Lockheed Translation

Oscillations and Stability of Numerical Solutions of the Heat Conduction Equation

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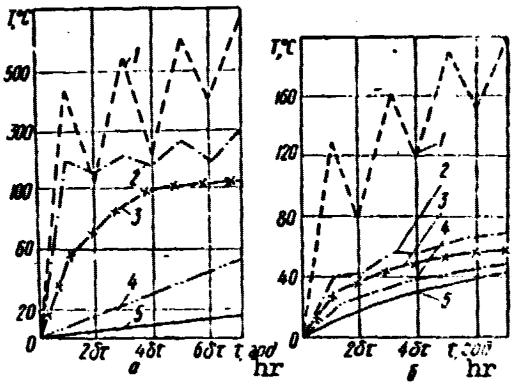
Four- and six-point explicit and implicit schemes of the mesh method /1/ are used extensively in numerical investigations of mathematical models which describe nonstationary heat conduction problems. The error of the numerical solution in the mesh method depends in all cases on the magnitude of the space (h) and time ($\delta\tau$) intervals. For explicit schemes, not only the error but also the stability of the solution depends on the relationship between the intervals h and $\delta\tau$. Numerical experiments for the solutions of linear and nonlinear nonstationary heat conduction problems by both explicit and implicit schemes have shown that oscillations in the values of the desired quantities (figure) occur for certain relationships between h and $\delta\tau$. The reason for these oscillations is a discrepancy in the heat balance. Oscillations ordinarily originate in the first steps of the solution as is noted (without explanation of the reasons for the oscillations) in /2,3/ (implicit scheme). However, these oscillations can also originate in the middle of the calculations for sharp changes in the coefficients (λ, c_v, q, q_v, α etc.) of the mathematical model /4,5/. Oscillations originate during calculations on both analog and digital computers /5,6/. It has been established that it is possible to avoid oscillations (implicit schemes), oscillations and instability (explicit schemes) by an appropriate selection of the relationship between the quantities h, $\delta\tau$ and the coefficients which enter into the mathematical model. These relationships for explicit schemes are sometimes called the stability criteria. By analogy, we call such relationships for implicit schemes oscillations criteria. Conserving the conditions that are imposed on the stability criteria permits avoidance of oscillations and instability and conserving the conditions imposed on the oscillations damps out the oscillations. Stability and oscillations criteria for explicit and implicit four- and six-point finite-difference schemes



Change in surface temperature with time in the first steps of the solution of the Fourier equation with different boundary conditions

a) 1 — Fo*(1+2Ki*)=2282, 2 — 352, 3 — 39, 4 — 4, 5 — 1; б) 1 — Fo*(1+2Bi*)=70, 2 — 3, 3 — 2, 4 — 1, 5 — 0б.

TRANSLATED BY MORRIS D. FRIEDMAN

STABILITY FOR EXPLICIT AND OSCILLATIONS CRITERIA FOR IMPLICIT SCHEMES

Math. Математична модель model		Explicit Явна схема Scheme	Implicit Scheme Неявна схема	
Fundamental Основне рівняння Equations	Kind of Рід граничних умов Boundary Conditions	чотириточкова Four-point	чотириточкова Four-point	шеститочкова Six-point
Фур'є Fourier	I	$Fo^* \leq 0,5$	$Fo^* \leq 0,5$	$Fo^* \leq 1$
	II	$Fo^*(1+Ki^*) \leq 0,5$	$Fo^*(1+Ki^*) \leq 0,5$	$Fo^*(1+2Ki^*) \leq 1$
	III	$Fo^*(1+Bi^*) \leq 0,5$	$Fo^*(1+Bi^*) \leq 0,5$	$Fo^*(1+2Bi^*) \leq 1$
Фур'є з нерухомими джерелами Fourier with fixed sources	I	$Fo^*(1 \pm Po^*) \leq 0,05$	$Fo^*(1 \pm Po^*) \leq 0,5$	$Fo^*(1 \pm 2Po^*) \leq 1$
	II	$Fo^*(1 \pm 0,5Po^* + Ki^*) \leq 0,5$	$Fo^*(1 \pm 0,5Po^* + Ki^*) \leq 0,5$	$Fo^*(1 \pm Po^* + 2Ki^*) \leq 1$
	III	$Fo^*(1 \pm 0,5Po^* + Bi^*) \leq 0,5$	$Fo^*(1 \pm 0,5Po^* + Bi^*) \leq 0,5$	$Fo^*(1 \pm Po^* + 2Bi^*) \leq 1$
Фур'є з нерухомими і рухомими джерелами Fourier with fixed and moving sources	I	$Fo^*(1 \pm Po^* \pm Pe^*) \leq 0,5$	$Fo^*(1 \pm Po^* \pm Pe^*) \leq 0,5$	$Fo^*(1 \pm 2Po^* \pm 2Pe^*) \leq 1$
	II	$Fo^*(1 \pm 0,5Po^* \pm 0,5Pe^* + Ki^*) \leq 0,5$	$Fo^*(1 \pm 0,5Po^* \pm 0,5Pe^* + Ki^*) \leq 0,5$	$Fo^*(1 \pm Po^* \pm Pe^* + 2Ki^*) \leq 1$
	III	$Fo^*(1 \pm 0,5Po^* \pm 0,5Pe^* + Bi^*) \leq 0,5$	$Fo^*(1 \pm 0,5Po^* \pm 0,5Pe^* + Bi^*) \leq 0,5$	$Fo^*(1 \pm Po^* \pm Pe^* + 2Bi^*) \leq 1$

