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On Signal Design By The R_o Criterion For Non-White Gaussian Noise Channels*

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ABSTRACT

of

"ON SIGNAL DESIGN BY THE

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GAUSSIAN NOISE CHANNELS"

by

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The use of the R_0 criterion for modulation system design is investigated for channels with non-white Gaussian noise. A signal space representation of the waveform channel is developed, and the cut-off rate R_0 for vector channels with additive non-white Gaussian noise and unquantized demodulation is derived. When the signal input to the channel is a continuous random vector, maximization of R_0 with constrained average signal energy leads to a water-filling interpretation of optimal energy distribution in signal space. The necessary condition for a finite signal set to maximize R_0 with constrained energy and an equally likely probability assignment of signal vectors is presented, and an algorithm is outlined for numerically computing the optimum signal set. A necessary condition on a constrained energy, finite signal set is found which maximizes a Taylor series approximation of R_0. This signal set and the finite signal set which has the water-filling average energy distribution are compared for some specific examples along with the computed optimum.

CHAPTER I

INTRODUCTION

A. Background

In the design of a data communication system, the engineer can employ several methods to enhance the efficiency and effectiveness of information transfer over noisy channels. Consider the block diagram of a coded digital communication system given in figure 1. The encoder emits a codeword whose n symbols $x_1, x_2, \ldots, x_n$ are each selected from an alphabet $\{a_1, a_2, \ldots, a_q\}$ of q letters. The q-ary modulator selects a signal waveform according to the symbol input to the modulator. Thus $s(t)$ is selected in each signaling interval from a set of q waveforms. While propogating through the waveform channel, the signal is corrupted by an additive random noise process. The demodulator acts upon the received waveform $r(t)$ according to some specified decision rule and emits a symbol selected from an alphabet $\{b_1, b_2, \ldots, b_{q'}\}$ of q' letters. In general, $q' \geq q$. It is assumed that, in each signaling interval, the demodulator output symbol is dependent only on the modulator input symbol and on the noise in that signaling interval. Thus the modulator, waveform channel, and the demodulator present to the coding and decoding system a discrete memoryless channel (DMC) with the q-ary input digit X and the q'-ary output digit Y. The

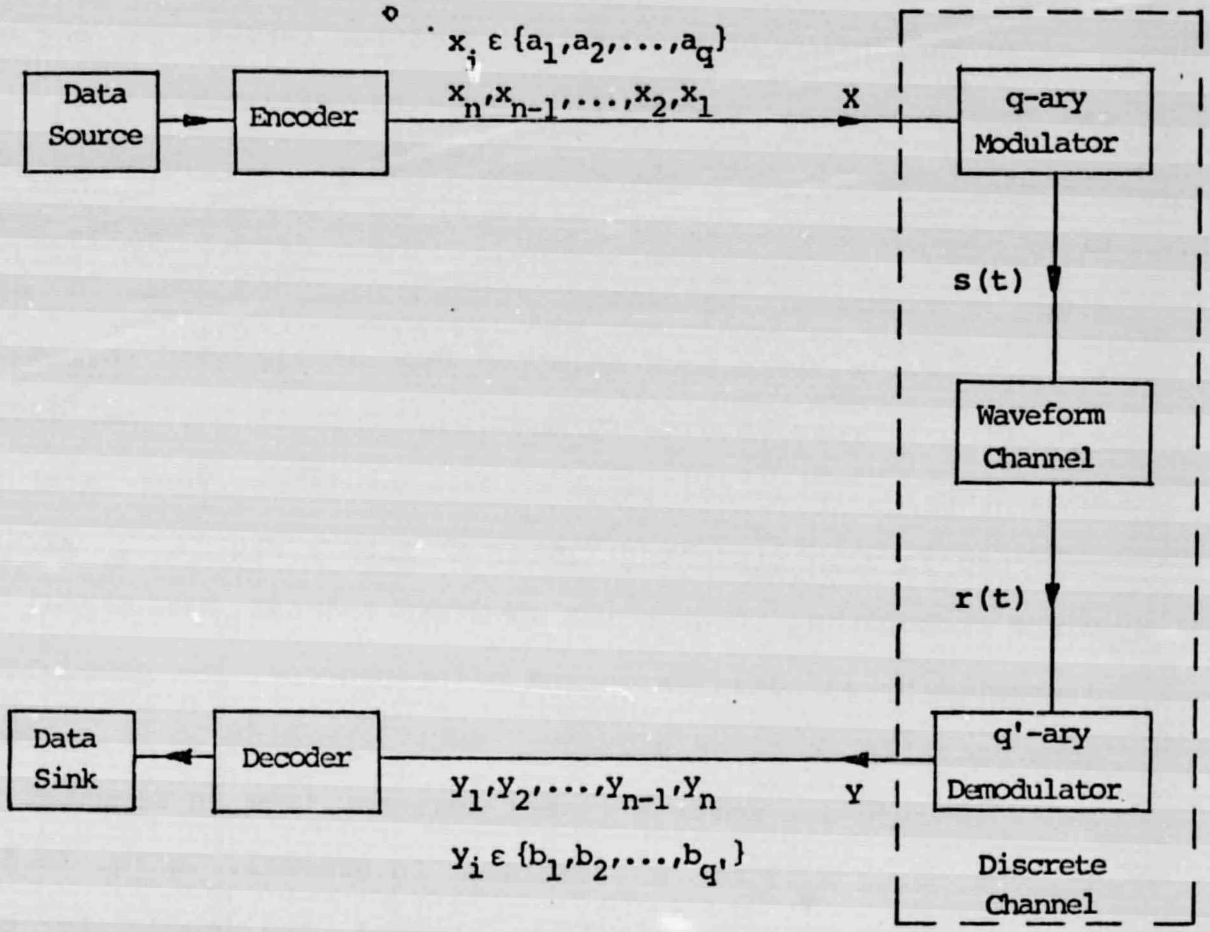

Figure 1. A coded digital communication system.

channel is completely described by the transition probabilities $f_{Y|X}(b_j|a_k)$ defined for each letter a_k in the input alphabet and each letter b_j in the output alphabet.

The modulation design problem reduces to selection of the modulator and demodulator to enhance the usefulness of the resulting DMC to the encoder and decoder. Wozencraft and Kennedy [1], and later Massey [2], proposed the "R_0 criterion" as a sensible design criterion for digital modulation systems. For a given average signal energy over the signaling interval, the "best" DMC which the modulator and demodulator can create is according to this criterion that which maximizes the cut-off rate R_0 of the DMC. Mathematically, R_0 is given by

$$R_0 = -\log_2\{ \min_Q \sum_y [\sum_x Q(x) \sqrt{f_{Y|X}(y|x)}]^2 \} , \qquad (1.1)$$

where the minimization is over all probability assignments $Q(x)$ for the q input letters. Supporting this choice of criterion, as Wozencraft and Kennedy pointed out, is the fact that the union bound on average error probability for the ensemble of random block codes of length n and rate R, the number of information bits per encoder output symbol, i.e., per use of the waveform channel, is

$$\overline{P}_e \leq 2^{-n(R_0 - R)} , \quad \text{for } R<R_0 . \qquad (1.2)$$

This exponential bound is equal at rate R_{crit} to Gallager's [3] random coding error exponent. Moreover, Wozencraft and

Kennedy noted further that R_0 is also the rate above which the average number of decoding steps per decoded digit becomes infinite for sequential decoding. More recently, it has been shown [4] that, if convolutional coding techniques are used on the DMC, then one can achieve

$$\bar{P}_e \leq c_R 2^{-nR_0} \quad , \quad \text{for} \quad R<R_0 \quad , \tag{1.3}$$

where c_R is a small constant and where n is the constraint length (in channel symbols) of the convolutional code. The one number R_0 then gives both a region of rates where it is possible to operate with arbitrarily small probability of error and an exponent of error probability (which Viterbi has shown is the best possible exponent for rates near R_0).

The R_0 criterion has been applied to modulation system design for communication over channels with additive white Gaussian noise [2]. Massey proved in this case that the simplex signal set is optimum for the R_0 criterion. Also, studies on demodulator design for the white noise case have been made [6] which demonstrate generally the superior performance of "soft-decision" demodulation ($q'>q$) over "hard-decision" demodulation ($q'=q$).

It is the purpose of the present thesis to explore the use of the R_0 criterion in modulation system design for channels with additive non-white Gaussian noise.

