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N-PLAYER STOCHASTIC DIFFERENTIAL GAMES*

by

Pravin Varaiya

ABSTRACT

The paper presents conditions which guarantee that the control strategies adopted by N players constitute an efficient solution, an equilibrium, or a core solution. The system dynamics are described by an Ito equation, and all players have perfect information. When the set of instantaneous joint costs and velocity vectors is convex the conditions are necessary.

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§1. INTRODUCTION

N players are simultaneously controlling the evolution of a system described by the Ito equation

$$dz_t = f(t,z,u_t^1,\ldots,u_t^N)\,dt + dB_t,\ t \in [0,1] \tag{1}$$

where (z_t) is the state process, (B_t) is Brownian motion and (u_t^i) is the control of the ith player. Player i chooses this control so as to minimize the cost

$$J^i(\mathcal{U}) = E[\int_0^1 c^i(t,z,u_t^i)dt + \gamma^i(z)], \tag{2}$$

where $\mathcal{U} = (u_t) = (u_t^1,\ldots,u_t^N)$.

Different solution concepts of the resulting game are studied. Sufficient conditions are given which guarantee that $\mathcal{U}^* = (u_t^{1*},\ldots,u_t^{N*})$ is a (Nash) equilibrium, a (Pareto) efficient solution, or a member of the core. When the set of admissible cost-drift vectors $(c^1,\ldots,c^N,f)$ possesses a certain convexity property, these sufficient conditions become necessary.

The next section gives a precise model of the game. The convexity property is stated, and its main implications are drawn out in section 3. The main results are given in section 4. A priori conditions on the c^i and f which imply the convexity property are examined in section 5.

§2. THE MODEL

2.1 Admissible Controls

The sample paths of the state process (z_t) are evidently continuous, hence members of the Banach space C of all continuous functions $\omega: [0,1] \to R^n$. Let ξ_t be the evaluation functional on C, that is, $\xi_t(\omega)=\omega(t)$. Let F_t be the σ-field of subsets of C generated by $\{\xi_s | 0 \le s \le t\}$. Let $F=F_1$.

For each i U_i is a compact metric space, the set of actions available to i. A function $u^i: [0,1] \times C \to U_i$ is an (admissible) control for i if

(i) u^i is jointly measurable,

(ii) $u^i_t = u^i(t,\cdot)$ is F_t- measurable for all t.

$\mathcal{U}_i$ denotes the set of controls for i.

Denote $U = U_1 x \ldots x U_N$ with elements $u = (u_1, \ldots, u_N)$ and $\mathcal{U} = \mathcal{U}^1 x \ldots x \mathcal{U}^N$ with elements $u = (u^1, \ldots, u^N)$. For $u \in U$, $v \in U$ and $i \in \{1, \ldots, N\}$ denote $(u_{\bar{i}}, v_i) = (u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_N)$. More generally, for $S \subset \{1, \ldots, N\}$ denote $(u_{\bar{S}}, v_S)$ to be the N-tuple obtained from u upon replacing u_i by v_i for each $i \in S$. In exactly the same way one defines $(u^{\bar{i}}, v^i)$ and $(u^{\bar{S}}, v^S)$ when u and v are in $\mathcal{U}$.

2.2 Dynamics

The function $f: [0,1] \times C \times U \to R^n$ in (1) satisfies the following conditions:

(i) f is jointly measurable,

(ii) $f(t,\cdot,u)$ is F_t- measurable for all t,u and $f(t,\omega,\cdot)$ is continuous for all t,ω,

(iii) There is a constant k such that $|f(t,\omega,u)| \le k(1+||\omega||)$ for all t,ω,u.

Let P denote Wiener measure on (C,F). Let (z_t) be the family of evaluation functionals on C so that (z_t, F_t, P) is a n-dimensional, standard, Brownian motion. For $u \in \mathcal{U}$ define the drift (ϕ_t^u, F_t, P) by

$$\phi_t^u = f(t,z, u(t,z))$$

and the density (ρ_t^u, F_t, P) by

$$\rho_t^u = \exp\left[\int_0^t \phi_s^u \, dz_s - 1/2 \int_0^t |\phi_s^u|^2 \, ds\right]$$

Denote $\rho^u = \rho_1^u$. The next result is well-known [1,2].

