^{-\xi})$$

The value of ϕ_{2n+1} can be found as in Article (4.9), ϕ_{2n+1} corresponds to λ_{2n}, $\phi_{2n+1} = -\dfrac{ce_{2n+1}(0,q)\,ce'_{2n+1}(\frac{\pi}{2},q)}{\pi k\left[A_1^{(2n+1)}\right]^2}$

On the same lines as these, the expansions for Se_{2n+1} and Se_{2n+2} are obtained and are given as follows:

$$Se_{2n+1}(\xi,q) = \frac{se'_{2n+1}(0,q)\,se_{2n+1}(\frac{\pi}{2},q)}{k\left[B_1^{(2n+1)}\right]^2}\sum_{m=0}^{\infty}(-1)^m B_{2m+1}^{(2n+1)} W_m$$

where

$$W_m = J_m(v_1)J_{m+1}(v_2) - J_{m+1}(v_1)J_m(v_2)$$

and

$$Se_{2n+2}(\xi,q) = -\frac{se'_{2n+2}(0,q)\,se'_{2n+2}(\frac{\pi}{2},q)}{k^2\left[B_2^{(2n+2)}\right]^2}\sum_{m=0}^{\infty}(-1)^m B_{2m+2}^{(2n+2)} W_m$$

where

$$W_m = J_m(v_1)J_{m+2}(v_2) - J_{m+2}(v_1)J_m(v_2)$$

These expansions of the modified Mathieu functions can also be obtained by another method - see Dougall 8, McLachlan 19.

4.11 As in Article 4.5, functions Je_{2n}, Je_{2n+1}, Jo_{2n+2}, Jo_{2n+1} can now be written down as

$$Je_{2n}(s,\xi) = \sqrt{\frac{\pi}{2}}\,\frac{(-1)^n}{De_0}\sum_{r=0}^{\infty}(-1)^r De_{2r}\, J_r(u)J_r(v)$$

$$Je_{2n+1}(s,\xi) = \frac{(-1)^n}{De_1}\sqrt{\frac{\pi}{2}}\sum_{r=0}^{\infty}(-1)^r De_{2r+1} \times \qquad (4.39)$$

$$\left[J_{r+1}(u)J_r(v) + J_r(u)J_{r+1}(v)\right]$$

$$Jo_{2n+2}(s,\xi) = \frac{(-1)^n}{Do_2}\sqrt{\frac{\pi}{2}}\sum_{r=0}^{\infty}(-1)^r Do_{2r+2} \times$$
$$\left[J_{r+2}(u)J_r(v) - J_{r+2}(v)J_r(u)\right]$$

$$Jo_{2n+1}(s,\xi) = \frac{(-1)^n}{Do_1}\sqrt{\frac{\pi}{2}}\sum_{r=0}^{\infty}(-1)^r Do_{2r+1} \times$$
$$\left[J_{r+1}(u)J_r(v) - J_r(u)J_{r+1}(v)\right]$$

where $u = \frac{\sqrt{s}}{2}e^{\xi}$ and $v = \frac{\sqrt{s}}{2}e^{-\xi}$

Though for computational work these expansions are not convenient, they are helpful in as much as they provide an independent means of evaluating the functions. In the next article it will be proved that these expansions converge much faster than expansions in Article 4.5.

4.12 Let u_r denote the rth term in the expansion of Je_{2n+p}, equation (4.20)

Then

$$u_r = \sqrt{\frac{\pi}{2}}\,(-1)^{r+n}\,De_{2r+p}\,J_{2r+p}(2k\cosh\xi)$$

and

$$u_{r+1} = \sqrt{\frac{\pi}{2}}\,(-1)^{r+n+1}\,De_{2r+p+2}\,J_{2r+2+p}(2k\cosh\xi)$$

When $2r$ is very large in comparison with the argument 2k cosh ξ of the Bessel function,

$J_{2r+p}(2k\cosh\xi)$ is (asymptotically) $= \frac{1}{(2r+p)!}(k\cosh\xi)^{2r+p}$ [18]

Hence $$\frac{u_{r+1}}{u_r} = -\frac{De_{2r+p+2}}{De_{2r+p}}\,\frac{(k\cosh\xi)^2}{(2r+p+2)(2r+p+1)}$$

But $$\left|\frac{De_{2r+p+2}}{De_{2r+p}}\right| \to \frac{|k|^2}{4(r+1)^2} \quad \text{as } r \to \infty$$

Hence $$\left|\frac{u_{r+1}}{u_r}\right| \to \frac{|k^4|\,|\cosh^2\xi|}{4(r+1)^2\cdot 4(r+1)^2}$$ whether p is 0 or 1. (4.40)

As $r \to \infty$, the above expression tends to zero, and so by the ratio test the series is absolutely convergent.

As $\dfrac{|k^4||\cosh^2\xi|}{16(r+1)^4} \to 0$ faster than $\dfrac{|k^2| e^{2|\xi|}}{4(r+1)^2}$ (see Article 4.3) expansions (4.20), (4.21) converge faster than expansions (4.6).

Now, in any finite closed region of the ξ -plane, for a given q , it is possible to find an 'M', indepent of ξ such that $|k^4||\cosh^2\xi| \leq 16M(r+1)^4$ as $r \to \infty$

Hence by 'M' test the series are uniformly convergent and term by term integration or differentiation with respect to ξ is permissible in any finite region of the ξ -plane.

Differentiating Je_{2n} with respect to ξ ,

$$\frac{d}{d\xi} Je_{2n}(\xi,q) = 2k\sinh\xi\,(-1)^n\sqrt{\frac{\pi}{2}}\sum_{m=0}^{\infty}(-1)^m\, 2m\, De_{2m}\, J'_{2m}(2k\cosh\xi)$$

Hence $\left|\dfrac{u_{r+1}}{u_r}\right| = \left|\dfrac{De_{2r+2}}{De_{2r}} \cdot \dfrac{(2r+2)}{2r} \cdot k^2\cosh^2\xi\,\dfrac{(2r)!}{(2r+2)!}\right|$

(when $2r$ is very large compared to $2k\cosh\xi$)

$$\simeq \frac{|k^4||\cosh\xi|^2}{16(r+1)^4}\left(1+\frac{1}{r}\right) \to 0 \text{ as } r \to \infty . \quad (4.41)$$

Hence the above series is absolutely convergent and as above it can also be proved that it is uniformly convergent.

In a similar manner, it can be proved that the other modified Mathieu functions Je_{2n+1}, Jo_{2n+1} and Jo_{2n+2} and their derivatives are absolutely and uniformly convergent.

As the functions Ce_m, Se_m differ from the above functions Je_m, Jo_m only by factors, which are independent of ξ, it can be proved that they and their derivatives are also uniformly and absolutely convergent.

Consider now the expansions given in a series of products of Bessel functions.

If W_r denotes the coefficient of A_{2r+p} or B_{2r+p} in the appropriate expansion,

$$\left|\frac{W_{r+1}}{W_r}\right| \simeq \frac{q}{4(r+1)^2}$$

Consider, for example $W_r = J_r(v_1)J_{r+2}(v_2) - J_{r+2}(v_1)J_r(v_2)$, $v_1 = ke^{-\xi}$, $v_2 = ke^{\xi}$ and $q = k^2$

If r is very large compared to v_1 or v_2

$$J_r(v_1) \simeq \frac{1}{r!}\left(\frac{k}{2}e^{-\xi}\right)^r, \quad J_{r+2}(v_2) \simeq \frac{1}{(r+2)!}\left(\frac{k}{2}e^{\xi}\right)^r$$

Hence $$|W_r| \simeq \frac{1}{r!}\,\frac{1}{(r+2)!}\,\frac{k^{2r+2}}{2^{2r}}\left|e^{2\xi} - e^{-2\xi}\right|$$

So $$\left|\frac{W_{r+1}}{W_r}\right| \simeq \frac{k^2}{4(r+1)^2}.$$

Therefore $$\left|\frac{u_{r+1}}{u_r}\right| \simeq \left|\frac{A_{2r+2}}{A_{2r}}\right|\frac{k^2}{4(r+1)^2}$$

$$\simeq \frac{k^4}{16(r+1)^4} \to 0 \text{ as } r \to \infty \qquad (4.42)$$

Comparing (4.40) and (4.42), it can be seen that the series (4.39) converge more rapidly than (4.20), (4.21).

The uniform convergence of the series and hence the absolute and uniform convergence of the derivatives is easily proved and it can also be proved that the derivatives of the series (4.39) converge faster than the derivatives of the series (4.20), (4.21).

5. METHOD OF FINDING THE HIGHER ORDER CUT-OFF FREQUENCIES.

5.1 The three sets of expansions for the modified Mathieu functions (and hence their derivatives) obtained in Chapter IV, are uniformly and absolutely convergent for a given q and for any given order of the function, so long as ξ remains finite. For the evaluation of the functions any of these expansions can be used. The expansions (4.6) converge but slowly and even though expansions (4.39) converge extremely rapidly, they are not so convenient for computation purposes. Hence the expansions (4.20), (4.21) are the most appropriate ones for any computational work. The expansions (4.39) are independently obtained. Hence they are most useful when the accuracy of the results which have been obtained earlier by using expansions (4.20), (4.21) is to be checked.

5.2 As mentioned in Chapter III, the calculation of the cut-off frequencies are obtained by finding the parametric zeros of

$$Ce_m'(\xi_0, q) = 0 \quad \text{and} \quad Se_{m+1}'(\xi_0, q) = 0, \quad m = 0, 1, 2, \ldots$$

where ξ_0 is such that if e is the eccentricity of the elliptic boundary, $\cosh \xi_0 = 1/e$.

In the expansions of $Ce_m(\xi, q)$ and $Se_{m+1}(\xi, q)$ the coeffieients themselves are functions of q and hence the functions can be evaluated for each ξ, only for a given q. So even if ξ_0 is known it is not practicable to find the parametric zeros directly. So we adopt the following method.

For a given order m, for each value of s, the value $\omega_0 = \sqrt{s}\cosh\xi_0$, which makes the function Ce_m or Se_{m+1} zero is calculated.

From equation (3.10) which gives

$$e = \frac{\sqrt{s}}{\omega_0}$$

the eccentricity e of the ellipse is calculated. Once e is known, from equation (3.11), $k = \omega_0 (1-e^2)^{1/4}$ the eigen values k (i.e. the cut-off frequencies) are found. But the e obtained here are not the required values, the values to be found being for ellipses of eccentricities $e = \cdot 1 (\cdot 1) \cdot 9$ and $e = \cdot 95$. Hence by linear iterated interpolation the values of k corresponding to the required e's are obtained. Tables A (1-15) and Tables B (1-15) give the e and k obtained for the various values of s. Tables C and D give the values of k for the e = .1 (.1) .9 and e = .95 for even and odd radial functions respectively of orders 1-15. These cut-off frequencies have been calculated for ellipses each of area π, so that while the eccentricity varies, the area has been kept constant. When the cut-off frequency in an elliptic duct of a particular eccentricity e of cross section, but not of area π, is required, equation (3.8) can be used to obtain the result.

5.3 From Chapter IV, Article 4.5, it is known that $Ce'_m(\xi, q)$ are proportional to $Je'_m(\xi, s)$ and $Se'_{m+1}(\xi, q)$ are proportional to $Jo'_{m+1}(\xi, s)$, where $s = 4q$; the constant of proportionality is a function of q. These are called 'joining factors' and are tabulated in 'Tables Relating to Mathieu Functions'.[22] The value of ξ which be considered to a fixed number of term depending ofcourse on the order of the function, the value of s and the needed accuracy of the result.

The coefficients De and Do, correct to nine significant figures, for given values of s (at certain intervals) up to 100 have been tabulated [22] for functions of up to order 15.

5.31 A computer program subroutine * (in Fortran IV) calculates the values (up to a required accuracy) of Bessel functions, up to a few orders less than 200 for the same argument.[1,15]

Let $J_n(x) = 0$ for a 'n' very large compared to the argument and let $J_{n-1}(x) = 10^{-70}$

With the help of the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

that is $J_{n-1}(x) = (2n/x)\, J_n(x) - J_{n+1}(x)$

the values of Bessel functions with decreasing n are succesively evaluated for the same argument x.

Then the factor A, such that

$$\frac{1}{A} = J_0(x) + 2J_2(x) + 2J_4(x) + \cdots$$

is obtained and each value of $J_n(x)$ obtained earlier is multiplied by A to give the required values of the Bessel functions of all orders up to a few orders less than n.

This method avoids the accumulation of rounding errors which occur when $n > x$ and the recurrence relation is used with ascending n.

5.32 It now becomes possible to compute the values of Je'_m (or Jo'_m) for a given s and a given $\sqrt{s}\cosh\xi\,(=\omega)$.

* used with the kind permission of Dr. M.V. Lowson.

makes Ce'_m vanish for a given value of q hence would not differ from that, which makes Je'_m zero for the same q. Hence without any loss of generality the functions $Je'_m(\xi, s)$ and $Jo'_{m+1}(\xi, s)$ can be considered.

Now $Je_{2n+p}(\xi, s) = (-1)^n \sqrt{\frac{\pi}{2}} \sum_{r=0}^{\infty} (-1)^r De_{2r+p} J_{2r+p}(\sqrt{s}\cosh\xi)$

p = 0 or 1 according as a function of period πi or $2\pi i$ is being considered.

Due to uniform convergence (Article 4.12), we can differentiate Je_{2n+p}, Jo_{2n+p} with respect to ξ, term by term.

Hence $Je'_{2n+p}(\xi, s) = (-1)^n \sqrt{\frac{\pi s}{2}} \sinh\xi \sum_{r=0}^{\infty} (-1)^r De_{2r+p} J'_{2r+p}(\sqrt{s}\cosh\xi)$

which can be written as [29]

$$Je'_{2n+p}(\xi, s) = (-1)^n \sqrt{\frac{\pi s}{2}} \sinh\xi \sum_{r=0}^{\infty} (-1)^r De_{2r+p} \times \left[J_{2r+p-1}(\sqrt{s}\cosh\xi) - J_{2r+p+1}(\sqrt{s}\cosh\xi)\right] \quad (5.1)$$

(from a recurrence relation for Bessel functions).

Similarly

$$Jo'_{2n+p}(\xi, s) = \frac{d}{d\xi}\left[(-1)^n \sqrt{\frac{\pi}{2}} \tanh\xi \sum_{r=0}^{\infty} (-1)^r (2r+p) Do_{2r+p} J_{2r+p}(\sqrt{s}\cosh\xi)\right]$$

$$= (-1)^n \sqrt{\frac{\pi s}{2}} \tanh\xi \sinh\xi \sum_{r=0}^{\infty} (-1)^r (2r+p) Do_{2r+p} J'_{2r+p}(\sqrt{s}\cosh\xi)$$

$$+ (-1)^n \sqrt{\frac{\pi}{2}} \frac{1}{\cosh^2\xi} \sum_{r=0}^{\infty} (-1)^r (2r+p) Do_{2r+p} J_{2r+p}(\sqrt{s}\cosh\xi)$$

$$= \frac{Jo_{2n+p}}{\sinh\xi\cosh\xi} + (-1)^n \sqrt{\frac{\pi s}{2}} \tanh\xi \sinh\xi \times$$

$$\sum_{r=0}^{\infty} (2r+p) Do_{2r+p} \left[J_{2r+p-1}(\sqrt{s}\cosh\xi) - J_{2r+p+1}(\sqrt{s}\cosh\xi)\right] \quad (5.2)$$

It is of importance to note here that though in principle the series is infinite, due to the rapid decrease of the coefficients to zero, for a given s, the series need only

A program evaluates the values of Je'_m (Jo'_m) for values of w starting with $\sqrt{s}$ (which means $\omega = \sqrt{s}\cosh\xi = \sqrt{s}$, or cosh ξ = 1 or ξ = 0) (we know the value of Je'_m for ξ = 0 is zero). The values of w increase by increments of 0.5 or 0.1 depending on the value of s and the order of the function. When the function changes sign from postive to negative, by chord iterative inverse interpolation [17] the value ω_0 of ω which makes the function Je'_m (Jo'_m) vanish is obtained.

Suppose $Je'(1)$ for a value w (1) of w is positive and $Je'(k)$ is negative for w (k). Then a better approximation to the zero of the function is obtained by the intersection of the line through ($\omega(1)$, $Je'(1)$) and ($\omega(k)$, $Je'(k)$) with the X-axis (where the ω is along the X-axis and Je' along the Y-axis).

The equation of the line is given by

$$Je' - Je'(k) = \frac{Je'(1) - Je'(k)}{\omega(1) - \omega(k)}\left[\omega - \omega(k)\right]$$

So that $$\omega(k+1) = \omega(k) - Je'(k)\,\frac{\omega(1) - \omega(k)}{Je'(1) - Je'(k)}$$

$$= \frac{\omega(k)\,Je'(1) - \omega(1)\,Je'(k)}{Je'(1) - Je'(k)}$$

gives a nearer value to the zero. We can find $Je'(k+1)$ for this $\omega(k+1)$ and apply the above formula again replacing k by $k+1$. This process can be continued until the required accuracy is obtained. The program provides for 18 such iterations and the result obtained is accurate to 8 signigicant figures. If even after 18 iterations the accuracy required is not obtained, it shows that ω has been incremented excessively. Hence a smaller increment to ω is to be given.

With this value of ω_0, e and k are found in a straight-forward manner from equations (3.10) and (3.11). Since the values of ω_0 are correct to eight significant figures and the values of $\sqrt{\Delta}$ are correct to eleven significant figures (due to computer limitations) the values of e are correct to eight significant figures at least and the values of k have an accuracy of seven significant figures. All this has been said on the assumption that the values of the function Je'_m (Jo'_m) are correct to eight significant figures.

With the help of expansions (4.39), the values of function Je_m (Jo'_m) were calculated and from these the values of ω_0, e and k were obtained as mentioned above. It was found that accuracy of eight significant figures has been maintained for ω_0 so that e and k are respectively accurate as mentioned above.

5.4 The values of e, obtained in the previous article, are at irregular intervals. They are, however, in an ascending order of magnitude. To obtain the values of k for e = .1 (.1) .9 and e = .95, the method of linear iterated interpolation is applied.[24]

Let x_i be the values of a variable and let values $f(x_i)$ of a function f be known for these x_i. Let x be an interior point (between the smallest and the largest of x_i) and we are to find the value of the function for this x. We calculate, successively, the functions $f_2(x_i, x_{i+1})$, $f_3(x_i, x_{i+1}, x_{i+2})$, ...

where $f_2(x_i, x_{i+1}) = \dfrac{(x_i - x) f(x_{i+1}) - (x_{i+1} - x) f(x_i)}{(x_i - x) - (x_{i+1} - x)}$

$$f_3(x_i, x_{i+1}, x_{i+2}) = \frac{(x_i - x) f(x_{i+1}, x_{i+2}) - (x_{i+2} - x) f(x_i, x_{i+1})}{(x_i - x) - (x_{i+2} - x)}$$

etc., until we obtain the final convergent. For example, if $i = 7$, we have to obtain $f_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. The final convergent gives the value of $f(x)$ at x. If the $x_i - x$ are so arranged that $|x_i - x|$ are in an ascending order, the convergence is rapid and a more accurate result is obtained.

The program from which the values of k for particular values of e are obtained, incorporates the above and the results so obtained are given in Tables C and D. These values are accurate to six significant figures.
The values of k corresponding to e = 0, (which is a circle) were obtained from Olver.[23]

5.5 The method described above was followed and the values of the cut-off frequencies for ellipses of the required eccentricities were obtained.[16] However, there were two main hurdles to be overcome.

(1) For higher values of e, but smaller orders of the function, the above program did not converge so fast so that the results obtained were not sufficiently accurate.

(2) For large orders of the function, the cut-off frequencies could not be found, as values of coefficients for Δ larger than 100 were needed in this case and these are not tabulated in National Bureau of Standards.[22]

In the first case, the values k decreased more rapidly as e increased. Hence it was necessary that more values of k be calculated nearer the required eccentricities.

Hence for larger values of s, smaller increments in s were to be considered. The characteristic values be_r, bo_r ($a_r = be_r - s/2$ and $b_r = bo_r - s/2$)* are tabulated at smaller intervals of s in National Bureau of Standards.[22] If the coefficients De (Do) could be generated from these characteristic values, then the functions could be evaluated by the method described above and then the values of w_o, e, and k computed.

In the second case, the characteristic values for s larger than 100 have been tabulated in Blanch and Rhodes 5. Here the tabulation has been done for values of t and the characteristic numbers $Be_r(t)$ and $Bo_r(t)$ are given, where

$$t = 1/\sqrt{s}$$

$$Be_r(t) = be_r(s) - \nu_1\sqrt{s} \quad , \qquad \text{where } \nu_1 = 2r+1$$

and

$$Bo_r(t) = bo_r(s) - \nu_2\sqrt{s} \; , \qquad \text{where } \nu_2 = 2r-1$$

The range of t is $0 \leq t \leq \cdot 1$ so that s ranges from ∞ to 100. The characteristic values a_r and b_r for a given s can be obtained from $a_r = be_r(s) - s/2$ and $b_r = bo_r(s) - s/2$, which are obtained from Be_r and Bo_r. Once again, when the characteristic values are known, the problem just reduces to one of generating the coefficients De_m (Do_m).

5.6 The method of obtaining the characteristic values a (or b) for the Mathieu's equation for a given value of $q(=4s)$ and then generating the coefficients from these values is given in Ince,[12] Goldstein,[9] McLachlan.[19] A detailed method by which the coefficients of expansion can

* See expansions 4.6

be found, once the characteristic values are known is given in Blanch 4, and it has been found most useful, when the coefficients have to be found for low order functions with not too high q and for high order functions with large q. The method is discussed in detail below.

Substituting the expansions (4.3) in the Mathieu's equation (4.1) and equating the coefficients of $\cos m\eta$ (or $\sin m\eta$) to zero, the following relations are obtained:

$$A_2 = v_0 A_0 , \; v_2 A_2 - 2A_0 = A_4 ; \; A_3 = A_1 (v_1 - 1) \qquad (5.3)$$

$$B_0 = 0 , \; B_4 = v_2 B_2 ; \; B_3 = B_1 (v_1 + 1) \qquad (5.4)$$

and

$$\left.\begin{aligned} &A_{m-2} + A_{m+2} = v_m A_m \\ &\text{or} \\ &B_{m-2} + B_{m+2} = v_m B_m \end{aligned}\right\} \quad m \geq 4 \qquad (5.5)$$

where

$$v_m = \frac{a - m^2}{q} \quad \text{or} \quad \frac{b - m^2}{q} \qquad (5.6)$$

a (or b) being the characteristic value of the function.

If $G_m = \dfrac{A_m}{A_{m-2}}$ or $\dfrac{B_m}{B_{m-2}}$, and $H_m = \dfrac{1}{G_m}$

then the above equations reduce to

$$G_2 = v_0 , \; G_4 = v_2 - \frac{2}{G_2} = v_2 - 2H_2 = \frac{1}{v_4 - G_6} \quad \text{for } ce_{2n} \qquad (5.7)$$

$$G_3 = v_1 - 1 = \frac{1}{v_3 - G_5} \quad \text{for } ce_{2n+1} \qquad (5.8)$$

$$H_2 = 0, \quad G_4 = v_2 = \frac{1}{v_4 - G_6} \qquad \text{for } se_{2n+1} \tag{5.9}$$

$$G_3 = v_1 + 1 = \frac{1}{v_3 - G_5} \qquad \text{for } se_{2n+2} \tag{5.10}$$

And in general $$G_m = \frac{1}{v_m - G_{m+2}}, \quad m \geq 3 \tag{5.11}$$

and $$G_m = v_{m-2} - \frac{1}{G_{m-2}} = v_{m-2} - H_{m-2}, \quad m \geq 5 \tag{5.12}$$

Since the series (4.3) are absolutely convergent for given values of q, there is a number M for each series such that

$$|G_m| < 1, \quad \text{for } m \geq M$$

Let m_1 be one such value. Then v_{m_1} has a very large numerical value.

Since
$$|G_{m_1}| = |v_{m_1-2} - H_{m_1-2}| < 1$$

the magnitude of H_{m_1-2} must be of the same order as that of v_{m_1-2} and also it must have the same sign as v_{m_1-2}. This means there will be a loss of considerable number of significant figures when H_{m_1-2} is subtracted from v_{m_1-2}. No doubt, the values of the coefficients do decrease rapidly in this region; however, the loss of accuracy of the result, if this formula were used is more and hence it is not possible to generate all the G_m from this formula. Once the G_2, G_4, ... (or G_1, G_3, ...) have been evaluated, the coefficients AD_0, AD_2, ... (or AD_1, AD_3, ...), can be found, A being an arbitrary constant. (Without any loss of generality, we use D_m to represent coefficient A_m or B_m, since for a characteristic value 'a' or 'b' gives rise to only one periodic solution, when q is not zero and no confusion arises as to the type of coefficients considered.)

Of the four type of solutions considered, only the first (even function with period π) is a bit complicated. The rest can be obtained without much difficulty. So we shall consider even functions of period π .

Initially the coefficients D_m are all positive and they increase up to a largest coefficient, say, D_ν and then they start decreasing so that

$$D_{\nu+n} < D_{\nu+n-2} \quad , \quad n \geq 2$$

This means

$$G_{\nu+n} < 1 \quad , \quad \forall \; n > 0$$

and

$$G_{\nu-n} > 1 \quad , \quad \forall \; n > 0$$

Since the coefficients decrease to zero, let us suppose that the series is terminated after a certain number of terms for the required accuracy. From the order of the function and the value of q , it is possible to determine this number without much difficulty. Let D_w be the last term of the series.

We first compute the values of $v_0, v_2, \cdots, v_w$ from equation (5.6) to the required number of significant figures, which is the number of significant figures needed in the largest coefficient D_m.

After this, the quantities G_2, H_2, G_4, H_4, ..., G_ν, H_ν are evaluated, where each G_m is computed from equations (5.7) or (5.12) and the H_m are obtained as reciprocals of G_m. The computation continues until the G_m are numerically greater than unity, G_ν being the last of these. The computation in this direction is now stopped after H_ν has been found. This is known as the 'forward' process.

Since D_w is the last term of the series, it will be very near zero, and we can write

$$G_\omega \simeq \frac{1}{v_\omega}$$

(Actually $G_w = \dfrac{1}{v_\omega - G_{\omega+2}}$, but since v_ω is very large and G_{w+2} negligible, we get the result)

From G_w, the values of H_{w-2}, H_{w-2}, ... H_ν are obtained from equation (5.12), as that equation gives

$$H_{m-2} = v_m - G_m$$

The G_m are now obtained as the reciprocals of H_m. $G_{w-2} = 1/H_{\omega-2}$ and so on. In these calculations the number of significant figures obtained increases in accuracy at each step until a maximum accuracy is reached. The last value obtained in this process is H_ν. This is known as the 'backward' process.

The value of H_ν obtained here is compared with the value of H_ν obtained earlier. If the agreement is satisfactory (if, for example, we desire to have the largest coefficient to nine significant figures, the two values of H_ν should agree to that extent), The coefficients can now be determined.

The coefficients so determined can then be normalised in the manner required.

We now write $D_{\nu} = 1.$

The remaining coefficients D_m are obtained by the relationships

$$D_{\nu+2} = G_{\nu+2}, \quad D_{\nu+4} = D_{\nu+2} G_{\nu+4}, \quad \cdots \quad D_{\omega} = D_{\omega-2} G_{\omega}$$

$$D_{\nu-2} = H_{\nu}, \quad D_{\nu-4} = D_{\nu-2} H_{\nu-2}, \quad \cdots\cdots, \quad D_0 = D_2 H_2, \quad \nu \geq 2$$

If $\nu = 0$, $D_0 = 1$, $D_2 = G_2$, and so on.

To obtain the coefficients De_{2m},

Let $A = \sum_{r=0}^{\omega} D_{2r} = 1$

Then each $De_{2r} = D_{2r}/A$.

Methods have to be found to increase the accuracy, when the values of H_{ν}, found by the two processes do not agree to the required number of significant figures.

The error in the ratios G_m is due to

(1) rounding off errors in the various calculations, which could be avoided by taking sufficient number of significant figures

and (2) error in the value of the characteristic value a.

Suppose $a = \bar{a} + \lambda$, where a is the real value, $\bar{a}$ is the assumed value and λ is the error.

Let the ratios G_m and H_m evaluated from the 'forward process' be denoted by $G_{m,1}$ and $H_{m,1}$ respectively and their errors from the actual value be $\epsilon_{m,1}$ and $\eta_{m,1}$ respectively.