B. Vector channel representation

To study the modulation system design problem for communication over the waveform channel as depicted in figure 1, a signal space representation of the channel, derived as in [5], is convenient. The signal waveform $s(t)$ is assumed to be expressible as a linear combination of N orthonormal waveforms, that is,

$$s(t) = \sum_{i=1}^{N} s_i \phi_i(t) \quad . \tag{1.4}$$

The vector of coefficients $\underline{s} = [s_1, s_2, \ldots, s_N]^T$ can be considered a vector in N-dimensional Euclidean space ("signal space"). We can project the non-white noise process n(t) onto these same orthonormal functions in such a way that the coefficients $[n_1, n_2, \ldots, n_N]^T$ are in general correlated zero-mean Gaussian random variables. Then, presuming $r(t)$ is reduced by the demodulator to its projection $\underline{r} = [r_1, r_2, \ldots, r_N]^T$ in signal space (the possible loss of optimality will be considered later), we can model a "one shot" use of the vector channel as shown in figure 2.

The Karhunen-Loéve expansion of the Gaussian noise process can be used to yield statistically independent noise components over the signaling interval, but the first N normalized eigenfunctions of the defining integral equation must then be used on the waveform channel for the orthonormal waveforms of (1.4). By using the fact that all the noise coefficients are

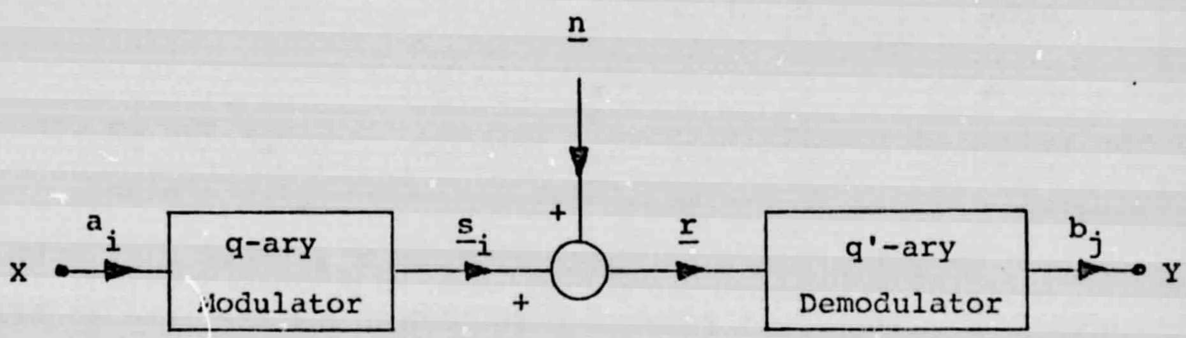

Figure 2. Vector non-white Gaussian noise channel model for "one-shot" use of the waveform channel (signal space representation).

statistically independent, and by invoking the Theorem of Irrelevance [5], we conclude that R_0 is not reduced by constraining the receiver to consider only the N-dimensional projection $\underline{r}$. Thus , the demodulation system can be made optimal for "one-shot" communication. We are severely restricted in this situation, however, by the requirement that signaling waveforms be expressible as a linear combination of the first N eigenfunctions of the Karhunen-Loéve expansion.

Alternatively, the vector channel representation can be achieved by <u>choosing</u> a convenient set of N signaling waveforms for use by the modulation system. It is evident that the receiver which processes only the N-dimensional projection $\underline{r}$ in this case discards relevant noise, and thus is in general sub-optimal. The R_0 obtained by using this modulation system, however, constitutes a lower bound on the performance achievable with an optimal receiver. In the remainder of this thesis, we shall allow the demodulator to consider only the projection $\underline{r}$ of $r(t)$ in signal space, being mindful of the non-optimality in doing so when an arbitrary signal set is used.

When the vector channel is used many times to transmit codewords to the decoder, noise vectors corresponding to different signaling intervals may be correlated, and intersymbol interference may be present. Thus, received signal vector components may be correlated with other components not only within bauds but also among many different bauds. If the memoryless assumption of section A is to remain valid, then the constraint time of the channel memory must be much less than the signaling interval. This can be realized by choosing a long baud length, assuming that we are free to do so, or by using a sufficiently long guard space between bauds. If the intersymbol interference is thus caused to be negligible, but the noise memory between successive uses of the channel is significant, we can interleave the signals at the transmitter and deinterleave the received signals to destroy the effect of the noise memory. However, the performance of this modulation system is inferior to one which makes use of the noise memory. In either case, the discrete channel seen by the coding system is memoryless, even though the waveform channel has memory because of the presence of non-white Gaussian noise therein.

C. Derivation of R_0 for vector non-white Gaussian noise channels and unquantized demodulation

In the design of the demodulator of figure 2, q' decision regions D_j are assigned which partition the received vector space in such a way that, when the input $\underline{r}$ falls in region D_j, symbol b_j is emitted from the demodulator. In order to bring the signal design problem to the surface, the demodulator decision regions are assumed to be unquantized, that is, the number of regions q' is infinite. It has been shown [2] that coarser quantization (finite q) reduces R_0 so that the cut-off rate with unquantized demodulation, denoted $(R_0)_{q'=\infty}$, overbounds that for any quantized demodulator. With this assumption, (1.1) reduces to

$$(R_0)_{q'=\infty} = -\log_2 \left[\min_Q \sum_{i=1}^{q} \sum_{k=1}^{q} Q(a_i)Q(a_k) \right.$$

$$\left. \cdot \int_{-\infty}^{\infty} \sqrt{p_{\underline{n}}(\underline{\alpha}-\underline{s}_i)\, p_{\underline{n}}(\underline{\alpha}-\underline{s}_k)}\; d\underline{\alpha} \right] . \tag{1.5}$$

where $p_{\underline{n}}(\ )$ is the probability density function of the noise vector $\underline{n}$, and the integration is N-fold. It was shown in section B that the noise components are in general correlated Gaussian random variables with zero mean. Thus, the probability density function for the noise vector becomes

$$p_{\underline{n}}(\underline{\alpha}) = [(2\pi)^{N/2} |\underline{\Lambda}|^{1/2}]^{-1} \exp(-\frac{1}{2}\, \underline{\alpha}^T \underline{\Lambda}^{-1} \underline{\alpha}) \ , \tag{1.6}$$

where $\underline{\Lambda}$ is the noise covariance matrix. Upon substitution of (1.6) into (1.5), the term under the square root sign in (1.5) becomes, after some rearrangement,

$$[(2\pi)^N |\underline{\Lambda}|]^{-1} \exp[-\underline{\alpha}^T \underline{\Lambda}^{-1} \underline{\alpha} \ -\frac{1}{2}(\underline{s}_i^T \underline{\Lambda}^{-1} \underline{s}_i + \underline{s}_k^T \underline{\Lambda}^{-1} \underline{s}_k)$$

$$+ \underline{\alpha}^T \underline{\Lambda}^{-1} (\underline{s}_i + \underline{s}_k)] \ . \tag{1.7}$$

Completing the square in the exponent and taking the square root in (1.5) yields

$$p_{\underline{n}}[\underline{\alpha} - \frac{1}{2}(\underline{s}_i + \underline{s}_k)] \cdot \exp[-\frac{1}{8}(\underline{s}_i - \underline{s}_k)^T \underline{\Lambda}^{-1} (\underline{s}_i - \underline{s}_k)] \ . \tag{1.8}$$

After performing the integration in (1.5), we obtain the following expression for the cut-off rate of the discrete memoryless channel with unquantized demodulation and with the signal set

$\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_q\}$ on the vector non-white Gaussian noise channel:

$$(R_0)_{q'=\infty} = -\log_2 \{\min_Q \sum_{i=1}^{q} \sum_{k=1}^{q} Q(a_i)Q(a_k)$$

$$\cdot \exp[-\tfrac{1}{8}(\underline{s}_i-\underline{s}_k)^T \underline{\Lambda}^{-1}(\underline{s}_i-\underline{s}_k)]\}. \qquad (1.9)$$

D. Plan of the thesis

In Chapter II, we consider the limit of (1.9) when q becomes infinite, that is, when the number of signals in the signal set becomes infinitely large, and thus the probability assignment Q for these signals becomes a probability density function. The resultant cut-off rate obtained for infinite q and q' overbounds that attainable with any finite q and q' for the same average signal energy. A necessary condition for the input probability density function to maximize $(R_0)_{q,q'=\infty}$ is found. When the input density is Gaussian, maximization of cut-off rate leads to a water-filling interpretation for the optimum assignment of input energy to the components of the signal vectors.

In Chapter III, we investigate the maximization of (1.9) when the signal set is constrained in average energy and in the number of signals, and when the input probability distribution is uniform. The calculus of variations is employed to give a necessary condition on the signal vectors maximizing cut-off rate with Q uniform. Also, an algorithm is outlined for numerically calculating the optimum signal set in N-dimensional signal space.

In Chapter IV, the necessary condition on the signal set which maximizes a Taylor series approximation to the cut-off rate with Q uniform is formulated, and signal sets are found for some specific values of q and N , as a function of average signal-to-noise ratio. Also, signal sets are found for q and N which have the same distribution of average signal energy among the N signal components as the water-filling set with infinite q described in Chapter II.