REMARK: By analogy with the known numbers Bi, Fo, Po, Ki, Pe, we introduce the following abbreviated notation in the expressions for the stability and oscillations criteria: $Fo^* = \frac{a\delta\tau}{h^2}$; $Bi^* = \frac{\alpha h}{\lambda}$; $Po^* = \frac{q_v h^2}{\lambda|\Delta T|}$; $Ki^* = \frac{q_n h}{\lambda|\Delta T|}$; $Pe^* = \frac{uh}{a}$. The initial temperature distribution is considered uniform, $|\Delta T| = 1$.

are presented in the table. The stability criteria of explicit four-point schemes are known for the Fourier equation with the boundary conditions of the first and third kinds. The other criteria are introduced for the first time (table).

The mathematical model and results of numerical solutions are given for the one-dimensional problem when the linear equations are written in a rectangular coordinate system. However, all the computations are easily realizable for two- and three-dimensional problems when the equations are written any coordinate system. Since à linear system of algebraic equations is considered in the numerical solutions utilizing iteration-free or iteration schemes to calculate the nonlinearities in each time step or in each approximation, then all the conclusions also apply for the analysis of the non-linear problems.

The equation of the main process has the form:

(1) $$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - c_V \frac{\partial T}{\partial \tau} \pm q_v \pm wc_V \frac{\partial T}{\partial x} = 0.$$

The boundary conditions (boundaries of the first, second, and third kinds) and the initial conditions are:

(2) $$T_n = T_1(\tau),$$

(3) $$q = -\lambda \frac{\partial T}{\partial x},$$

(4) $$\alpha(T_c - T_n) = -\lambda \frac{\partial T}{\partial x},$$

(5) $$T_0 = T(x, 0).$$

Let us consider a plate of thickness l with symmetric or nonsymmetric boundary conditions (2)-(4) of the I - III kind. Equation (1) is the equation of nonstationary heat conduction with internal q_v and moving (at the velocity w) heat sources.

Examples of numerical solutions performed for explicit schemes when the values of Fo^* are greater than, equal to, and less than 0.5 are presented in /9/. An analysis of the stability zone for explicit schemes of solving the Fourier equation with boundary conditions of the third kind is given in /8/. Let us note that the mode of giving the boundary conditions alters the writing of the expression for the criteria. The stability zones for various modes of giving the boundary conditions of the third kind are analyzed in /8/. Values of the stability and oscillations criteria are given in the table for some of the modes of giving the boundary conditions of the second and third kinds. Shown in the figure are curves of the change in plate surface temperature obtained for numerical solutions by a six-point implicit scheme with different values of the oscillations criteria. The values of the criterion (Fig. a) changed for different values of $\delta\tau$. The values of $\delta\tau$ along the horizontal axis for each curve number and accuracy of the solution emerge from the theory of difference schemes $0(h^2+\delta\tau)$ /1/. It is seen from the figure that the amplitudes of the surface temperature oscillations diminish with the decrease in the values of the oscillation criteria. And finally, let us note yet another characteristic fact, that confirms the expediency of the balance approach to estimate the reasons for the instability and oscillations. It has been determined comparatively long ago (for instance, /10/) that although $Fo^* > 0.5$ there can be no instability in explicit schemes. This means that $\delta\tau$ could be increased; the time and cost of the solution on an analog or digital computer will hence diminish. We also observed an analogous phenomenon. The oscillations criteria for curves 3 and 4 are greater than 1, and there are no oscillations or they are almost imperceptible. A computation of the heat balance shows that the balance is not disturbed although the criterion is greater than 1. It is required to select the maximum values of h and $\delta\tau$ for which there will be no instability nor oscillation by starting from the balance relationships. Selecting the quantities h and $\delta\tau$, by starting from the conditions presented in the table, permits obtaining stable numerical solutions without oscillations. Examination of the heat balance at each step permits increasing h and (or) $\delta\tau$, i.e., disturbing the conditions of the table, but permits diminishing the time and cost of the solution. We obtain the stability and oscillation criteria from the balance relationships under any assumptions (that diminish h and $\delta\tau$) about the temperature drops in space and time at each step of the solution.

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