Hence

$$\epsilon_{m,1} = G_m - G_{m,1} \quad \text{and} \quad \eta_{m,1} = H_m - H_{m,1}$$

Since

$$G_2 = \frac{a}{q} \quad , \quad \epsilon_{2,1} = \frac{\lambda}{q}$$

Again since

$$H_2 = 1/(G_{2,1} + \epsilon_{2,1}) \quad ,$$

$\eta_{2,1} \simeq \epsilon_{2,1} H_{2,1}^2 \simeq -\frac{\lambda}{q} H_2^2$ neglecting powers of λ/q greater than 1 and replacing $H_{2,1}$ by H_2.

From the methods used to obtain the ratios G_m and H_m in the forward process, we obtain

$$\epsilon_{2,1} = \frac{\lambda}{q} \quad , \quad \eta_{2,1} = -\frac{\lambda}{q} H_2^2$$

$$\epsilon_{4,1} = \frac{\lambda}{q}(1+2H_2^2) \quad , \quad \eta_{4,1} = -\frac{\lambda}{q}(H_4^2 + 2H_4^2 H_2^2)$$

. . . .

$$\epsilon_{m,1} = \frac{\lambda}{q}(1 + H_{m-2}^2 + H_{m-2}^2 H_{m-4}^2 + \cdots\cdots + 2H_{m-2}^2 H_{m-4}^2 \cdots H_2^2), \quad m \geq 6$$

$$\eta_{m,1} = -\frac{\lambda}{q} H_m^2 \epsilon_{m,1}$$

If

$$R_{m,1} = H_m^2 + H_m^2 H_{m-2}^2 + \cdots\cdots + 2H_m^2 H_{m-2}^2 \cdots\cdots H_2^2 \quad m \geq 4$$

$$R_{2,1} = H_2^2 \quad , \quad R_{0,1} = 0 .$$

it at once follows that

$$\left.\begin{aligned} \epsilon_{m,1} &= \frac{\lambda}{q}(1 + R_{m-2,1}) \qquad m \geq 6 \\ \epsilon_{4,1} &= \frac{\lambda}{q}(1 + 2R_{2,1}) \\ \epsilon_{2,1} &= \frac{\lambda}{q} \\ \eta_{m,1} &= -\frac{\lambda}{q} R_{m,1} \end{aligned}\right\} \qquad (5.13)$$

Let $H_{m,2}$ and $G_{m,2}$ denote the values of H_m, G_m evaluated from the 'backward process' and let $\eta_{m,2}$, $\epsilon_{m,2}$ denote the respective errors from the real values H_m and G_m so that $\eta_{m,2} = H_m - H_{m,2}$ and $\epsilon_{m,2} = G_m - G_{m,2}$.

If G_w is the last ratio evaluated $G_\omega \simeq 1/v_\omega$

Hence $\epsilon_{\omega,2} = -\frac{\lambda}{q} G_\omega^2$

With the help of the method used in generating the ratios G_m and H_m in the 'backward process' we obtain

$$\eta_{\omega-2,2} = \frac{\lambda}{q}(1+G_\omega^2), \quad \epsilon_{\omega-2,2} = -\frac{\lambda}{q}(G_{\omega-2}^2 + G_{\omega-2}^2 \cdot G_\omega^2)$$

- -

$$\eta_{m,2} = \frac{\lambda}{q}(1+G_{m+2}^2 + G_{m+2}^2 G_{m+4}^2 + \cdots + G_{m+2}^2 G_{m+4}^2 \cdots G_\omega^2), \quad m \geq 4 \qquad (5.14)$$

$$\eta_{2,2} = \frac{\lambda}{q}\cdot\frac{1}{2}(1+G_4^2 + G_4^2 G_6^2 + \cdots + G_4^2 G_6^2 \cdots G_\omega^2)$$

$$\epsilon_{m,2} = -G_m^2 \eta_{m,2}.$$

If

$$R_{m,2} = 1 + G_{m+2}^2 + G_{m+2}^2 G_{m+4}^2 + \cdots + G_{m+2}^2 G_{m+4}^2 \cdots G_\omega^2 \quad m \geq 4$$

$$R_{2,2} = \frac{1}{2}(1+G_4^2 + G_4^2 G_6^2 + \cdots + G_4^2 G_6^2 \cdots G_\omega^2)$$

$$R_{0,2} = 2 + G_2^2 + G_2^2 G_4^2 + \cdots + G_2^2 G_4^2 G_6^2 \cdots G_\omega^2$$

it follows that

$$\eta_{m,2} = \frac{\lambda}{q} R_{m,2}, \quad m \geq 2 \qquad (5.15)$$

$$\epsilon_{m,2} = -\frac{\lambda}{q}(R_{m-2,2} - 1), \quad m \geq 6 \qquad (5.16)$$

$$\epsilon_{4,2} = -\frac{\lambda}{q}(2R_{2,2} - 1)$$

If H_ν is the real value,

$$H_\nu = H_{\nu,1} - \frac{\lambda}{q} R_{\nu,1} \qquad \text{(from equation (5.13))}$$

$$H_\nu = H_{\nu,2} + \frac{\lambda}{q} R_{\nu,2} \qquad \text{(from equation (5.15))}$$

Hence $\frac{\lambda}{q} = (H_{\nu,1} - H_{\nu,2}) / (R_{\nu,1} + R_{\nu,2})$

and $H_\nu = (R_{\nu,2}\, H_{\nu,1} + R_{\nu,1}\, H_{\nu,2}) / (R_{\nu,1} + R_{\nu,2})$

Since the ratios G_m and H_m are so obtained that $G_m > 1$ for $m \leq \nu$, and $G_m < 1$ for $m > \nu$, from the expressions for $R_{\nu,1}$ and $R_{\nu,2}$ it is obvious that $R_{\nu,1}$ is generally < 1 and $R_{\nu,2} > 1$ when $\nu > 2$. Hence, generally $H_{\nu,1}$ will be nearer H_ν than $H_{\nu,2}$.

With this corrected value of H_ν, we now proceed to correct a few of the ratios G_m, H_m, m having values near ν. (As the coefficients rapidly reduce in magnitude when m is much greater than ν, only a few of the ratios H_m, G_m, m near ν, need correction).

Evaluate $H_m = H_{m,1} - \frac{\lambda}{q} R_{m,1} \qquad (m < \nu)$

(as H_m generates the value G_m, only H_m need be corrected)

and

$$G_m = G_{m,2} - \frac{\lambda}{q} (R_{m-2,2} - 1) \qquad (m > \nu)$$

(as G_m generates H_m, only G_m, for a few values of m need be corrected)

After these corrections, it can be checked whether the H_ν obtained above agrees with $H_{\nu,1}$ to the required number of significant figures (which, has been found to be true).

The coefficients D_m are computed and the values of D_m can be checked by the formula

$$v_{m-2} D_{m-2} = D_{m-4} + D_m \qquad m \geq 5$$

$$v_2 D_2 = D_4 + 2 D_0$$

which gives the accuracy of the values found. Now the D_m could be normalised to obtain the coefficients De_m.

The above method could be applied to all the four periodic functions with the appropriate initial relations given by equations (5.7) to (5.10).

Once the coefficients are found the required values are obtained by a method described earlier in this chapter.

CONCLUSION.

The cut-off frequencies for high order circumferential modes have been calculated for various eccentricities of the elliptic duct section.

The cut-off frequencies for even functions decrease with increasing eccentricity as was expected. For odd functions, though for the lowest two orders the eigen frequencies increase with eccentricity, it is of interest to note that the third order eigen frequencies are oscillatory as the eccentricity increases. For higher order odd functions, the eigen frequencies decrease in as much as, for high orders, they assume the same values as those for even functions. As a matter of fact, we find that deforming a circular pipe into an elliptic one of sufficiently large eccentricity produces only a small reduction in the cut-off frequency, provided the area of the pipe section is kept invariable.

TABLE A ORDER 1

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	1.8411838
0.20	0.24260034	1.8156734
0.40	0.34266693	1.7889420
0.50	0.38287512	1.7750756
0.60	0.41915605	1.7608497
0.80	0.48338714	1.7312303
1.00	0.53974928	1.6998831
1.20	0.59049548	1.6665634
1.40	0.63696520	1.6309672
1.50	0.65888119	1.6121994
1.60	0.68003176	1.5927107
1.80	0.72030214	1.5512972
2.00	0.75821848	1.5060681
2.20	0.79411472	1.4561193
2.40	0.82825039	1.4001573
2.50	0.84472419	1.3693482
2.60	0.86083194	1.3362262
2.80	0.89202680	1.2611360
3.00	0.92197298	1.1690700
3.20	0.95078536	1.0472905
3.40	0.97856305	0.8551235
3.50	0.99208932	0.6681237

TABLE A ORDER 2

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	3.0542369
0.20	0.14565701	3.0539037
0.40	0.20495449	3.0529044
0.60	0.24980620	3.0512406
0.80	0.28711696	3.0489144
1.00	0.31958267	3.0459289
1.20	0.34859671	3.0422881
1.40	0.37499295	3.0379964
1.60	0.39931845	3.0330588
1.80	0.42195555	3.0274805
2.00	0.44318412	3.0212673
2.40	0.48221761	3.0069580
2.60	0.50031972	2.9988729
2.80	0.51762964	2.9901742
3.00	0.53423551	2.9808661
3.40	0.56561791	2.9604364
3.60	0.58050941	2.9493195
3.80	0.59493085	2.9376028
4.00	0.60892157	2.9252859
4.20	0.62251533	2.9123674
4.40	0.63574368	2.8988442
4.80	0.66120301	2.8699634
5.00	0.67347879	2.8545916
5.40	0.69721609	2.8219347
5.60	0.70870939	2.8046231
5.80	0.71997108	2.7866341
6.00	0.73101349	2.7679483
6.60	0.76293233	2.7074656
6.80	0.77320043	2.6857302
7.00	0.78329649	2.6631475
7.20	0.79322778	2.6396737
7.40	0.80300100	2.6152594
7.60	0.81262237	2.5898481
8.00	0.83143217	2.5357682
9.00	0.87615681	2.3772493
9.40	0.89321267	2.3016476
9.50	0.89740765	2.2813449
9.60	0.90157582	2.2604100
9.80	0.90983316	2.2164700
10.00	0.91798750	2.1694252
10.20	0.92604152	2.1187662
10.40	0.93399774	2.0638329
10.50	0.93793992	2.0345004
10.60	0.94185854	2.0037481
10.80	0.94962620	1.9373059
11.00	0.95730286	1.8627783
11.20	0.96489059	1.7775484
11.40	0.97239132	1.6773398

TABLE A ORDER 3

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	4.2011889
0.20	0.10614918	4.2011487
0.40	0.14969727	4.2010261
0.60	0.18283122	4.2008184
0.80	0.21053260	4.2005224
1.00	0.23473776	4.2001355
1.20	0.25644327	4.1996545
1.40	0.27624330	4.1990765
1.60	0.29452641	4.1983985
1.80	0.31156358	4.1976175
2.00	0.32755288	4.1967303
2.20	0.34264447	4.1957340
2.40	0.35695552	4.1946253
2.60	0.37057961	4.1934013
3.00	0.39605872	4.1905946
3.20	0.40802988	4.1890056
3.40	0.41955150	4.1872886
3.60	0.44139583	4.1834582
3.80	0.43066229	4.1854405
4.00	0.45178148	4.1813384
4.40	0.47160961	4.1766739
4.60	0.48109551	4.1741230
4.80	0.49032113	4.1714220
5.00	0.49930304	4.1685680
5.20	0.50805621	4.1655578
5.40	0.51659428	4.1623883
5.60	0.52492968	4.1590564
7.00	0.57845435	4.1309375
7.20	0.58551256	4.1262002
7.40	0.59244388	4.1212744
7.60	0.59925460	4.1161572
7.80	0.60595016	4.1108459
8.00	0.61253564	4.1053378
8.20	0.61901582	4.0996302
10.40	0.68446581	4.0229262
10.60	0.68996891	4.0146196
10.80	0.69540892	4.0060792
11.00	0.70078799	3.9973023
11.20	0.70610818	3.9882859
11.40	0.71137143	3.9790272
11.60	0.71657957	3.9695232
14.40	0.78454679	3.8087513
14.60	0.78909683	3.7951187
14.80	0.79361147	3.7811717
15.00	0.79809150	3.7669039
15.20	0.80253769	3.7523082
15.40	0.80695074	3.7373773
15.60	0.81133136	3.7221035
19.20	0.88526328	3.3755673
19.40	0.88912529	3.3512839
19.60	0.89296384	3.3263074
19.80	0.89677921	3.3006012
20.00	0.90057168	3.2741253
20.50	0.90995440	3.2042668
21.00	0.91919953	3.1284798

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
21.50	0.92831063	3.0456770
22.00	0.93729099	2.9543919
22.50	0.94614363	2.8525757
23.00	0.95487134	2.7372327
23.50	0.96347671	2.6037041
24.00	0.97196214	2.4440897
24.50	0.98032987	2.2430914

TABLE A ORDER 4

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	5.3175531
0.50	0.13239212	5.3174442
1.00	0.18641619	5.3171171
1.50	0.22732803	5.3165708
2.00	0.26137446	5.3158037
2.50	0.29098854	5.3148136
3.00	0.31742494	5.3135975
3.50	0.34143320	5.3121520
4.00	0.36350449	5.3104728
4.50	0.38398202	5.3085551
5.00	0.40311719	5.3063934
5.50	0.42110101	5.3039815
6.00	0.43808285	5.3013128
6.50	0.46949707	5.2951743
7.50	0.48410822	5.2916879
8.00	0.49808405	5.2879112
8.50	0.51148282	5.2838342
9.00	0.52435472	5.2794464
10.00	0.54868738	5.2696933
11.00	0.57137285	5.2585557
11.50	0.58217170	5.2524351
12.00	0.59264141	5.2459283
13.00	0.61267989	5.2316972
13.50	0.62228718	5.2239426
14.00	0.63164282	5.2157409
16.00	0.66684223	5.1781307
17.00	0.68328426	5.1562008
17.50	0.69125338	5.1443916
18.00	0.69906844	5.1319948
18.50	0.70673655	5.1189916
19.00	0.71426628	5.1053626
19.50	0.72166515	5.0910884
24.00	0.78338480	4.9301936
24.50	0.78979345	4.9083079
25.00	0.79612707	4.8855310
25.50	0.80238882	4.8618351
26.00	0.80858165	4.8371908
26.50	0.81470828	4.8115668
27.00	0.82077120	4.7849297
32.50	0.88380106	4.4121113
33.00	0.88923469	4.3693114
33.50	0.89462334	4.3246304
34.00	0.89996776	4.2779338
34.50	0.90526859	4.2290677
35.00	0.91052645	4.1778553
35.50	0.91574187	4.1240917
37.00	0.93113831	3.9448857
37.50	0.93618859	3.8780535
38.00	0.94119853	3.8069319
38.50	0.94616845	3.7309226
39.00	0.95109865	3.6492764
39.50	0.95598939	3.5610367
40.00	0.96084092	3.4649520

TABLE A ORDER 5

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	6.4156164
0.50	0.10988328	6.4155583
1.00	0.15493138	6.4153840
1.50	0.18918446	6.4150934
2.00	0.21780251	6.4146865
2.50	0.24279148	6.4141632
3.00	0.26518383	6.4135232
3.50	0.28559510	6.4127665
4.00	0.30442781	6.4118926
4.50	0.32196256	6.4109012
5.00	0.33840444	6.4097921
5.50	0.35390881	6.4085647
6.00	0.36859638	6.4072185
6.50	0.38256538	6.4057529
7.00	0.39589303	6.4041672
7.50	0.40864498	6.4024607
8.00	0.42087589	6.4006324
8.50	0.43263218	6.3986815
10.50	0.47563345	6.3896287
11.00	0.48552085	6.3870466
11.50	0.49510912	6.3843339
12.00	0.50441705	6.3814888
12.50	0.51346156	6.3785096
13.00	0.52225796	6.3753941
13.50	0.53082019	6.3721402
16.50	0.57791967	6.3495727
17.00	0.58514678	6.3452776
17.50	0.59221750	6.3408216
18.00	0.59913871	6.3362009
18.50	0.60591685	6.3314119
19.00	0.61255798	6.3264505
19.50	0.61906730	6.3213127
24.00	0.67254667	6.2663566
24.50	0.67799729	6.2591849
25.00	0.68336281	6.2517784
26.00	0.69385009	6.2362351
26.50	0.69897955	6.2280846
27.00	0.70403485	6.2196721
28.00	0.71393622	6.2020318
36.50	0.78920174	5.9991536
37.00	0.79324230	5.9837049
37.50	0.79724926	5.9677992
38.00	0.80122380	5.9514251
38.50	0.80516707	5.9345706
39.00	0.80908015	5.9172236
39.50	0.81296409	5.8993714
50.50	0.89256410	5.3463365
51.00	0.89596381	5.3118799
51.50	0.89934705	5.2763549
52.00	0.90271400	5.2397126
52.50	0.90606478	5.2018992
53.00	0.90939953	5.1628561
53.50	0.91271837	5.1225190
57.50	0.93870502	4.7428451
58.00	0.94188353	4.6864868

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
58.50	0.94504658	4.6276009
59.00	0.94819418	4.5659528
59.50	0.95132632	4.5012691
60.00	0.95444299	4.4332291
60.50	0.95754418	4.3614518

TABLE A ORDER 6

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	7.5012661
1.00	0.13272230	7.5011267
2.00	0.18687588	7.5007084
3.00	0.22788138	7.5000109
4.00	0.26200134	7.4990341
5.00	0.29167481	7.4977774
6.00	0.31815944	7.4962403
7.00	0.34220639	7.4944221
8.00	0.36430773	7.4923219
9.00	0.38480716	7.4899386
10.00	0.40395629	7.4872710
11.00	0.42194612	7.4843176
12.00	0.43892586	7.4810769
14.00	0.47031033	7.4737252
15.00	0.48489296	7.4696097
16.00	0.49883049	7.4651973
17.00	0.51218056	7.4604849
18.00	0.52499273	7.4554691
19.00	0.53731000	7.4501458
22.00	0.57164733	7.4322842
23.00	0.58232041	7.4256817
24.00	0.59264887	7.4187443
25.00	0.60265407	7.4114648
26.00	0.61235541	7.4038352
27.00	0.62177056	7.3958469
28.00	0.63091574	7.3874002
34.00	0.68087828	7.3289732
35.00	0.68849505	7.3176963
36.00	0.69503534	7.3059411
37.00	0.70320779	7.2936886
38.00	0.71032058	7.2809189
39.00	0.71728145	7.2676106
40.00	0.72409779	7.2537410
50.00	0.78571713	7.0781346
51.00	0.79134719	7.0561698
52.00	0.79690001	7.0332539
53.00	0.80237979	7.0093439
54.00	0.80779057	6.9843951
55.00	0.81313619	6.9583611
56.00	0.81842029	6.9311935
68.00	0.87809873	6.4959515
72.00	0.89683980	6.2927007
73.00	0.90145358	6.2359082
74.00	0.90604000	6.1764261
75.00	0.91059948	6.1140736
76.00	0.91513233	6.0486460
77.00	0.91963875	5.9799100
80.00	0.93300027	5.7509484
82.00	0.94177605	5.5755085
83.00	0.94612393	5.4793951
84.00	0.95044483	5.3767322
85.00	0.95473845	5.2665906
86.00	0.96715938	4.8340809
87.00	0.97155769	4.6718040

TABLE A ORDER 7

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	8.5778365
1.00	0.11618530	8.5777460
2.00	0.16375864	8.5774743
3.00	0.19989264	8.5770214
4.00	0.23004984	8.5763873
5.00	0.25635479	8.5755717
6.00	0.27990146	8.5745745
7.00	0.30134234	8.5733954
8.00	0.32110431	8.5720342
9.00	0.33943509	8.5704906
10.00	0.35670233	8.5687641
11.00	0.37292105	8.5668545
12.00	0.38826997	8.5647611
13.00	0.40285188	8.5624836
14.00	0.41675048	8.5600214
15.00	0.43003503	8.5573739
16.00	0.44276366	8.5545403
19.00	0.47807415	8.5449160
20.00	0.48900925	8.5413306
21.00	0.49957721	8.5375549
22.00	0.50980302	8.5335879
23.00	0.51970898	8.5294283
24.00	0.52931503	8.5250751
25.00	0.53863917	8.5205268
29.00	0.57342055	8.5003524
30.00	0.58155190	8.4948049
31.00	0.58947993	8.4890519
32.00	0.59721426	8.4830912
33.00	0.60476384	8.4769204
34.00	0.61213696	8.4705370
35.00	0.61934134	8.4639381
45.00	0.68349300	8.3853433
46.00	0.68923371	8.3761283
47.00	0.69486926	8.3666455
48.00	0.70040337	8.3568886
49.00	0.70583958	8.3468505
50.00	0.71118130	8.3365241
51.00	0.71643178	8.3259016
65.00	0.78194046	8.1404809
66.00	0.78614750	8.1241109
67.00	0.79030336	8.1072413
68.00	0.79441001	8.0898536
69.00	0.79846940	8.0719280
70.00	0.80248343	8.0534442
71.00	0.80645394	8.0343809
94.00	0.88932407	7.3721437
95.00	0.89268005	7.3300240
96.00	0.89602230	7.2863950
97.00	0.89935125	7.2411899
98.00	0.90266728	7.1943368
99.00	0.90597070	7.1457580
100.00	0.90926177	7.0953695
104.12	0.92270513	6.8663640
106.28	0.92966086	6.7309014
108.51	0.93677739	6.5777960

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
110.80	0.94405724	6.4031149
113.17	0.95150197	6.2013427
115.62	0.95911198	5.9643286
118.15	0.96688648	5.6791437

TABLE A ORDER 8

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	9.6474217
1.00	0.10337725	9.6473594
2.00	0.14580815	9.6471725
3.00	0.17810443	9.6468611
4.00	0.20511471	9.6464250
5.00	0.22872306	9.6458642
6.00	0.24989872	9.6451787
7.00	0.26921929	9.6443683
8.00	0.28706227	9.6434329
9.00	0.30369073	9.6423724
10.00	0.31929681	9.6411867
11.00	0.33402602	9.6398756
12.00	0.34799179	9.6384389
13.00	0.36128461	9.6368765
14.00	0.37397809	9.6351881
15.00	0.38613309	9.6333734
16.00	0.39780066	9.6314323
17.00	0.40902409	9.6293644
18.00	0.41984054	9.6271696
19.00	0.43028215	9.6248474
20.00	0.44037700	9.6223976
24.00	0.47774232	9.6113147
25.00	0.48642291	9.6082211
26.00	0.49486994	9.6049975
27.00	0.50309624	9.6016434
28.00	0.51111354	9.5981583
29.00	0.51893251	9.5945418
30.00	0.52656300	9.5907931
38.00	0.58177546	9.5559747
39.00	0.58804573	9.5510080
40.00	0.59419364	9.5459020
41.00	0.60022369	9.5406559
42.00	0.60614010	9.5352686
43.00	0.61194684	9.5297391
44.00	0.61764771	9.5240664
58.00	0.68805278	9.4268565
59.00	0.69250990	9.4208612
60.00	0.69690185	9.4126956
61.00	0.70123035	9.4043571
62.00	0.70549709	9.3958432
63.00	0.70970369	9.3871510
64.00	0.71385171	9.3782776
85.00	0.78977039	9.1428461
86.00	0.79294891	9.1288355
87.00	0.79609544	9.1145153
88.00	0.79921090	9.0998764
89.00	0.80229618	9.0849090
90.00	0.80535218	9.0696032
91.00	0.80837976	9.0539485
118.15	8.82636570	8.4431612
120.76	0.88924195	8.3580389
123.46	0.89601088	8.2632593
126.25	0.90295427	8.1572811
129.13	0.91003282	8.0382244
132.12	0.91740672	7.9037568

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
135.21	0.92493527	7.7509195
138.41	0.93267647	7.5758536
141.72	0.94063663	7.3733442
145.16	0.95120489	7.0358673
148.72	0.96206046	6.6211722
152.42	0.97310912	6.0889508

TABLE A ORDER 9

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	10.7114340
2.00	0.13145648	10.7112550
4.00	0.18510830	10.7107180
6.00	0.22574309	10.7098230
8.00	0.25956171	10.7085690
10.00	0.28897834	10.7069570
12.00	0.31523939	10.7049850
14.00	0.33908680	10.7026520
16.00	0.36100805	10.6999590
18.00	0.38134318	10.6969040
20.00	0.40034111	10.6934870
22.00	0.41819077	10.6897050
24.00	0.43503970	10.6855580
26.00	0.45100537	10.6810440
28.00	0.46618544	10.6761620
30.00	0.48065806	10.6709100
32.00	0.49449067	10.6652850
34.00	0.50774015	10.6592880
36.00	0.52045540	10.6529140
38.00	0.53267880	10.6461610
46.00	0.57733300	10.6153170
48.00	0.58757457	10.6066330
50.00	0.59749270	10.5975520
52.00	0.60710626	10.5880720
54.00	0.61643239	10.5781880
56.00	0.62548673	10.5678970
58.00	0.63428361	10.5571930
70.00	0.68232848	10.4840190
72.00	0.68964085	10.4702730
74.00	0.69677756	10.4560650
76.00	0.70374600	10.4413850
78.00	0.71055311	10.4262260
80.00	0.71720545	10.4105750
82.00	0.72370924	10.3944230
102.03	0.78182200	10.2017750
104.12	0.78726421	10.1779600
106.28	0.79276982	10.1525590
108.51	0.79834075	10.1254190
110.80	0.80397922	10.0963640
113.17	0.80968780	10.0651930
115.62	0.81546949	10.0316790
145.16	0.87848537	9.4797409
148.72	0.88550162	9.3875735
152.42	0.89270119	9.2838500
156.25	0.90010194	9.1664634
160.23	0.90772244	9.0328054
164.37	0.91609356	8.8617310
168.66	0.92641672	8.6017187
173.13	0.93687494	8.3048276
175.43	0.94215657	8.1388781
177.78	0.94747406	7.9585287
180.17	0.95282792	7.7609791
182.62	0.95821842	7.5424990
185.11	0.96364560	7.2979409
187.65	0.96910914	7.0198755

TABLE A ORDER 10

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	11.7708770
2.00	0.11971369	11.7707440
4.00	0.16869653	11.7703440
5.00	0.18827317	11.7700450
6.00	0.20587741	11.7696790
8.00	0.23688872	11.7687470
10.00	0.26392167	11.7675490
12.00	0.28810482	11.7660830
14.00	0.31011148	11.7643510
15.00	0.32044226	11.7633840
16.00	0.33038226	11.7623510
18.00	0.34922452	11.7600830
20.00	0.36686305	11.7575460
22.00	0.38346829	11.7547400
24.00	0.39917323	11.7516650
25.00	0.40672213	11.7500260
26.00	0.41408411	11.7483190
28.00	0.42828745	11.7447020
35.00	0.47336383	11.7298950
36.00	0.47929917	11.7275050
38.00	0.49084218	11.7225190
40.00	0.50197546	11.7172560
42.00	0.51272823	11.7117140
44.00	0.52312637	11.7058930
45.00	0.52819976	11.7028770
55.00	0.57489827	11.6688130
56.00	0.57920904	11.6650120
58.00	0.58765701	11.6571910
60.00	0.59588317	11.6490770
62.00	0.60389832	11.6406680
64.00	0.61171238	11.6319620
65.00	0.61554692	11.6274970
85.00	0.68359921	11.5219760
86.00	0.68662934	11.5158600
88.00	0.69259791	11.5033760
90.00	0.69844778	11.4905550
92.00	0.70418301	11.4773920
94.00	0.70980747	11.4638820
95.00	0.71257932	11.4569960
120.76	0.77599115	11.2467830
126.25	0.78781939	11.1928490
129.13	0.79384008	11.1630100
132.12	0.79993406	11.1309950
135.21	0.80610342	11.0965750
138.41	0.81235064	11.0594840
141.72	0.81867873	11.0194110
182.62	0.88780980	10.3258700
185.11	0.89179567	10.2618820
187.65	0.89678874	10.1606970
190.25	0.90178710	10.0551970
192.90	0.90679047	9.9449219
195.61	0.91179891	9.8293304
198.37	0.91681287	9.7077798
210.04	0.93694711	9.1441117
213.12	0.94200917	8.9775317

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
216.26	0.94708705	8.7970697
219.48	0.95218305	8.6000578
222.77	0.95729933	8.3830187
226.13	0.96243762	8.1412860
229.57	0.96759911	7.8683559