Finally, in Chapter V we compare the performance of the Taylor series and the water-filling signal sets with the numerically calculated optimum sets.

CHAPTER II

OPTIMUM SIGNAL DESIGN WITH LARGE SIGNAL SETS AND CONSTRAINED ENERGY

A. R_0 for infinite q

The number of symbols q in the codeword symbol alphabet is assumed to be arbitrarily large. This in turn requires an equally large number q of signals in the signal set. In Chapter I, it was shown that the expression (1.9) for R_0 with unquantized demodulation is a minimization over all probability assignments of the q codeword symbols. For the same average signal-to-noise ratio, $(R_0)_{q'=\infty}$ increases as q increases because of the greater degree of freedom in choosing the q probabilities for the channel input symbols. Thus the value of the cut-off rate with infinite q and q', denoted $(R_0)_{q,q'=\infty}$, overbounds the attainable cut-off rate for any finite q and q'.

Now consider the probability assignments $Q(a_i)$ as an appropriate partitioning of an N-dimensional input signal space so that, when a vector $\underline{\alpha}$ falls in the i^{th} partition ξ_i letter a_i is chosen, and signal $\underline{s}_i$ is transmitted. If $p(\underline{\alpha})$ is a probability density function defined on the input space for choosing signal vectors, then

$$Q(a_i) = \int_{\xi_i} p(\underline{\alpha})\, d\underline{\alpha} \,. \qquad (2.1)$$

As the number of letters q becomes large and the partitioning becomes finer, the signal vectors become continuously distributed in signal space, and the distribution is described by the probability density function $p(\underline{\alpha})$. Thus

$$(R_0)_{q,q'=\infty} = -\log_2 \{\min_p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\underline{\alpha})p(\underline{\beta}) \cdot \exp[-\frac{1}{8}(\underline{\alpha} - \underline{\beta})^T \underline{\Lambda}^{-1} (\underline{\alpha} - \underline{\beta})] d\underline{\alpha}\, d\underline{\beta}\}. \tag{2.2}$$

To illustrate the fact that the finite-q signal set is a special case of the continuously distributed ($q=\infty$) signal set, consider choosing in the right of (2.2)

$$p(\underline{\alpha}) = \sum_{i=1}^{q} Q(a_i)\, \delta(\underline{\alpha} - \underline{s}_i) \quad . \tag{2.3}$$

Then substitution of (2.3) into (2.2) yields (1.9). The average energy of a signal vector is constrained to be

$$\overline{|\underline{\alpha}|^2} = E = \overline{\alpha_1^{\,2}} + \overline{\alpha_2^{\,2}} + \ldots + \overline{\alpha_N^{\,2}} = E_1 + E_2 + \ldots + E_N , \tag{2.4}$$

where E_1, E_2, ..., E_N are the average energies of a signal vector on each of the N coordinates.

We now investigate the conditions for which a probability density function $p(\underline{\alpha})$ minimizes the double integral of (2.2), and thus maximizes $(R_0)_{q,q'=\infty}$. Consider the one-dimensional case in which α is a zero mean random variable, that is, the signals assume values among the real numbers. Let

$$F[p(\alpha)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\alpha)p(\beta)\ \exp[-\frac{1}{8}\ \frac{(\alpha-\beta)^2}{\sigma^2}]\, d\alpha\, d\beta$$

$$+ \lambda_1\int_{-\infty}^{\infty} p(\alpha)\ d\alpha + \lambda_2\int_{-\infty}^{\infty} \alpha\ p(\alpha)\ d\alpha + \lambda_3\int_{\infty}^{\infty} \alpha^2\ p(\alpha)\, d\alpha \quad . \tag{2.5}$$

where σ^2 is the noise variance, and λ_1, λ_2, and λ_3 are the Lagrange multipliers for the constraints

$$\int_{-\infty}^{\infty} p(\alpha)\ d\alpha = 1,\ \int_{-\infty}^{\infty} \alpha\ p(\alpha)\ d\alpha = 0, \int_{-\infty}^{\infty} \alpha^2\ p(\alpha)\ d\alpha = E\ ,$$

respectively. If the function $p(\alpha)$ maximizes (2.2), then

$$\frac{\partial}{\partial\varepsilon} F[p(\alpha) + \varepsilon h(\alpha)]\ \Big|_{\varepsilon=0} \tag{2.6}$$

must be zero for all choices of $h(\alpha)$. Carrying out the indicated differentiation in (2.6), we obtain

$$\int_{-\infty}^{\infty} p(\beta)\ \exp[-\frac{1}{8}\ \frac{(\alpha-\beta)^2}{\sigma^2}]\, d\beta + \lambda_1 + \lambda_2\alpha + \lambda_3\alpha^2 = 0 \tag{2.7}$$

as a necessary condition for $p(\ )$ to maximize $(R_0)_{q,q'=\infty}$.

It is well known [3] that the Gaussian density function maximizes the entropy of continuous random variables, and thus the Gaussian random variable achieves capacity when used as the input distribution for the continuous additive Gaussian noise channel with an average energy constraint. However, when we substitute into (2.7)

$$p(\beta) = \frac{1}{\sqrt{2\pi E}} \exp(-\frac{\beta^2}{2E}) , \quad (2.8)$$

we find that, if the Gaussian density does indeed maximize $(R_0)_{q,q'=\infty}$, then we must have, for some value of λ_1, λ_2, and λ_3,

$$c_1 e^{c_2\alpha^2} + \lambda_1 + \lambda_2\alpha + \lambda_3\alpha^2 = 0 , \quad (2.9)$$

where c_1 and c_2 are constants depending on the signal-to-noise ratio. Thus we conclude that the Gaussian density function does not in general satisfy the necessary condition (2.7) for the maximization of $(R_0)_{q,q'=\infty}$.

B. Water-filling interpretation of signal energy distribution

It is instructive to substitute the Gaussian density function (2.8) into the double integral in (2.2), in order to obtain a lower bound on $(R_0)_{q,q'=\infty}$. The signal is the N-vector whose components are assumed to be statistically independent Gaussian random variables with density function

$$p(\underline{\alpha}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi E_i}} \exp(-\frac{\alpha_i^2}{2E_i}) . \quad (2.10)$$

The average signal energy is again constrained as in (2.4).

We also assume that the components of the non-white Gaussian noise vector $\underline{n}$ are statistically independent, which can be achieved by rotating the axes of signal space so that the projection of the noise process onto this signal space yields uncorrelated components. Thus the covariance matrix $\underline{\Lambda}$ is diagonal with diagonal elements σ_i^2, $i = 1, 2, \ldots, N$, the noise variances on each axis. Substitution of (2.10) into (2.2) yields

$$(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}} = -\log_2\left\{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{2\pi E_i} \exp\left[-\frac{\alpha_i^2}{2E_i} - \frac{\beta_i^2}{2E_i} - \frac{(\alpha_i-\beta_i)^2}{8\sigma_i^2}\right] d\alpha_1 \ldots d\alpha_N \, d\beta_1 \ldots d\beta_N\right\}, \quad (2.11)$$

where $(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}$ denotes the cut-off rate for a continuous statistically independent Gaussian random variable input with unquantized demodulation. Completing the square in the exponent in (2.11), and interchanging the order of integration and product gives

$$(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}} = -\sum_{i=1}^{N} \log_2\left\{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi E_i}} \exp\left[-\frac{\left(\alpha_i - \frac{2X_i^2}{8\sigma_i^2}\beta_i\right)^2}{2X_i^2}\right] \cdot \frac{1}{\sqrt{2\pi E_i}} \exp\left(-\frac{\beta_i^2}{2Y_i^2}\right) d\alpha_i \, d\beta_i\right\}, \quad (2.12)$$

where

$$X_i^2 = \frac{2E_i}{1 + \frac{E_i}{4\sigma_i^2}}, \quad Y_i^2 = \frac{X_i^2}{1 - \frac{X_i^4}{16\sigma_i^4}}.$$

The result of the integration within the braces in (2.12) is $X_i Y_i / E_i$. After some manipulation, (2.12) becomes

$$(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}} = \sum_{i=1}^{N} 1/2 \log_2(1 + E_i/2\sigma_i^2). \tag{2.13}$$

The implication of (2.13) is that, with statistically independent zero-mean Gaussian random vector inputs, the vector channel reduces to N parallel scalar channels, whose respective cutoff rates are

$$[(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}]_i = 1/2 \log_2(1 + E_i/2\sigma_i^2) \ , \ i=1,\ldots,N \tag{2.14}$$

when a Gaussian random variable with variance E_i is input to the i^{th} additive Gaussian noise channel with noise variance σ_i^2 and unquantized demodulation.