Theorem 1 (Beneš) $E(\rho_t^u) \equiv 1$. Hence P^u is a probability measure on (C, F) where

$$P^u(F) = \int_F \rho^u(z) P(dz) \quad , F \in F$$

Furthermore, the process (w_t^u, F_t, P^u) defined by

$$w_t^u = z_t - \int_0^t \phi_s^u \, ds$$

is a Brownian motion.

This theorem justifies the following definition. The solution of (1) corresponding to $u \in \mathcal{U}$ is the process (z_t, F_t, P^u).

2.3 Solutions of the Game

Conditions analogous to those imposed on f are also imposed on the functions c^i in (2). The functions γ^i: $C \to R$ are F- measurable and integrable with respect to P^u for all u. In addition, the c^i and γ^i are non-negative.

The cost to player i of $u \in \mathcal{U}$ is defined to be

$$J^i(u) = E^u \left[\int_0^1 c^i(t, \cdot, u_t) dt + \gamma^i(\cdot)\right] ,$$

where E^u denotes expectation with respect to P^u

Recall the following definitions. $u^* = (u^{1*}, \ldots u^{N*})$ is

a) an equilibrium if there is no i and no u such that

$$J^i(u^{*\bar{i}}, u^i) < J^i(u^*)$$

b) efficient if there is no u such that

$$J^i(u) < J^i(u^*) \text{ for all } i$$

c) in the <u>core</u> if there is no S and no u such that

$$J^i(u^{*\bar{S}}, u^S) < J^i(u^*) \text{ for all } i\epsilon S$$

To avoid confusion it should be pointed out that the definitions b) and c) are not the standard ones. Usually, u^* is said to be efficient if there is no u such that $J^i(u) \leq J^i(u^*)$ for all i with the strict inequality holding for at least one i. If one adopts this definition, then the observation at the beginning of section 4.3 below needs to be modified and so do the subsequent results; these modifications are slight but clumsy, and the definition given here avoids the clumsiness. In any case the difference is slight. The core is usually defined only for games where comparison of inter-personal utilities is permitted and where side payments are allowed. For games where such comparison is not permitted, as is the normal posture in mathematical economics, one is naturally led to the definition given here.

§3. THE CONVEXITY PROPERTY

Let $g(t,z,u) = (c^1(t,z,u),\ldots,c^N(t,z,u),\ f(t,z,u))$. g is an $(N+n)$-dimensional vector.

The game is said to have the convexity property if for all t,z

$$\{g(t,z,u)\,|\,u\varepsilon U\}$$

is a convex set. It is said to have the strong convexity property if for all t,z,u and for all S

$$\{g(t,z,(u_{\bar{S}},v_S))\,|\,v\varepsilon U\}$$

is a convex set.

In [1] and [2] it is shown that the convexity property implies that the set of densities obtained by using all possible admissible controls is convex. The two lemmas below follow readily from these results.

Lemma 1 Suppose the game has the convexity property. Then

$$J = \{(J^1(u),\ldots,J^N(u)\,|\,u\varepsilon\mathcal{U}\}$$

is a convex subset of R^N.

Lemma 2 Suppose the game has the strong convexity property. Let $u\varepsilon\mathcal{U}$ and $S\subset\{1,\ldots,N\}$. Then

$$J(u^{\bar{S}}) = \{(J^1(u^{\bar{S}},v^S),\ldots,\ J^N(u^{\bar{S}},v^S)\,|\,v\varepsilon\mathcal{U}\}$$

is a convex subset of R^N.

§4. THE MAIN RESULTS

4.1 A Result from Control Theory

Suppose N=1 so that the game is simply an optimal control problem. Dropping superscripts and subscripts, the control problem is to find $\mathcal{u}^*\varepsilon\mathcal{U}$ so as to minimize

$$J(\mathcal{u}) = E^{\mathcal{u}}\left[\int_0^1 c(t,\cdot,u_t)dt + \gamma(\cdot)\right]$$

A minimizing control is said to be <u>optimal</u>.

The result below has been proved in [3] in a slightly more restrictive form than necessary.

