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GEOMETRIZATION OF THE DIRAC THEORY OF THE ELECTRON

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16. Abstract Using the concept of parallel displacement of a half vector, the Dirac equations are generally written in invariant form. The energy tensor is formed and both the macroscopic and quantum mechanic equations of motion are set up. The former have the usual form: divergence of the energy tensor equals the Lorentz force and the latter are essentially identical with those of the geodesic line. The occurrence of of the four-potential ϕ_1 together with the Ricci coefficient γ_{ikl} in the formula for parallel displacement on the one hand gives a simple geometrical reason for the occurrence of the expression $p_1 - \frac{e}{c}\phi_1$ in the wave equation, and on the other hand it shows that the ϕ_1 potentials, differing from Einstein's idea, have an independent place in the geometric world picture and must not, for example, be functions of γ_{ikl}.

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[Summary]

Using the concept of parallel displacement of a half vector, /261*
the Dirac equations are generally written in invariant form. The
energy tensor is formed and both the macroscopic and quantum
mechanic equations of motion are set up. The former have the
usual form: divergence of the energy tensor equals the Lorentz
force and the latter are essentially identical with those of
the geodesic line. The occurrence of the four-potential ϕ_1
together with the Ricci coefficient γ_{ikl} in the formula for parallel
displacement on the one hand gives a simple geometrical reason
for the occurrence of the expression $p_1 - \frac{e}{c}\phi_1$ in the wave equation,
and on the other hand it shows that the ϕ_1 potentials, differing
from Einstein's idea, have an independent place in the geometric
world picture and must not, for example, be functions of γ_{ikl}.

In a work by D. Iwaneko and the author [7] it was suggested
that the Dirac matrices have a purely geometric significance.
In another work by these authors [6] the concept of parallel
shift of a half vector, i.e. of a quadruple of quantities which
are transformed like Dirac ψ functions, was proposed.

In another note [3] the author used this concept to set up
the general relativity wave equation of the electron and derived
the macroscopic equation of motion in the Einsteinian form.

The present work is a complete presentation of the observations
made in the note cited above.

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* Numbers in the margin indicate pagination in the foreign text.

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1.   The transformation properties of the Dirac ψ functions have been studied in detail by F. Möglich [9] and J. von Neumann [10]. The transformation equation has an especially simple form if for the first three Dirac α matrices one chooses the expression

$$\alpha_1 = \sigma_1, \qquad \alpha_2 = \rho_3\sigma_2, \qquad \alpha_3 = \sigma_3 \qquad (1)$$

and for the fourth matrix one chooses

$$\alpha_4 = \rho_2\sigma_2, \qquad \alpha_5 = \rho_1\sigma_2, \qquad (1^*)$$

where $\rho_1, \rho_2, \rho_3; \sigma_1, \sigma_2, \sigma_3$ are the four-rowed matrices introduced by Dirac [2].

The the following transformation of the ψ functions corresponds to a general Lorentz transformation:

$$\left.\begin{array}{ll} \psi_1' = \alpha\,\psi_1 + \beta\,\psi_2; & \psi_3' = \bar{\alpha}\,\psi_3 + \bar{\beta}\,\psi_4; \\ \psi_2' = \gamma\,\psi_1 + \delta\,\psi_2; & \psi_4' = \bar{\gamma}\,\psi_3 + \bar{\delta}\,\psi_4. \end{array}\right\} \qquad (2)$$

The complex quantities α, β, γ, δ satisfy the condition

$$\alpha\delta - \beta\gamma = 1 \qquad (3)$$

and in the case of a purely spatial rotation, are transformed into the usual parameters of Cayley and Klein.

If α_0 is used to designate the unit matrix, then the quantities

$$A_i = \bar{\psi}\alpha_i\psi \quad (i = 0, 1, 2, 3) \qquad (4)$$

form the components of a four-vector, and the quantities

$$A_4 = \bar{\psi}\alpha_4\psi, \quad A_5 = \bar{\psi}\alpha_5\psi \qquad (4^*)$$

are in variance.   This fact is expressed in equations as follows. If S is used to designate Transformations (2):

$$\psi' = S\psi, \quad \bar{\psi}' = \bar{\psi}S^+, \tag{5}$$

where their S^+ stands for the adjoint (transposed-conjugated) matrix to S, then the following equations hold:

$$S^+\alpha_i S = \sum_{k=0}^{3} a_{ik}\alpha_k; \quad S^+\alpha_4 S = \alpha_4; \quad S^+\alpha_5 S = \alpha_5, \tag{6}$$

where a_{ik} are the coefficients of a general Lorentz transformation. Because

$$\bar{\psi}'\alpha_i\psi' = \bar{\psi}S^+\alpha_i S\psi$$

the quantities (4) and (4*) are therefore transformed according to the equations

$$A'_i = \sum_{k=0}^{3} a_{ik}A_k; \quad A'_4 = A_4; \quad A'_5 = A_5, \tag{7}$$

i.e. like a four-vector of like invariance. Since the A_i (i = 0, 1, 2, 3) in the ψ quadratic quantities form a four-vector, we want to designate the ψ quantities with transformation properties (2) as "half vectors."[1]