Following the method of Gallager for achieving capacity over parallel channels, we maximize the sum in (2.13) over the N signal variances E_i with the constraint on average energy given by (2.4) by application of the Kuhn-Tucker conditions with μ as the Lagrange multiplier of the constraint. The resulting necessary and sufficient conditions for the maximum are

$$\frac{\partial}{\partial E_i}\left[\sum_{i=1}^{N} \frac{1}{2} \log_2\left(1 + \frac{E_i}{2\sigma_i^2}\right)\right] \leq \mu, \ i=1,\ldots,N \ , \tag{2.15}$$

with equality if $E_i > 0$. Performing the differentiation yields

$$\frac{1}{2(E_i + 2\sigma_i^2)} \leq \mu \ , \ i=1,\ldots,N \ , \tag{2.16}$$

and, by choosing $\mu = 1/2B$, we arrive at the following necessary and sufficient conditions on the optimizing E_i's :

$$E_i + 2\sigma_i^2 = B \quad \text{for} \quad 2\sigma_i^2 < B \ ;$$
$$E_i = 0 \quad \text{for} \quad 2\sigma_i^2 \geq B \ , \tag{2.17}$$

where B is chosen so that the constraint (2.4) is satisfied. This has the water-filling interpretation depicted in figure 3. The N blocks of height $2\sigma_i^2$ form the bottom of a container into which the average signal energy is "poured." The container is connected so that energy is distributed among each of the N components of the signal vector in the amount of the depth E_i below the surface B.

It was shown by Gallager [3, Theorem 7.5.1] that capacity is achieved on N parallel additive Gaussian noise channels with noise variances σ_i^2, i=1,...,N, by choosing the inputs to be statistically independent, zero-mean, Gaussian random variables of variance E_i with

$$E_i + \sigma_i^2 = B' \quad \text{for} \quad \sigma_i^2 < B' \ ;$$
$$E_i = 0 \quad \text{for} \quad \sigma_i^2 \geq B' \ , \tag{2.18}$$

where B' is chosen so that (2.4) is satisfied. The capacity C of the parallel combination is given by

$$C = \sum_{i=1}^{N} \frac{1}{2} \log_2\left(1 + \frac{E_i}{\sigma_i^2}\right) . \tag{2.19}$$

This gives the familiar capacity-achieving water-filling inter-

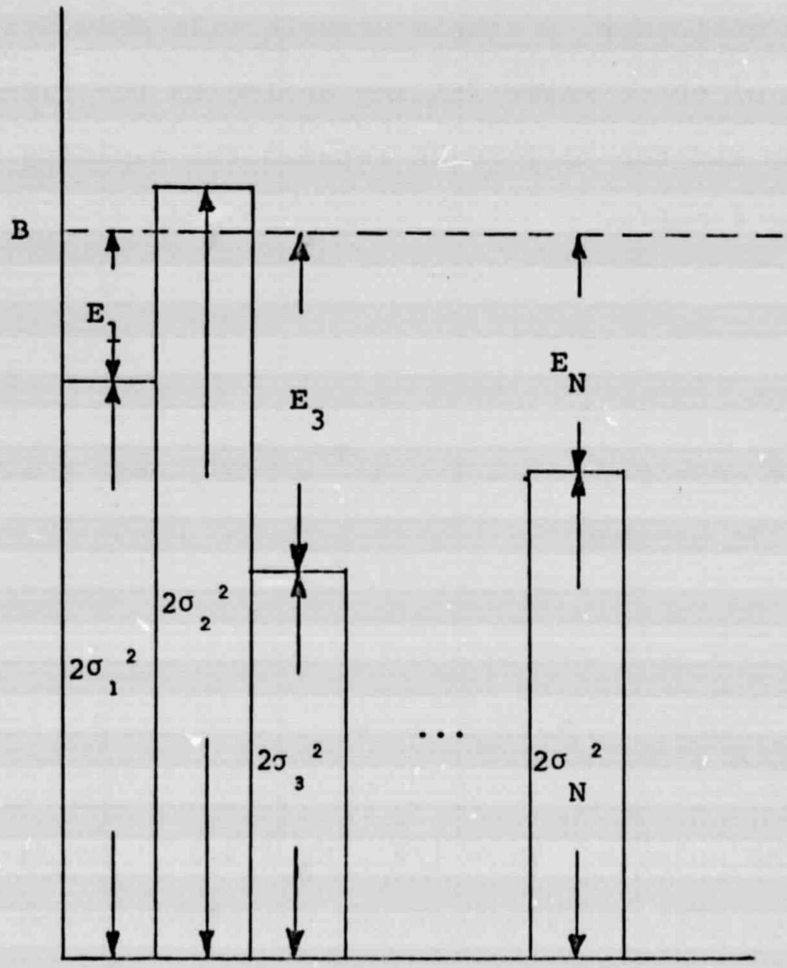

Figure 3. Water-filling interpretation of achieving maximum over E_n, $n = 1, \ldots, N$, of $(R_0)_{q,q'=\infty}^{\text{gaussian}}$.

pretation of signal energy distribution. The difference between the water-filling energy assignments for the maximization of $(R_0)_{q,q'=\infty \atop \text{gaussian}}$ and of capacity C is in the weighting of the noise variances σ_i^2, and thus in the height of the N blocks forming the container. A simple example will demonstrate the application of these water-filling analogies for signal energy distribution and the difference between the capacity and the cut-off rate results.

C. Example

Suppose $N = 2$, $\sigma_1^2 = 1$, $\sigma_2^2 = 1/2$, and $E = 2$. Equation (2.17) becomes

$$E_1 + 2 = B$$

$$E_2 + 1 = B$$

$$E_1 + E_2 = 2$$

which has the solution $E_1 = 1/2$, $E_2 = 3/2$, being the signal energy distribution over the two signal space coordinates that maximizes $(R_0)_{q,q'=\infty \atop \text{gaussian}}$ for this example. Equation (2.13) then gives

$$(R_0)_{q,q'=\infty \atop \text{gaussian}} = 1/2 \log_2(1 + 1/4) + 1/2 \log_2(1 + 3/2) = .8219 \,. \qquad (2.20)$$

Similarly, (2.18) becomes

$$E_1 + 1 = B'$$
$$E_2 + 1/2 = B'$$
$$E_1 + E_2 = 2$$

which has the solution $E_1 = 3/4$, $E_2 = 5/4$. This choice of signal energies on the two coordinates achieves capacity on this combination of $N = 2$ discrete channels. Equation (2.19) then gives

$$C = 1/2 \log_2(1 + 3/4) + 1/2 \log_2(1 + 5/2) = 1.3074 \ . \quad (2.21)$$

The water-filling interpretation for this example is shown in figure 4. It can be seen that optimum distribution for achieving capacity will divide energy more symmetrically than the corresponding optimum distribution for maximizing $(R_0)_{q,q'=\infty \atop \text{gaussian}}$ because of the greater weighting of the noise variances in the latter case. Another difference is in the fact that the water-filling distribution in figure 4(b) does achieve the maximum mutual information (capacity) over E_1 and E_2 for this example, but the water-filling distribution shown in figure 4(a) does not represent the maximum achievable cut-off rate for a continuous random vector input with unquantized demodulation, which is given by (2.2) with $N = 2$ in this example. Thus the number given in (2.20) is only the lower bound on the value of (2.2) with the maximizing probability density function, rather than the Gaussian.

It is interesting to note that when the capacity-achieving energy distribution, viz., $E_1 = 3/4$, $E_2 = 5/4$, is used in (2.13), we obtain $(R_0)_{q,q'=\infty \atop \text{gaussian}} = .8147$. Thus for this example the