<u>Theorem 2</u> $\mathcal{u}^*$ is an optimal control if and only if there exist a constant J^*, and processes (ΛV_t), (∇V_t) with values in R, R^n respectively such that

$$\text{(i)} \qquad J^* + \int_0^1 \Lambda V_t\, dt + \int_0^1 \nabla V_t dz_t = \gamma \quad \text{a.s.}$$

$$\text{(ii)} \qquad \Lambda V_t + \min_{u\varepsilon U} \{\nabla V_t f(t,z,u) + c(t,z,u)\} = 0,$$

and the minimum is achieved at $\mathcal{u}^*(t,z)$ for almost all t,z. Furthermore, $J^*=J(\mathcal{u}^*)$ is the minimum cost; in fact,

$$J^* + \int_0^t \Lambda V_s ds + \int_0^t \nabla V_s\, dz_s = \min_{\mathcal{u}\varepsilon\mathcal{U}} E^{\mathcal{u}}\{\int_t^1 c^i(s,z,u_s)ds + \gamma^i|F_t\}$$

4.2 Conditions for Equilibrium

The controls $\mathcal{u}^* = (\mathcal{u}^{*1},\ldots,\mathcal{u}^{*N})$ constitute an equilibrium if and only if for each i $\mathcal{u}^{*i}$ minimizes $J^i(\mathcal{u}^{*\bar{i}},\mathcal{u}^i)$ over the set $\mathcal{U}^i$. Theorem 2, therefore, immediately yields the next result.

<u>Theorem 3</u> $\mathcal{u}^* = (\mathcal{u}^{*1},\ldots,\mathcal{u}^{*N})$ is an equilibrium if and only if for each i there exist a constant J^{*i}, and processes (ΛV_t^i), (∇V_t^i) such that

$$\text{(i)} \qquad J^* + \int_0^1 \Lambda V_t^i\, dt + \int_0^1 \nabla V_t^i\, dz_t = \gamma^i \quad \text{a.s.}$$

$$\text{(ii)}\quad \Lambda V_t^i + \min_{u_i \in U_i} \{ \nabla V_t^i\, f(t,z,(u^{*\bar{i}}(t,z),\, u_i)) + c^i(t,z,(u^{*\bar{i}}(t,z),u_i))\} = 0$$

and the minimum is achieved at $u^{*i}(t,z)$ a.s. Furthermore, $J^{*i}=J^i(u^*)$.

4.3 Conditions for Efficiency

Consider the set $\mathcal{J} = \{J(u) = (J^1(u),\ldots,J^N(u))\,|\,u\in\mathcal{U}\}$, the set of attainable cost vectors. Suppose there exists a non-negative vector $\lambda=(\lambda_1,\ldots,\lambda_N)\neq 0$ and u^* such that

$$\lambda J(u^*) \leq \lambda J \quad \text{for all } J\in\mathcal{J} \tag{3}$$

It is then immediate that u^* is an efficient solution. It is also well-known that (3) is a necessary condition in the event that $\mathcal{J}$ is a convex set. This observation, in conjunction with Theorem 2 and Lemma 1, imply the next result.

Theorem 4 a) u^* is an efficient solution if there exist $\lambda\geq 0$, $\lambda\neq 0$, and for each i a constant J^{*i}, and processes (ΛV_t^i), (∇V_t^i) such that

$$\text{(i)}\quad \Sigma\, \lambda_i [J^{*i} + \int_0^1 \Lambda V_t^i\, dt + \int_0^1 \nabla V_t^i\, dz_t] = \Sigma\, \lambda_i \gamma^i \quad \text{a.s.}$$

$$\text{(ii)}\quad \Sigma\, \lambda_i\, \Lambda V_t^i + \min_{u\in U} \Sigma\, \lambda_i \{\nabla V_t^i\, f(t,z,u) + c^i(t,z,u)\} = 0,$$

and the minimum is achieved at $u^*(t,z)$ a.s.

b) If the game has the convexity property, then the conditions above are necessary for efficiency.

From a game-theoretic viewpoint an efficient solution is of interest only insofar as it is also an equilibrium. The combination of the results above gives the first intriguing result. Its proof is given in the Appendix.

Theorem 5 a) u^* is an efficient equilibrium if there exist for each i a constant J^{*i}, and processes (ΛV_t^i), (∇V_t^i) such that

(i) $J^{*i} + \int_0^1 \Lambda V_t^i \, dt + \int_0^1 \nabla V_t^i \, dz_t = \gamma^i$ a.s.

(ii) $\Lambda V_t^i + \underset{u_i \varepsilon U_i}{\text{Min}} \{\nabla V_t^i f(t,z,(u^{*\bar{i}}(t,z),u_i)) + c^i(t,z,(u^{*\bar{i}}(t,z),u_i))\} = 0,$

and the minimum is achieved at $u^{*i}(t,z)$ a.s.,

(iii) there exist $\lambda \geq 0$, $\lambda \neq 0$ such that

$$\Sigma \lambda_i \{\nabla V_t^i f(t,z,u^*(t,z)) + c^i(t,z,u^*(t,z))\} = \underset{u \varepsilon U}{\text{Min}} \Sigma \lambda_i \{\nabla V_t^i f(t,z,u) + c^i(t,z,u)\} \text{a.s.}$$

b) If the game has the convexity property, then the conditions above are also necessary.