The explicit expressions for the quantities A_i (i = 0, 1, 2, 3, 4, 5) are as follows:

$$
\begin{aligned}
A_0 &= \quad \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 + \bar{\psi}_3\psi_3 + \bar{\psi}_4\psi_4, \\
A_1 &= \quad \bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1 + \bar{\psi}_3\psi_4 + \bar{\psi}_4\psi_3, \\
A_2 &= -i\bar{\psi}_1\psi_2 + i\bar{\psi}_2\psi_1 + i\bar{\psi}_3\psi_4 - i\bar{\psi}_4\psi_3, \\
A_3 &= \quad \bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 + \bar{\psi}_3\psi_3 - \bar{\psi}_4\psi_4, \\
A_4 &= -\quad \bar{\psi}_1\psi_4 + \bar{\psi}_2\psi_3 + \bar{\psi}_3\psi_2 - \bar{\psi}_4\psi_1, \\
A_5 &= -i\bar{\psi}_1\psi_4 + i\bar{\psi}_2\psi_3 - i\bar{\psi}_3\psi_2 + i\bar{\psi}_4\psi_1.
\end{aligned}
$$

Using these expressions, we confirm the following identical

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[1] This term was introduced by L. Landau.

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relationship between the quantities A_i:

$$A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 = A_6^2.$$

<div align="right">(8)</div>

2.   We have considered the transformation properties of ψ functions for a Lorentz transformation in the context of the special relativity theory. If we now assume the standpoint of the general relativity theory, then in order to be able to introduce the half vector concept, we must have an orthogonal (more precisely pseudo-orthogonal) reference system in each space-time point. For this purpose we introduce a network of four orthogonal curve congruences and, after Einstein, designate the directions of these congruences as "legs." The observations made in the above paragraphs then also remain valid for the general relativity theory case if by A_i we mean the components of a vector along the legs.

We number the legs with Roman indices and the coordinates with Greek indices which everywhere run through the values 0, 1, 2, 3. In the summation according to the Roman indices, the sum symbol is given explicitly, whereas in the summation using the Greek indices, it is suppressed. We designate the parameters of the curve congruences with small h_k^α and the moments with $h_{k,\alpha}$. Since we are dealing here with an indefinite metric, we introduce along with Eisenhart[1] the quantities $e_1 = e_2 = e_3 = -1$; $e_0 = +1$. The components of a vector along the coordinate directions (A_σ) and along the legs (A'_k)[2] are then expressed in terms of each

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[1] See [1] and also the excellent collection of the most important equations and facts in the study of T. Levi-Civita [8], p. 3.

[2] In what follows, tne leg and coordinate components are more frequently designated with one and the same letter; in order to avoid confusions, the leg components are provided witn a prime accent.

other as follows:

$$A'_k = A_\sigma h^\sigma_k; \quad A_\sigma = \sum_k e_k A'_k h_{k,\sigma} \tag{9}$$

If we designate the leg components of an infinitesimal   shift with ds_k, then for the change of the components of a vector in a parallel shift, from the following equation

$$\delta A_\alpha = \Gamma^\beta_{\alpha\sigma} A_\beta dx^\sigma; \quad \Gamma^\beta_{\alpha\sigma} = \left\{ {}^{\alpha\,\sigma}_{\beta} \right\} \tag{10}$$

results the following expression for the change of its leg components:

$$\delta A'_i = \sum_{kl} e_k e_l \gamma_{ikl} A'_k ds_l \tag{11}$$

where γ_{ikl} are the rotation coefficients introduced by Ricci:

$$\gamma_{ikl} = (\nabla_\sigma h^\beta_i) h_{k,\beta} h^\sigma_l = (\nabla_\sigma h_{i,\beta}) h^\beta_k h^\sigma_l. \tag{12}$$

Here ∇_σ stands for the covariant derivation with respect to x^σ.

3. We now want to consider the change of the components of a half vector ψ for an infinitesimal parallel shift. For this change, we set up the equation

$$\delta\psi = \sum_l e_l C_l ds_l \psi. \tag{13}$$

The C_l's are matrices with the elements $(C_l)_{mn}$, and by $C_l\psi$ we mean four functions whose mth degree is given by the equation

$$(C_l\psi)_m = \sum_{n=1}^4 (C_l)_{mn} \psi_n$$

The conjugate complex equation for (13) is as follows:

$$\delta \bar{\psi} = \bar{\psi} \sum_l e_l C_l^+ \, ds_b \tag{13*}$$

where C_1^+ stands for the adjoint matrix. Now by means of Eq. (13) for the parallel shift of a half vector, the equation of a vector is already determined; specifically we must have:

$$\begin{aligned}
\delta A_i' = \delta (\bar{\psi} \alpha_i \psi) &= \delta \bar{\psi} \alpha_i \psi + \bar{\psi} \alpha_i \delta \psi \\
&= \bar{\psi} \sum_l e_l (C_l^+ \alpha_i + \alpha_i C_l) \, ds_l \psi.
\end{aligned} \tag{14}$$

Should this change agree with that given by (11), then the C_1's must satisfy the conditions

$$C_l^+ \alpha_i + \alpha_i C_l = \sum_k e_k \alpha_k \gamma_{ikl} \tag{15}$$

Furthermore, since $A'_4 = \bar{\psi} \alpha_4 \psi$ and $A'_5 = \bar{\psi} \alpha_5 \psi$ are invariants, then

$$\delta A_i' = \bar{\psi} \sum_l e_l (C_l^+ \alpha_4 + \alpha_4 C_l) \, ds_l \psi \tag{16}$$

and likewise $\delta A'_5$ must disappear, from which follow the further conditions

$$C_l^+ \alpha_4 + \alpha_4 C_l = 0; \quad C_l^+ \alpha_5 + \alpha_5 C_l = 0 \tag{17}$$

to Eqs. (15) and (17) is given by the equation

$$C_l = \tfrac{1}{4} \sum_{mk} \alpha_m \alpha_k e_k \gamma_{mkl} + i \Phi_l' \tag{18}$$

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in which Φ'_1 's are Hermitian matrices, which must be inter-
changeable with all α_1's as well as with α_4 and α_5.   If we
remain in the region of the four-row matrices, then proportion-
ality with the unit matrix follows from the interchangeability
with all α matrices. By contrast, if we consider matrices with
more than four rows,[1] then the case is not excluded that the
Φ'_1 are not proportional to the unit matrix. We want to stay
with the four-row matrices and consider the Φ'_1 's as real
numbers.