TABLE A ORDER 11

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	12.8264910
2.00	0.10992353	12.8263890
4.00	0.15498727	12.8260840
6.00	0.18925124	12.8255760
8.00	0.21787723	12.8248640
10.00	0.24287136	12.8239480
12.00	0.26526789	12.8228280
14.00	0.28568102	12.8215040
16.00	0.30451385	12.8199770
18.00	0.32204705	12.8182440
20.00	0.33848570	12.8163080
22.00	0.35398520	12.8141660
24.00	0.36866672	12.8118190
26.00	0.38262699	12.8092660
28.00	0.39504472	12.8065070
30.00	0.40868501	12.8035420
32.00	0.42090248	12.8003690
34.00	0.43264350	12.7969900
42.00	0.47556435	12.7813860
44.00	0.48542646	12.7769600
46.00	0.49498719	12.7723240
48.00	0.50426523	12.7674760
50.00	0.51327742	12.7624150
52.00	0.52203894	12.7571410
54.00	0.53056361	12.7516520
64.00	0.57003930	12.7209640
66.00	0.57737343	12.7141700
68.00	0.58454002	12.7071540
70.00	0.59154623	12.6999170
72.00	0.59839873	12.6924550
74.00	0.60510374	12.6847690
76.00	0.61166710	12.6768560
100.00	0.68108229	12.5636930
102.03	0.68628047	12.5525220
104.12	0.69154418	12.5407370
106.28	0.69687425	12.5282950
108.51	0.70227151	12.5151490
110.80	0.70773677	12.5012490
113.17	0.71327084	12.4865410
145.16	0.77891999	12.2491380
148.72	0.78534433	12.2177750
152.42	0.79184909	12.1840730
156.25	0.79843570	12.1477880
160.23	0.80510585	12.1086380
164.37	0.81186159	12.0662970
168.66	0.81870547	12.0203840
213.12	0.88386696	11.2959330
216.26	0.88879688	11.2011890
210.48	0.89369307	11.1039020
222.77	0.89855258	11.0039250
226.13	0.90337346	10.9009950
229.57	0.90815492	10.7947730
233.09	0.91289765	10.6848150
248.00	0.93155609	10.1936050
251.95	0.93617761	10.0526550

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
256.00	0.94080188	9.9013426
260.15	0.94544103	9.7374431
264.39	0.95010764	9.5581804
268.74	0.95481415	9.3600527
273.21	0.95957217	9.1385525

TABLE A ORDER 12

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	13.8788430
2.00	0.10163350	13.8787640
4.00	0.14336132	13.8785250
6.00	0.17513082	13.8781270
8.00	0.20170737	13.8775700
10.00	0.22494243	13.8768540
12.00	0.24578840	13.8759790
14.00	0.26481272	13.8749440
16.00	0.28238625	13.8737490
18.00	0.29876737	13.8723950
20.00	0.31414478	13.8708820
22.00	0.32866138	13.8692080
24.00	0.34242852	13.8673740
28.00	0.35553505	13.8653800
30.00	0.36805322	13.8632250
32.00	0.38004277	13.8609090
34.00	0.39155377	13.8584330
36.00	0.40262874	13.8557950
38.00	0.41330411	13.8529960
40.00	0.42361142	13.8500340
50.00	0.47906240	13.8288470
52.00	0.48781040	13.8247420
54.00	0.49554145	13.8204720
56.00	0.50346703	13.8160370
58.00	0.51119762	13.8114350
60.00	0.51874284	13.8066680
62.00	0.52611155	13.8017330
80.00	0.58565780	13.7497000
82.00	0.59162710	13.7430600
84.00	0.59748428	13.7362450
86.00	0.60323323	13.7292550
88.00	0.60887765	13.7220890
90.00	0.61442100	13.7147450
92.00	0.61986658	13.7072240
118.15	0.68331973	13.5920550
120.76	0.68807338	13.5787810
123.46	0.69470491	13.5647130
126.25	0.70051384	13.5497880
129.13	0.70640166	13.5339430
132.12	0.71236935	13.5171050
135.21	0.71841789	13.4991950
168.66	0.77664650	13.2719380
173.13	0.78355421	13.2365930
177.78	0.79055296	13.1984560
182.62	0.79764390	13.1572230
187.65	0.80482838	13.1125450
192.90	0.81210807	13.0640170
198.37	0.81948503	13.0111610
248.00	0.88410829	12.1761600
251.95	0.88885251	12.0879920
256.00	0.89350227	11.9998510
260.15	0.89805346	11.9117710
264.39	0.90250518	11.8236650
268.74	0.90686057	11.7352610
273.21	0.91112757	11.6460760

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
302.46	0.93598333	11.0245970
307.79	0.94027463	10.8861910
313.26	0.94469579	10.7290690
318.88	0.94927531	10.5485540
324.65	0.95403660	10.3389920
330.58	0.95899717	10.0932700

TABLE A ORDER 13

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	14.9283740
2.00	0.09452135	14.9283110
4.00	0.13337550	14.9281210
6.00	0.16298830	14.9278040
8.00	0.18778640	14.9273600
10.00	0.20948893	14.9267890
12.00	0.22897980	14.9260910
14.00	0.24678545	14.9252660
16.00	0.26324976	14.9243140
18.00	0.27861224	14.9232350
20.00	0.29304769	14.9220280
22.00	0.30668845	14.9206950
24.00	0.31963760	14.9192340
26.00	0.33197732	14.9176450
28.00	0.34377446	14.9159280
30.00	0.35508421	14.9140840
34.00	0.37641966	14.9100120
36.00	0.38651824	14.9077830
38.00	0.39627772	14.9054260
40.00	0.40572349	14.9029400
42.00	0.41487786	14.9003260
44.00	0.42376056	14.8975820
60.00	0.48682029	14.8709580
62.00	0.49387232	14.8670420
64.00	0.50076912	14.8629930
66.00	0.50751762	14.8588130
68.00	0.51412425	14.8545000
70.00	0.52059500	14.8500550
72.00	0.52693542	14.8454760
94.00	0.58935726	14.7862040
96.00	0.59446058	14.7799940
98.00	0.59948153	14.7736450
100.00	0.60442258	14.7671570
102.03	0.60935941	14.7604260
104.12	0.61436692	14.7533360
106.28	0.61944632	14.7458640
132.12	0.67446578	14.6433030
135.21	0.68041789	14.6293750
138.41	0.68645639	14.6145660
141.72	0.69258402	14.5988090
145.16	0.69880050	14.5820260
148.72	0.70510735	14.5641330
152.42	0.71150637	14.5450390
156.25	0.71799812	14.5246400
198.37	0.78078342	14.2587740
204.08	0.78826186	14.2163790
210.04	0.79584413	14.1703480
216.26	0.80353128	14.1202560
222.77	0.81132454	14.0656080
229.57	0.81922545	14.0058210
236.69	0.82723605	13.9402040
292.21	0.88933272	12.9975590
297.27	0.89346814	12.9323460
302.46	0.89744130	12.8709670
307.79	0.90126979	12.8127920

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
313.26	0.90498178	12.7566360
318.88	0.90861696	12.7006870
324.65	0.91222487	12.6424930
360.82	0.94151808	11.8565620
377.04	0.94683027	11.6323850
384.47	0.95249148	11.3607090
388.26	0.95560657	11.1927000
392.12	0.95932218	10.9675600
396.03	0.96308749	10.7207830
400.00	0.96690236	10.4482210

TABLE A ORDER 14

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	15.9754390
2.00	0.08835126	15.9753870
4.00	0.12470420	15.9752330
6.00	0.15243433	15.9749760
8.00	0.17567541	15.9746160
10.00	0.19603225	15.9741540
12.00	0.21432971	15.9735880
14.00	0.23105889	15.9729200
16.00	0.24654048	15.9721480
18.00	0.26099772	15.9712740
20.00	0.27459353	15.9702960
22.00	0.28745118	15.9692160
24.00	0.29966665	15.9680320
26.00	0.31131646	15.9667450
28.00	0.32246278	15.9653550
30.00	0.33315697	15.9638610
38.00	0.37218249	15.9568510
40.00	0.38114855	15.9548390
42.00	0.38984458	15.9527230
44.00	0.39828890	15.9505030
46.00	0.40649779	15.9481780
48.00	0.41448584	15.9457490
50.00	0.42226614	15.9432160
66.00	0.47824934	15.9191640
68.00	0.48458477	15.9156820
70.00	0.49079423	15.9120940
72.00	0.49688236	15.9083980
74.00	0.50285546	15.9045970
76.00	0.50871652	15.9006880
78.00	0.51447024	15.8966710
106.28	0.58623961	15.8282360
108.51	0.59124748	15.8219140
110.80	0.59633121	15.8152440
113.17	0.60149220	15.8082050
115.62	0.60673189	15.8007710
118.15	0.61205170	15.7929140
120.76	0.61745309	15.7846060
152.42	0.67628422	15.6680020
156.25	0.68268256	15.6518340
160.23	0.68918067	15.6345670
164.37	0.69578005	15.6161050
168.66	0.70248214	15.5963470
173.13	0.70928841	15.5751760
177.78	0.71620025	15.5524660
222.77	0.77544821	15.2942620
229.57	0.78336489	15.2487490
236.69	0.79139812	15.1991470
244.14	0.79954869	15.1449640
251.95	0.80781746	15.0856260
260.15	0.81620539	15.0204620
264.39	0.82044442	14.9854500
336.67	0.88792346	14.0153110
342.94	0.89131162	13.9898550
349.38	0.89456832	13.9686050
356.00	0.89776619	13.9482650

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
362.81	0.90098941	13.9247160
369.82	0.90432708	13.8932470
377.04	0.90786589	13.8488320
425.12	0.93709623	12.9996310
429.54	0.94075852	12.8284400
434.03	0.94443089	12.6437860
438.58	0.94826192	12.4440780
443.21	0.95210233	12.2272000
447.92	0.95600257	11.9905280
452.69	0.95996274	11.7307510

TABLE A ORDER 15

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	17.0203230
2.00	0.08294662	17.0202810
4.00	0.11710281	17.0201540
6.00	0.14317550	17.0199430
8.00	0.16504257	17.0196470
10.00	0.18420905	17.0192670
12.00	0.20144833	17.0188020
14.00	0.21722070	17.0182530
16.00	0.23182668	17.0176190
18.00	0.24547539	17.0169000
20.00	0.25831939	17.0160970
24.00	0.28202937	17.0142360
26.00	0.29305677	17.0131780
28.00	0.30361446	17.0120360
30.00	0.31375048	17.0108090
32.00	0.32350506	17.0094960
34.00	0.33291227	17.0080990
36.00	0.34200134	17.0066170
44.00	0.37563579	16.9998360
46.00	0.38345580	16.9979280
48.00	0.39107015	16.9959340
50.00	0.39849104	16.9938540
52.00	0.40572952	16.9916890
54.00	0.41279558	16.9894380
56.00	0.41969832	16.9871000
78.00	0.48681539	16.9556900
80.00	0.49225313	16.9523120
82.00	0.49759779	16.9488470
84.00	0.50285231	16.9452950
86.00	0.50802124	16.9416540
120.76	0.58645214	16.8641410
123.46	0.59179266	16.8569790
126.25	0.59721939	16.8493980
129.13	0.60273401	16.8413670
132.12	0.60833825	16.8328540
135.21	0.61403385	16.8238230
138.41	0.61982256	16.8142360
173.13	0.67640949	16.6943950
177.78	0.68322919	16.6760980
182.62	0.69016201	16.6564700
187.65	0.69720974	16.6353920
192.90	0.70437410	16.6127290
198.37	0.71165681	16.5883290
204.08	0.71905950	16.5620250
260.15	0.78277332	16.2545380
268.74	0.79131658	16.1988790
273.21	0.79563749	16.1690450
277.78	0.79999136	16.1377600
282.47	0.80437825	16.1049320
287.27	0.80951222	16.0428580
292.21	0.81534671	15.9525810
416.49	0.89608144	15.1737860
420.78	0.89800084	15.1520670
425.12	0.90002891	15.1237840
429.54	0.90217150	15.0883710

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
434.03	0.90443303	15.0452960
438.58	0.90681664	14.9940500
443.21	0.90932433	14.9341390
483.03	0.93603007	13.9290280
488.39	0.93986338	13.7417840
493.83	0.94376230	13.5391860
499.36	0.94772662	13.3191350
504.99	0.95175780	13.0790680
510.71	0.95585639	12.8157870
516.53	0.96002254	12.5252080

TABLE B ORDER 1

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	1.8411838
0.10	0.16937246	1.8535189
0.20	0.23629806	1.8655918
0.30	0.28560255	1.8774147
0.40	0.32556206	1.8889990
0.50	0.35944292	1.9003552
0.60	0.38894995	1.9114932
0.70	0.41511274	1.9224222
0.80	0.43861242	1.9331508
0.90	0.45992878	1.9436871
1.00	0.47941562	1.9540387
1.10	0.49734308	1.9642127
1.20	0.51392315	1.9742158
1.30	0.52932593	1.9840545
1.40	0.54369037	1.9937346
1.60	0.56974693	2.0126414
1.70	0.58161822	2.0218785
1.80	0.59281622	2.0309780
1.90	0.60340199	2.0399443
2.00	0.61342876	2.0487817
2.10	0.62294324	2.0574945
2.20	0.63198663	2.0660865
2.60	0.66412438	2.0993184
2.70	0.67128930	2.1073590
2.80	0.67815286	2.1152987
2.90	0.68473455	2.1231405
3.00	0.69105210	2.1308869
3.50	0.71920057	2.1682766
4.00	0.74308271	2.2017635
4.50	0.76261606	2.2371885
5.00	0.77973611	2.2691442
5.50	0.79461023	2.2996642
6.00	0.80765502	2.3288892
6.50	0.81918877	2.3569397
7.00	0.82945932	2.3839197
12.00	0.89208191	2.6103241
12.50	0.89597243	2.6296956
13.00	0.89960210	2.6486101
13.50	0.90299593	2.6670923
14.00	0.90617582	2.6851650
14.50	0.90919061	2.7025521
15.00	0.91199632	2.7198786
26.50	0.94901976	3.0454551
27.00	0.94905033	3.0572938
27.50	0.95084845	3.0689683
28.00	0.95170987	3.0806534
28.50	0.95257038	3.0916437
29.00	0.95336406	3.1032460
29.50	0.95414781	3.1144025

TABLE B ORDER 2

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	3.0542369
0.10	0.10325981	3.0542514
0.20	0.14563855	3.0542948
0.30	0.17788860	3.0543672
0.40	0.20485130	3.0544684
0.50	0.22840909	3.0545985
1.00	0.31861749	3.0556798
1.50	0.38484009	3.0574702
2.00	0.43818033	3.0599525
2.50	0.48301844	3.0631036
3.00	0.52164759	3.0668952
3.50	0.55546229	3.0712945
4.00	0.58539515	3.0762657
4.50	0.61211249	3.0817706
5.00	0.63611391	3.0877694
5.50	0.65778799	3.0942219
6.00	0.67744551	3.1010883
6.50	0.69534053	3.1083294
7.00	0.71168440	3.1159073
7.50	0.72665529	3.1237860
8.00	0.74040508	3.1319313
9.50	0.77554991	3.1576589
10.00	0.78556109	3.1665738
10.50	0.79485603	3.1756181
11.00	0.80350189	3.1847708
11.50	0.81155815	3.1940130
12.00	0.81907769	3.2033277
12.50	0.82610762	3.2126095
19.50	0.89054981	3.3443387
20.00	0.89354967	3.3535303
20.50	0.89640850	3.3626755
21.00	0.89913535	3.3717728
21.50	0.90173852	3.3808207
22.00	0.90420853	3.3900264
22.50	0.90660389	3.3987639
38.50	0.94964230	3.6574258
39.00	0.95039531	3.6647078
39.50	0.95112788	3.6719467
40.00	0.95184078	3.6791428
40.50	0.95253476	3.6862966
41.00	0.95321054	3.6934088
41.50	0.95386877	3.7004796

TABLE B ORDER 3

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	4.2011889
0.50	0.16713281	4.2009478
1.00	0.23473138	4.2002530
1.50	0.28552961	4.1991467
2.00	0.32748375	4.1976695
2.50	0.36370066	4.1958605
3.00	0.39578543	4.1937573
3.50	0.42469782	4.1913959
4.00	0.45106689	4.1888110
4.50	0.47533190	4.1860354
5.00	0.49781421	4.1831005
5.50	0.51875750	4.1800363
6.00	0.53835188	4.1768712
6.50	0.55674918	4.1736319
7.00	0.57407306	4.1703437
7.50	0.59042594	4.1670305
8.00	0.60589386	4.1637146
8.50	0.62055010	4.1604168
9.00	0.63445775	4.1571564
9.50	0.64767171	4.1539515
10.50	0.67220604	4.1477725
11.00	0.68360735	4.1448271
11.50	0.69447853	4.1419947
12.00	0.70485074	4.1392864
12.50	0.71475237	4.1367118
13.00	0.72420946	4.1342795
13.50	0.73324600	4.1319967
16.50	0.77977776	4.1217141
17.00	0.78641605	4.1205947
17.50	0.79277602	4.1196472
18.00	0.79887108	4.1188711
18.50	0.80471391	4.1182648
19.00	0.81031657	4.1178267
19.50	0.81569045	4.1175543
30.50	0.89320698	4.1460359
31.00	0.89545334	4.1485249
31.50	0.89762468	4.1510875
32.00	0.89972420	4.1537204
32.50	0.90175493	4.1564206
33.00	0.90371976	4.1591850
33.50	0.90562143	4.1620109
52.50	0.94764256	4.3207288
53.00	0.94826016	4.3259195
53.50	0.94886340	4.3311329
54.00	0.94945273	4.3363681
54.50	0.95002850	4.3416240
55.00	0.95059142	4.3468998
55.50	0.95114161	4.3521946

TABLE B ORDER 4

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	5.3175531
0.50	0.13239212	5.3174442
1.00	0.18641616	5.3171177
1.50	0.22732738	5.3165742
2.00	0.26137395	5.3158146
2.50	0.29098715	5.3148402
3.00	0.31742182	5.3136526
3.50	0.34142705	5.3122540
4.00	0.36349344	5.3106466
4.50	0.38396349	5.3088334
5.00	0.40308784	5.3068172
5.50	0.42105658	5.3046015
6.00	0.43801805	5.3021901
6.50	0.45409075	5.2995869
7.00	0.46937110	5.2967963
7.50	0.48393887	5.2938230
8.00	0.49786094	5.2906717
8.50	0.51119404	5.2873476
9.00	0.52398678	5.2838561
9.50	0.53628118	5.2802028
11.00	0.57051893	5.2683313
11.50	0.58114487	5.2640910
12.00	0.59141730	5.2597198
12.50	0.60135640	5.2552244
13.00	0.61098033	5.2506116
13.50	0.62030552	5.2458883
14.00	0.62934636	5.2410615
17.00	0.67831970	5.2103250
17.50	0.68569730	5.2049734
18.00	0.69287336	5.1995754
18.50	0.69985505	5.1941382
19.00	0.70664905	5.1886688
19.50	0.71326164	5.1831743
20.00	0.71969871	5.1776614
26.50	0.78958348	5.1071996
27.00	0.79403693	5.1020694
27.50	0.79837552	5.0970048
28.00	0.80260220	5.0920097
28.50	0.80671933	5.0870877
29.00	0.81073119	5.0822424
29.50	0.81463900	5.0774770
44.50	0.89493703	4.9790747
45.00	0.89669310	4.9773024
45.50	0.89840616	4.9756188
46.00	0.90007744	4.9740225
46.50	0.90170813	4.9725122
47.00	0.90329937	4.9710865
47.50	0.90485230	4.9697438
70.50	0.94876537	4.9746814
71.00	0.94933284	4.9758103
71.50	0.94983940	4.9769683
72.00	0.95043581	4.9781547
72.50	0.95097230	4.9793687
73.00	0.95149912	4.9806099
73.50	0.95201650	4.9818774

TABLE B ORDER 5

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	6.4156164
1.00	0.15493138	6.4153840
2.00	0.21780251	6.4146866
3.00	0.26518381	6.4135238
4.00	0.30442770	6.4118949
5.00	0.33840408	6.4097993
6.00	0.36859593	6.4072364
7.00	0.39589084	6.4042059
8.00	0.42087142	6.4007078
9.00	0.44394528	6.3967426
10.00	0.46541129	6.3923113
11.00	0.48549655	6.3874156
12.00	0.50437863	6.3820579
13.00	0.52219951	6.3762416
14.00	0.53907490	6.3699707
15.00	0.55510058	6.3632506
16.00	0.57035689	6.3560873
17.00	0.58491194	6.3484882
18.00	0.59882408	6.3404617
19.00	0.61214367	6.3320173
20.00	0.62491453	6.3231658
21.00	0.63717500	6.3139189
24.00	0.67121261	6.2839429
25.00	0.68173277	6.2732568
26.00	0.69187752	6.2622519
27.00	0.70166591	6.2509467
28.00	0.71111515	6.2393607
29.00	0.72024077	6.2275141
30.00	0.72905690	6.2154280
37.00	0.78306361	6.1260231
38.00	0.78979024	6.1128508
39.00	0.79629374	6.0996433
40.00	0.80258021	6.0864334
41.00	0.80865628	6.0732412
42.00	0.81452810	6.0600894
43.00	0.82020157	6.0470001
60.00	0.89135668	5.8508400
61.00	0.89432542	5.8415167
62.00	0.89718658	5.8324770
63.00	0.89994410	5.8237218
64.00	0.90260182	5.8152511
65.00	0.90516348	5.8070640
66.00	0.90763273	5.7991593
92.00	0.94886719	5.6794850
93.00	0.94984745	5.6775506
94.00	0.95079794	5.6757674
95.00	0.95171979	5.6741309
96.00	0.95261411	5.6726369
97.00	0.95348193	5.6712812
98.00	0.95432426	5.6700596

TABLE R ORDER 6

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	7.5012661
1.00	0.13272230	7.5011267
2.00	0.18687588	7.5007084
3.00	0.22788138	7.5000109
4.00	0.26200134	7.4990341
5.00	0.29167481	7.4977775
6.00	0.31815943	7.4962406
7.00	0.34220636	7.4944227
8.00	0.36430767	7.4923232
9.00	0.38480703	7.4899413
10.00	0.40395604	7.4872761
11.00	0.42194566	7.4843267
12.00	0.43892506	7.4810922
13.00	0.45501351	7.4775716
14.00	0.47030821	7.4737638
15.00	0.48488969	7.4696679
16.00	0.49882558	7.4652829
17.00	0.51217339	7.4606079
18.00	0.52498249	7.4556421
19.00	0.53729567	7.4503847
22.00	0.57161189	7.4328569
23.00	0.58227384	7.4264278
24.00	0.59258845	7.4197055
25.00	0.60257656	7.4126900
26.00	0.61225702	7.4053818
27.00	0.62164689	7.3977818
28.00	0.63076169	7.3898908
34.00	0.68038793	7.3365358
35.00	0.68791360	7.3266660
36.00	0.69524965	7.3165263
37.00	0.70240339	7.3061212
38.00	0.70938159	7.2954557
39.00	0.71619052	7.2845349
40.00	0.72283741	7.2733435
51.00	0.78647089	7.1361784
52.00	0.79149056	7.1226400
53.00	0.79639535	7.1089691
54.00	0.80118761	7.0951773
55.00	0.80586960	7.0812766
56.00	0.81044344	7.0672791
57.00	0.81491119	7.0531973
81.00	0.89474471	6.7218998
82.00	0.89709964	6.7095441
83.00	0.89939013	6.6973750
84.00	0.90161771	6.6853975
85.00	0.90378390	6.6736159
86.00	0.90589021	6.6620339
87.00	0.90793812	6.6506549
100.00	0.92000056	6.5219259
104.12	0.93534336	6.4885201
108.51	0.94048474	6.4567155
118.15	0.94985191	6.3991205
123.46	0.95408153	6.3738892
129.13	0.95801662	6.3513746
135.21	0.96166600	6.3318067

TABLE B ORDER 7

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	8.5778365
1.00	0.11618530	8.5777459
2.00	0.16375864	8.5774743
3.00	0.19989264	8.5770214
4.00	0.23004984	8.5763873
5.00	0.25635479	8.5755717
6.00	0.27900146	8.5745745
7.00	0.30134234	8.5733954
8.00	0.32110431	8.5720342
9.00	0.33948509	8.5704906
10.00	0.35670233	8.5687642
11.00	0.37292104	8.5668546
12.00	0.38826996	8.5647614
13.00	0.40285136	8.5624842
14.00	0.41675044	8.5600223
15.00	0.43003497	8.5573752
16.00	0.44276356	8.5545424
19.00	0.47807380	8.5449232
20.00	0.48900874	8.5413409
21.00	0.49957648	8.5375694
22.00	0.50980200	8.5336079
23.00	0.51970757	8.5294557
24.00	0.52931312	8.5251118
25.00	0.53863661	8.5205756
29.00	0.57361312	8.5004896
30.00	0.58154244	8.4949785
31.00	0.58946797	8.4892699
32.00	0.59719928	8.4833629
33.00	0.60474521	8.4772568
34.00	0.61211395	8.4709507
35.00	0.61931311	8.4644439
45.00	0.68333046	8.3882119
46.00	0.68904472	8.3794648
47.00	0.69465031	8.3705132
48.00	0.70015060	8.3613575
49.00	0.70554876	8.3519982
50.00	0.71084777	8.3424359
51.00	0.71605048	8.3326713
68.00	0.79189601	8.1373801
69.00	0.79571322	8.1243248
70.00	0.79946710	8.1111197
71.00	0.80315863	8.0977697
72.00	0.80678876	8.0842800
73.00	0.81035839	8.0706561
74.00	0.81386837	8.0569037
96.00	0.87737293	7.7354505
100.00	0.88636707	7.6766013
104.12	0.89490627	7.6170624
108.51	0.90321325	7.5555190
113.17	0.91123715	7.4925074
118.15	0.91902896	7.4286755
123.46	0.92624425	7.3647630
129.13	0.93314573	7.3015705
135.21	0.93960540	7.2399224
141.72	0.94560579	7.1806286

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
148.72	0.95114030	7.1244507
156.25	0.95621253	7.0720784
164.37	0.96083468	7.0241172
173.13	0.96502679	6.9810866

TABLE B ORDER 8

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	9.6474217
2.00	0.14580815	9.6471725
4.00	0.20511471	9.6464250
6.00	0.24989872	9.6451787
8.00	0.28706227	9.6434329
10.00	0.31929681	9.6411667
12.00	0.34799179	9.6384390
14.00	0.37397809	9.6351881
16.00	0.39780065	9.6314323
18.00	0.41984053	9.6271697
20.00	0.44037699	9.6223976
22.00	0.45962226	9.6171141
24.00	0.47774227	9.6113158
26.00	0.49486985	9.6049995
28.00	0.51111337	9.5981619
30.00	0.52656271	9.5907993
32.00	0.54129343	9.5829077
34.00	0.55536983	9.5744828
36.00	0.56884720	9.5655201
38.00	0.58177350	9.5560151
40.00	0.59419067	9.5459628
42.00	0.60613569	9.5353580
44.00	0.61764130	9.5241957
46.00	0.62873671	9.5124705
54.00	0.66950247	9.4598341
56.00	0.67889195	9.4452149
58.00	0.68799513	9.4300018
60.00	0.69682667	9.4149037
62.00	0.70540002	9.3977766
64.00	0.71372752	9.3807571
66.00	0.72182050	9.3631289
82.00	0.77922006	9.2001334
84.00	0.78558718	9.1770522
86.00	0.79179432	9.1533912
88.00	0.79784610	9.1291612
90.00	0.80374674	9.1043751
92.00	0.80950015	9.0790470
94.00	0.81510990	9.0531950
120.76	0.87762232	8.6691909
123.46	0.88273134	8.6282264
126.25	0.88779959	8.5858369
129.13	0.89281716	8.5420731
132.12	0.89777354	8.4970059
135.21	0.90265778	8.4507237
141.72	0.91216454	8.3549972
168.66	0.94156410	8.0051089
173.13	0.94515737	7.9561054
177.78	0.94858349	7.9080349
182.62	0.95184111	7.8611132
187.65	0.95493041	7.8155427
192.90	0.95785297	7.7715094
198.37	0.96061165	7.7291842