Remark Define the Hamiltonian $H^i(t,z,u) = \nabla V_t^i f(t,z,u) + c^i(t,z,u)$. Condition (ii) above says that ith Hamiltonian must be minimized along the ith "coordinate" u_i. Condition (iii) says that in order that the "private" minimization (implied in the equilibrium concept) also be "socially" efficient this private minimization should lead to the "global" minimization of the social cost obtained as a weighted combination of the private costs. The intriguing part of the result is that these weights, the λ_i, are constant, that is, they do not depend on time t or the random state z.

4.4 Conditions for the Core

The result for the core follows in the same way as Theorem 5.

Theorem 6 a) u^* is in the core if there exist for each i a constant J^{*i}, and processes (ΛV_t^i), (∇V_t^i) such that

(i) $J^{*i} + \int_0^1 \Lambda V_t^i \, dt + \int_0^1 \nabla V_t^i \, dz_t = \gamma^i$ a.s.

(ii) $\Lambda V_t^i + \underset{u_i \varepsilon U_i}{\text{Min}} \{\nabla V_t^i f(t,z,(u^{*\bar{i}}(t,z),u_i) + c^i(t,z,(u^{*\bar{i}}(t,z),u_i))\} = 0,$

and the minimum is achieved at $u^{*i}(t,z)$ a.s.

(iii) for each S there exist constants $\lambda_i^S \geq 0$, $i \in S$, not all zero, such that

$$\sum_{i \in S} \lambda_i^S \{\nabla V_t^i f(t,z,u^*(t,z)) + c^i(t,z,u^*(t,z))\} = \min_{u \in U} \sum_{i \in S} \lambda_i^S \{\nabla V_t^i f(t,z,(u^{*\bar{S}}(t,z),u_S)) + c^i(t,z,(u^{*\bar{S}}(t,z),u_S)\} \text{ a.s.}$$

b) If the game has the strong convexity property, then the conditions above are also necessary.

<u>Remark</u> It may appear reasonable, at first sight, to conjecture that the weights, λ_i^S, should not depend upon S. However, upon further reflection, the reader should become convinced that this is unlikely. Thus the weights associated with different players will vary with the coalition S in which they are being considered as members.

§5. RANDOMIZED STRATEGIES

The convexity property is evidently quite restrictive. However, if one permits randomized controls, then convexity is guaranteed. To see this, define M_i as the set of all probability measures on U_i. U_i can then be regarded as a subset of M_i and the function f can be extended to the domain $[0,1] \times C \times M_1 \times \ldots \times M_N$ by setting

$$f(t,z,m_1,\ldots,m_N) = \int_{U_1}\ldots\int_{U_N} f(t,z,u_1,\ldots,u_N)m_1(du_1)\ldots m_N(du_N) \qquad (4)$$

The cost functions c^i can be extended analogously. The spaces M_i can be made compact and metrizable in a standard manner and $f(t,z,\cdot)$ remains continuous on $M=M_1 \times \ldots \times M_N$. The controls for i are now randomized controls that is functions $m^i: [0,1] \times C \to M_i$. The previous results continue to hold for this "extended" game. But notice from (4) that this extended game enjoys the convexity property and if joint randomization is allowed it also enjoys the strong convexity property.

APPENDIX

Proof of Theorem 5

Part a of the theorem follows immediately from theorem 3 and Part a of theorem 4. Hence it only remains to prove Part b.

By theorem 3 there exist for each i J^{*i}, and processes (ΛV_t^i), (∇V_t^i) such that

$$\left.\begin{aligned} & J^{*i} + \int_0^1 \Lambda V_t^i \, dt + \int_0^1 \nabla V_t^i \, dz_t = \gamma^i \quad \text{a.s.} \\ & \Lambda V_t^i + \min_{u_i \in U_i} \{\nabla V_t^i f(t,z,(u^{*\bar{i}}(t,z),u_i)) + c^i(t,z,(u^{*\bar{i}}(t,z),u_i))\} = 0 \end{aligned}\right\} \quad \text{(A1)}$$