It must be borne in mind that the C_1's do not contain the
α_4 and α_5 matrices so that the first two ψ functions are trans-
formed between themselves and the last two between themselves.
Because of Eq. (2) this was also to be expected a priori.

4.   After we have proposed the concept of the parallel shift
of a half vector, we can define the covariant derivation $D'_1\psi$
of a half vector ψ along leg direction 1 by the following
equation

$$D'_l\psi = \frac{\partial\psi}{\partial s_l} - C_l\psi$$

(19)

where $\frac{\partial\psi}{\partial s_l} = h_1^\sigma \cdot \frac{\partial\psi}{\partial x^\sigma}$ stands for the derivation in the direction

of the 1th leg. We designate the covariant derivation of a half
vector along coordinate x^σ as follows

$$D_\sigma\psi = \frac{\partial\psi}{\partial x^\sigma} - \Gamma_\sigma\psi.$$

(19*)

where for purposes of abbreviation

$$\Gamma_\sigma = \sum_k e_k h_{k,\sigma} C_k$$

(20)

---

[1] Such matrices could perhaps occur in certain generalizations
of the Dirac equation, for example as applied to the two body
problem.

If for a moment we consider the space as pseudo-euclidian and set γ_{ikl} equal to zero, then Eq. (20) for D'_i equals

$$D'_i \psi = \frac{\partial \psi}{\partial s_i} - i \Phi'_i \psi.$$

But this is the same expression which appears in the Dirac equation if Φ'_i is taken to mean the quantity

$$\Phi'_i = \frac{2\pi e}{hc} \varphi'_i \qquad (21)$$

where φ'_i stands for the leg components of the vector potential. In what follows we intend to adhere to this physical interpretation of the geometric quantities Φ'_i. Thus we have obtained a geometric interpretation for the occurrence of the vector potential in the Dirac equation, and indeed this interpretation is such that the potential can also be distinguished from zero when the gravitation terms containing the quantities γ_{ikl} disappear.

If we now turn to Eq. (13) for $\delta\psi$, we see that precisely here the Weylian linear differential form appears:

$$\sum_i e_i \varphi'_i ds_i = \varphi_\sigma dx^\sigma \quad .$$

This is in agreement with the assumption expressed by Weyl.[1] The occurrence of the Weylian differential form in the equation for the parallel shift of a half vector is closely related to the fact noted by the author [4] and also by Weyl (loc. cit.) that the addition of a gradient to the four-potential corresponds to the multiplication of the ψ-function by a factor with an absolute value of 1. This fact was designated by Weyl as the "principle of gauge invariance."

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[1] [13] Section 19, page 88.

8

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[1] [13] Section 19, page 88.

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5.   Using the concept of covariant derivation of a half vector,
it is possible to set up the Dirac wave equation for the electron
in the general relativity theory. For this purpose we consider
the operator

$$F\psi = \frac{h}{2\pi i} \sum_k e_k \alpha_k \left(\frac{\partial \psi}{\partial \dot{a}_k} - C_k \psi\right) - mc\alpha_4 \psi.$$

(22)

We want to show that it is a self-adjoint operator.[1] In order
to see this, we pass from the legs to the coordinates and intro-
duce the matrices

$$\gamma^\sigma = \sum_k e_k \alpha_k h_k^\sigma$$

(23)

and the Γ_σ matrices defined by (20). Analogous relationships
follow from Eqs. (15) for the just-introduced matrices

$$\Gamma_\sigma^* \gamma^\sigma + \gamma^\sigma \Gamma_\sigma = -\nabla_\sigma \gamma^\sigma.$$

(24)

This equation can easily be proved by going back to Definition
(12) of γ_{1kl}.

Expressed in terms of the coordinates, the operator F
appears as follows:

$$F\psi = \frac{h}{2\pi i} \gamma^\sigma \left(\frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi\right) - mc\alpha_4 \psi.$$

(25)

Taking into consideration (24) one can easily prove the identity

$$\overline{\psi} F\psi - \overline{(F\psi)} \psi = \frac{h}{2\pi i} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} (\overline{\psi} \sqrt{g} \gamma^\sigma \psi),$$

(26)

---

[1] The word "self-adjoint" is meant here in a somewhat extended
sense. Specifically, we mean that the expression $\overline{\psi} F\psi - \overline{(F\psi)}\psi$
can be written in the form of a "in general four-dimensional"
divergence.