TABLE 8 ORDER 9

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	10.7114340
2.00	0.13145658	10.7112550
4.00	0.18510830	10.7107180
6.00	0.22574309	10.7098230
8.00	0.25956171	10.7085690
10.00	0.28897884	10.7069570
12.00	0.31523939	10.7049850
14.00	0.33908680	10.7026520
16.00	0.36100805	10.6999590
18.00	0.38134318	10.6969040
20.00	0.40034111	10.6934870
22.00	0.41819077	10.6897050
24.00	0.43503969	10.6855580
26.00	0.45100586	10.6810440
28.00	0.46618543	10.6761620
30.00	0.48065806	10.6709100
32.00	0.49449066	10.6652860
34.00	0.50774013	10.6592880
36.00	0.52045536	10.6529140
38.00	0.53267874	10.6461630
44.00	0.56674692	10.6236140
46.00	0.57733266	10.6153250
48.00	0.58757407	10.6066440
50.00	0.59749198	10.5975680
52.00	0.60710523	10.5880950
54.00	0.61643094	10.5782210
56.00	0.62548472	10.5679420
70.00	0.68231367	10.4843460
72.00	0.68962185	10.4706920
74.00	0.69675338	10.4565990
76.00	0.70371543	10.4420620
78.00	0.71051473	10.4270760
80.00	0.71715759	10.4116380
82.00	0.72364990	10.3957440
102.03	0.78143897	10.2107030
104.12	0.78681054	10.1886070
106.28	0.79223200	10.1652740
108.51	0.79770262	10.1406260
110.80	0.80322144	10.1145790
113.17	0.80878723	10.0870450
115.62	0.81439845	10.0579320
156.25	0.88976716	9.4910325
160.23	0.89549723	9.4302107
164.37	0.90115751	9.3668839
168.66	0.90673061	9.3011815
173.13	0.91219820	9.2332857
177.78	0.91754128	9.1634380
182.62	0.92274074	9.0919366
204.08	0.94174786	8.7971803
210.04	0.94597905	8.7235746
216.26	0.94998212	8.6509663
222.77	0.95375272	8.5798236
229.57	0.95728977	8.5105846
236.69	0.96059533	8.4436475
244.14	0.96367416	8.3793644

TABLE B ORDER 10

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	11.7708770
2.00	0.11971369	11.7707440
4.00	0.16869654	11.7703440
5.00	0.18827317	11.7700450
6.00	0.20587741	11.7696790
8.00	0.23688872	11.7687470
10.00	0.26392167	11.7675490
12.00	0.28810482	11.7660830
14.00	0.31011148	11.7643510
15.00	0.32044226	11.7633840
16.00	0.33038226	11.7623510
18.00	0.34922452	11.7600830
20.00	0.36686305	11.7575460
22.00	0.38346829	11.7547400
24.00	0.39917323	11.7516650
25.00	0.40672213	11.7500260
26.00	0.41408411	11.7483190
28.00	0.42828745	11.7447020
30.00	0.44185485	11.7408130
35.00	0.47336383	11.7298950
36.00	0.47929917	11.7275050
38.00	0.49084218	11.7225190
40.00	0.50197546	11.7172560
42.00	0.51272823	11.7117140
44.00	0.52313636	11.7058930
45.00	0.52819975	11.7028770
56.00	0.57920897	11.6650140
58.00	0.58765690	11.6571930
60.00	0.59583303	11.6490810
62.00	0.60389811	11.6406730
64.00	0.61171210	11.6319690
65.00	0.61554659	11.6275050
66.00	0.61933420	11.6229670
85.00	0.68359449	11.5220910
86.00	0.68662405	11.5159880
88.00	0.69259127	11.5035370
90.00	0.69843951	11.4907560
92.00	0.70417276	11.4776410
94.00	0.70979484	11.4641900
95.00	0.71256532	11.4573370
123.46	0.78169975	11.2250870
126.25	0.78760948	11.1982720
129.13	0.79358131	11.1697490
132.12	0.79961466	11.1393890
135.21	0.80570869	11.1070510
138.41	0.81186224	11.0725840
141.72	0.81807378	11.0358290
182.62	0.88251861	10.5007790
187.65	0.88902310	10.4263320
192.90	0.89548297	10.3475350
198.37	0.90187628	10.2644040
204.08	0.90817829	10.1770500
210.04	0.91436180	10.0856930
216.26	0.92039765	9.9906754
236.69	0.93731824	9.6891373

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
244.14	0.94246836	9.5855807
251.95	0.94733428	9.4819647
260.15	0.95190012	9.3792327
268.74	0.95615631	9.2783115
277.78	0.96009972	9.1800690
287.27	0.96373339	9.0852829

TABLE B ORDER 11

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	12.8264910
2.00	0.10902353	12.8263900
4.00	0.15498727	12.8260840
6.00	0.18925124	12.8255760
8.00	0.21787723	12.8248640
10.00	0.24287186	12.8239480
12.00	0.26526789	12.8228280
14.00	0.28568102	12.8215040
16.00	0.30451335	12.8199770
18.00	0.32204705	12.8182440
20.00	0.33848570	12.8163080
22.00	0.35398520	12.8141660
24.00	0.36866672	12.8118190
26.00	0.38262699	12.8092660
28.00	0.39594472	12.8065070
30.00	0.40868501	12.8035420
32.00	0.42090248	12.8003690
34.00	0.43264350	12.7969900
36.00	0.44394736	12.7934020
42.00	0.47556435	12.7813860
44.00	0.48542646	12.7769600
46.00	0.49498719	12.7723240
48.00	0.50426523	12.7674760
50.00	0.51327742	12.7624150
52.00	0.52203894	12.7571410
54.00	0.53056361	12.7516520
68.00	0.58454000	12.7071550
70.00	0.59154620	12.6999170
72.00	0.59339869	12.6924560
74.00	0.60510369	12.6847700
76.00	0.61166704	12.6768580
78.00	0.61809419	12.6687180
80.00	0.62439027	12.6603490
102.03	0.68627887	12.5525650
104.12	0.69154219	12.5407900
106.28	0.69687176	12.5283610
108.51	0.70226840	12.5152310
110.80	0.70773237	12.5013530
113.17	0.71326595	12.4866710
115.62	0.71886840	12.4711280
145.16	0.77885285	12.2510080
148.72	0.78525815	12.2201950
152.42	0.79173828	12.1872110
156.25	0.79329295	12.1518700
160.23	0.80492163	12.1139610
164.37	0.81162346	12.0732580
168.66	0.81839717	12.0295120
216.26	0.88106907	11.4469510
222.77	0.88916724	11.3547190
229.57	0.89633237	11.2558010
236.69	0.90343621	11.1500630
244.14	0.91044544	11.0375250
251.95	0.91732179	10.9184060
260.15	0.92402270	10.7931740
287.27	0.94261728	10.3897400

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
292.21	0.94543906	10.3201370
297.27	0.94816915	10.2503790
302.46	0.95080406	10.1806820
307.79	0.95334098	10.1112600
313.26	0.95577785	10.0423290
318.88	0.95811332	9.9740955

TABLE B ORDER 12

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	13.8788430
2.00	0.10163350	13.8787640
4.00	0.14336132	13.8785250
6.00	0.17513082	13.8781270
8.00	0.20170737	13.8775700
10.00	0.22494243	13.8768540
12.00	0.24578840	13.8759790
14.00	0.26481272	13.8749440
16.00	0.28238625	13.8737490
18.00	0.29876737	13.8723950
20.00	0.31414478	13.8708820
22.00	0.32866138	13.8692080
24.00	0.34242852	13.8673740
28.00	0.36805322	13.8632250
30.00	0.38004277	13.8609090
32.00	0.39155377	13.8584330
34.00	0.40262874	13.8557950
36.00	0.41330411	13.8529960
38.00	0.42361142	13.8500340
40.00	0.43357820	13.8469110
50.00	0.47906240	13.8288470
52.00	0.48741040	13.8247420
54.00	0.49554145	13.8204720
56.00	0.50346703	13.8160370
58.00	0.51119762	13.8114350
60.00	0.51874284	13.8066680
62.00	0.52611155	13.8017330
78.00	0.57957226	13.7561660
80.00	0.58565780	13.7497000
82.00	0.59162709	13.7430600
84.00	0.59748427	13.7362450
86.00	0.60323322	13.7292550
88.00	0.60887764	13.7220890
90.00	0.61442098	13.7147460
120.76	0.68807332	13.5787970
123.46	0.69470420	13.5647330
126.25	0.70051291	13.5498150
129.13	0.70640044	13.5339780
132.12	0.71236776	13.5171500
135.21	0.71841581	13.4992550
138.41	0.72454551	13.4802060
168.66	0.77662002	13.2727340
173.13	0.78351864	13.2376720
177.78	0.79050507	13.1999220
182.62	0.79757926	13.1592210
187.65	0.80474094	13.1152800
192.90	0.81198949	13.0677720
198.37	0.81932339	13.0163350
244.14	0.87280407	12.5067550
251.95	0.88066375	12.4056700
260.15	0.88853525	12.2955880
268.74	0.89639225	12.1759310
277.78	0.90420120	12.0462510
287.27	0.91192031	11.9063180
297.27	0.91949908	11.7562250

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
330.58	0.94077708	11.2529650
336.67	0.94402336	11.1634350
342.94	0.94716269	11.0730530
349.38	0.95018843	10.9821700
356.00	0.95309490	10.8911520
362.81	0.95587754	10.8003730
369.82	0.95853305	10.7102050

TABLE 8, ORDER 13

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	14.9283740
2.00	0.09452135	14.9283110
4.00	0.13337550	14.9281210
6.00	0.16298830	14.9278040
8.00	0.18778640	14.9273600
10.00	0.20948893	14.9267890
12.00	0.22897980	14.9260910
14.00	0.24678545	14.9252660
16.00	0.26324976	14.9243140
18.00	0.27861224	14.9232350
20.00	0.29304769	14.9220280
22.00	0.30668845	14.9206950
24.00	0.31963760	14.9192340
26.00	0.33197732	14.9176450
28.00	0.34377446	14.9159280
34.00	0.37641966	14.9100120
36.00	0.38651824	14.9077830
38.00	0.39627772	14.9054260
40.00	0.40572349	14.9029400
42.00	0.41487786	14.9003260
44.00	0.42376056	14.8975820
46.00	0.43238918	14.8947090
58.00	0.47960553	14.8747430
60.00	0.48682029	14.8709580
62.00	0.49387232	14.8670420
64.00	0.50076912	14.8629930
66.00	0.50751762	14.8588130
68.00	0.51412425	14.8545000
70.00	0.52059500	14.8500550
94.00	0.58935726	14.8786204
96.00	0.59446058	14.7799940
98.00	0.59948153	14.7736450
100.00	0.60442258	14.7671570
102.03	0.60935941	14.7604260
104.12	0.61436692	14.7533360
106.28	0.61944631	14.7458640
135.21	0.68041778	14.6293780
138.41	0.68645674	14.6145710
141.72	0.69258381	14.5988150
145.16	0.69880022	14.5820340
148.72	0.70510717	14.5641450
152.42	0.71150585	14.5450540
160.23	0.72458298	14.5028550
198.37	0.78076964	14.2592220
204.08	0.78824236	14.2170190
210.04	0.79581647	14.1712660
216.26	0.80349192	14.1215780
222.77	0.81126834	14.0675200
229.57	0.81914498	14.0086010
236.69	0.82712052	13.9442610
297.27	0.88534953	13.2785050
302.46	0.88961447	13.2114860
307.79	0.89387967	13.1411640
313.26	0.89814052	13.0674160
318.88	0.90239166	12.9901340

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
324.65	0.90662697	12.9092250
330.58	0.91083945	12.8246260
337.04	0.93891686	12.1320940
384.47	0.94259436	12.0209600
392.12	0.94615272	11.9078280
400.00	0.94960867	11.7912850
408.12	0.95416481	11.5827440
416.49	0.95843832	11.3734050
425.12	0.96241719	11.1645200

TABLE B ORDER 14

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	15.9754390
2.00	0.08835126	15.9753870
4.00	0.12470420	15.9752330
6.00	0.15243433	15.9749760
8.00	0.17567541	15.9746160
10.00	0.19603225	15.9741540
12.00	0.21432971	15.9735880
14.00	0.23105889	15.9729200
16.00	0.24654048	15.9721480
18.00	0.26099772	15.9712740
20.00	0.27459353	15.9702960
22.00	0.28745118	15.9692160
24.00	0.29966665	15.9680320
26.00	0.31131646	15.9667450
28.00	0.32246278	15.9653550
30.00	0.33315697	15.9638610
38.00	0.37218249	15.9568510
40.00	0.38114855	15.9548390
42.00	0.38984458	15.9527230
44.00	0.39828890	15.9505030
46.00	0.40649779	15.9481780
48.00	0.41448584	15.9457490
50.00	0.42226614	15.9432160
68.00	0.48458477	15.9156820
70.00	0.49079423	15.9120940
72.00	0.49688286	15.9083980
74.00	0.50285546	15.9045970
76.00	0.50871652	15.9006880
78.00	0.51447024	15.8966710
80.00	0.52012056	15.8925480
104.12	0.58130622	15.8342350
106.28	0.58623961	15.8282360
108.51	0.59124748	15.8219140
110.80	0.59633121	15.8152440
113.17	0.60149220	15.8082050
115.62	0.60672544	15.8009870
118.15	0.61205170	15.7929140
156.25	0.68268252	15.6518350
160.23	0.68918062	15.6345680
164.37	0.69577997	15.6161080
168.66	0.70248204	15.5963500
173.13	0.70928826	15.5751810
177.78	0.71620003	15.5524730
182.62	0.72321871	15.5280850
222.77	0.77544363	15.2944210
229.57	0.78335807	15.2489880
236.69	0.79138792	15.1995070
244.14	0.79953339	15.1455100
251.95	0.80779440	15.0864610
260.15	0.81617051	15.0217440
268.74	0.82466080	14.9506510
336.67	0.88238582	14.2638600
342.94	0.88696664	14.1886700
349.38	0.89155557	14.1092650
356.00	0.89614807	14.0254260

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
362.81	0.90073868	13.9369420
369.82	0.90532096	13.8436170
377.04	0.90988728	13.7452810
425.12	0.93635339	13.0469340
434.03	0.94123180	12.8637360
443.21	0.94653730	12.6327250
452.69	0.95153985	12.3998290
462.48	0.95621087	12.1671500
472.59	0.96052574	11.9372640
483.03	0.96446315	11.7134180

TABLE B ORDER 15

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
0.00	0.00000000	17.0203230
2.00	0.08294662	17.0202810
4.00	0.11710281	17.0201540
6.00	0.14317550	17.0199430
8.00	0.16504257	17.0196470
10.00	0.18420905	17.0192670
12.00	0.20144833	17.0188020
14.00	0.21722070	17.0182530
16.00	0.23182668	17.0176190
18.00	0.24547539	17.0169000
20.00	0.25831939	17.0160970
22.00	0.27047411	17.0152090
24.00	0.28202937	17.0142360
26.00	0.29305677	17.0131780
28.00	0.30361446	17.0120360
30.00	0.31375048	17.0108090
32.00	0.32350506	17.0094960
44.00	0.37563579	16.9998360
46.00	0.38345580	16.9979280
48.00	0.39107015	16.9959340
50.00	0.39849104	16.9938540
52.00	0.40572952	16.9916890
54.00	0.41279558	16.9894380
56.00	0.41969832	16.9871000
76.00	0.48128173	16.9589800
78.00	0.48681559	16.9556900
80.00	0.49225313	16.9523120
82.00	0.49759779	16.9488470
84.00	0.50285281	16.9452950
86.00	0.50802124	16.9416540
88.00	0.51310598	16.9379250
118.15	0.58119615	16.8709120
120.76	0.58645214	16.8641410
123.46	0.59179266	16.8569790
126.25	0.59721939	16.8493980
129.13	0.60273401	16.8413670
132.12	0.60833825	16.8328540
135.21	0.61403385	16.8238230
173.13	0.67640948	16.6943950
177.78	0.68322918	16.6760980
182.62	0.69016200	16.6564710
187.65	0.69720971	16.6353930
192.90	0.70437407	16.6127300
198.37	0.71165675	16.5883310
204.08	0.71905941	16.5620290
251.95	0.77435908	16.3054170
260.15	0.78277042	16.2546450
268.74	0.79131200	16.1990520
277.78	0.79998406	16.1380380
287.27	0.80878657	16.0709100
297.27	0.81771909	15.9968640
307.79	0.82678066	15.9149590
377.04	0.87877903	15.2642840
384.47	0.88365929	15.1819830
392.12	0.88855534	15.0945680

PARAMETER S	ECCENTRICITY	EIGENFREQUENCY
400.00	0.89346455	15.0017010
408.12	0.89838035	14.9030380
416.49	0.90329696	14.7982390
425.12	0.90820674	14.6869830
472.59	0.93410191	13.9054790
483.03	0.93906265	13.6594030
493.83	0.94551320	13.4106950
504.99	0.95071008	13.1622340
516.53	0.95551190	12.9177630
528.47	0.95987977	12.6821280
540.83	0.96377610	12.4616970

TABLES C - EIGEN FREQUENCIES FOR EVEN FUNCTIONS.

m \ e	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
1	1.84118	1.83697	1.82399	1.80163	1.76857	1.72255	1.65971	1.57293	1.44713	1.23890	1.05117
2	3.05424	3.05416	3.05303	3.04785	3.03291	2.99902	2.93325	2.81784	2.62290	2.26842	1.93390
3	4.20119	4.20116	4.20065	4.19816	4.19047	4.16834	4.11558	3.99860	3.76069	3.27818	2.80370
4	5.31755	5.31751	5.31697	5.31444	5.30677	5.28735	5.24099	5.13046	4.87102	4.27764	3.66806
5	6.41562	6.41558	6.41496	6.41211	6.40364	6.38287	6.33561	6.22642	5.95653	5.26935	4.52909
6	7.50127	7.50122	7.50053	7.49734	7.48786	7.46480	7.41345	7.29919	7.01989	6.25412	5.38760
7	8.57784	8.57779	8.57702	8.57348	8.56295	8.53740	8.48086	8.35762	8.06497	7.23217	6.24420
8	9.64742	9.64737	9.64652	9.64262	9.63104	9.60293	9.54085	9.40676	9.09609	8.20352	7.07837
9	10.7114	10.7114	10.7105	10.7062	10.6936	10.6629	10.5951	10.4494	10.1170	9.16815	7.86750
10	11.7709	11.7708	11.7698	11.7652	11.7515	11.7182	11.6448	11.4871	11.1306	10.0932	8.68659
11	12.8265	12.8264	12.8253	12.8204	12.8056	12.7697	12.6907	12.5208	12.1388	10.9734	9.56250
12	13.8788	13.8788	13.8776	13.8723	13.8564	13.8180	13.7332	13.5511	13.1429	10.9733	10.5182
13	14.9284	14.9283	14.9271	14.9214	14.9045	14.8635	14.7730	14.5787	14.1437	12.8320	11.4845
14	15.9754	15.9754	15.9740	15.9680	15.9500	15.9064	15.8103	15.6038	15.1418	13.9325	12.3478
15	17.0203	17.0202	17.0188	17.0124	16.9934	16.9472	16.8454	16.6267	16.1377	15.1242	13.1859

TABLES D - EIGEN FREQUENCIES FOR ODD FUNCTIONS.

m \ e	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
1	1.84118	1.84543	1.85850	1.88137	1.91598	1.96578	2.03702	2.14225	2.31144	2.65073	3.05788
2	3.05424	3.05425	3.05445	3.05536	3.05806	3.06463	3.07914	3.11039	3.18097	3.37472	3.66087
3	4.20119	4.20116	4.20069	4.19870	4.19344	4.18280	4.16500	4.14055	4.11874	4.15408	4.34136
4	5.31755	5.31752	5.31697	5.31447	5.30717	5.29016	5.25585	5.19402	5.09509	4.97410	4.97720
5	6.41562	6.41558	6.41496	6.41212	6.40368	6.38337	6.33975	6.25292	6.09191	5.82354	5.67726
6	7.50127	7.50122	7.50053	7.49734	7.48786	7.46489	7.41455	7.30968	7.09864	6.69411	6.39822
7	8.57784	8.57779	8.57702	8.57348	8.56295	8.53741	8.48114	8.36161	8.10922	7.57976	7.13612
8	9.64742	9.64737	9.64652	9.64262	9.63104	9.60293	9.54093	9.40881	9.12025	8.47614	7.88778
9	10.7114	10.7114	10.7105	10.7062	10.6936	10.6629	10.5952	10.4499	10.1299	9.38011	8.65064
10	11.7709	11.7708	11.7698	11.7652	11.7515	11.7182	11.6448	11.4872	11.1374	10.2894	9.42266
11	12.8265	12.8264	12.8253	12.8204	12.8056	12.7697	12.6907	12.5208	12.1423	11.2022	10.2022
12	13.8788	13.8788	13.8776	13.8723	13.8564	13.8180	13.7332	13.5511	13.1447	12.1174	10.9879
13	14.9284	14.9283	14.9271	14.9214	14.9045	14.8635	14.7732	14.5787	14.1446	13.0341	11.7759
14	15.9754	15.9754	15.9740	15.9680	15.9500	15.9064	15.8103	15.6038	15.1423	13.9515	12.4731
15	17.0203	17.0202	17.0188	17.0124	16.9934	16.9472	16.8454	16.6267	16.1379	14.8692	13.1971

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APPENDIX I.

SOME PROPERTIES OF BESSEL FUNCTIONS.

Bessel functions J_n (x) of the first kind are solutions of the Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

where n is a positive integer.

A few of the important properties of these functions are listed below. These properties have been used frequently in the text.

(1) $J_{-n}(x) = (-1)^n J_n(x)$

(2) $J_n(-x) = (-1)^n J_n(x)$

(3) $J_n'(x) = n J_n(x)/x - J_{n+1}(x)$

(4) $J_n'(x) = -n J_n(x)/x - J_{n-1}(x)$

(5) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(6) $2n J_n(x)/x = J_{n+1}(x) + J_{n-1}(x)$

The formulae (3) - (6) are recurrence relationships.

(7) Addition formulae:

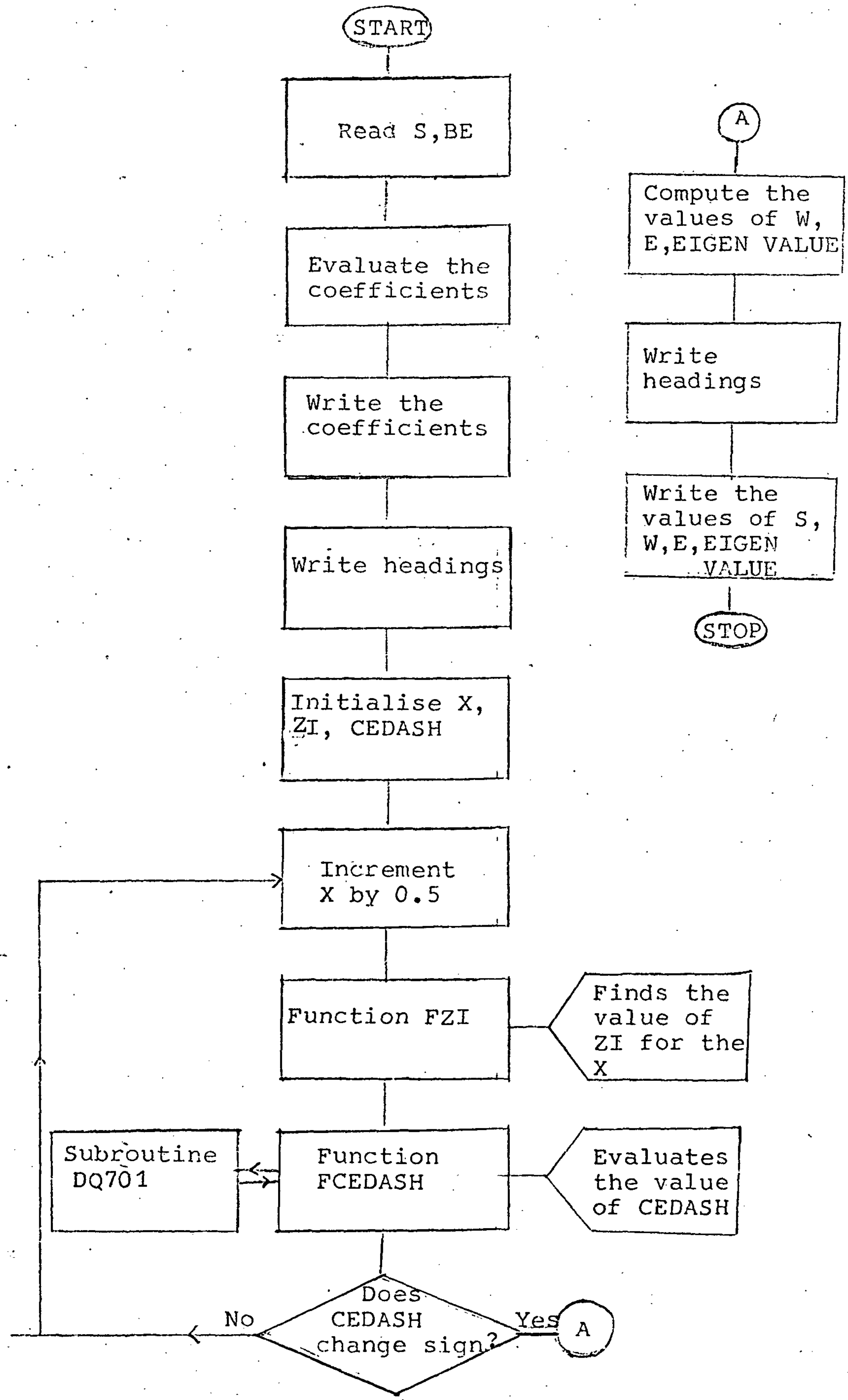
If $R = (x^2 + y^2 - 2xy \cos a)^{1/2}$ then

(i) $J_0(R) = J_0(x) J_0(y) + 2 \sum_{r=1}^{\infty} J_r(x) J_r(y) \cos ra$

and (ii) $R^{-n} J_n(R) = (xy/2)^{-n} \Gamma(n) \sum_{r=0}^{\infty} (r+n) J_{r+n}(x) J_{r+n}(y) C_r^n(\cos a)$,

where C_r^n represents Gegenbauer polynomial and $|x| < |ye^{\pm ia}|$

APPENDIX II. Computer Program Flow-chart.



APPENDIX III. About the Computer Programs.

A In programs EJ and OJ, the eccentricities of the ellipses and the corresponding eigen values for given values of the parameter s were obtained. Programs EJ were for even radial Mathieu functions and Programs OJ were for odd radial Mathieu functions.

The input variables were

(i) S = parameter s

(ii) BE = the characteristic number 'be' or 'bo' according as the function is even or odd

The variables used in the programs were

(i) DE = the coefficients De or Do of expansions, which are computed in the program for each particular s and the corresponding characteristic number.

(ii) X = w = $\sqrt{\Delta} \cosh \xi$

(iii) ZI = ξ

(iv) CEDASH or SEDASH = the value of Je' or Jo' for the X.

The values obtained were

(i) W = W_o = the value $\sqrt{\Delta} \cosh \xi_0$ of w which makes the function zero.

(ii) E = the eccentricity of the ellipse

(iii) EIGEN VALUE = the cut-off frequency corresponding to the eccentricity.

B In program LCM, the cut-off frequencies in elliptic ducts of eccentricities e = .1 (.1) .9 and e = .95 were obtained.

The input variables were

(i) X = the eccentricities obtained from Program EJ or OJ

(ii) Y = the eigen values corresponding to the above eccentricities, also obtained from the Program EJ or OJ

The variables used in the program were

(i) XL = seven values of X, nearest to the eccentricity for which the eigen value is to be evaluated.

(ii) YL = seven values of Y, corresponding to the seven values of X above.

Subroutine INTER evaluates the successive convergents for linear iterated interpolation.

The values obtained were

FUN = the successive convergents and hence also the final convergent for this interpolation.