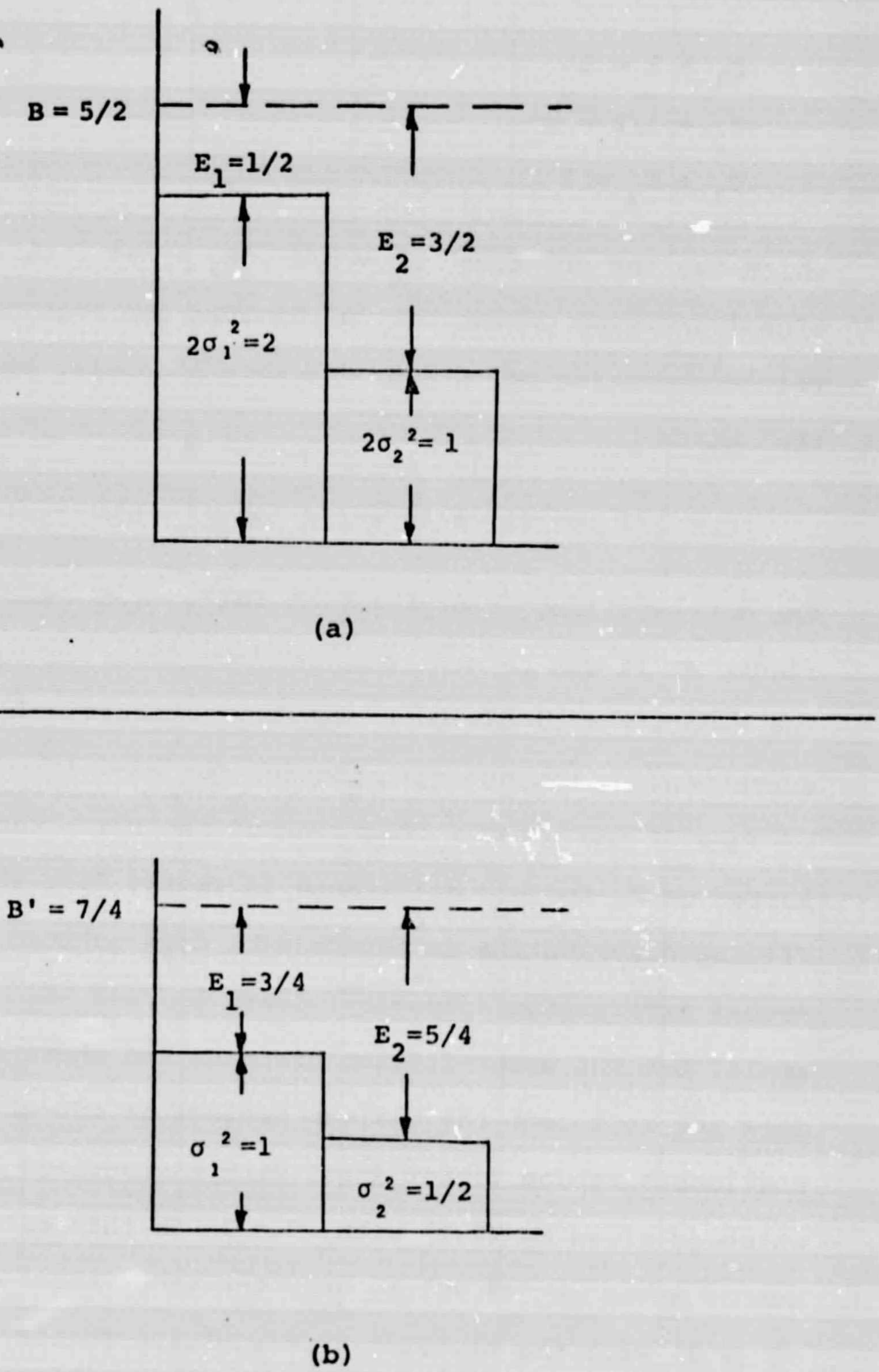

Figure 4. Water-filling interpretation for (a) achieving maximum of $(R_0)_{q,q'=\infty}$ and (b) achieving capacity over the set of $N=2$ parallel gaussian channels with $\sigma_1{}^2 = 1$, $\sigma_2{}^2 = 1/2$, and $E = 2$.

difference in the cut-off rates obtained by using the capacity-achieving optimum signal energy distribution and the distribution which maximizes $(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}$ is very small.

CHAPTER III

OPTIMUM SIGNAL DESIGN WITH SMALL SIGNAL SETS AND CONSTRAINED ENERGY

With the modulator constrained to emit only a finite number of signals, (1.9) is the starting point for a consideration of the optimum choice of the signal vectors used to transmit the q letters of the modulator alphabet. To avoid the awkward minimization in (1.9), the symmetric cut-off rate [6] is employed, which is the value of the right of (1.9) when Q is the uniform distribution $Q(a_i) = 1/q$, $i = 1, 2, \ldots, q$, rather than the minimizing distribution. Thus, the symmetric cut-off rate, denoted $\tilde{R}_0$, is less than or equal to the actual cut-off rate of the system, with equality if the uniform is indeed the minimizing distribution, as it is for many practical cases including the case $q = 2$. The expression for the symmetric cut-off rate for the Gaussian channel with unquantized demodulation is

$$(\tilde{R}_0)_{q'=\infty} = -\log_2 \{\frac{1}{q^2} \sum_{i=1}^{q} \sum_{k=1}^{q} \exp[-\frac{1}{8}(\underline{s}_i-\underline{s}_k)^T \underline{\Lambda}^{-1} (\underline{s}_i-\underline{s}_k)]\}. \tag{3.1}$$

We now seek to maximize the right side of (3.1) or, equivalently, to minimize the double summation by choice of the signal set $\{s_1, s_2, \ldots, s_q\}$ with the following constraint on signal energy:

$$\sum_{i=1}^{q} \underline{s}_i^T \underline{s}_i = qE \ , \tag{3.2}$$

where E is the average signal energy. The problem can be stated : minimize

$$f(\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_q) = \sum_{i=1}^{q} \sum_{k=i}^{q} \exp[-\tfrac{1}{8}(\underline{s}_i-\underline{s}_k)^T \underline{\Lambda}^{-1} (\underline{s}_i-\underline{s}_k)] \tag{3.3}$$

subject to the constraint

$$g(\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_q) = \sum_{i=1}^{q} \underline{s}_i^T \underline{s}_i - qE = 0 \ . \tag{3.4}$$

In order to formulate the necessary condition for a stationary point of (3.3), the vector gradient, defined as

$$\nabla_{\underline{s}_1}(f) = \begin{bmatrix} \frac{\partial f}{\partial s_{11}} \\ \frac{\partial f}{\partial s_{12}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial s_{1N}} \end{bmatrix} , \tag{3.5}$$

proves to be useful. A necessary condition for a set of vectors $\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_q\}$ to be a local minimum of (3.3) subject to the constraint (3.4) is that

$$\nabla_{\underline{s}_i}(f) + \lambda \nabla_{\underline{s}_i}(g) = \underline{0} \ , \quad i = 1, 2, \ldots, q \ , \tag{3.6}$$

where λ is the Lagrange multiplier corresponding to the constraint (3.4). To evaluate the first term in (3.6), we note that

$$\nabla_{\underline{x}}(e^{-\underline{x}^T \underline{A}\, \underline{x}}) = -2\underline{A}\, \underline{x}\, e^{-\underline{x}^T \underline{A}\, \underline{x}} \quad .$$

Thus,

$$\nabla_{\underline{s}_i}(f) = - \sum_{k=1}^{q} \frac{1}{2} \underline{\Lambda}^{-1}(\underline{s}_i - \underline{s}_k) \exp[- \frac{1}{8}(\underline{s}_i - \underline{s}_k)^T \underline{\Lambda}^{-1}(\underline{s}_i - \underline{s}_k)] . \quad (3.7)$$

The second term in (3.6) is just $2\lambda \underline{s}_i$, so we are left with, as necessary conditions on the signal vectors $\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_q\}$ to maximize $(\tilde{R}_0)_{q'=\infty}$,

$$\lambda' \underline{s}_i + \sum_{k=1}^{q} \underline{\Lambda}^{-1}(\underline{s}_i - \underline{s}_k) \exp[- \frac{1}{8}(\underline{s}_i - \underline{s}_k)^T \underline{\Lambda}^{-1}(\underline{s}_i - \underline{s}_k)]$$

$$= \underline{0} \ , \ i = 1, 2, \ldots, q \ . \quad (3.8)$$

A closed form solution of the system of equations (3.8) is not in general practicable. However, one can use the results of non-linear programming techniques to compute solutions to the minimization of (3.3), the non-linear objective, with the non-linear constraint (3.4). The method employed here will be the gradient projection method as given in Luenberger [7]. Let

$$\nabla f(\underline{S}) = [\nabla_{\underline{s}_1}^T(f), \nabla_{\underline{s}_2}^T(f), \ldots, \nabla_{\underline{s}_q}^T(f)] \quad (3.9)$$

$$\nabla g(\underline{S}) = [\nabla_{\underline{s}_1}^T(g), \nabla_{\underline{s}_2}^T(g), \ldots, \nabla_{\underline{s}_q}^T(g)] \quad (3.10)$$

be the gradient vectors of $f(\underline{S})$ and $g(\underline{S})$ which are, in turn, the scalar valued functions of the qN vector

$$\underline{S} = [s_{11}, \ldots, s_{1N}, s_{21}, \ldots, s_{2N}, \ldots, s_{q1}, \ldots, s_{qN}] \, . \tag{3.11}$$

From an initial test point $\underline{S}_0$ which satisfies the constraint (3.4), $\nabla f(\underline{S})$ and $\nabla g(\underline{S})$ as in (3.9) and (3.10), respectively, are computed. The vector $\nabla f(\underline{S})$ is geometrically projected onto the plane tangent to the surface $g(\underline{S}_0) = 0$ at the point $\underline{S}_0$, producing a vector $\underline{d}$. A step is taken along this vector, and then the return to the constraint surface is achieved by stepping in the direction of $-\nabla g(\underline{S})$. Thus a new point

$$\underline{S}_1 = \underline{S}_0 + a\,\underline{d} - b\,\nabla g(\underline{S}_0 + a\,\underline{d}) \tag{3.12}$$

is generated, where a is a small increment and b is chosen so that (3.4) is satisfied at $\underline{S}_1$. This process is repeated until $\underline{d} = \underline{0}$, in which case (3.6) is satisfied and the algorithm is terminated.