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where g stands for the sbsolute value of the determinant $\|g_{\rho\sigma}\|$.
This identity expresses the fact that the operator F is a self-adjoint operator. This fact permits us to set up the equation

$$F\psi = 0 \qquad (27)$$

for the Dirac equation in the general relativity theory. If ψ
satisfies this equation, then it follows from identity (26) that
the divergence of the current vector

$$S^\nu = \tilde\psi\gamma^\nu\psi, \qquad (28)$$

which is obviously real because of the Hermitian character of
the γ^ρ matrices, disappears:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}(\sqrt{g}\,S^\nu) = 0. \qquad (29)$$

It is easy to prove[1] that Eqs. (25) and (27) are invariant
(more precisely, covariant) for the Dirac equation not only with
respect to the choice of coordinates but also with respect to
the choice of orthogonal curve congruences.

By way of proof, let us first observe that the values of
Γ_σ can be uniquely defined in agreement with the above Definitions
(18), (20), and (21) by the following equations:

$$\left.\begin{array}{c}\Gamma_\sigma^*\gamma^\nu + \gamma^\nu\Gamma_\sigma = -\nabla_\sigma\gamma^\nu\\[4pt]\frac{1}{4}\mathrm{trace}\,\Gamma_\sigma = \frac{2\pi i e}{hc}\varphi_\sigma\end{array}\right\} \qquad (30)$$

If we now introduce any new network whatsoever of curve congruences

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[1] This section (to the end of section 5.) was added during the
corrections.

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and designate the quantities on this network with a star, then the new Γ_σ^* solutions to the analogous equations are as follows:

$$\left.\begin{array}{l} \Gamma_a^{*+}\gamma^\sigma + \gamma^{*\sigma}\Gamma_a^* = -\nabla_a\gamma^{*\sigma} \\[2mm] \frac{1}{4}\text{trace}\Gamma_\sigma^* = \frac{2\pi i e}{hc}\varphi_\sigma. \end{array}\right\} \tag{30*}$$

However the transition to the new leg directions in each space-time point looks like a local Lorentz transformation. Therefore the new components of the half vector ψ^* and the new matrices $\gamma^{*\sigma}$ are related to the old ψ and γ^σ by equations of the form

$$\psi^* = S\psi; \qquad \gamma^\sigma = S^+\gamma^{*\sigma}S \tag{31}$$

[cf. Eqs. (5) and (6)], where S stands for a matrix of the form

$$S = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \bar{\alpha} & \bar{\beta} \\ 0 & 0 & \bar{\gamma} & \bar{\delta} \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1$$

with variable elements.

But the transformation equation for the Γ_σ coefficients of the parallel displacement is as follows:

$$\Gamma_\sigma^* = S\Gamma_\sigma S^{-1} + \frac{\partial S}{\partial x^\sigma}S^{-1}, \tag{32}$$

for this expression is the unique solution of (30^*)[1].

---

[1] It holds that: trace $\frac{\partial S}{\partial x\sigma} S^{-1} = 0$.

Furthermore it holds that

$$\frac{\partial \psi^*}{\partial x^\sigma} - \Gamma_\sigma \psi^* = S\left(\frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi\right).$$

(33)

If we let F*ψ* stand for the analogous terms in (25) which we get if in that equation a star is assigned to γ^σ, Γ_σ and ψ, then from (31) and (33) it follows that

$$F\psi = S^+ F^* \psi^*.$$

(34)

The equation Fψ = 0 is thus equivalent to F*ψ* = 0, which was to be proved.

6. In this section we want to present the operator F in another form in which we calculate the sum $\sum_k e_k \alpha_k C_k$ in Eq. (22).

In order to present the result in a clear form, we proceed as follows. We introduce the quantities ε_{ijkl}, which should disappear if among the ijkl indices two appear which are identical, and in case different indices are equal to +1 or -1, according as the ijkl sequence arises from 0 1 2 3 by means of an even or odd permutation. With the help of these quantities we form the "leg vector"

$$f_i = \frac{1}{2} \sum_{jkl} e_j e_k e_l \varepsilon_{ijkl} \gamma_{jkl}$$

(35)

with the components

$$\left.\begin{array}{l} f_0 = -e_0(\gamma_{123} + \gamma_{231} + \gamma_{312}), \\ f_1 = -e_1(\gamma_{302} + \gamma_{023} + \gamma_{230}), \\ f_2 = -e_2(\gamma_{301} + \gamma_{013} + \gamma_{130}), \\ f_3 = -e_3(\gamma_{102} + \gamma_{021} + \gamma_{210}). \end{array}\right\}$$

(35*)

If we consider the identities

$$
\left.\begin{array}{c}
\alpha_1\alpha_2\alpha_3 = i\varrho_3\alpha_4, \\
\alpha_2\alpha_3 = i\varrho_3\alpha_1, \\
\alpha_3\alpha_1 = i\varrho_3\alpha_2, \\
\alpha_1\alpha_2 = i\varrho_3\alpha_3,
\end{array}\right\} \tag{1**}
$$

which follow from Definition (1) of the α_i matrices, then we can write the sum $\sum_k e_k \alpha_k C_k$ in the form

$$
\sum_i e_i \alpha_i C_i = \sum_i e_i \alpha_i \left(i\Phi_i' - \frac{1}{2}\sum_j e_j \gamma_{jij} - \frac{i}{2}\varrho_3 f_i \right) \tag{*}
$$

We let

$$
k_i = -\sum_j e_j \gamma_{jij} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^o}(\sqrt{g}\,h_i^o) \tag{36}
$$

and introduce the term (*) into (22). We then obtain

$$
F\psi = \sum_j e_j \alpha_j \left(\frac{h}{2\pi i}\frac{\partial\psi}{\partial s_j} - \frac{e}{c}\varphi_j'\psi + \frac{h}{4\pi i}k_j\psi \right) \\
+ \frac{h}{4\pi}\varrho_3 \sum_j e_j\alpha_j f_j\,\psi - mc\,\alpha_4\psi. \tag{22*}
$$

We note that in this equation the first and second sums are single self-adjoint operators.