APPENDIX IV COMPUTER PROGRAM LISTING.

```
      MASTER FJ10
      DOUBLE PRECISION BE,ALPHA,V,G,H,FREM,SREM,H1,H2,ERROR,A,B
      COMMON S(70),DE(70,60),I,PI
      DIMENSION BE(60),ALPHA(60),V(50),G(50),H(50),X(100),ZI(100),
     1CEDASH(100),B(60),FREM(50),SREM(50),U(60),E(60),EIGENVALUE(60),
     2CED(20),Y(20),T(60),BET(60)
      PI=3.1415926536
      READ(1,101)((S(I),BE(I)),I=1,37)
101   FORMAT(F5.2,F12.8)
      READ(1,102)((T(I),BET(I)),I=38,58)
102   FORMAT(F6.5,F12.8)
      DO 30 I=1,58
      IF(I.LT.38)GO TO 47
      S(I)=1/T(I)/T(I)
      BE(I)=BET(I)+21*SQRT(S(I))
47    ALPHA(I)=BE(I)-S(I)/2
      DO 8 M=2,46,2
8     V(M)=4*(ALPHA(I)-(M-2)*(M-2))/S(I)
      G(4)=V(2)
      H(4)=1/G(4)
      G(6)=V(4)-2*H(4)
      H(6)=1/G(6)
      H1=H(4)
      MK=4
      IF(ABS(G(6)).LT.1)GO TO 9
      DO 7 M=8,20,2
      G(M)=V(M-2)-H(M-2)
      H(M)=1/G(M)
      MK=M-2
      H1=H(M-2)
      IF(ABS(G(M)).LT.1)GO TO 9
7     CONTINUE
9     G(46)=1/V(46)
      H(46)=V(46)
      DO 10 M=2,46-MK-2,2
      H(46-M)=V(46-M)-G(48-M)
10    G(46-M)=1/H(46-M)
      H(MK)=V(MK)-G(MK+2)
      IF(MK.EQ.4)H(MK)=0.5*(V(4)-G(6))
      WRITE(2,203)(G(M),M=4,46,2)
      WRITE(2,203)(H(M),M=4,46,2)
203   FORMAT(/4D30.20)
      IF(ABS(H1-H(MK))-.1E-10)80,80,81
81    WRITE(2,203)H1,H(MK)
      H2=H(MK)
      FREM(4)=H(4)*H(4)
      IF(MK.EQ.4)GO TO 21
      FREM(6)=H(6)*H(6)*(1+2*FREM(4))
      DO 221 M=8,MK,2
221   FREM(M)=H(M)*H(M)*(1+FREM(M-2))
```

```
   21 SREM(24)=1+G(26)*G(26)
      DO 22 M=4,26-MK,2
   22 SREM(26-M)=1+G(26-M+2)*G(26-M+2)*SREM(26-M+2)
      H(MK)=(FREM(MK)*H2+SREM(MK)*H1)/(FREM(MK)+SREM(MK))
      ERROR =(H1-H2)/(FREM(MK)+SREM(MK))
      DO 23 M=MK+2,MK+8,2
   23 G(M)=G(M)-ERROR*(SREM(M-2)-1)
      IF(MK.EQ.4)GO TO 28
      DO 13 M=4,MK-2,2
   13 H(M)=H(M)-ERROR*FREM(M)
   28 H(MK)=H1-ERROR*FREM(MK)
      WRITE(2,203)(G(M),M=MK+2,MK+8,2)
       WRITE(2,203)(H(M),M=4,MK,2)
   80 B(MK)=1.0
      DO 12 M=2,MK-2,2
   12 B(MK-M)=H(MK-M+2)*B(MK-M+2)
      DO 14 M=MK,44,2
   14 B(M+2)=B(M)*G(M+2)
      A=B(2)
      DO 11 M=4,46,2
   11 A=A+B(M)
      DO 16 M=2,46,2
   16 DE(I,M)=B(M)/A
      WRITE(2,201)(DE(I,M),M=2,40,2)
  201 FORMAT(1X,6E20.10)
      WRITE(2,901)
  901 FORMAT(1X,'X',20X,'ZI',20X,'CEDASH')
      X(1)=SQRT(S(I))
      ZI(1)=FZI(X(1))
      CEDASH(1)=0.5
      WRITE(2,900)X(1),ZI(1),CEDASH(1)
      DO 20 J=1,49
      X(J+1)=X(J)+0.5
       IF(I.GE.45)X(J+1)=X(J)+0.1
       MAX=J+1
      ZI(J+1)=FZI(X(J+1))
      CEDASH(J+1)=FCEDASH(X(J+1))
      WRITE(2,900) X(J+1),ZI(J+1),CEDASH(J+1)
  900 FORMAT(3E20.8)
       IF(CEDASH(J+1))370,371,20
  370 IF((J+1).EQ.2)GO TO 281
   43 CED(1)=CEDASH(MAX-1)
        CED(2)=CEDASH(MAX)
        Y(1)=X(MAX-1)
       Y(2)=X(MAX)
   45  DO 5 K=1,18
      Y(K+2)=(Y(K+1)*CED(1)-Y(1)*CED(K+1))/(CED(1)-CED(K+1))
       W(I)=Y(K+2)
      CED(K+2)=FCEDASH(Y(K+2))
       IF(ABS(CED(K+2))-.1E-03) 42,42,6
```

```
    6  IF(K.EQ.18)WRITE(2,904) W(I),CED(K+2 )
  904  FORMAT(1X,'VALUE OF CEDASH NOT ACCURATE',2E20.8)
    5  CONTINUE
       GO TO 42
  371  W(I)=X(MAX)
   42  E(I)=SQRT(S(I))/F(I)
       EIGENVALUE(I)=W(I)*SQRT(SQRT(1.-E(I)*F(I)))
       WRITE(2,902)
  902  FORMAT(3X,1HS,12X,1HW,18X,1HE,14X,10HEIGENVALUE)
       WRITE(2,903)S(I),W(I),E(I),EIGENVALUE(I)
  903  FORMAT(1X,F5.2,3E20.8)
       GO TO 30
  281  X(1)=X(J)
       DO 500 J1=1,9
       MAX=J1+1
       X(J1+1)=X(J1)+0.01
       CEDASH(J1+1)=FCEDASH(X(J1+1))
       WRITE(2,906)X(MAX),ZI(MAX),CEDASH(MAX)
       IF(CEDASH(J1+1))43,371,51
   51  IF(J1.EQ.9)CED(1)=CEDASH(MAX)
       IF(J1.LT.9)GO TO 500
       CED(2)=CEDASH(J+1)
       Y(1)=X(MAX)
       Y(2)=CEDASH(J+1)
       GO TO 45
  500  CONTINUE
   20  CONTINUE
   30  CONTINUE
       STOP
       END
       FUNCTION FZI(X)
         COMMON S(70),DE(70,60),I,PI
       FZI=ACOSH(X/SQRT(S(I)))
       RETURN
        END
       FUNCTION FCEDASH(X)
         COMMON S(70),DE(70,60),I,PI
         DIMENSION SOL(46),BJ(200)
       ZI=FZI(X)
        CALL DG701(X,56,BJ,.1E-09,IER)
       SOL(2)=-2*DE(I,2)*BJ(2)
       SOL(4)=SOL(2)-DE(I,4)*(BJ(2)-BJ(4))
       SOL(6)=SOL(4)+DE(I,6)*(BJ(4)-BJ(6))
       SOL(8)=SOL(6)-DE(I,8)*(BJ(6)-BJ(8))
        SOL(10)=SOL(8)+DE(I,10)*(BJ(8)-BJ(10))
         DO 8 K=12,46,2
    8   SOL(K)=SOL(K-2)-ABS(DE(I,K))*(BJ(K-2)-BJ(K))
        FCEDASH=-0.5*SQRT(S(I)*PI/2)*SINH(ZI)*SOL(46)
       RETURN
       END
```

```
      SUBROUTINE DO701(X,N,BJ,A,IER)
      DIMENSION ,BJ(200)
      D=A
      DO 4 K=1,200
 4    BJ(K)=0.0
      Y=X
      IF(X)1,2,40
 2    BJ(1)=1.0
      RETURN
 1    X=-X
   40 IER=0
      N1=N+1
      BPREV=.0
C     COMPUTE STARTING VALUE OF M
       AN=N
      IF(X-AN) 20,20,21
 20   MZERO=N+10
      GO TO 74
 21    MZERO=X+20
   74 MMAX=199
  100 DO 190 M=MZERO,MMAX,3
C     OVERFLOW TRAP  BASED ON 1.E70 .
 23   XOMIN=EXP(ALOG(1.E-70)/M)
      XOM=X/M
      IF(XOM-XOMIN)22,22,130
 22   M=M-3
      D=10.
      IER=M
      GO TO 23
C     SET F(M),F(M-1)
 130  BJ(M+1)=0.
      BJ(M)=1.0E-70
      M2=M-1
      DO 160 K=1,M2
      MK=M-K
 160  BJ(MK)=2.* MK    *BJ(MK+1)/X-BJ(MK+2)
      SUM=-BJ(1)
      DO 161 K=1,M,2
 161  SUM=2.0*BJ(K)+SUM
      DO 162 K=1,M
 162  BJ(K)=BJ(K)/SUM
      IF(ABS(BJ(N1)-BPREV)-ABS(D*BJ(N1))) 10,10,190
 190  BPREV=BJ(N1)
      IER=199
 10   IF(Y-X) 201,200,201
 201  DO 202 K=2,M,2
  202 BJ(K)=-BJ(K)
 200  RETURN
      END
      FINISH
```

```
      MASTER OJ1
      DOUBLE PRECISION BE,ALPHA,V,G,H,FREM,SREM,H1,H2,A,B
      COMMON S(50),DE(50,50),I,PI
      DIMENSION BE(50),ALPHA(50),V(50),G(50),H(50),X(100),ZI(100),
     1SEDASH(100),W(50),E(50),EIGENVALUE(50),CED(20),Y(20),B(50),
     2FREM(30),SREM(40),SE(100),GE(50),DELTA(50)
      PI=3.1415926536
      READ(1,100)(S(I),I=1,48)
  100 FORMAT(16F5.2)
      READ(1,101)(BE(I),I=1,31,5)
      READ(1,101)(BE(I),I=32,50)
  101 FORMAT(6F12.8)
      READ(1,102)(DELTA(I),I=1,31,5)
  102 FORMAT(6F12.8)
      DO 30 I=2,50
      IF(I.LT.6)GO TO 51
      IF(I.EQ.6)GO TO 47
      IF(I.LT.11.AND.I.GT.6)GO TO 52
      IF(I.EQ.11)GO TO 47
      IF(I.GT.11.AND.I.LT.16)GO TO 53
      IF(I.EQ.16)GO TO 47
      IF(I.GT.16.AND.I.LT.21)GO TO 54
      IF(I.EQ.21)GO TO 47
      IF(I.GT.21.AND.I.LT.26)GO TO 55
      IF(I.EQ.26)GO TO 47
      IF(I.GT.28.AND.I.LT.31)GO TO 56
      IF(I.GE.31)GO TO 47
   51 M=6
      GO TO 58
   52 M=11
      GO TO 58
   53 M=16
      GO TO 58
   54 M=21
      GO TO 58
   55 M=26
      GO TO 58
   56 M=31
   58 P=(S(I)-S(M-5))/0.5
      E1=P*(1-P*P)/6
      F1=(1-P)*(1-(1-P)*(1-P))/6
      BE(I)=(1-P)*BE(M-5)+P*BE(M)-E1*DELTA(M)-F1*DELTA(M-5)
   47 ALPHA(I)=BE(I)-S(I)/2
      DO 8 M=1,35,2
    8 V(M)=4*(ALPHA(I)-M*M)/S(I)
      G(3)=V(1)+1
      H(3)=1/G(3)
      H1=H(3)
      MK=3
      IF(ABS(G(3)).LT.1)GO TO 9
```

```
      DO 7 M=5,21,2
      G(M)=V(M-2)-H(M-2)
      H(M)=1/G(M)
      H1=H(M-2)
      MK=M-2
      IF(ABS(G(M)).LT.1)GO TO 9
    7 CONTINUE
    9 G(35)=1/V(35)
      H(35)=V(35)
      DO 10 M=2,35-MK-2,2
      H(35-M)=V(35-M)-G(37-M)
   10 G(35-M)=1/H(35-M)
      H(MK)=V(MK)-G(MK+2)
      WRITE(2,203)(G(M),M=3,35,2)
      WRITE(2,203)(H(M),M=3,35,2)
  203  FORMAT(1X,4D30.20)
      IF(ABS(H1-H(MK))-.1E-10)80,80,81
   81 WRITE(2,203)H1,H(MK)
      H2=H(MK)
      FREM(3)=H(3)*H(3)
     IF(MK.EQ.3)GO TO 21
     DO 28 M=5,MK,2
   28 FREM(M)=H(M)*H(M)*(1+FREM(M-2))
   21 SREM(11)=1+G(13)*G(13)
      DO 22 M=4,13-MK,2
   22 SREM(13-M)=1+G(13-M+2)*G(13-M+2)*SREM(13-M+2)
      H(MK)=(FREM(MK)*H2+SREM(MK)*H1)/(FREM(MK)+SREM(MK))
      ERROR=(H1-H2)/(FREM(MK)+SREM(MK))
      DO 23 M=MK+2,MK+8,2
   23 G(M)=G(M)-ERROR*(SREM(M-2)-1)
      IF(MK.EQ.3)GO TO 25
      DO 26 M=5,MK,2
   26 H(M)=H(M)-ERROR*FREM(M)
   25 H(MK)=H1-ERROR*FREM(MK)
      WRITE(2,203)(G(M),M=MK+2,MK+8,2)
      WRITE(2,203)(H(M),M=3,MK,2)
   80 B(MK)=1.0
      DO 78 M=2,MK+1,2
   78 B(MK-M)=B(MK-M+2)*H(MK-M+2)
      DO 14 M=MK,33,2
   14 B(M+2)=B(M)*G(M+2)
      A=B(1)
      DO 11 M=3,35,2
   11  A=A+M*B(M)
      DO 16 M=1,35,2
   16  DE(I,M)=B(M)/A
      WRITE(2,201)(DE(I,M),M=1,35,2)
  201 FORMAT(1X,6E20.10)
     WRITE(2,901)
  901 FORMAT(1H ,11X,1HX,18X,2HZI,17X,4HSE01,17X,6HSERASH)
```

```
      X(1)=SQRT(S(I))
      ZI(1)=FZI(X(1))
      SE(1)=0.
       GE(I)=FGE(S(I))
      SEDASH(1)=SQRT(S(I))/GE(I)
      WRITE(2,900)X(1),ZI(1),SE(1),SEDASH(1)
      DO 20 J=1,49
      X(J+1)=X(J)+0.50
      ZI(J+1)=FZI(X(J+1))
       SE(J+1)=FSE(X(J+1))
      SEDASH(J+1)=FSODASH(X(J+1))
      WRITE(2,900)X(J+1),ZI(J+1),SE(J+1),SEDASH(J+1)
  900 FORMAT(1H ,4E20.8)
      IF(SEDASH(J+1))40,41,20
   40 CED(1)=SEDASH(J)
      CED(2)=SEDASH(J+1)
      Y(1)=X(J)
      Y(2)=X(J+1)
       DO 5 L=1,18
       Y(L+2)=(Y(L+1)*CED(1)-Y(1)*CED(L+1))/(CED(1)-CED(L+1))
       W(I)=Y(L+2)
       CED(L+2)=FSODASH(Y(L+2))
       IF(ABS(CED(L+2))-.1E-08)42,42,6
    6  IF(L.EQ.18)WRITE(2,904)W(I),CED(L+2)
  904 FORMAT(1H ,'VALUE OO SEDASH IS NOT ACCURATE',2E20.10)
    5 CONTINUE
      GO TO 42
   41 W(I)=X(J+1)
   42 E(I)=SQRT(S(I))/W(I)
      EIGENVALUE(I)=W(I)*(SQRT(SQRT(1.-E(I)*E(I))))
      WRITE(2,902)
  902 FORMAT(1H ,2X,1HS,12X,1HW,18X,1HE,14X,10HEIGENVALUE)
      WRITE(2,903)S(I),W(I),E(I),EIGENVALUE(I)
  903 FORMAT(1H ,F5.2,3E20.8)
      GO TO 30
   20 CONTINUE
   30 CONTINUE
      STOP
      END
      FUNCTION FZI(X)
       COMMON S(50),DE(50,50),I,PI
      FZI=ACOSH(X/SQRT(S(I)))
      RETURN
      END
        FUNCTION FSE(X)
       COMMON S(50),DE(50,50),I,PI
       DIMENSION SO(50),BJ(200)
      ZI=FZI(X)
      CALL DO701(X,35,BJ,0.1E-09,IER)
       SO(1)=DE(I,1)*BJ(2)
```

```
      DO 6 K=3,29,2
    6  SO(K)=SO(K-2)+ABS(DE(I,K))*K*BJ(K+1)
      FSE=SQRT(PI/2)*TANH(ZI)*SO(29)
      RETURN
      END
      FUNCTION FSODASH(X)
       DIMENSION SOL(50),BJ(200)
       COMMON S(50),DE(50,50),I,PI
      ZI=FZI(X)
      CALL DO701(X,35,BJ,0.1E-09,IER)
       SOL(1)=DE(I,1)*(BJ(1)-BJ(3))
       DO 8 K=3,29,2
    8  SOL(K)=SOL(K-2)+K*ABS(DE(I,K))*(BJ(K)-BJ(K+2))
       SE=FSE(X)
       FSODASH=SE/SINH(ZI)/COSH(ZI)+0.5*SQRT(PI*S(I)/2)*SINH(ZI)*
     1 TANH(ZI)*SOL(29)
      RETURN
      END
       FUNCTION FGE(X)
       COMMON S(50),DE(50,50),I,PI
       DIMENSION SOL(50)
       SOL(1)=DE(I,1)
       DO 6 K=3,29,2
    6   SOL(K)=SOL(K-2)+ABS(DE(I,K))
      FGE=  SQRT(2/PI)*2*SOL(29)/DE(I,1)
      RETURN
      END
      SUBROUTINE DO701(X,N,BJ,A,IER)
      DIMENSION  BJ(200)
      D=A
      DO 4 K=1,200
    4 BJ(K)=0.0
      Y=X
      IF(X)1,2,40
    2 BJ(1)=1.0
      RETURN
    1 X=-X
   40 IER=0
      N1=N+1
      BPREV=.0
C     COMPUTE STARTING VALUE OF M
       AN=N
      IF(X-AN) 20,20,21
   20 MZERO=N+10
      GO TO 74
   21  MZERO=X+20
   74 MMAX=199
  100 DO 190 M=MZERO,MMAX,3
C     OVERFLOW TRAP  BASED ON 1.E70
   23 XOMIN=EXP(ALOG(1.E-70)/N)
```

```
      XOM=X/M
      IF(XOM-XOMTP)22,22,130
22    M=M-3
      D=10.
      IER=M
      GO TO 23
C     SET F(M),F(M-1)
130   BJ(M+1)=0.
      BJ(M)=1.0E-70
      M2=M-1
      DO 160 K=1,M2
      MK=M-K
160   BJ(MK)=2.* MK    *BJ(MK+1)/X-BJ(MK+2)
      SUM=-BJ(1)
      DO 161 K=1,M,2
161   SUM=2.0*BJ(K)+SUM
      DO 162 K=1,M
162   BJ(K)=BJ(K)/SUM
      IF(ABS(BJ(N1)-BPREV)-ABS(D*BJ(N1))) 10,10,190
190   BPREV=BJ(N1)
      IER=199
10    IF(Y-X) 201,200,201
201   DO 202 K=2,M,2
 202  BJ(K)=-BJ(K)
200   RETURN
      END
      FINISH
```

```
      MASTER OJ9
      DOUBLE PRECISION BE,ALPHA,V,G,H,FREM,SREM,H1,H2,A,B
       COMMON S(60),PE(60,60),I,PI
      DIMENSION BE(60),ALPHA(60),V(60),G(60),H(60),X(100),ZI(100),
     1SEDASH(100),W(60),E(60),EIGENVALUE(60),CFD(20),Y(20),B(60),
     2 FREM(30),SREM(60),T(60),BET(60),SF(100),GE(60)
      PI=3.1415926536
       READ(1,101)((S(I),PE(I)),I=1,34)
  101 FORMAT(F5.2,F12.8)
       READ(1,102)((T(I),BET(I)),I=35,55)
  102 FORMAT(F6.5,F12.8)
       DO 30 I=1,55
       IF(I.LE.34)GO TO 47
       S(I)=1/T(I)/T(I)
       BE(I)=BET(I)+1/*SQRT(S(I))
   47 ALPHA(I)=BE(I)-S(I)/2
      DO 8 M=1,51,2
    8 V(M)=4*(ALPHA(I)-M*M)/S(I)
      G(3)=V(1)+1
      H(3)=1/G(3)
      H1=H(3)
      MK=3
      IF(ABS(G(3)).LT.1)GO TO 9
      DO 7 M=5,21,2
      G(M)=V(M-2)-H(M-2)
      H(M)=1/G(M)
      H1=H(M-2)
      MK=M-2
      IF(ABS(G(M)).LT.1)GO TO 9
    7 CONTINUE
    9 G(51)=1/V(51)
      H(51)=V(51)
      DO 10 M=2,51-MK-2,2
      H(51-M)=V(51-M)-G(53-M)
   10 G(51-M)=1/H(51-M)
      H(MK)=V(MK)-G(MK+2)
      N1=39
      WRITE(2,203)(G(M),M=3,N1,2)
      WRITE(2,203)(H(M),M=3,N1,2)
  203 FORMAT(1X,4D30.20)
      IF(ABS(H1-H(MK))-.1E-10)80,80,81
   81 WRITE(2,203)H1,H(MK)
      H2=H(MK)
      FREM(3)=H(3)*H(3)
      IF(MK.EQ.3)GO TO 21
      DO 28 M=5,MK,2
   28 FREM(M)=H(M)*H(M)*(1+FREM(M-2))
   21 SREM(MK+10)=1+G(MK+12)*G(MK+12)
      DO 22 M=4,12,2
   22 SREM(MK+12-M)=1+G(MK+12-M)*G(MK+12-M)*SREM(MK+12-M+2)
```

```
      H(MK)=(FREM(MK)*H2+SREM(MK)*H1)/(FREM(MK)+SREM(MK))
      ERROR=(H1-H2)/(FREM(MK)+SREM(MK))
      DO 23 M=MK+2,MK+8,2
   23 G(M)=G(M)-ERROR*(SREM(M-2)-1)
      IF(MK.EQ.3)GO TO 25
      DO 26 M=5,MK,2
   26  H(M)=H(M)-ERROR*FREM(M)
   25 H(MK)=H1-ERROR*FREM(MK)
      WRITE(2,203)(G(M),M=MK+2,MK+8,2)
      WRITE(2,203)(H(M),M=3,MK,2)
   80 B(MK)=1.0
      DO 78 M=2,MK-1,2
   78 B(MK-M)=B(MK-M+2)*H(MK-M+2)
      DO 14 M=MK,51,2
   14  B(M+2)=B(M)*G(M+2)
      A=B(1)
      DO 11 M=3,51,2
   11 A=A+M*B(M)
      DO 16 M=1,51,2
   16  DE(I,M)=B(M)/A
      WRITE(2,201)(DE(I,M),M=1,N1,2)
  201 FORMAT(1X,6E20.10)
      WRITE(2,901)
  901 FORMAT(1H ,11X,1HX,15X,2HZI,17X,4HSE09,17X,6HSEDASH)
      X(1)=SQRT(S(I))
      ZI(1)=FZI(X(1))
      SE(1)=0.
      GE(I)=FGE(S(I))
      SEDASH(1)=SQRT(S(I))/GE(I)*S(I)*S(I)*S(I)*S(I)
      WRITE(2,900)X(1),ZI(1),SE(1),SEDASH(1)
  900 FORMAT(1H ,4E20.8)
      DO 20 J=1,49
      X(J+1)=X(J)+0.50
      ZI(J+1)=FZI(X(J+1))
      SE(J+1)=FSE(X(J+1))
      SEDASH(J+1)=FSODASH(X(J+1))
      WRITE(2,900)X(J+1),ZI(J+1),SE(J+1),SEDASH(J+1)
      IF(SEDASH(J+1))40,41,20
   40 CED(1)=SEDASH(J)
      CED(2)=SEDASH(J+1)
      Y(1)=X(J)
      Y(2)=X(J+1)
      DO 5 L=1,18
      Y(L+2)=(Y(L+1)*CED(1)-Y(1)*CED(L+1))/(CED(1)-CED(L+1))
      CED(L+2)=FSODASH(Y(L+2))
      W(I)=Y(L+2)
      IF(ABS(CED(L+2))-.1E-08)42,42,6
    6 IF(K.EQ.18)WRITE(2,904)W(I),CED(L+2)
  904  FORMAT(1X,'VALUE OF SEDASH IS NOT ACCURATE',2E20.10)
    5 CONTINUE
```

```
      GO TO 42
   41 W(I)=X(J+1)
   42 E(I)=SQRT(S(I))/K(I)
      EIGENVALUE(I)=W(I)*SQRT(SQRT(1.-F(I)*E(I)))
      WRITE(2,902)
  902 FORMAT(1H ,2X,1HS,12X,1HW,13X,1HF,14X,10HEIGENVALUE)
      WRITE(2,903)S(I),W(I),E(I),EIGENVALUE(I)
  903 FORMAT(1H ,F5.2,3E20.8)
      GO TO 30
   20 CONTINUE
   30 CONTINUE
      STOP
      END
      FUNCTION FZI(X)
        COMMON S(60),DE(60,60),I,PI
      FZI=ACOSH(X/SQRT(S(I)))
      RETURN
      END
       FUNCTION FSE(X)
       COMMON S(60),DE(60,60),I,PI
       DIMENSION SO(60),BJ(200)
      ZI=FZI(X)
       CALL DG701(X,57,BJ,.1E-10,IER)
       SO(1)=DE(I,1)*BJ(2)
       SO(3)=SO(1)-3*DE(I,3)*BJ(4)
       SO(5)=SO(3)+5*DE(I,5)*BJ(6)
       SO(7)=SO(5)-7*DE(I,7)*BJ(8)
       SO(9)=SO(7)+9*DE(I,9)*BJ(10)
       DO 6 K=11,39,2
    6  SO(K)=SO(K-2)+K*ABS(DE(I,K))*BJ(K+1)
       K=39
       FSE= SQRT(PI/2)*TANH(ZI)*SO(K)
      RETURN
      END
      FUNCTION FSODASH(X)
        COMMON S(60),DE(60,60),I,PI
          DIMENSION SOL(60),BJ(200)
      ZI=FZI(X)
       CALL DG701(X,57,BJ,.1E-10,IER)
       SOL(1)=DE(I,1)*(BJ(1)-BJ(3))
       SOL(3)=SOL(1)-3*DE(I,3)*(BJ(3)-BJ(5))
       SOL(5)=SOL(3)+5*DE(I,5)*(BJ(5)-BJ(7))
       SOL(7)=SOL(5)-7*DE(I,7)*(BJ(7)-BJ(9))
       SOL(9)=SOL(7)+9*DE(I,9)*(BJ(9)-BJ(11))
        DO 8 K=11,39,2
    8   SOL(K)=SOL(K-2)+K*ABS(DE(I,K))*(BJ(K)-BJ(K+2))
       SE=FSE(X)
       K=39
       FSODASH=SE/SINH(ZI)/COSH(ZI)+0.5*SQRT(PI*S(I)/2)*SINH(ZI) *
     1  TANH(ZI)*SOL(K)
```

```
      RETURN
      END
      FUNCTION FGE(X)
       COMMON S(60),DE(60,60),I,PI
       DIMENSION SOL(60)
      SOL(1)=DE(I,1)
      SOL(3)=SOL(1)-DE(I,3)
      SOL(5)=SOL(3)+DE(I,5)
      SOL(7)=SOL(5)-DE(I,7)
       SOL(9)=SOL(7)+DE(I,9)
       DO 6 K=11,39,2
    6 SOL(K)=SOL(K-2)+ABS(DE(I,K))
      FGE=SQRT(2/PI)*2*SOL(39)*X*X*X*X/DE(I,1)
      RETURN
      END
      SUBROUTINE DO701(X,N,BJ,A,IER)
      DIMENSION  BJ(200)
      D=A
      DO 4 K=1,200
    4 BJ(K)=0.0
      Y=X
      IF(X)1,2,40
    2 BJ(1)=1.0
      RETURN
    1 X=-X
   40 IER=0
      N1=N+1
      BPREV=.0
C     COMPUTE STARTING VALUE OF M
      AN=N
      IF(X-AN) 20,20,21
   20 MZERO=N+10
      GO TO 74
   21  MZERO=X+20
   74 MMAX=199
  100 DO 190 M=MZERO,MMAX,3
C     OVERFLOW TRAP  BASED ON 1.E70
   23 XOMIN=EXP(ALOG(1.E-70)/M)
      XOM=X/M
      IF(XOM-XOMIN)22,22,130
   22 M=M-3
      D=10.
      IER=M
      GO TO 23
C     SET F(M),F(M-1)
  130 BJ(M+1)=0.
      BJ(M)=1.0E-70
      M2=M-1
      DO 160 K=1,M2
      MK=M-K
```

```
160   BJ(MK)=2.* MK    *BJ(MK+1)/X-BJ(MK+2)
      SUM=-BJ(1)
      DO 161 K=1,M,2
161   SUM=2.0*BJ(K)+SUM
      DO 162 K=1,M
162   BJ(K)=BJ(K)/SUM
      IF(ABS(BJ(N1)-BPREV)-ABS(D*BJ(N1))) 10,10,190
190   BPREV=BJ(N1)
      IER=199
10    IF(Y-X) 201,200,201
201   DO 202 K=2,M,2
202   BJ(K)=-BJ(K)
200   RETURN
      END
      FINISH
```

```
      MASTER LCMO 07
      DIMENSION X(70),Y(70),XL(7),YL(7),FUN(7,7)
      M=59
      READ(1,100)((X(I),Y(I)),I=1,M)
  100 FORMAT(2(2F20.8))
      X1=0.1
      X2=0.2
      X3=0.3
      X4=0.4
      X5=0.5
      X6=0.6
      X7=0.7
      X8=0.8
      X9=0.9
      X10=0.95
      DO 1 I=1,7
      XL(I)=X(I)-X1
    1 YL(I)=Y(I)
      S=X1
      GO TO 30
   92 DO 2 I=1,7
      XL(I)=X(I)-X2
    2 YL(I)=Y(I)
      S=X2
      GO TO 30
   93 DO 3 I=1,7
      XL(I)=X(I+4)-X3
    3 YL(I)=Y(I+4)
      S=X3
      GO TO 30
   94 DO 4 I=1,7
      XL(I)=X(I+10)-X4
    4 YL(I)=Y(I+10)
      S=X4
      GO TO 30
   95 DO 5 I=1,7
      XL(I)=X(I+17)-X5
    5 YL(I)=Y(I+17)
      S=X5
      GO TO 30
   96 DO 6 I=1,7
      XL(I)=X(I+24)-X6
    6 YL(I)=Y(I+24)
      S=X6
      GO TO 30
   97 DO 7 I=1,7
      XL(I)=X(I+31)-X7
    7 YL(I)=Y(I+31)
      S=X7
      GO TO 30
```

```
 98 DO 8 I=1,7
    XL(I)=X(I+38)-X8
  8 YL(I)=Y(I+38)
    S=X8
    GO TO 30
 99  DO 9 I=1,7
    XL(I)=X(I+45)-X9
  9 YL(I)=Y(I+45)
    S=X9
    GO TO 30
995 DO 10 I=1,7
    XL(I)=X(I+52)-X10
 10 YL(I)=Y(I+52)
    S=X10
 30 CALL INTER(XL,YL,FUN)
    WRITE(2,200)S
200 FORMAT(1X,'E=',F3.1)
    IUL=7
    DO 20 J=1,7
 20  WRITE(2,201)(FUN(J,I),I=1,IUL-J+1)
201 FORMAT(6F20.8)
     ES=FUN(7,1)*FUN(7,1)*S*S/SQRT(1.-S*S)
    WRITE(2,202)ES
202  FORMAT(1X,'S','=',F20.8)
    IF(S.EQ.X1)GO TO 92
    IF(S.EQ.X2)GO TO 93
   IF(S.EQ.X3)GO TO 94
   IF(S.EQ.X4)GO TO 95
    IF(S.EQ.X5)GO TO 96
    IF(S.EQ.X6)GO TO 97
    IF(S.EQ.X7)GO TO 98
    IF(S.EQ.X8)GO TO 99
    IF(S.EQ.X9)GO TO 995
    STOP
    END
    SUBROUTINE INTER(X,Y,F)
    DIMENSION X(7),Y(7),F(7,7)
    DO 80 I=1,6
    DO 80 J=1,7-I
    IF(ABS(X(I)).GT.ABS(X(I+J)))GO TO 70
    GO TO 80
 70 A=X(I+J)
   C=X(I)
    B=Y(I+J)
    D=Y(I)
    X(I+J)=C
   X(I)=A
    Y(I+J)    =D
    Y(I)=B
 80 CONTINUE
```

```
      DO 10 I=1,7
   10 F(1,I)=Y(I)
      IUL=7
      DO 9 J=2,IUL
      DO 9 I=1,IUL-J+1
    9 F(J,I)=(X(I+J-1)*F(J-1,I)-X(I)*F(J-1,I+1))/(X(I+J-1)-X(I))
      RETURN
      END
      FINISH
```

ORGANISATION OF COMPRESSOR NOISE STUDY.