The above method was used to obtain computer solutions to the maximization of (3.1), for the values $q = 3$, $N = 2$, and for $q = 4$, $N = 2$. The optimum signal sets and the values of $(\tilde{R}_0)_{q'=\infty}$ obtained, for various values of average signal-to-noise ratio and the noise variance ratio in the two dimensions are summarized in Tables 1 to 4 in Chapter V. These cut-off rates for the optimum signal sets will be used to evaluate the quality of the sub-optimal signal sets which maximize approximations to R_0 derived in Chapter IV.

CHAPTER IV

OPTIMUM SIGNAL DESIGN USING APPROXIMATIONS TO R_0

A. Introduction

Since an analytic solution of (3.6) for any interesting choices of q and of N is intractable, certain approximations will be made in order to discover a general rule-of-thumb for "good" signal selection by the R_0 criterion. Our first approach will be to expand $(\tilde{R}_0)_{q'=\infty}$ in a Taylor series, and to find signal sets which maximize the second order approximation. Our second method will use the water-filling signal energy distribution for the continuously distributed infinite-q signal set, developed in Chapter II, for assigning an energy distribution to small-q signal sets.

B. Taylor series expansion and optimum signal sets

The following expansion of the exponential term in (3.1) will be used:

$$\exp[-\frac{1}{8}(\underline{s}_i-\underline{s}_k)^T\underline{\Lambda}^{-1}(\underline{s}_i-\underline{s}_k)] = 1 - \frac{1}{8}(\underline{s}_i-\underline{s}_k)^T\underline{\Lambda}^{-1}(\underline{s}_i-\underline{s}_k)$$

$$+ \frac{1}{16}[(\underline{s}_i-\underline{s}_k)^T\underline{\Lambda}^{-1}(\underline{s}_i-\underline{s}_k)]^2 - \dots \quad . \tag{4.1}$$

When the first order term only is retained in the expansion of

$(\tilde{R}_0)_{q'=\infty}$, we find only trivial solutions to the necessary condition for a stationary point of the approximation. Interesting results occur only when the second term is also used. The necessary condition for a signal set $\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_q\}$ to maximize the second order approximation to $(\tilde{R}_0)_{q'=\infty}$ becomes

$$\sum_{k=1}^{q} [(\underline{s}_i-\underline{s}_k)^T \underline{\Lambda}^{-1}(\underline{s}_i-\underline{s}_k) - 1]\underline{\Lambda}^{-1}(\underline{s}_i-\underline{s}_k)$$

$$= \lambda'' \underline{s}_i \ , \ i = 1, 2, \ldots, q \ . \tag{4.2}$$

Thus, we are left with a set of equations to solve which involve at most cubic terms in the unknown signal components.

We now apply this technique to a specific example.

C. Example

Consider $q = 3$ and $N = 2$. We choose, without loss of optimality, the signal vector constellation shown in figure 5, where a and b are parameters to be determined. The average signal energy constraint demands also that

$$2a^2 + 6b^2 = 3 \ . \tag{4.3}$$

The noise is Gaussian and has statistically independent components with variances σ_1^2 and σ_2^2 on the first and second coordinates, respectively. Then (4.2) reduces to the two independent equations

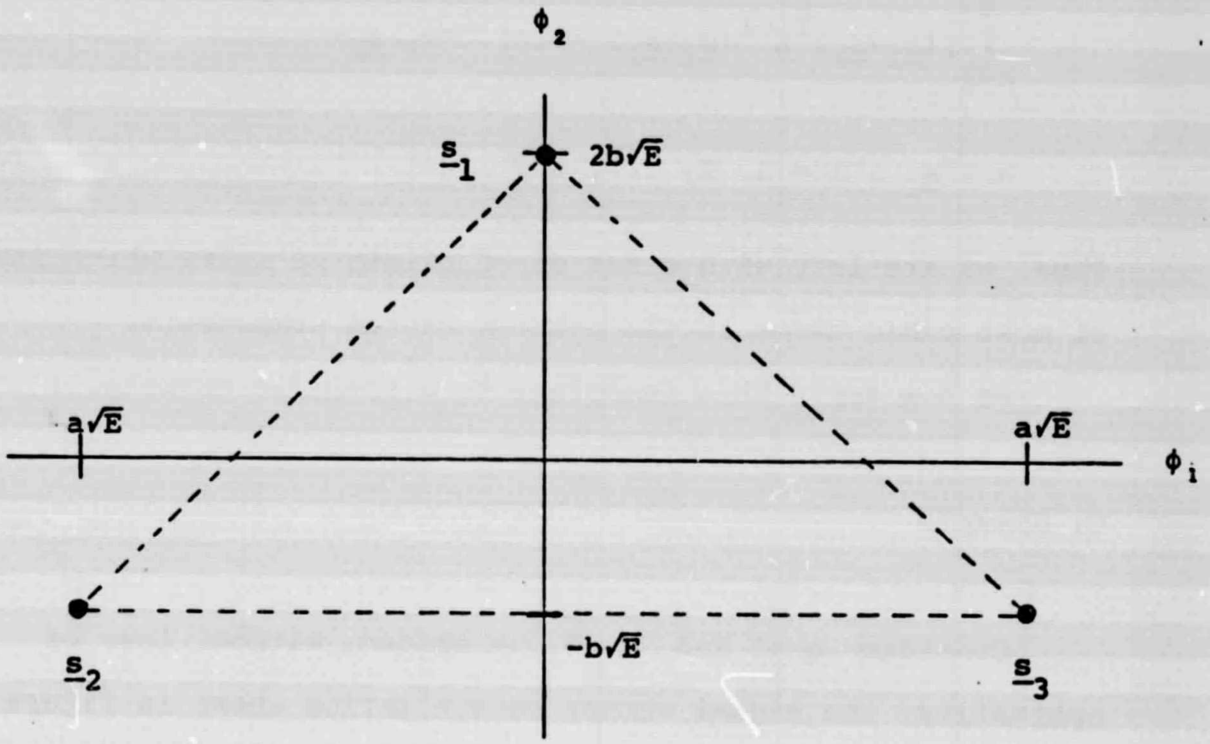

Figure 5. Signal constellation assumed for the q = 3, N = 2 signal sets considered for the example.

$$\frac{3a^2}{\sigma_1^2} + \frac{3b^2}{\sigma_2^2} - 1 = \lambda''\sigma_1^2$$

$$\frac{a^2}{\sigma_1^2} + \frac{9b^2}{\sigma_2^2} - 1 = \lambda''\sigma_2^2 \quad . \tag{4.4}$$

Let

$$SNR_1 = \frac{E}{2\sigma_1^2}$$

denote the average signal-to noise ratio on the first coordinate, and let

$$\gamma = \frac{\sigma_2^2}{\sigma_1^2}$$

denote the ratio of noise variances which also gives an indication of the assymetry of the noise in two-dimensional signal space. The solution to (4.4), from (4.3), is

$$b^2 = \frac{\gamma - 1 - SNR_1(3\gamma - 1)}{2\,SNR_1(2 - 3\gamma - 3/\gamma)}$$

$$a^2 = \frac{3}{2}(1 - 2b^2) \quad . \tag{4.5}$$

With $\gamma = 1$, the white noise case, the solution is $b = .5$, $a = .866$, which is the simplex set for $N = 2$. Massey [2] has already shown of course that the simplex is optimal for the R_0 criterion in Gaussian white noise. Thus, the solution (4.4) is asymptotically optimum for $\gamma \to 1$. As $\gamma \to 0$, evaluation of

(4.5) in the limit shows that $a = 0$, which is obviously an optimal solution when there is no noise on the second coordinate.

D. Water-filling signal sets

It was shown in Chapter II that, for a continuously distributed, infinite-q signal set with constrained average energy, described by an N-dimensional Gaussian density function, maximization of capacity C and of $(R_0)_{q,q'=\infty \atop \text{gaussian}}$ over the possible distribution of available signal energy led to a water-filling interpretation of these optimal distributions. We now consider the use of this water-filling interpretation for finite-q signal sets.