In the event that all these congruences are normal congruences, then the "leg" factor f_i disappears, since each Ricci symbol γ_{ikl} disappears individually with three different indices. Furthermore, we can then choose the hypersurfaces, whose perpendiculars are given by the curve congruences, as coordinate surfaces. We then have

$$
ds^2 = \sum_j e_j H_j^2\,dx_j^2; \quad \sqrt{g} = H_0 H_1 H_2 H_3; \tag{37}
$$

$$
h_i^i = \frac{1}{H_i}; \quad h_{i,i} = e_i H_i; \quad f_i = 0, \tag{37*}
$$

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while all $h_{1,\sigma}$ and h_1^σ parameters disappear with different indices. The equation for the operator F is then as follows

$$F\psi = \sum_j e_j\alpha_j \frac{1}{H_j}\left(\frac{h}{2\pi i}\frac{\partial\psi}{\partial x_j} - \frac{e}{c}\varphi_j\psi + \frac{h}{4\pi i}\frac{\partial}{\partial x_j}\left(\lg\frac{\sqrt{g}}{H_j}\right)\psi\right) - mc\alpha_4\psi.$$

(38)

This equation permits us to immediately write the Dirac equation in any curvilinear orthogonal coordinates. In so doing, the following must be borne in mind. If for example in the case of an ordinary euclidian space, we write Eq. (38) first in cartesian coordinates and then in curvilinear coordinates, then the ψ functions occurring in both cases in (38) are not identical, but they are related to one another by a transformation of the form (2) with variable coefficients α, β, γ, δ. This fact must be kept in mind in setting up the uniqueness postulates for the ψ functions.

In concluding this section it should be noted that as everyone knows, it is not always possible in a general Riemann space to select all the curve congruences as normal congruences. This is possible, however, in the important special cases of a static gravitation field with central and axial symmetry as has been shown by the solutions to the Einsteinian equations found by Schwarzschild and Levi-Civita.

7. We now want to try to find the energy tensor. To do this we consider the tensor

$$A_{.\alpha}^\sigma = \bar\psi\gamma^\sigma\left(\frac{\partial\psi}{\partial x^\alpha} - \Gamma_\alpha\psi\right) = \bar\psi\gamma^\sigma D_\alpha\psi$$

(39)

and calculate its divergence.[1]

---

[1] The results of this section could also be derived in a more elegant form by considering an infinitesimal transformation (cf. [11]). However, we prefer to proceed in a more elementary manner.

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We write the Dirac equation with its conjugate complexes
in the following form:

$$\gamma^\sigma\left(\frac{\partial\psi}{\partial x^\sigma}-\Gamma_\sigma\psi\right)-\frac{2\pi i}{h}mc\alpha_4\psi=0,\tag{40}$$

$$\left(\frac{\partial\bar\psi}{\partial x^\sigma}-\bar\psi\Gamma_\sigma^+\right)\gamma^\sigma+\frac{2\pi i}{h}mc\bar\psi\alpha_4=0.\tag{40*}$$

We differentiate (40) with respect to x^α and multiply on
the left by $\bar\psi$. We multiply Eq. (40*) on the right with $\dfrac{\partial\psi}{\partial x^\alpha}$
and add the results. If we take into account the following
equation, resulting from (24)

$$\Gamma_\sigma^+\gamma^\sigma+\gamma^\sigma\Gamma_\sigma=-\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\,\gamma^\sigma}{\partial x^\sigma},\tag{41}$$

then we can write the sum in the following form:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\sigma}\left(\bar\psi\,\gamma_i\,\gamma^\sigma\frac{\partial\psi}{\partial x^\alpha}\right)+\bar\psi\frac{\partial\gamma^\sigma}{\partial x^\alpha}D_\sigma\psi-\bar\psi\gamma^\sigma\frac{\partial\Gamma_\sigma}{\partial x^\alpha}\psi=0\tag{42}$$

Furthermore, we multiply (40) on the left with $-\bar\psi\Gamma_\alpha^+$, (40*) on
the right with $\Gamma_\alpha\psi$ and add the results. In the sum we replace
the terms $\Gamma_\alpha^+\gamma^\sigma$ and $\Gamma_\sigma^+\gamma^\sigma$ by their equivalent terms from Eqs. (24)
and (41). In this way we get

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\sigma}\left(\bar\psi\sqrt{g}\,\gamma^\sigma\Gamma_\alpha\psi\right)+\bar\psi(\nabla_\alpha\gamma^\sigma)D_\sigma\psi$$
$$-\bar\psi\gamma^\sigma\frac{\partial\Gamma_\alpha}{\partial x^\sigma}\psi+\bar\psi\gamma^\sigma(\Gamma_\sigma\Gamma_\alpha-\Gamma_\alpha\Gamma_\sigma)\psi=0.\tag{43}$$