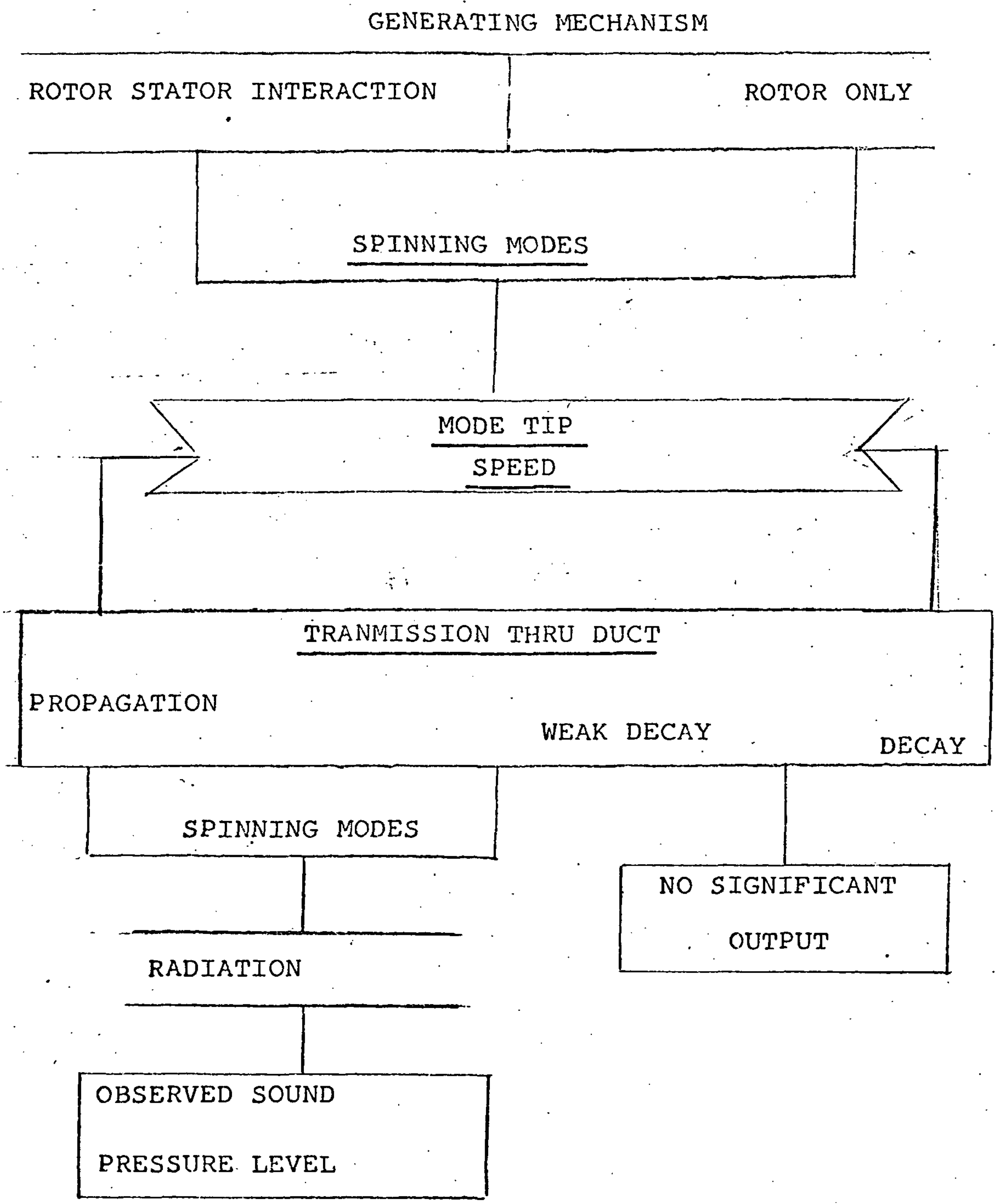


Figure 1

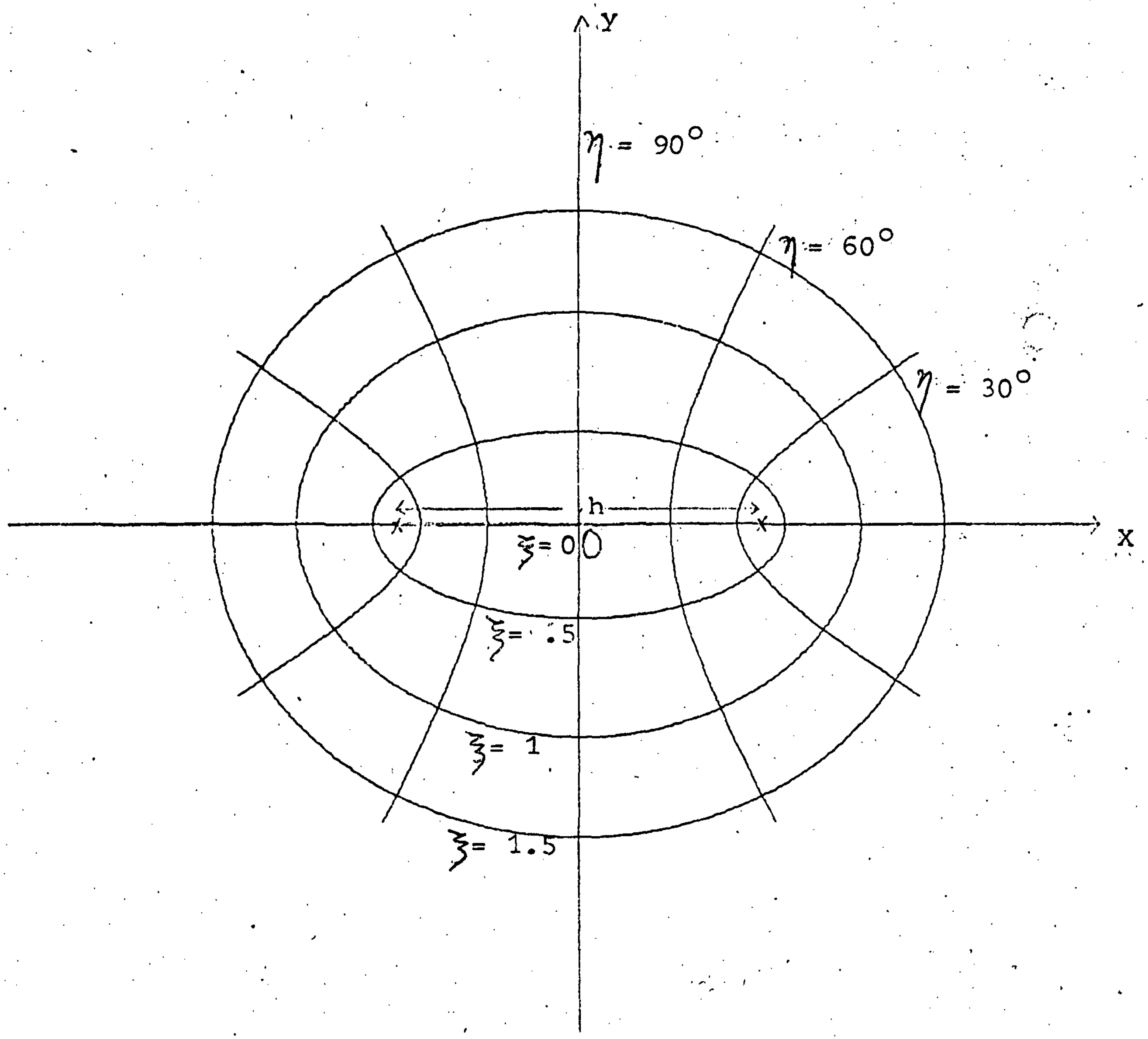


Figure 2: COORDINATE SYSTEM FOR ELLIPTIC DUCT.

CRITICAL WAVE-LENGTHS OF WAVES IN ELLIPTICAL PIPES.

Here the H-curves are the critical magnetic wavelengths and E-curves are the critical electrical wavelengths. Thus ${}_oH_1$ and ${}_eH_1$ are respectively odd and even curves of order one. ${}_eH_o$ is an even curve of order zero.

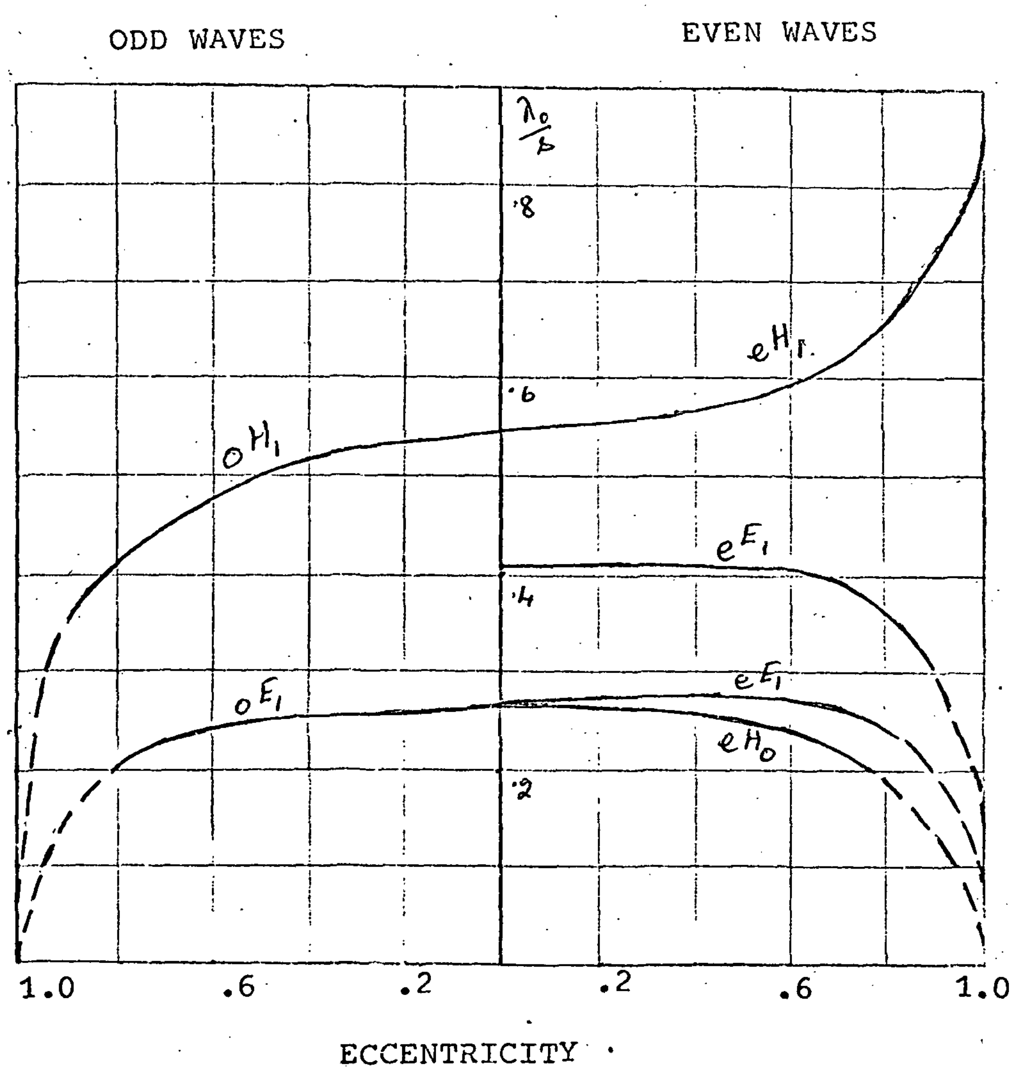
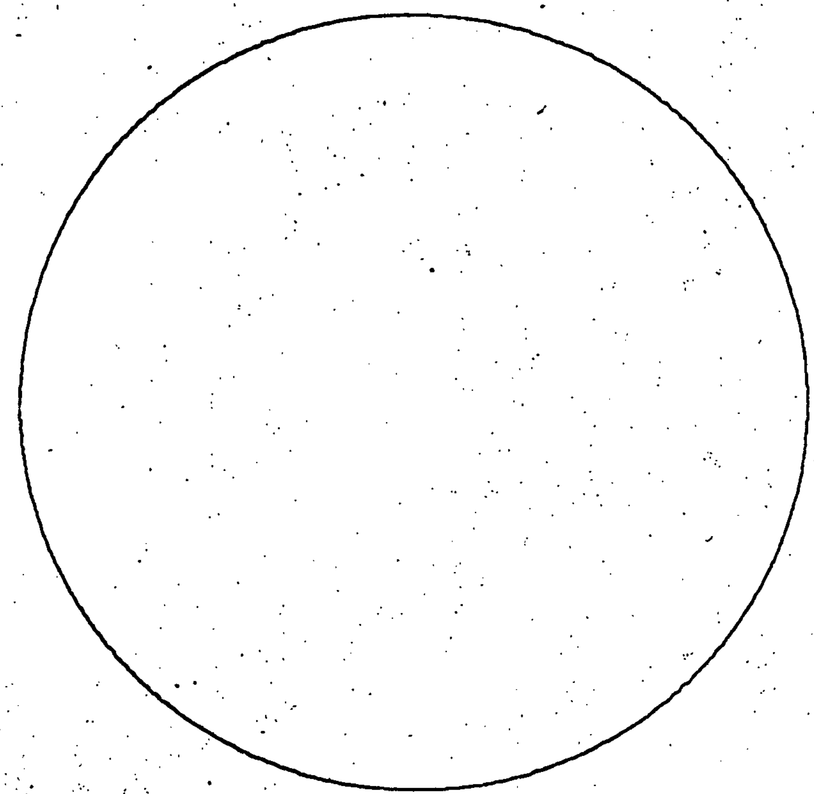


Figure 3

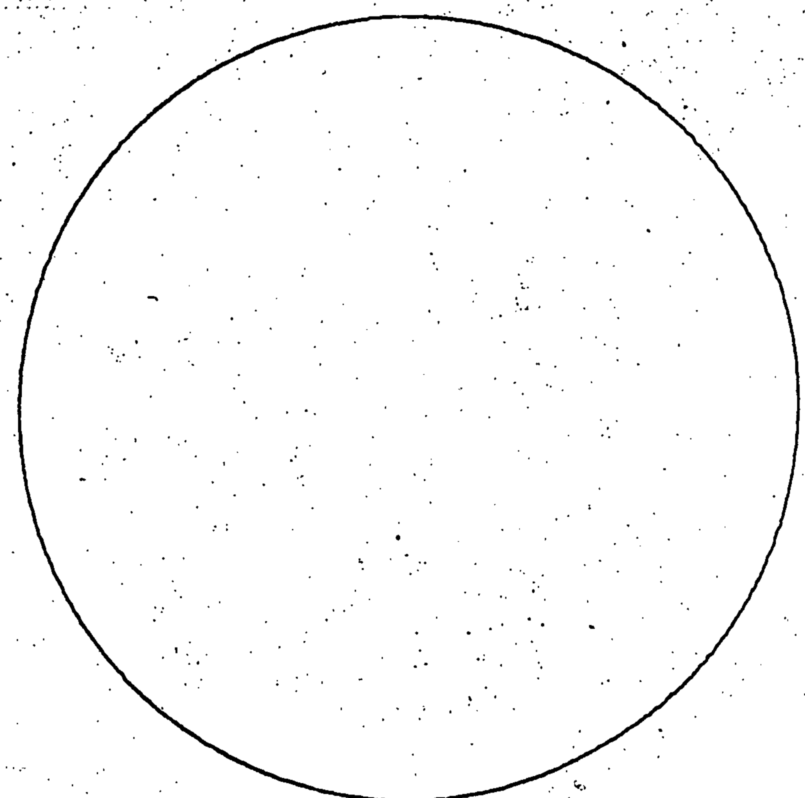
Ellipses of equal area and eccentricities .1, .2, .3, .4, .5, .6, .7, .8, .9 and .95

Figure 4 - (i) - (x)



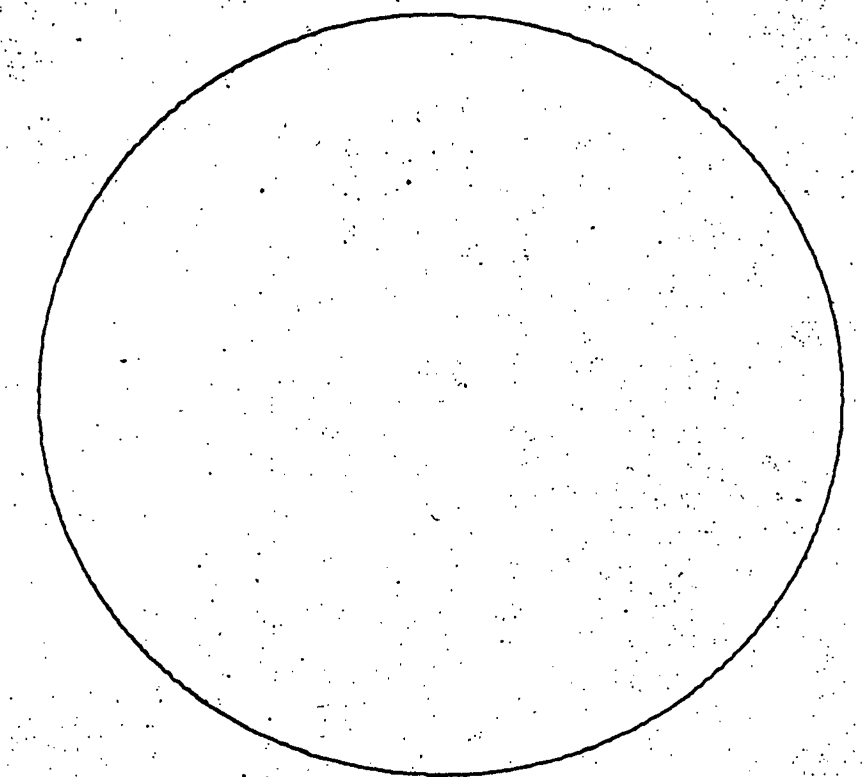
e = 0.1

Figure 4 - (i)



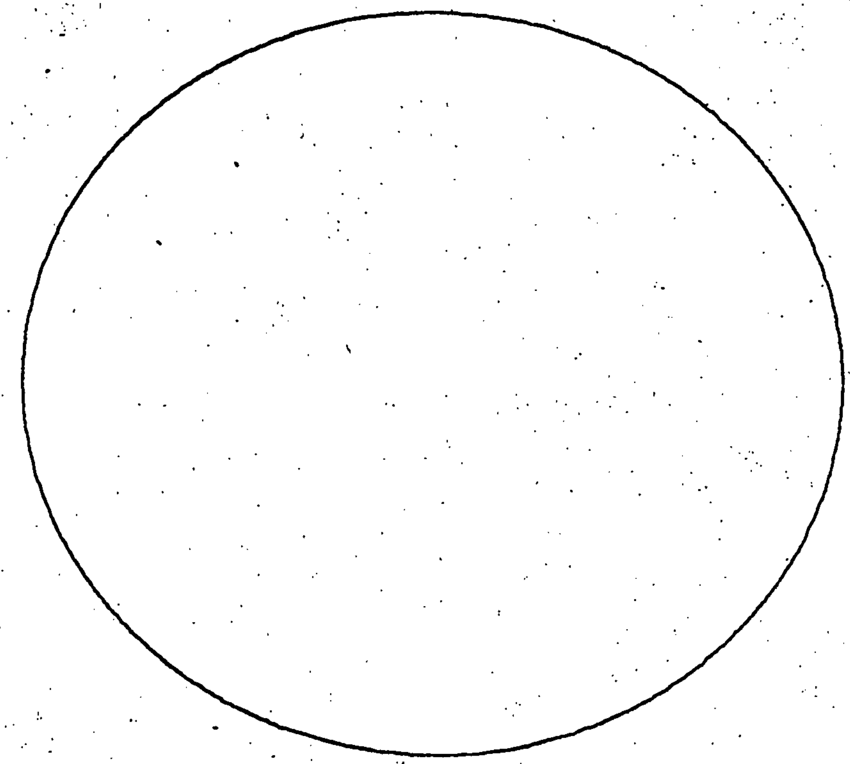
e = 0.2

Figure 4 - (ii)



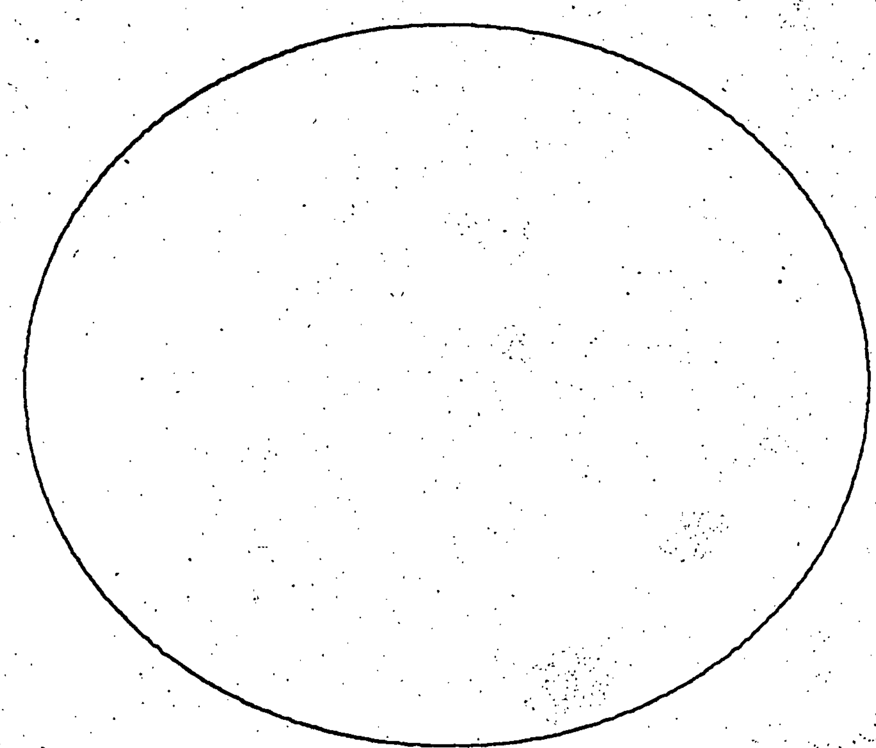
$e = 0.3$

Figure 4 – (iii)



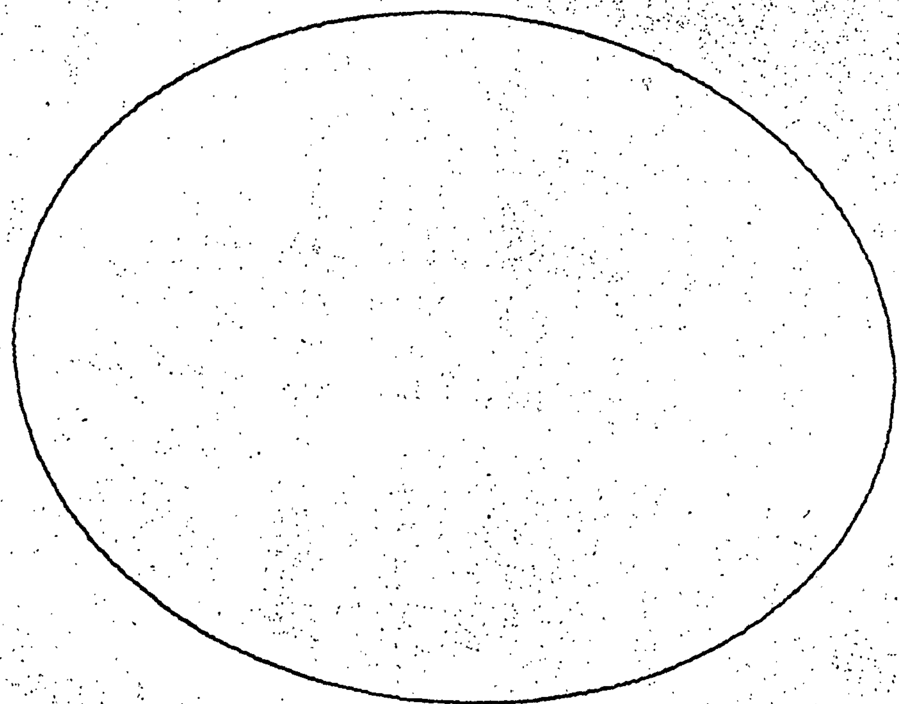
$e = 0.4$

Figure 4 - (iv)