and the minimum is achieved at $u_i^*(t,z)$. On the other hand, by Part b of theorem 4 there exist $\lambda \geq 0$, $\lambda \neq 0$ and for each i K^{*i}, and processes (ΛW_t^i), (∇W_t^i) such that

$$\left.\begin{aligned} & \Sigma \lambda_i \left[K^{*i} + \int_0^1 \Lambda W_t^i \, dt + \int_0^1 \nabla W_t^i \, dz_t\right] = \Sigma \lambda_i \gamma^i \quad \text{a.s.} \\ & \Sigma \lambda_i \Lambda W_t^i + \min_{u \in U} \Sigma \lambda_i \{\nabla W_t^i f(t,z,u) + c^i(t,z,u)\} = 0 \end{aligned}\right\} \quad \text{(A2)}$$

and the minimum is achieved at $u^*(t,z)$ a.s.

Comparison of these two sets of conditions reveals that it is enough to show that whenever (A1) and (A2) are both satisfied, then (A2) is also satisfied by choosing

$$K^{*i} = J^{*i}, \quad \Lambda W_t^i = \Lambda V_t^i, \text{ and } \nabla W_t^i = \nabla V_t^i$$

Now, by the last part of theorem 2,

$$J^{*i} + \int_0^t \Lambda V_t^i \, ds + \int_0^t \nabla V_t^i \, dz_s$$

$$= \min_{u \in \mathcal{U}} E^{u}\{\int_t^1 c^i(s,z_s,(u_s^{*\bar{i}}, u_s^i)\, ds + \gamma^i | F_t\}$$

$$= E^{u^*}\{\int_t^1 c^i(s,z_s,u_t^*) + \gamma^i | F_t\}$$

and similarly,

$$\Sigma \lambda_i [K^{*i} + \int_0^t \Lambda W_s^i \, ds + \int_0^t \nabla W_s^i \, dg_s]$$

$$= E^{u^*}\{\Sigma \lambda_i [\int_t^1 c^i(s,z_s,u^*_t) + \gamma^i] | F_t\}$$

Hence,

$$\Sigma \lambda_i [J^{*i} + \int_0^t \Lambda V_s^i \, ds + \int_0^t \nabla V_s^i \, dz_s] = \Sigma \lambda_i [K^{*i} + \int_0^t \Lambda W_s^i \, ds + \int_0^t \nabla W_s^i \, dz_s]$$

Setting t=0, gives $\Sigma \lambda_i J^{*i} = \Sigma \lambda_i K^{*i}$ and so

$$\int_0^t (\Sigma \lambda_i V_s^i - \Sigma \lambda_i W_s^i)\, ds = \int_0^t (\Sigma \lambda_i \nabla W_s^i - \Sigma \lambda_i \nabla V_s^i)\, dz_s .$$

But, under the measure P (z_t) is a Brownian motion so that the term on the right is a continuous martingale whereas the term on the left is a process with integrable variation. It follows that both terms must vanish so that $\Sigma \lambda_i V_s^i = \Sigma \lambda_i W_s^i$ and $\Sigma \lambda_i \nabla V_s^i = \Sigma \lambda_i \nabla W_s^i$ and the result follows.

§6. CONCLUSIONS

These remarks are mainly suggestions for further research.

It is known that for deterministic differential games the condition that the weights λ_i are constant is sufficient but not necessary even when the game has the convexity property. The results presented here therefore convey surprise. However, it is not evident that these results should be regarded as curiosities or as significant. To decide this it is necessary to clarify the precise rôle played by the Brownian motion in (1). Such clarification should also aid in restoring a measure of unity to the currently disparate traditions in the literature on deterministic and stochastic differential games. In the cases of control problems and two-player zero-sum games this has been achieved by the important work of Fleming [4,5] and subsequent work of Danskin [6] and Friedman [7], but it is not clear that these directions will prove useful for the many-player games.

This paper is not addressed to the important question of existence of solutions. For efficient controls, this question is immediately settled by known results on existence of optimal controls. A recent study [8] has nicely resolved the problem of existence of saddle points and value for two-player, zero-sum, stochastic differential games. It seems likely that the methods used in that study combined with the usual fixed-point arguments will help in proving existence of equilibrium solutions and the core.

Finally, the condition of complete information is a serious a priori restriction on the family of games considered in this paper. It is likely that results similar to those obtained here hold when all players have the same information even if it is incomplete [9]. The game is enormously more complicated when different players have different information. In the context of static games many important insights are provided by the results reported in [10,11].

Note: Reference [9] contains several incorrect statements.

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