Here if we replace $\nabla_\alpha\gamma^\sigma$ with

$$\nabla_\alpha\gamma^\sigma=\frac{\partial\gamma^\sigma}{\partial x^\alpha}+\Gamma_{\alpha\varrho}^\sigma\gamma^\varrho,$$

then subtracting (43) from (42) gives

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\sigma}(\sqrt{g}\,A^\varrho_{\cdot\alpha}) - \Gamma^\sigma_{\alpha\varrho}A^\varrho_{\cdot\sigma} =: \bar\psi\gamma^\sigma D_{\sigma\alpha}\psi, \tag{44}$$

where by way of abbreviation we set

$$D_{\sigma\alpha} = \frac{\partial \Gamma_\sigma}{\partial x^\alpha} - \frac{\partial \Gamma_\alpha}{\partial x^\sigma} + \Gamma_\sigma\Gamma_\alpha - \Gamma_\alpha\Gamma_\sigma \tag{45}$$

We must now calculate the $D_{\sigma\alpha}$ matrix. We have

$$D_{\sigma\alpha} = D_\sigma D_\alpha - D_\alpha D_\sigma = \sum_{kl} e_k e_l h_{k,\sigma} h_{l,\alpha} D'_{kl}, \tag{46}$$

where we set

$$D'_{kl} = D'_k D'_l - D'_l D'_k + \sum_m e_m (\gamma_{mlk} - \gamma_{mkl}) D'_m \tag{47}$$

The operator (47) is equal to

$$D'_{kl} = \frac{1}{4}\sum_{ij} \alpha_i \alpha_j e_j \gamma_{ijkl} + \frac{2\pi i e}{hc} M'_{kl}, \tag{48}$$

where γ_{ijkl} stand for the leg components of the Riemann tensor:

$$\gamma_{ijkl} = \frac{\partial \gamma_{ijk}}{\partial s_l} - \frac{\partial \gamma_{ijl}}{\partial s_k}$$
$$+ \sum_m e_m \{\gamma_{ijm}(\gamma_{mkl} - \gamma_{mlk}) + \gamma_{mil}\gamma_{mjk} - \gamma_{mik}\gamma_{mjl}\}, \tag{49}$$

and the skewsymmetric tensor M'_{kl}:

$$M'_{kl} = \frac{\partial \varphi'_k}{\partial s_l} - \frac{\partial \varphi'_l}{\partial s_k} + \sum_m e_m(\gamma_{mlk} - \gamma_{mkl})\varphi'_m, \tag{50}$$

represents the electromagnetic field.

First of all we express the $\gamma^\sigma D_{\sigma\alpha}$ matrix in terms of D'_{kl}:

$$\gamma^\sigma D_{\sigma\alpha} = \sum_{kl} e_k e_l \alpha_k h_{l\alpha} D'_{kl}. \tag{51}$$

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The sum $\sum_k e_k \alpha_k D'_{kl}$ which appears here can be calculated with the aid of (48), whereby the cyclic symmetry of the Riemann tensor must be taken into account. We get

$$\sum_k e_k \alpha_k D'_{kl} = \sum_k e_k \alpha_k \left(-\frac{1}{2} R'_{kl} + \frac{2\pi i e}{hc} M'_{kl} \right),$$ (52)

where

$$R'_{kl} = -\sum_i e_i \gamma_{ikil}$$ (53)

denote the leg components of the reduced Riemann tensor. If we set (52) in (51) then we get

$$\gamma^\varrho D_{\sigma\alpha} = \gamma^\varrho \left(-\frac{1}{2} R_{\varrho\alpha} + \frac{2\pi i e}{hc} M_{\varrho\alpha} \right).$$ (51*)

Thus for the divergence of the $A^{\delta\cdot}_{\cdot\alpha}$ we get the following equation

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} \left(\sqrt{g}\, A^\sigma_{\cdot\alpha} \right) - \Gamma^\sigma_{\alpha\varrho} A^\varrho_{\cdot\sigma} = S^\varrho \left(-\frac{1}{2} R_{\varrho\alpha} + \frac{2\pi i e}{hc} M_{\varrho\alpha} \right).$$ (54)

If we set

$$\frac{ch}{2\pi i} A^\sigma_{\cdot\alpha} = W^\sigma_{\cdot\alpha} = T^\sigma_{\cdot\alpha} + i U^\sigma_{\cdot\alpha},$$ (55)

where $T^{\sigma\cdot}_{\cdot\alpha}$ and $U^{\sigma\cdot}_{\cdot\alpha}$ stand for the real and the imaginary component of the complex tensor $W^{\sigma\cdot}_{\cdot\alpha}$, then Eq. (54) can be written in the following form:

$$\nabla_\sigma W^\sigma_{\cdot\alpha} = S^\varrho \left(e M_{\varrho\alpha} - \frac{hc}{4\pi i} R_{\varrho\alpha} \right)$$ (56)

or if we separate the real and the imaginary components:

$$\left. \begin{array}{l} \nabla_\sigma T^\sigma_{\cdot\alpha} = e\, S^\varrho M_{\varrho\alpha}, \\[2mm] \nabla_\sigma U^\sigma_{\cdot\alpha} = \dfrac{hc}{4\pi} S^\varrho R_{\varrho\alpha}. \end{array} \right\}$$ (57)

The second of these two equations is an easy-to-prove identity, for the $U^{\sigma \cdot}_{\cdot \alpha}$ tensor is equal to

$$U^{\bullet}_{\cdot \alpha} = - \frac{hc}{4\pi} \nabla_{\bullet} S^{\bullet},\tag{58}$$

and the divergence of the S^{σ} vector disappears according to (29).