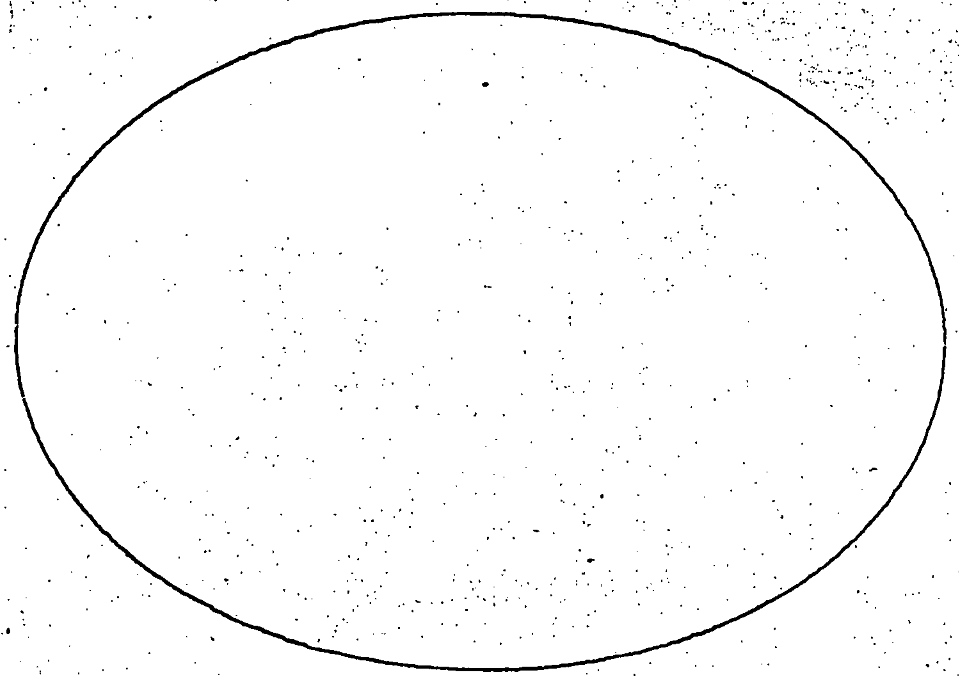
$e = 0.5$

Figure 4 - (v)



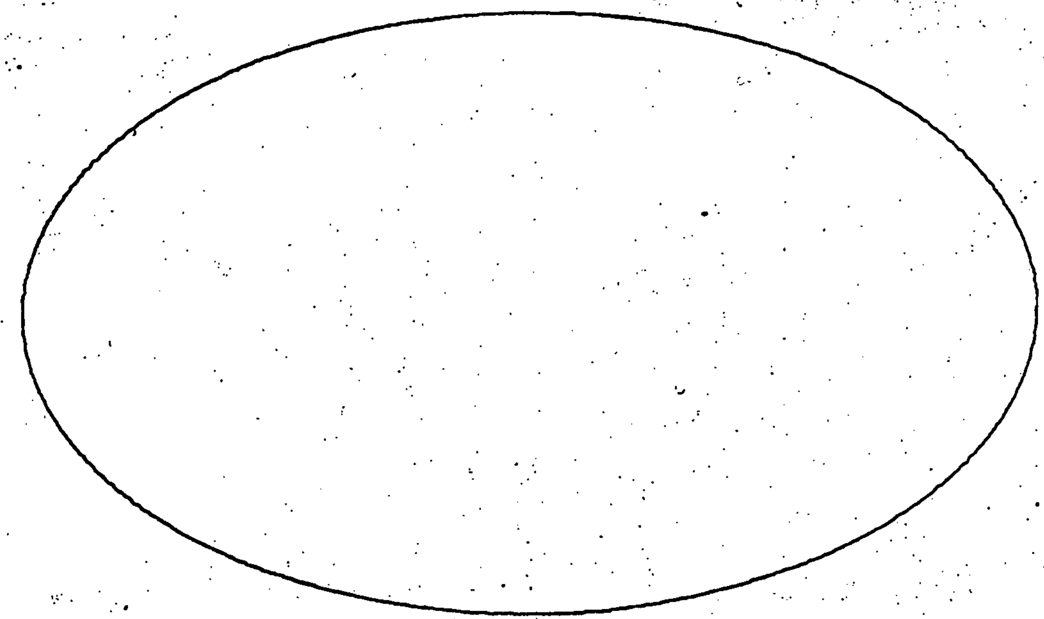
e = 0.6

Figure 4 - (vi)



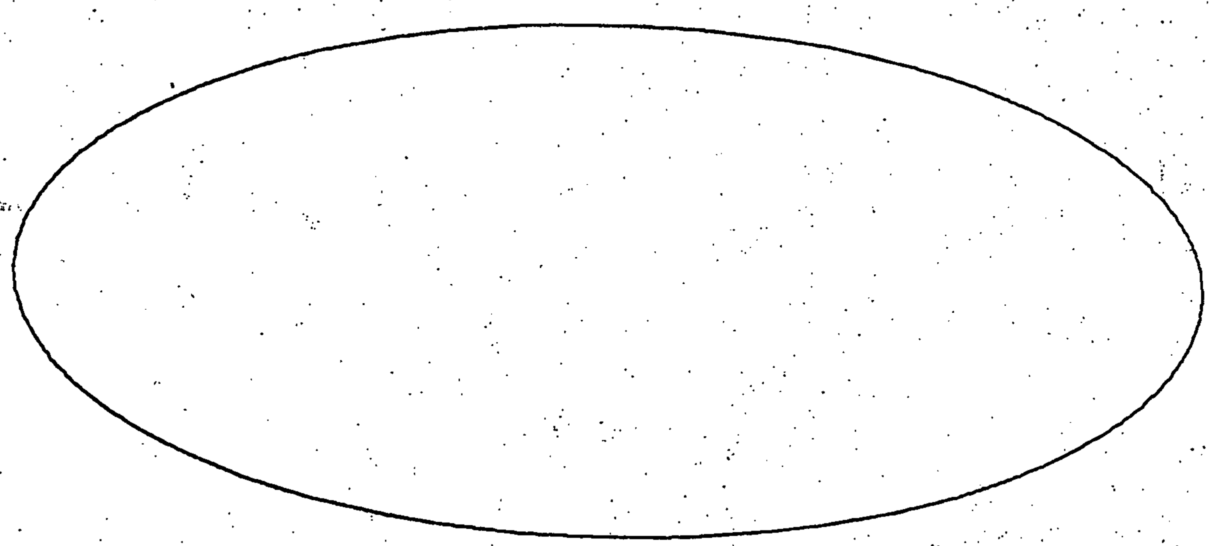
e = 0.7

Figure 4 - (vii)



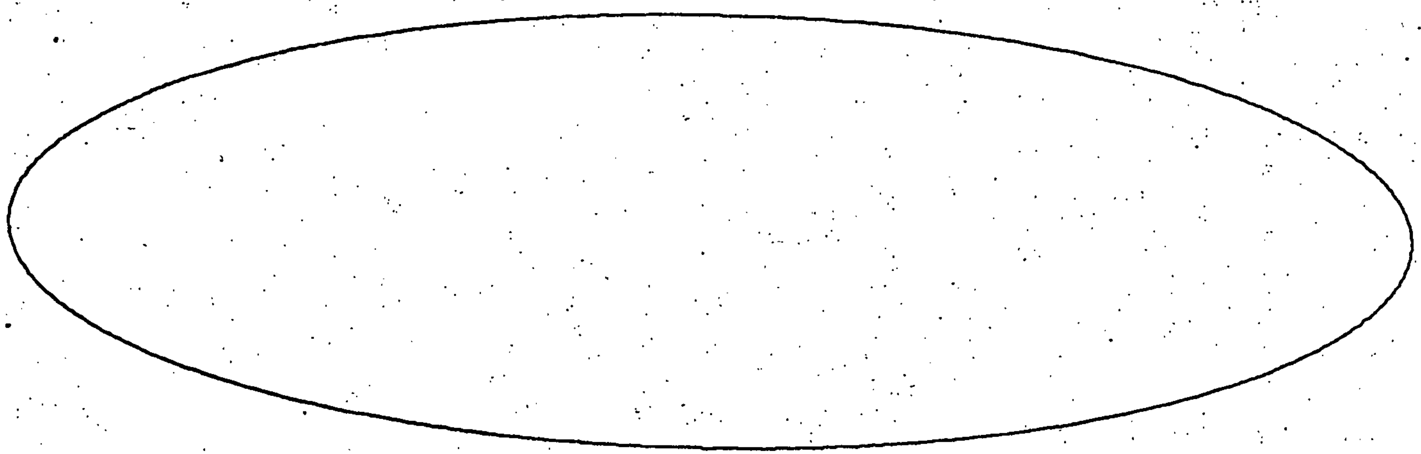
e = 0.8

Figure 4 - (viii)



$e = 0.9$

Figure 4 - (ix)



e = 0.95

Figure 4 - (x)

RADIAL PRESSURE FUNCTIONS FOR A CIRCULAR DUCT.

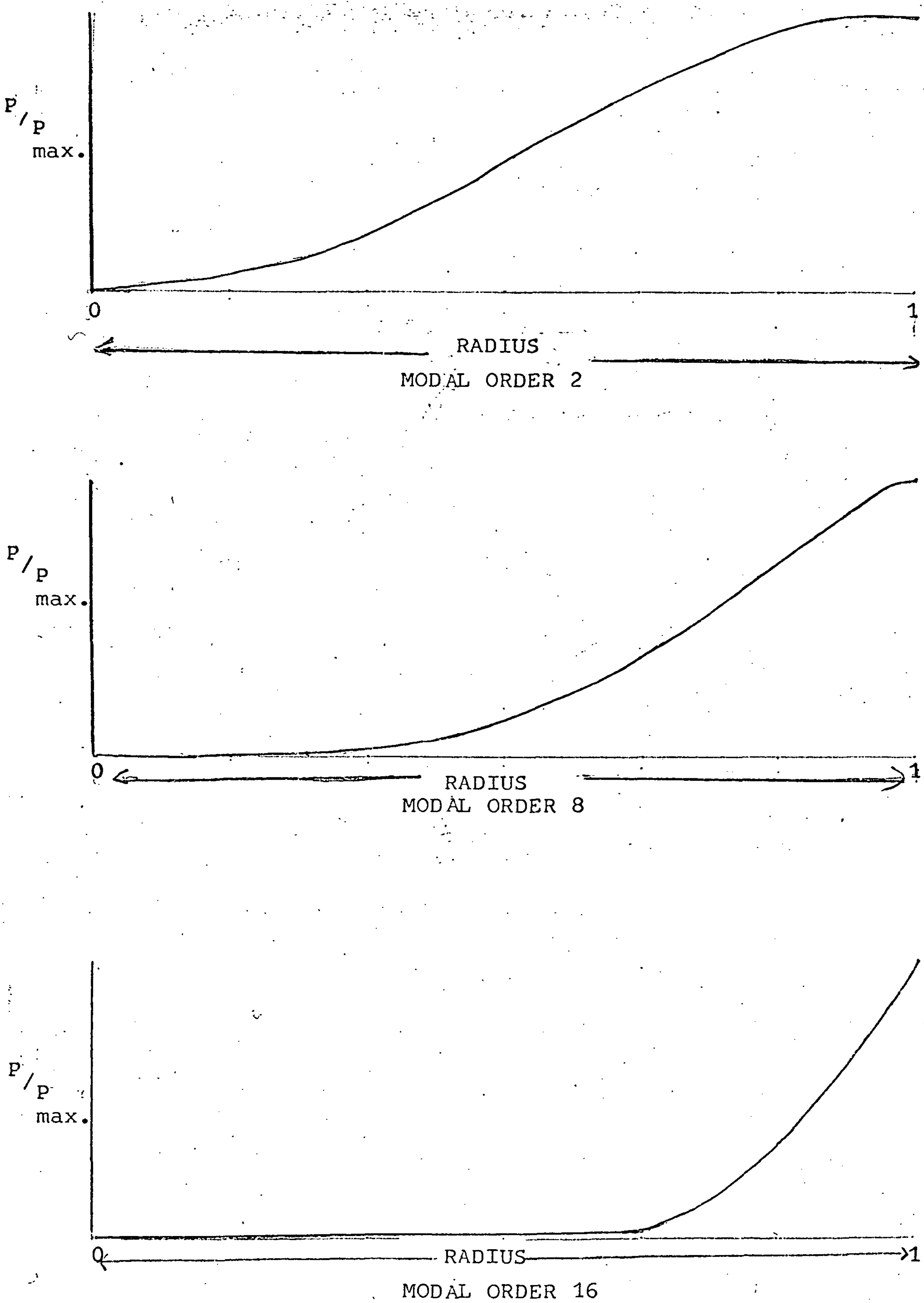


Figure 5

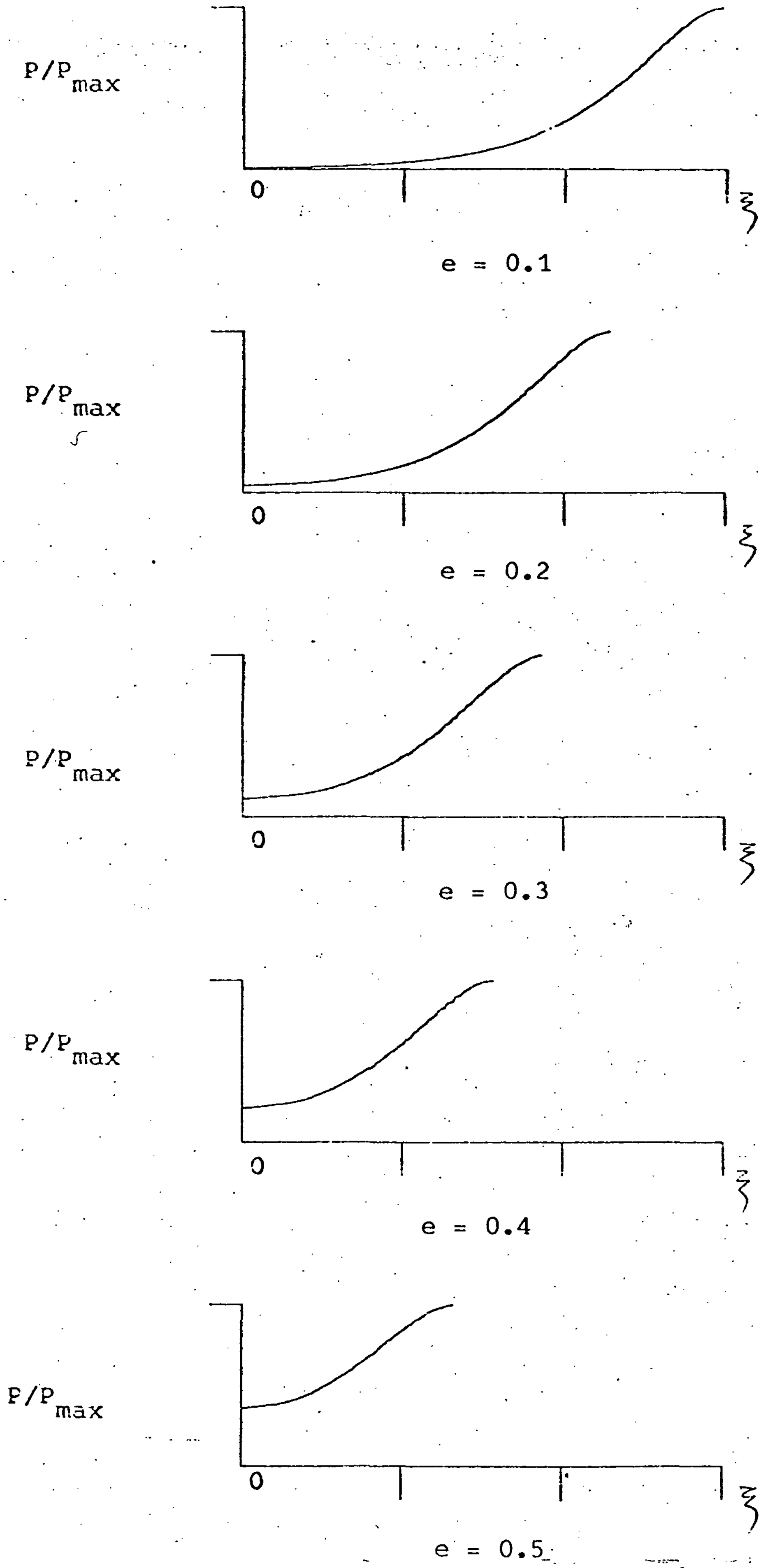


Figure 6 (a)

Even Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 2)

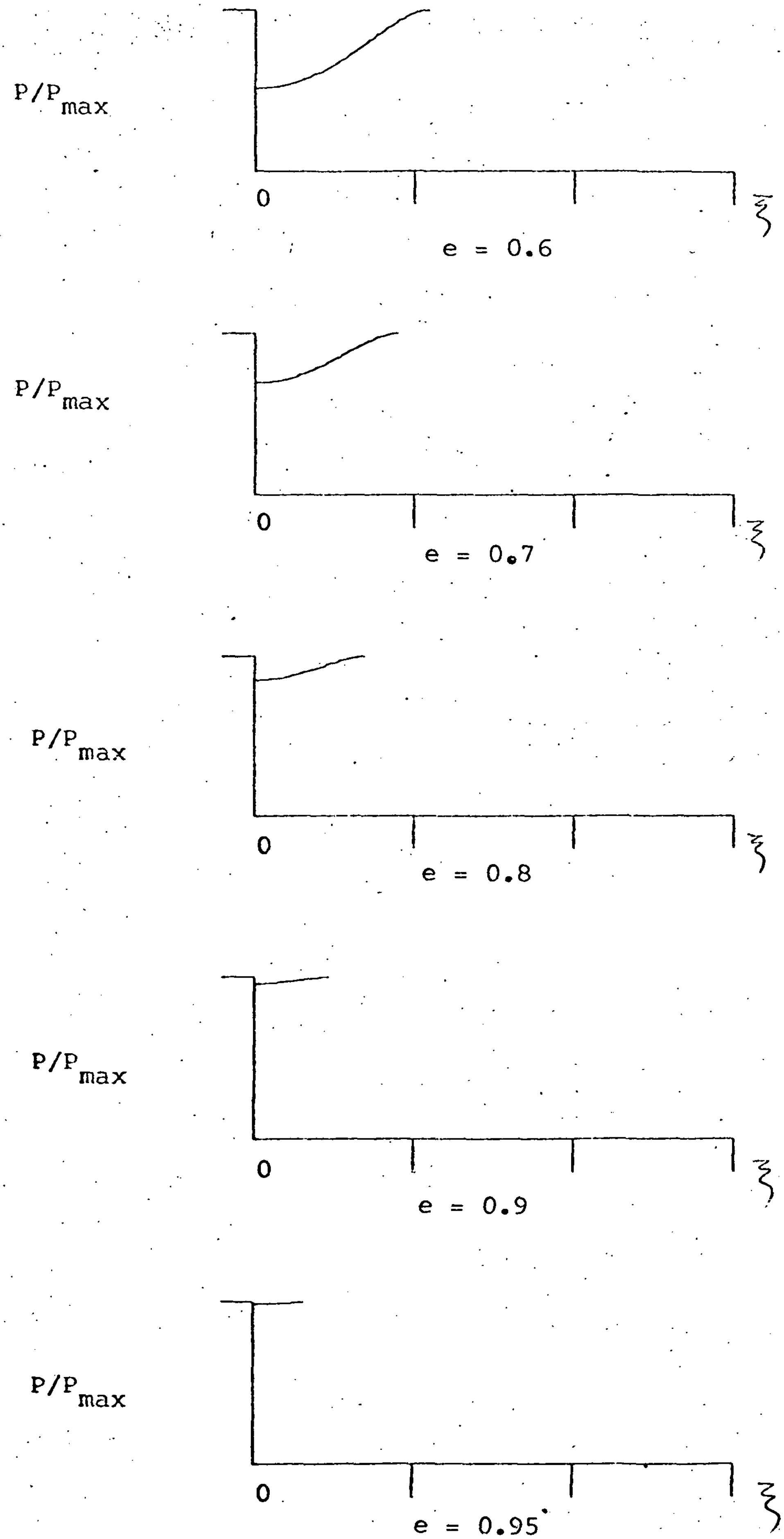


Figure 6 (b)

Even Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 2)

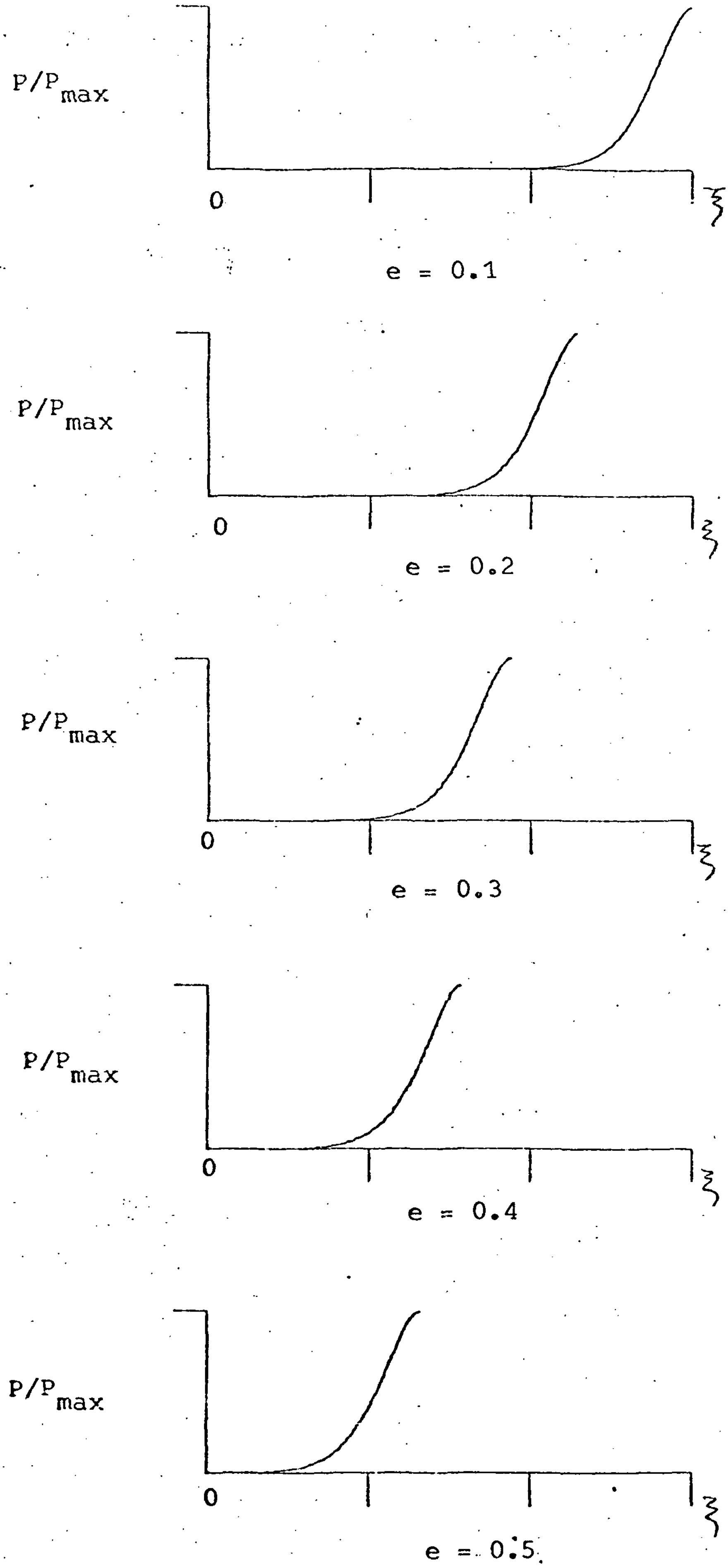


Figure 7 (a)

Even Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 8)

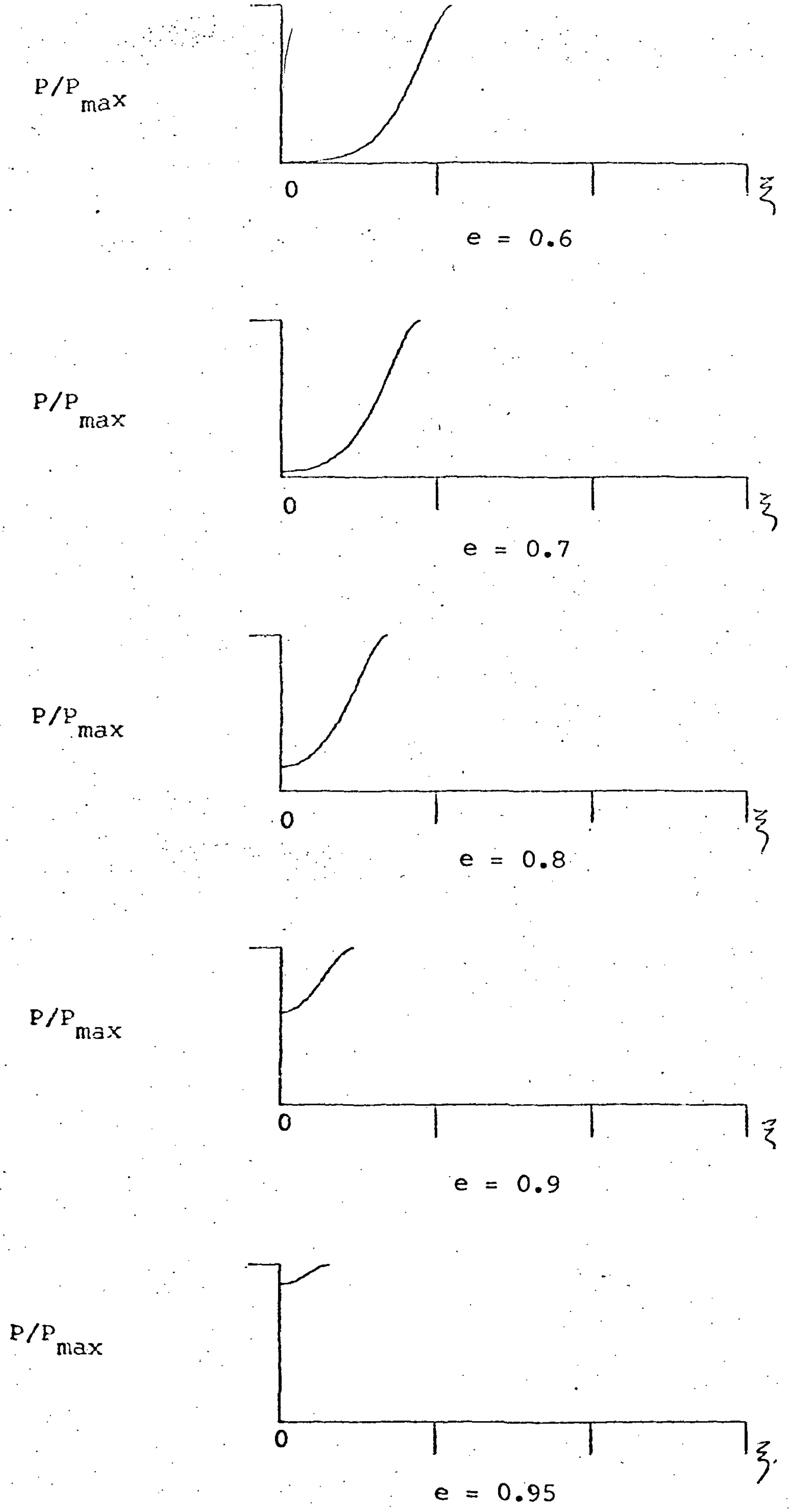


Figure 7 (b)

Even Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 8)