The motivation behind the application to finite-q signal sets of the water-filling method of optimizing capacity and cut-off rate for continuous random variable inputs can be described as follows. Although it is certainly not possible to form a Gaussian density from a uniformly distributed impulsive density, as in (2.3) with Q uniform, we can match the optimum statistics, that is, the mean and the variance of the optimal Gaussian density function, in choosing the signal vectors and the distribution of signal energy for finite-q signal sets. Thus, we choose the signal vectors, as in figure 5, so that the centroid is zero, and we choose the parameters a and b to satisfy the water-filling requirement.

At very low signal-to-noise ratios, the water-filling results obtained by maximizing capacity and $(R_0)_{q,q'=\infty \atop \text{gaussian}}$ are identical since all the signal energy must be assigned along the low-noise coordinate. The value of $(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}$ is half of capacity, as inspection of (2.15) and (2.19) show, when one of the E_i's is non-zero and $E_i/\sigma_i^2 \ll 1$. Since the Gaussian density function maximizes capacity, we expect that the water-filling signal set will be asymptotically optimum for small signal-to-noise ratios.

We now apply the water-filling results found in Chapter II, given by (2.17) and (2.18), to some specific examples.

E. Examples

Consider again $q = 3$, and $N = 2$. The signal vectors are shown in figure 5 and constraint (4.3) holds. Then (2.17) together with (4.3) gives the solution

$$a^2 = .75 + \frac{3(\gamma - 1)}{4\,SNR_1}\ , \quad b^2 = .25 - \frac{\gamma - 1}{4\,SNR_1} \tag{4.6}$$

as the optimum parameters for maximizing $(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}$ by the water-filling method.

Using (2.18) with (4.3) gives

$$a^2 = .75 + \frac{3(\gamma - 1)}{8\,SNR_1}\ , \quad b^2 = .25 - \frac{\gamma - 1}{8\,SNR_1} \tag{4.7}$$

as the optimum parameters for maximizing capacity by the water-filling method. We can use either of the sets of parameters

(4.6) or (4.7) in (2.13), which for this example gives

$$(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}} = 1/2\ \log_2(1 + 2\,a^2\,SNR_1) + 1/2\ \log_2(1 + 6\,b^2\,SNR_1/\gamma) \quad . \tag{4.8}$$

As a second example, let $q = 4$, and $N = 2$. The signal vector constellation of figure 6 is assumed. The energy constraint requires also that

$$4\,a^2 + 4\,b^2 = 4 \ . \tag{4.9}$$

Again the noise is assumed to be statistically independent and Gaussian with variances σ_1^2 and σ_2^2. Then (2.17) together with (4.9) gives the solution

$$a^2 = .5 + \frac{\gamma - 1}{8\,SNR_1} \quad , \quad b^2 = .5 - \frac{\gamma - 1}{8\,SNR_1} \tag{4.10}$$

as the water-filling parameters for maximizing $(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}$.

Using (2.18) with (4.9) gives

$$a^2 = .5 + \frac{\gamma - 1}{16\,SNR_1} \quad , \quad b^2 = .5 - \frac{\gamma - 1}{16\,SNR_1} \tag{4.11}$$

as the water-filling parameters for maximizing capacity for this example. Then (2.13) reduces to

$$(R_C)_{\substack{q,q'=\infty \\ \text{gaussian}}} = 1/2\ \log_2(1 + 4\,a^2\,SNR_1) + 1/2\ \log_2(1 + 4\,b^2\,SNR_1/\gamma) \quad . \tag{4.12}$$

In the next chapter, we compute and tabulate R_0 for the signal sets found in this chapter.

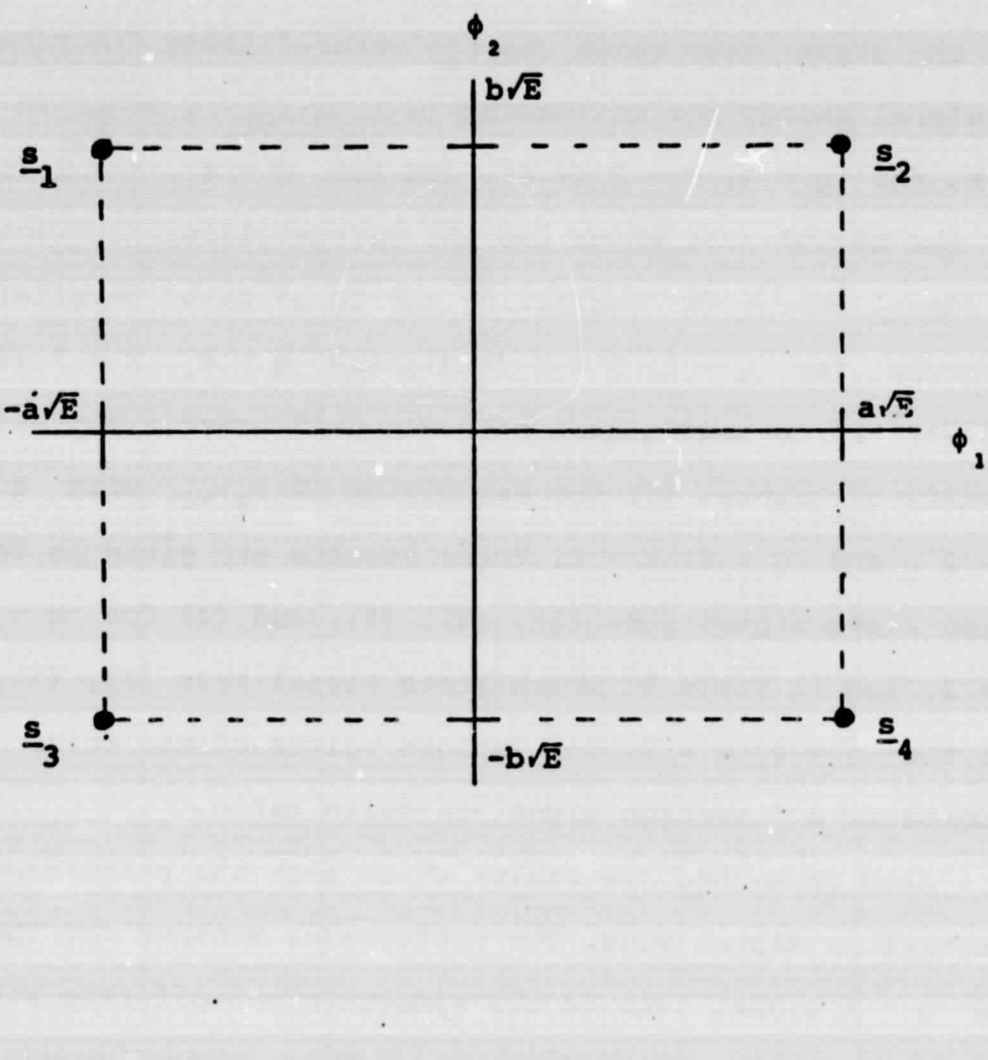

Figure 6. Signal constellation assumed for the $q = 4$, $N = 2$ signal sets considered for the example.

CHAPTER V

RESULTS AND CONCLUSIONS

In this chapter, the following signal sets are tabulated: (1) the signal set which maximizes the Taylor series approximation of $(\tilde{R}_0)_{q'=\infty}$ found in section IV C, equation (4.5); (2) the signal sets which use the water-filling distribution of signal energy for maximizing $(R_0)_{q,q'=\infty \atop \text{gaussian}}$, found in section IV E, for $q = 3$, $N = 2$ (4.6), and for $q = 4$, $N = 2$ (4.10); (3) the signal sets which use the water-filling distribution of signal energy for maximizing the capacity of parallel channels, for $q = 3$, $N = 2$ (4.7), and $q = 4$, $N = 2$ (4.11); and (4) the optimum signal sets computed numerically using the results of Chapter III for maximizing $(\tilde{R}_0)_{q'=\infty}$ with $q = 3$, $N = 2$, and $q = 4$, $N = 2$. These results are given in Table 1, which lists signal sets (1), (2), (3), and (4) for $q = 3$, $N = 2$, and in Table 2, which lists signal sets (2), (3), and (4) for $q = 4$, $N = 2$, for various values of the noise asymmetry ratio and average signal-to-noise ratio.