Eq. (57) states that the divergence of tensor $T^{\alpha}_{\ \alpha}$ is equal to the Lortentz force. We can therefore interpret $T^{\sigma \cdot}_{\cdot \alpha}$ as an energy tensor. Eq. (57) are then the equations of motion of the general relativity theory. Perhaps it would be more consistent to interpret not just the real component $T^{\sigma \cdot}_{\cdot \alpha}$ but the entire complex tensor $W^{\sigma \cdot}_{\cdot \alpha}$ as an energy tensor. We will not discuss here which of these interpretations is preferable.

What is surprizing here is the appearance of the electro-magnetic sensor $M_{\rho \alpha}$ along with the Riemann tensor $R_{\rho \alpha}$ in the form of a Hermitian matrix.

$$-\left\| R^{\cdot}_{\varrho \alpha} - \frac{4\pi i e}{hc} M_{\varrho \alpha} \right\|.$$

8. From the results obtained in order to derive the quantum mechanic equations of motion which correspond to those of a mass point (geodesic), we proceed as follows.

In the range of spatial variables x_1, x_2, x_3 we choose a complete system of functions:

$$\psi_n (x_0 x_1 x_2 x_3; \zeta) \quad (\zeta = 1, 2, 3, 4),\tag{59}$$

each of which satisfies the Dirac equation[1] and is normalized by the postulate:

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[1] Cf [5] and [2].

$$\iiint \bar{\psi}_n \gamma^0 \psi_n \sqrt{g}\, dx_1\, dx_2\, dx_3 = 1 \qquad (60)$$

Because of (26) and (27) it follows from the existence of this equation for a special value of x_0 that it is valid for any other value of x_0. We define the matrix element for an operator L by the equation

$$L_{mn} = \iiint \bar{\psi}_m L \psi_n \sqrt{g}\, dx_1\, dx_2\, dx_3. \qquad (61)$$

We bear in mind that the operations performed in the preceding section and in particular Eq. (54) remain unchanged if in $A^{\sigma\cdot}{}_{\cdot\alpha}$ and in S^ρ we replace ψ by ψ_n and $\bar{\psi}$ by $\bar{\psi}_m$, hence by two __different__ solutions of the Dirac equation. We now write Eq. (54) in the form

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}(\bar{\psi}_m \sqrt{g}\, \gamma^0 D_a \psi_n) = \Gamma^\sigma_{a\varrho}\bar{\psi}_m \gamma^\varrho D_\sigma \psi_n$$
$$+ \bar{\psi}_m \gamma^\varrho\left(-\frac{1}{2}R_{\varrho a} + \frac{2\pi i e}{h c}M_{\varrho a}\right)\psi_n. \qquad (62)$$

If we multiply (62) with $\sqrt{g}\,dx_1 dx_2 dx_3$ and integrate over the entire space, then only one single term remains left over from the sum on the left side of (62), and we obtain $\underline{/274}$

$$\frac{d}{dx^0}\left\{\iiint \bar{\psi}_m \gamma^0 D_a \psi_n \sqrt{g}\, dx_1\, dx_2\, dx_3\right\}$$
$$= \iiint \bar{\psi}_m\left[\Gamma^\sigma_{a\varrho}\gamma^\varrho D_\sigma + \gamma^\varrho\left(-\frac{1}{2}R_{\varrho a} + \frac{2\pi i e}{h c}M_{\varrho a}\right)\right]\psi_n \cdot \sqrt{g}\, dx_1\, dx_2\, dx_3, \qquad (63)$$

which we can also write symbolically in the form

$$\frac{d}{dx^0}(\gamma^0 D_a) = \Gamma^\sigma_{a\varrho}\gamma^\varrho D_\sigma + \gamma^\varrho\left(-\frac{1}{2}R_{\varrho a} + \frac{2\pi i e}{h c}M_{\varrho a}\right) \qquad (64)$$

or also, if we set

$$P_a = \frac{h}{2\pi i}D_a \qquad (65)$$

in the form

$$\frac{d}{dx^\rho}(\gamma^\rho P_\alpha) = \Gamma^\omega_{\alpha\rho}\gamma^\rho P_\omega + \gamma^\rho\left(\frac{e}{c}M_{\rho\alpha} - \frac{h}{4\pi i}R_{\rho\alpha}\right).\tag{66}$$

Now we can interpret the γ^ρ operators as representing the classical velocity $\frac{dx^\sigma}{dx^0}$ and P_σ as that particular quantity of the covariant motion quantities $mg_{\sigma\rho}\frac{dx^\sigma}{ds}$. This interpretation makes it possible to complete the transition to the classical theory. If we do this and consistently ignore the h term on the right side, then we obtain precisely the classical equation of motion for a charged mass point in a gravitation field and in particular -- when no electromagnetic field is present -- the differential equation of the geodesic.

9. The pure covariant tensor

$$W_{\sigma\alpha} = g_{\sigma\rho}W^\rho_{\cdot\alpha} = c\bar{\psi}\gamma_\sigma P_\alpha\psi\tag{55*}$$

is not symmetric with respect to its indices. Because of the significance of the $c\gamma_\sigma$ and P_α operators (velocity and momentum) the quantum mechanic parameter $W_{\sigma\alpha}$ corresponds to the classic parameter $\rho_0 u_\sigma u_\alpha$

$$W_{\sigma\alpha} \rightarrow \rho_0 u_\sigma u_\alpha,\tag{67}$$

where u_α stands for the classical covariant component of the four-velocity and ρ_0 stands for the rest density of the matter. However, the parameter $\rho_0 u_\sigma u_\alpha$ is symmetric with respect to the indices.