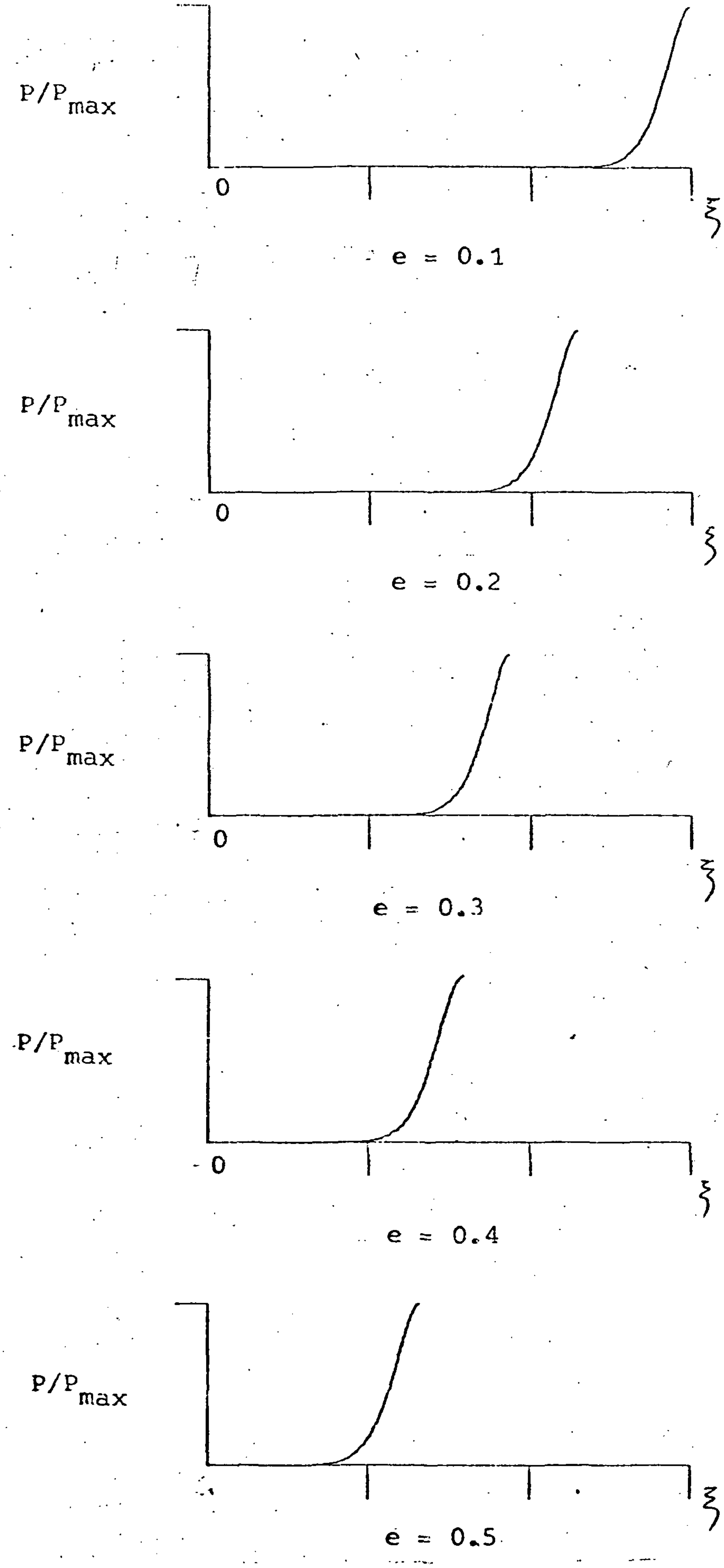
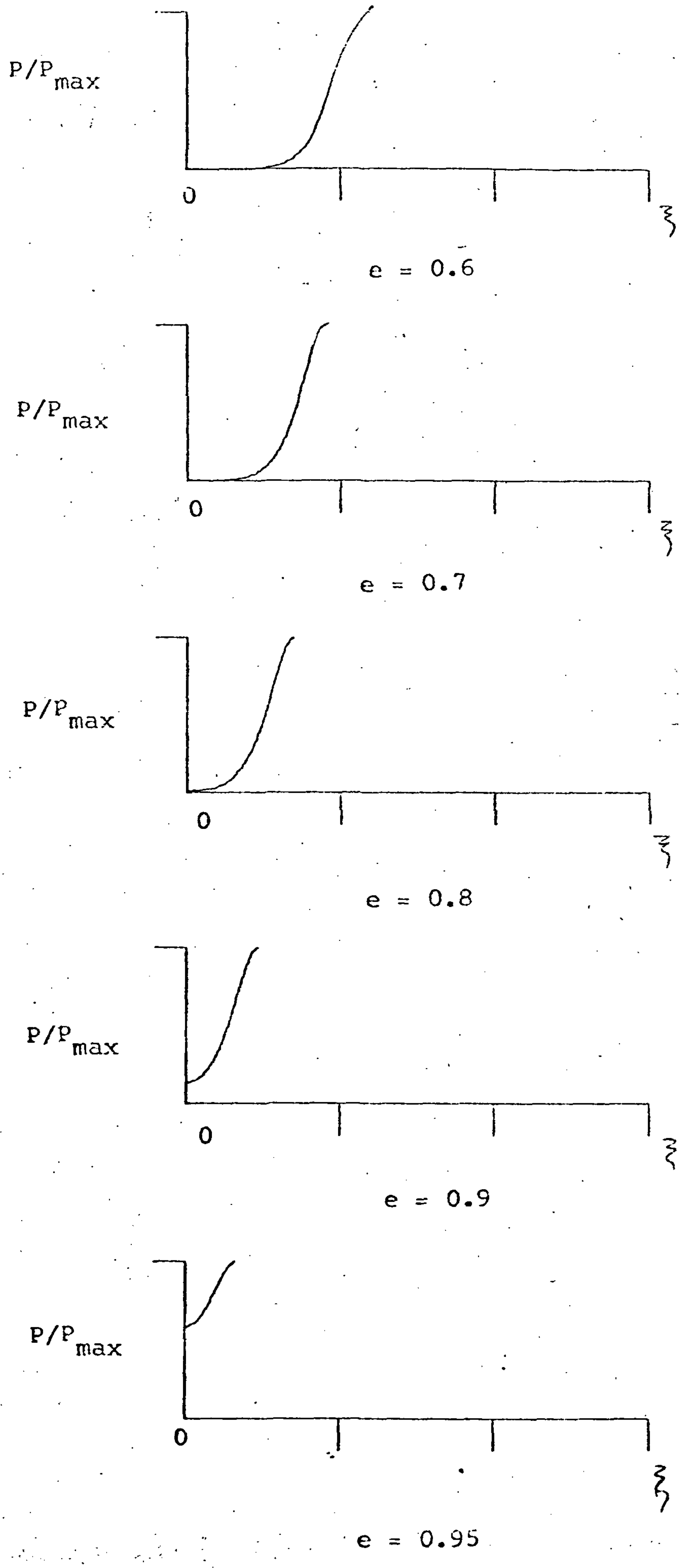


Figure 8 (a)

Even Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 15)



<u>Figure 8 (b)</u>

Even Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 15)

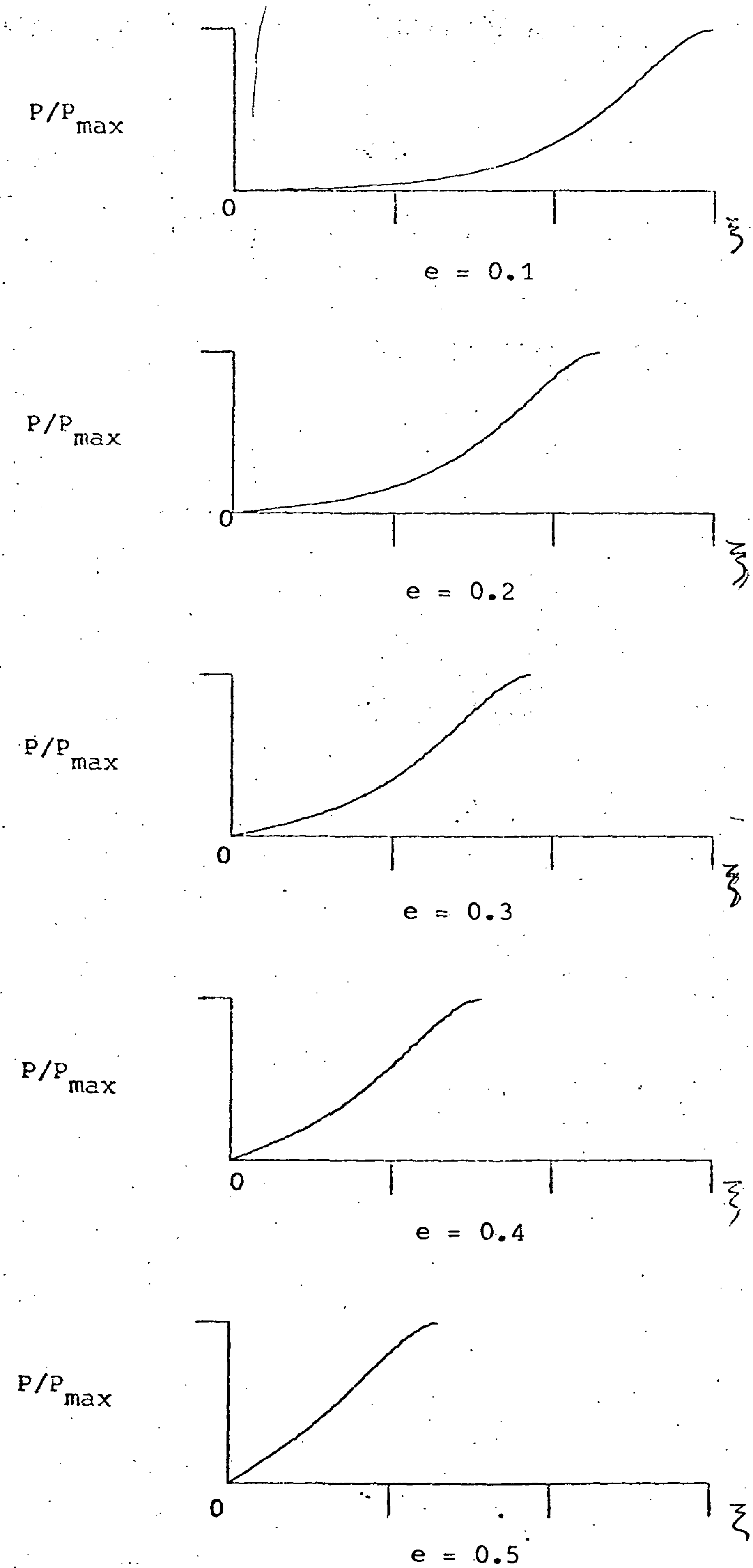
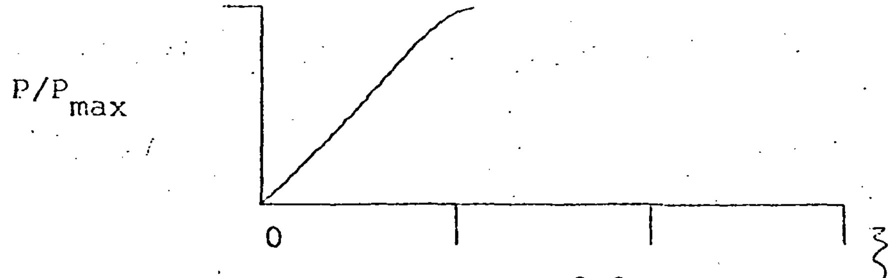


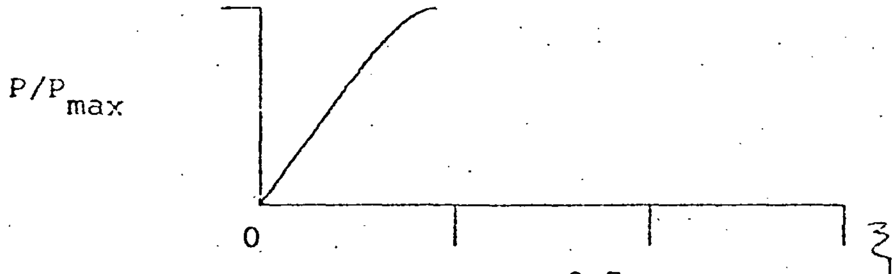
Figure 9 (a)

Odd Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

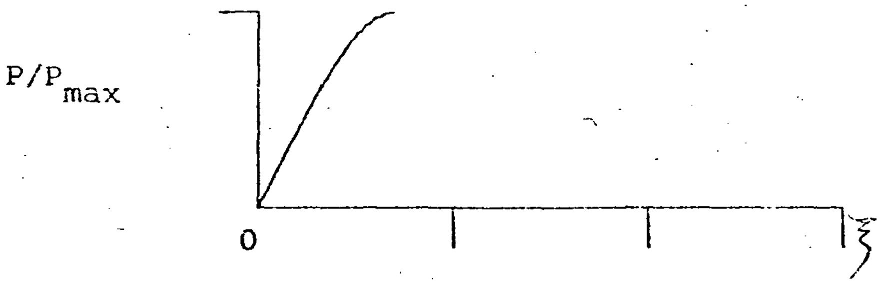
(mode = 2)



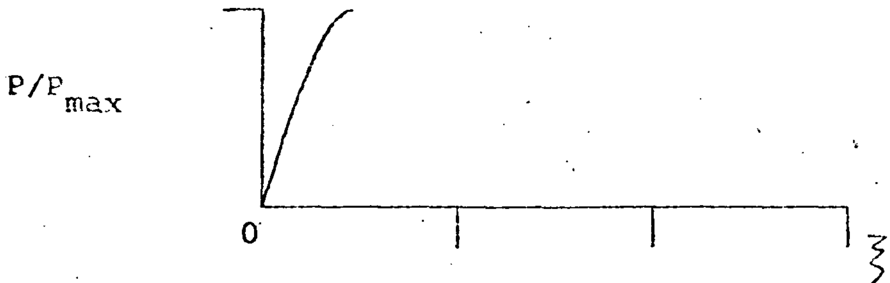
e = 0.6



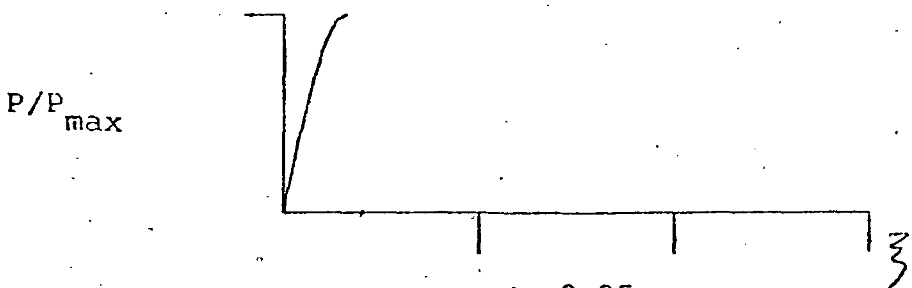
e = 0.7



e = 0.8



e = 0.9



e = 0.95

<u>Figure 9 (b)</u>

Odd Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 2)

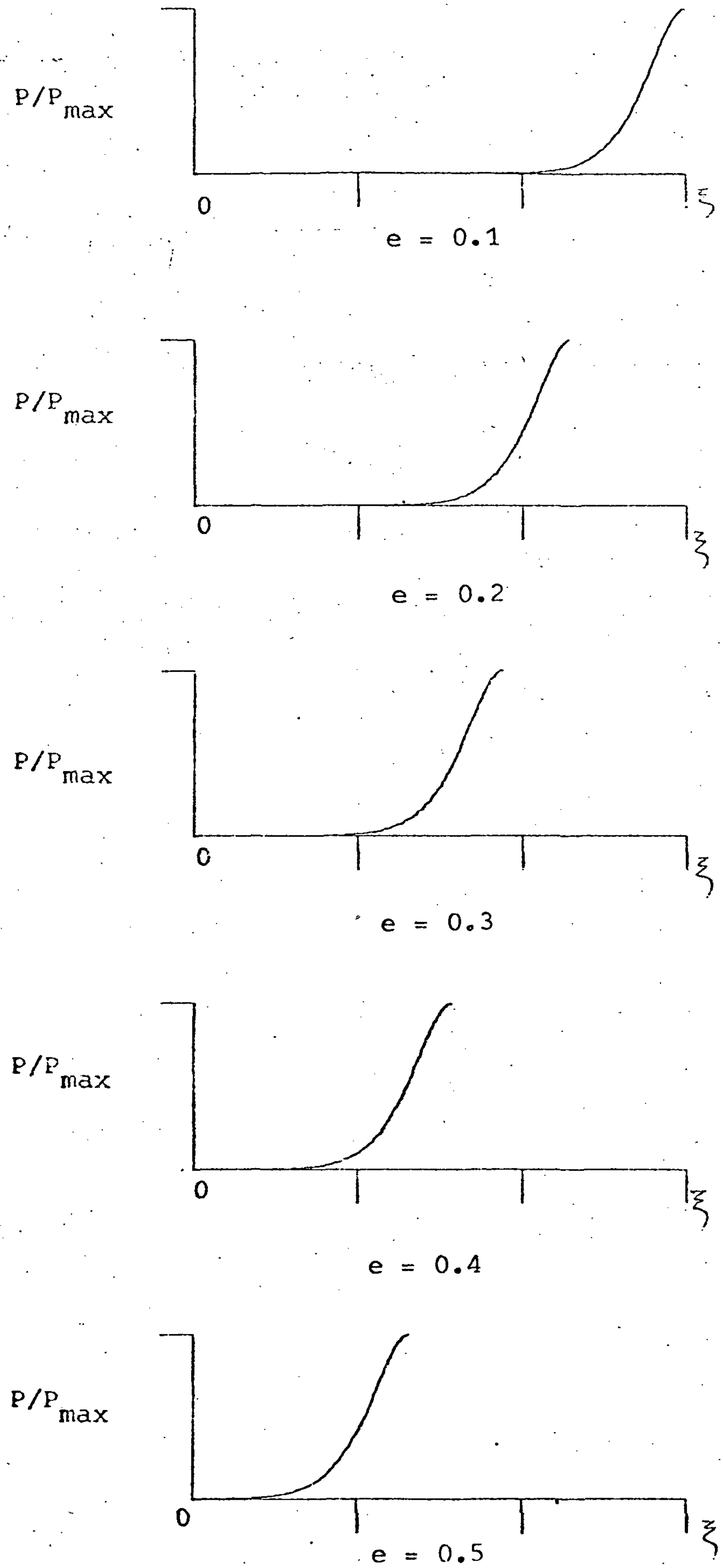


Figure 10(a)

Odd Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 8)

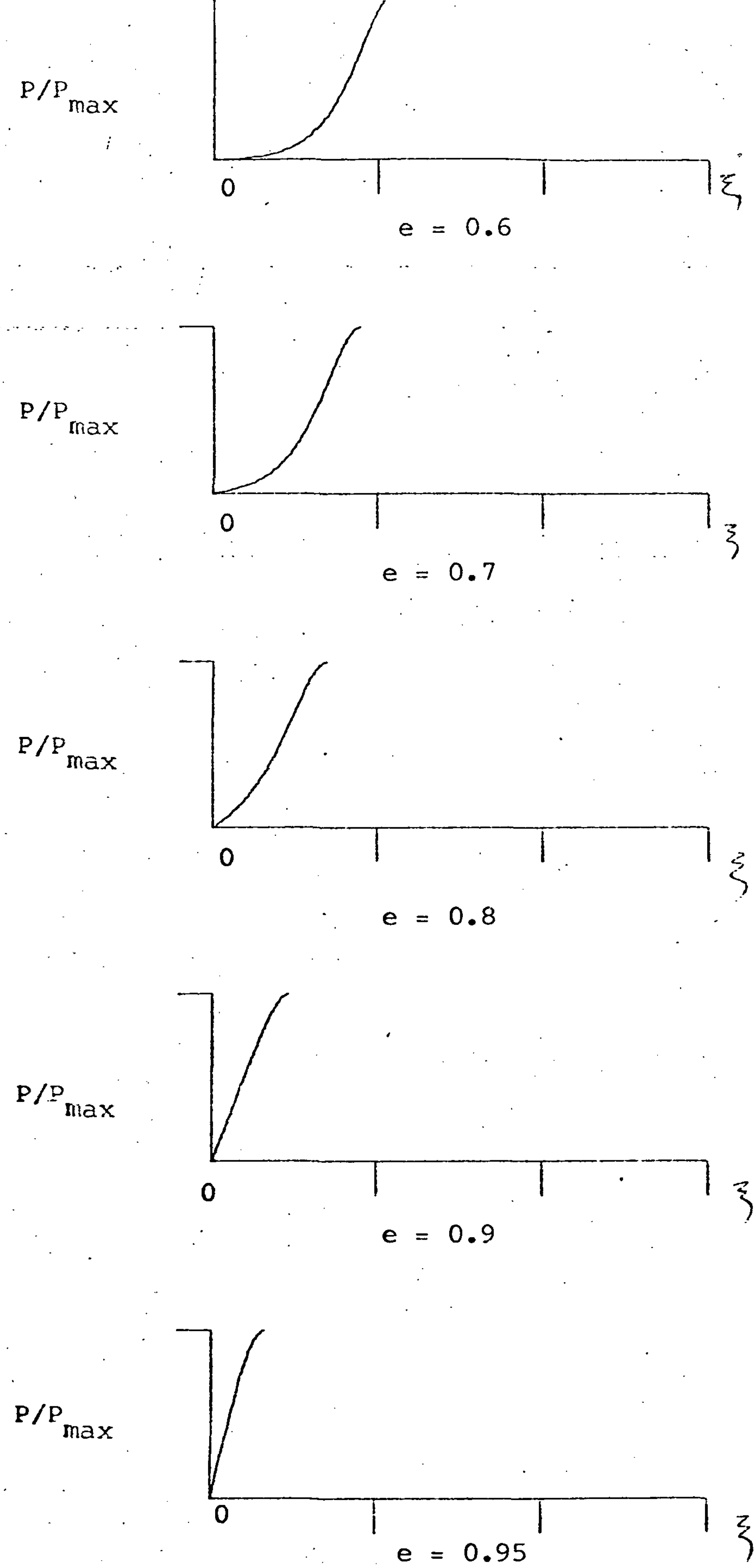


Figure 10 (b)

Odd Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 8)

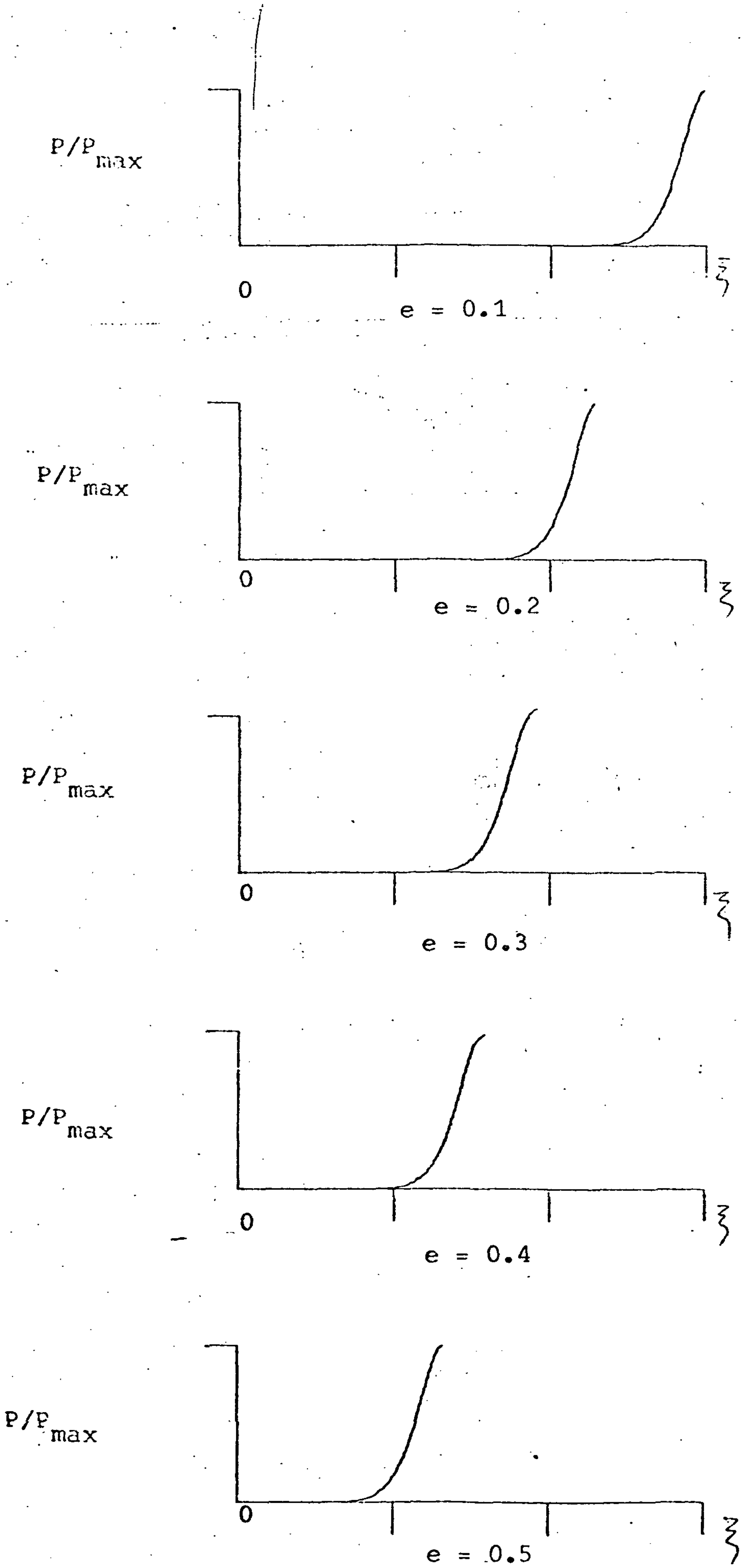


Figure 11 (a)

Odd Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.
(mode = 15)

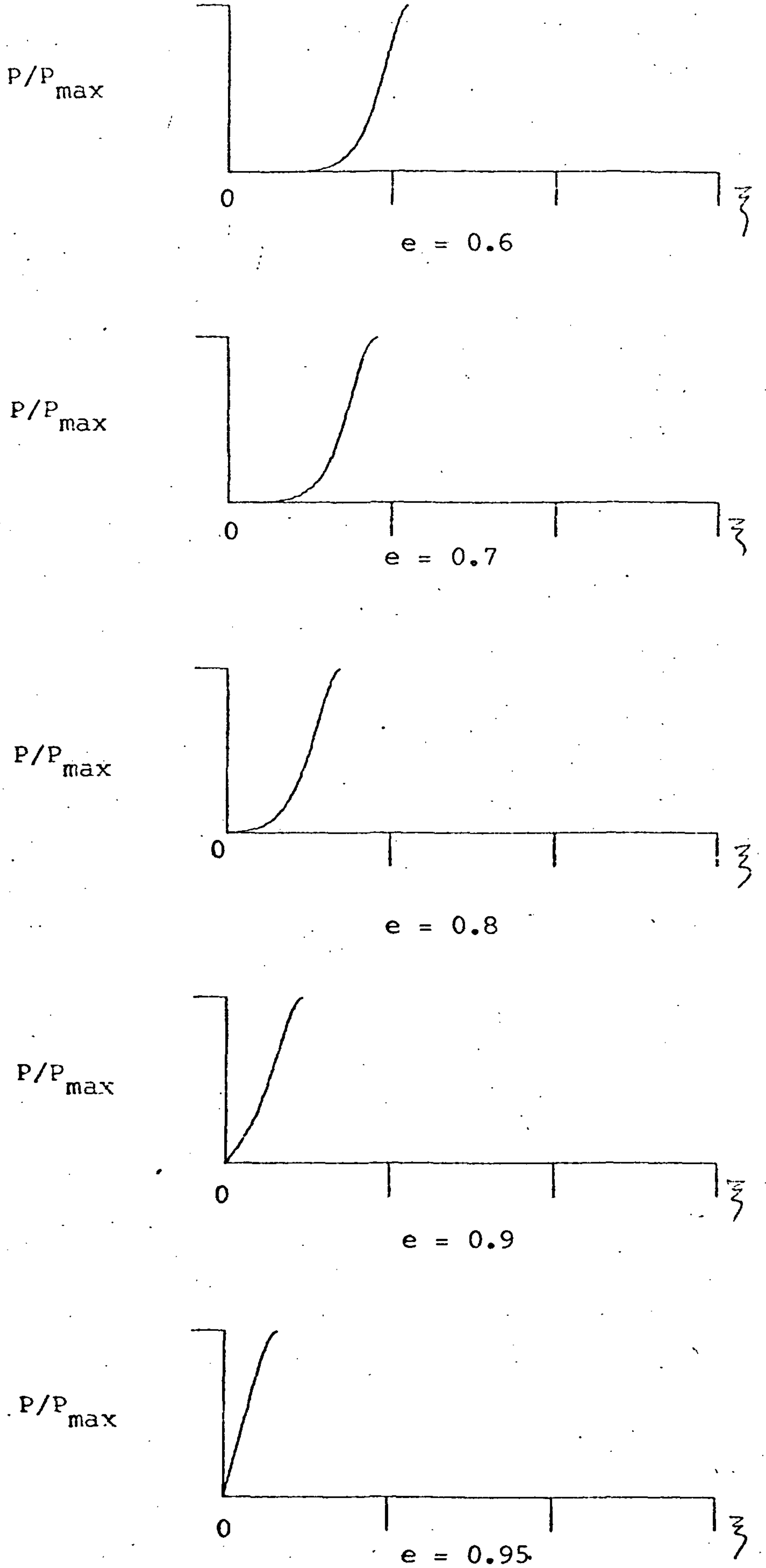


Figure 11 (b)

Odd Radial Pressure Functions of Lowest Cross-mode in an Elliptic Duct.

(mode = 15)