Also presented are tables which show the performance of the various signal sets. The performance measure for the $q = 3$ and $q = 4$ signal sets is the symmetric cut-off rate with unquantized demodulation, $(\tilde{R}_0)_{q'=\infty}$. Thus, the value of $(\tilde{R}_0)_{q'=\infty}$ was computed using (3.1) with the appropriate values for the signal vectors in the signal sets, and the results are tabulated in Tables 3 and 4 for the signal sets in Tables 1 and

		Taylor series optimal		$(R_0)_{q,q'=\infty}$ gaussian water-filling		Capacity water-filling		$(\tilde{R}_0)_{q'=\infty}$ optimal	
γ	SNR_1	a^2	b^2	a^2	b^2	a^2	b^2	a^2	b^2
1.0	.5	.75	.25	.75	.25	.75	.25	.75	.25
	1.0	.75	.25	.75	.25	.75	.25	.75	.25
	1.5	.75	.25	.75	.25	.75	.25	.75	.25
	2.0	.75	.25	.75	.25	.75	.25	.75	.25
1.1	.5	.7178	.2607	.8000	.2333	.7750	.2417	.9166	.1945
	1.0	.6806	.2731	.7750	.2417	.7625	.2458	.8173	.2275
	1.5	.6682	.2773	.7667	.2444	.7583	.2472	.7831	.2390
	2.0	.6620	.2793	.7625	.2458	.7563	.2479	.7651	.2450
1.5	.5	.6667	.2778	1.0000	.1667	.8750	.2083	1.5000	0
	1.0	.5000	.3333	.8750	.2083	.8125	.2292	1.5000	0
	1.5	.4444	.3519	.8333	.2222	.7917	.2361	.9705	.1764
	2.0	.4167	.3611	.8125	.2292	.7813	.2396	.8769	.2077
2.0	.5	.6818	.2727	1.2500	.0833	1.0000	.1667	1.5000	0
	1.0	.4091	.3636	1.0000	.1667	.8750	.2083	1.5000	0
	1.5	.3182	.3939	.9167	.1944	.8333	.2222	1.3233	.0589
	2.0	.2727	.4091	.8750	.2083	.8125	.2292	1.1168	.1277
3.0	.5	.7500	.2500	1.5000	0	1.2500	.0833	1.5000	0
	1.0	.3750	.3750	1.2500	.0833	1.0000	.1667	1.5000	0
	1.5	.2500	.4167	1.0833	.1389	.9167	.1944	1.5000	0
	2.0	.1875	.4375	1.0000	.1667	.8750	.2083	1.5000	0

Table 1. Approximately optimal and optimal signal sets for $q = 3$ and $N = 2$. The values for the parameters a and b refer to the signal constellation of figure 4.

		$(R_0)_{q,q'=\infty}$ gaussian water-filling		Capacity water-filling		$(\tilde{R}_0)_{q'=\infty}$ optimal	
γ	SNR_1	a^2	b^2	a^2	b^2	a^2	b^2
1.0	1.0	.5	.5	.5	.5	.5	.5
1.1	1.0	.5125	.4875	.5063	.4937	.5571	.4429
1.5	1.0	.5625	.4375	.5313	.4687	.7967	.2033
2.0	1.0	.6250	.3750	.5625	.4375	1.0000	0

Table 2. Approximately optimum and optimum signal sets for $q = 4$ and $N = 2$. The values for the parameters a and b refer to the signal constellation of figure 5.

		Taylor series optimal	$(R_0)_{q,q'=\infty}$ gaussian water-filling	Capacity water-filling	q = 3 optimal	q = ∞ upper bound
γ	SNR_1	$(\tilde{R}_0)_{q'=\infty}$	$(\tilde{R}_0)_{q'=\infty}$	$(\tilde{R}_0)_{q'=\infty}$	$(\tilde{R}_0)_{q'=\infty}$	$(R_0)_{q,q'=\infty}$ gaussian
1.0	.5	.3373	.3373	.3373	.3373	.8074
	1.0	.6254	.6254	.6254	.6254	1.3219
	1.5	.8631	.8631	.8631	.8631	1.7004
	2.0	1.0526	1.0526	1.0526	1.0526	2.0000
1.1	.5	.3224	.3236	.3233	.3243	.7792
	1.0	.5989	.6015	.6013	.6018	1.2818
	1.5	.8279	.8335	.8334	.8335	1.6536
	2.0	1.0166	1.0211	1.0211	1.0211	1.9491
1.5	.5	.2783	.2965	.2906	.3098	.7076
	1.0	.5022	.5411	.5371	.5488	1.1669
	1.5	.6946	.7509	.7484	.7544	1.5148
	2.0	.8571	.9270	.9256	.9287	1.7950
2.0	.5	.2491	.2951	.2771	.3098	.6699
	1.0	.4199	.5104	.4972	.5387	1.0850
	1.5	.5689	.6978	.6885	.7158	1.4068
	2.0	.6976	.8585	.8524	.8680	1.6699
3.0	.5	.2268	.3098	.2850	.3098	.6610
	1.0	.3373	.5095	.4728	.5387	1.0148
	1.5	.4358	.6658	.6377	.7131	1.2950
	2.0	.5228	.8025	.7812	.8512	1.5295

Table 3. Symmetric cut-off rates for the $q = 3$, $N = 2$ signal sets of Table 1, and the infinite-q, Gaussian density upper bound, using the water-filling signal energy distribution.

		$(R_0)_{q,q'=\infty}$ gaussian water-filling	Capacity water-filling	q = 4 optimal	q = ∞ upper bound
γ	SNR_1	$(\tilde{R}_0)_{q'=\infty}$	$(\tilde{R}_0)_{q'=\infty}$	$(\tilde{R}_0)_{q'=\infty}$	$(R_0)_{q,q'=\infty}$ gaussian
1.0	1.0	.6321	.6321	.6321	1.5850
1.1	1.0	.6074	.6072	.6080	1.5401
1.5	1.0	.5455	.5412	.5576	1.4080
2.0	1.0	.5104	.4986	.5481	1.3074

Table 4. Symmetric cut-off rates for the $q = 4$, $N = 2$ signal sets of Table 2, and the infinite-q, Gaussian density upper bound, using the water-filling signal energy distribution.

2 respectively. Also tabulated is the value of $(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}$, the cut-off rate with an infinite number of signals, and thus is an upper bound on the performance of the finite-q signal sets. This value was computed using (4.8) and the optimum water-filling energy distribution maximizing this cut-off rate for $N = 2$.

It would be useful at this point to recapitulate the introduction of the various cut-off rates employed in this work, and to relate these to the results of this chapter. It was shown that, for constrained average signal energy, a large signal set obtains a large value of $(R_0)_{q'=\infty}$. Thus, (2.2) gives, upon substitution of the optimum density function, the theoretically largest cut-off rate attainable for a given energy and dimensionality, which we denote $(R_0)_{\substack{q,q'=\infty \\ \text{optimal}}}$. Although this optimum density was not found, it was shown that, when a Gaussian density was used, a simple expression resulted for the cut-off rate for infinite q, denoted $(R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}}$, which is a lower bound on the optimal cut-off rate. Then, when we constrained the number of signals and assigned them equally likely probabilities, we sought to maximize the symmetric cut-off rate $(\tilde{R}_0)_{q'=\infty}$. An algorithm was presented for computing optimum signal sets and the maximum achievable symmetric cut-off rate, which we denote $(\tilde{R}_0)_{\substack{q'=\infty \\ \text{optimal}}}$. Thus, from theoretical considerations, we have

$$(R_0)_{\substack{q,q'=\infty \\ \text{optimal}}} > (R_0)_{\substack{q,q'=\infty \\ \text{gaussian}}} > (\tilde{R}_0)_{\substack{q'=\infty \\ \text{optimal}}} .$$

For selected values of q and N , the optimal signal sets were computed, along with the various sub-optimal signal sets considered in Chapter IV. In order to obtain these approximately optimum signal sets, the locus of signal vectors in signal space was constrained, as given in figures 5 and 6. Since the optimum signal vectors did have this same assumed constellation, the performance of the approximately optimum signal sets was close to the maximum for the examples investigated. It can be seen from the data in Tables 1 through 4 that, among the sub-optimal sets, the water-filling signal sets display better performance than the Taylor series approximation set, and that the signal set with signal energy distribution maximizing $(R_0)_{q,q'=\infty}$ gaussian is better than that which achieves capacity. Also evident is the fact that the sub-optimum signal sets are indeed optimum when the noise is white, which is a consequence of choosing signal vectors so that maximum R_0 is achieved for white noise, that is, so that signal energy is equally divided among the N coordinates. As the noise assymetry increases, the water-filling signal sets do not depart appreciably from the optimum $(\tilde{R}_0)_{q'=\infty}$, despite the fact that the optimum signal set distributes energy less symmetrically. This is seen in the entries where $b = 0$ for the optimum with large γ and small SNR_1, whereas the sub-optimal sets retain energy on the first coordinate as well. This did not cause a very large difference in the cut-off rates. It can be concluded that, for small values of q and N, the R_0 water-filling distribution of signal energy is a convenient and nearly optimal rule-of-thumb for modulation system design with non-white Gaussian noise channels.

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