The Dirac equation (27) can be derived from a variation principle which can be formulated as follows with the aid of

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the energy tensor:

$$\delta \iiint (W^{\sigma}_{.\sigma} - m c^2 \overline{\psi} \alpha_4 \psi) \sqrt{g}\, dx_0\, dx_1\, dx_2\, dx_3 = 0. \qquad (68)$$

This equation produces a simple physical interpretation of the invariance $m\overline{\psi}\alpha_4\psi$ as the rest density of the matter.

10. We now want to summarize the results of our study.

Our starting point is the concept of the parallel shift of a half vector. By means of this concept, the appearance of the ϕ_α potentials along with the p_α impulses in the Dirac equation can be interpreted purely geometrically. The purely formal transfer of the expression $p_\alpha - \frac{e}{c} \phi_\alpha$ from classical mechanics into quantum mechanics thus became superfluous. Furthermore, this concept allowed us to easily arrange the potentials in the geometrical scheme of the general relativity theory, and this can be useful for setting up a unified theory of electricity and gravitation.

Moreover, the Dirac equations were set up in the general relativity theory. These are invariant with respect to the choice of coordinates and "legs." This produced a secondary result, namely an explicit representation of the Dirac operator in curvilinear orthogonal coordinates. A tensor was constructed whose divergence is equal to the Lorentz force. This tensor was interpreted as an energy tensor and the equation satisfying it was interpreted as a macroscopic equation of motion. In addition, the quantum mechanic equations of motion for the electron were derived which correspond to the classical equations for a charged mass point or -- in the absence of an electromagnetic field -- the equations of a geodesic. Finally, the variation pricniple from which the Dirac equation can be derived was written.

Our aim was to geometrize the Dirac theory of the electron
and integrate it into the general relativity theory. In so
doing, the difficulties attached to the Dirac theory -- such as
the occurrence of negative energy values and a non-disappearing
probability of charge exchange of the electron -- were not at all
encountered. But perhaps our observations can contribute indi-
rectly to solving these difficulties by showing what the original,
unchanged Dirac theory can accomplish.

Leningrad, University Physics Institute, May/June 1929.

After completing this paper, I discoverd the very interesting paper of H. Weyl [12]. Weyl's basic mathematical idea is essentially identical to the concept of the parallel shift of a half vector. However the physical content of Weyl's work is completely different from mine.

The essential features of Weyl's approach can be summarized as follows.

1. Weyl considers the Dirac equation to be a wave equation not for the electron but for the electron-proton system.

2. In the additive gravitation terms, Weyl thinks he has found a substitute for the $mc\alpha_4$ term, the latter simply being cancelled.

In my view, both of these theses can hardly be supported, for they run into considerable difficulties which I would here like to draw attention to.

The quantum mechanic equations of motion resulting from the Dirac equation are completely analogous with the classic equations of motion for a charged particle (and not by the way for a two-body system), as has already been shown in my earlier work [2].

The Dirac equation, and indeed with the $mc\alpha_4$ terms, is perfectly suited for describing the force-free motion of an electron as a wave in the sense of the original de Broglie view.

The splitting up of the current vector S by Weyl into two summands $S^{(+)}$ and $S^{(-)}$, which are interpreted as positive current and negative electricity, cannot be upheld, for these

summands are null vectors, and only their sum $S = S^{(+)} + S^{(-)}$ is a time-like vector.[1] However, the current is a static-macroscopic quantity and as such must have the same character as in the classical theory, hence must necessarily be time-like.

Weyl's equations are supposed to describe the electron-proton system. We may therefore demand that they accurately reflect the energy level of the hydrogran atom. However, since the $mc\alpha_4$ term has been left out, this is hardly possible and in any case is not proven.

The gravitation terms ["leg vector" f_1 in our equation (35)] interpreted by Weyl as a substitute for the mass can be made to disappear as soon as a system of normal congruences exists and especially in the case of spherical symmetry as well as in the static case of axial symmetry. However, one can expect a large degreee of symmetry from the electron-proton system.

Finally, it remains completely unclear just how the constants m and M -- the mass of the electron and of the proton -- should be produced from the gravitation terms.

Because of these difficulties, I cannot consider Weyl's attempt to tackle the quantum mechanics problem of mass and the two body problem as successful. On the other hand, I gladly concur with Weyl's general idea that both problems are closely

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[1] Proof: The time-like character of S follows from identity (8) (where now S_1 is to be read instead of A_1), for it gives

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = S_4^2 + S_5^2 \qquad (*)$$

$S_1^{(+)}$ or $S_1^{(-)}$ is obtained from S_1 if ψ_3 and ψ_4 or ψ_1 and ψ_2 are set equal to zero. In both cases S_4 and S_5 disappear, thus also the left side of (*), which was to be proved.

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related to one another and to the problem of gravity.

In conclusion, I would like to make a few general remarks on the physical content of the Dirac equations and on the two body problem in quantum mechanics.

In my opinion, the Dirac equation describes only the electron in terms of quantum mechanics while it describes the rest of the world (perhaps also the mass of the electron) macroscopically. In this case the rest of the world also includes the proton. The solution to the two body problem must consist in finding a quantum mechanical description of the electron, the proton, the electromagnetic field and the mass. The quantum mechanical problem of mass seems to me to defy solution as long as only one body is considered. By contrast, for the macroscopic description of gravity and electricity, the quantum mechanical one body problem seems to render good service.

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