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AVAILABILITY AND MEAN TIME BETWEEN FAILURES OF REDUNDANT SYSTEMS WITH RANDOM MAINTENANCE OF SUBSYSTEMS

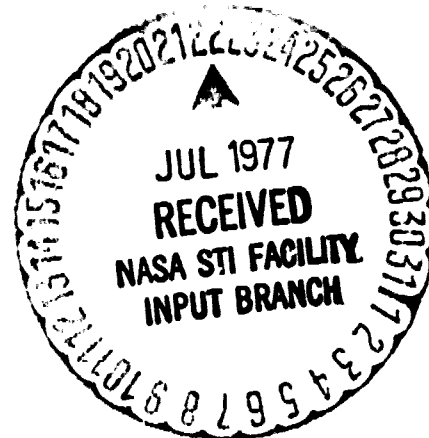
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16. Abstract  It is shown how the availability and MTBF of a redundant system with subsystems maintained at the points of so-called stationary renewal processes can be determined from the distributions of the intervals between maintenance actions and of the failure-free operating intervals of the subsystems. The results make it possible, for example, to determine the frequency and duration of hidden failure states in computers which are incidentally corrected during the repair of observed failures..			
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# AVAILABILITY AND MEAN TIME BETWEEN FAILURES OF REDUNDANT SYSTEMS WITH RANDOM MAINTENANCE OF SUBSYSTEMS

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## 1. Introduction

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The distinction between "self-indicating" and "hidden" failures is extremely important in computer practice. In most cases the "self-indicating" failures can be repaired relatively quickly, e.g. by the replacement of subsassemblies. This maintenance is performed at statistically-describable points in time and is independent of hidden faults, although hidden faults are often corrected during the course of such maintenance. This kind of maintenance strategy can be considered stochastic with respect to hidden failures and derives its feasibility solely from the correction of self-indicating faults.

Fig. 1 shows the time sequence of hidden failures and maintenance for a subsystem no.  $i$  which is in stationary operation, i.e. has been operating for a long period of time. At  $t = t_v$  (cf. the crosses on the time axis in Fig. 1) maintenance is performed; at  $t = t_{A,v}$  a failure occurs which is not repaired until  $t_{v+1}$ .

We shall define  $W_i$  as the interval between successive maintenance actions.

If  $A_i$  is the down-time,  $B_i$  is the time between failures and

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\* Numbers in the margin indicate pagination in the foreign text.

E is the symbol (functional) for the formation of the expectation, then the availability of subsystem i is given by the known formula [2]

$$V_i = \frac{EB_i}{E(A_i + B_i)} \quad (1)$$

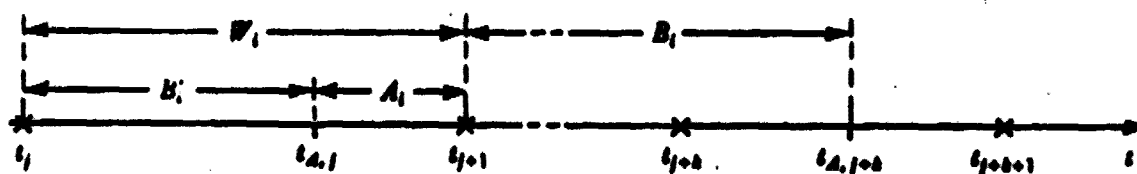


Fig. 1. Failure and maintenance, where  $t_{A,j}$  and  $t_{A,j+k}$  are successive instances of failure.

The main purpose of the investigation which follows is to determine the mean duration of a hidden fault (here called the mean down-time  $EA_s$ ) and the mean interval  $EB_s$  between such states in the case of complex redundant systems (such as multiple-computer systems). It is assumed that the brief maintenance actions do not interfere with the operation of the system.

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Although we shall be concerned primarily with the determination of  $EA_i$  for subsystems no. i, we shall first show how  $EA_s$  and  $EB_s$ , as well as the availability  $V_s$  of the whole system, can be determined from  $EA_i$  and  $EB_i$ .

## 2. The Formulas of Isphording [1]

Let the system state in terms of failure and operational readiness be described by a monotonically-increasing Boolean function, i.e., the system function of Störmer [2]:

$$X_i = \sum_{l=1}^m \left[ c_l \prod_{k=1}^{n_l} X_{l_k} \right] \quad (2)$$

where  $c_i$  is an integral constant,

$$X_i^{1)} = \begin{cases} 1 & \text{for intact subsystem } i, \\ 0 & \text{for defective subsystem } i \end{cases} \quad (3)$$

is the so-called state-indicator variable, and  $l_{ik}$  is the subscript numbering the  $M$  subsystems, i.e.

$$l_{ik} \in \{1, 2, \dots, M\}.$$

Then, according to Isphording [1] (cf. new findings of Schneeweiss [3]) if all subsystems are intact or defective stochastically independently of one another, we have for the system availability  $V_s$

$$V_s = \sum_{l=1}^m \left[ c_l \prod_{k=1}^{n_l} V_{l_k} \right], \quad (4)$$

and, with  $B_i$  the duration of an error-free period of operation of subsystem no.  $i$ , we have for the mean time between failures (MTBF) of the entire system  $EB_s$

$$\frac{V_s}{EB_s} = \sum_{l=1}^m \left[ c_l \left( \prod_{k=1}^{n_l} V_{l_k} \right) \sum_{k=1}^{n_l} \frac{1}{EB_{l_k}} \right]. \quad (5)$$

---

1) Colons are always placed alongside the newly-defined quantity in our defining equations.

### 3. MTBF of a Subsystem

This problem is almost trivial if the distribution function  $F_i(\tau)$  of the operating interval  $B_i$  of subsystem no.  $i$ , i.e. the probability  $P(B_i \leq \tau)$ <sup>1)</sup>, is known (though this will sometimes be difficult to achieve). In this case

$$EB_i = \int_0^{\infty} \tau f_i(\tau) d\tau = \int_0^{\infty} [1 - F_i(\tau)] d\tau. \quad (6)$$

If it happens that  $B_i$  is exponentially distributed according to

$$F_i(\tau) = 1 - \exp(-\gamma_i \tau), \quad (7)$$

then it can be quickly confirmed by substitution in eqn. (6) that

$$EB_i = \frac{1}{\gamma_i}. \quad (8)$$

### 4. Mean Down-Time of a Subsystem

It is known (cf. Fig. 1) that

$$F_i(\tau) = P(B_i \leq \tau); \quad f_i(\tau) = \frac{d}{d\tau} F_i(\tau)$$

and, with  $W_i$  denoting the maintenance interval of subsystem no.  $i$ ,

$$F_i(\tau) = P(W_i \leq \tau); \quad f_i(\tau) = \frac{d}{d\tau} F_i(\tau). \quad (9)$$

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1)  $P(A)$  is the probability of the random event  $A$ .

Since this is to be true for all maintenance intervals of this subsystem, the points at which maintenance is performed form a stationary renewal process (cf. Cox [4] or Störmer [2], for example). We are now considering only such maintenance intervals, i.e. the time intervals between successive maintenance actions (with the actual checking and renewal procedure carried out in a negligibly short time), which contain a subsystem failure ( $t_{A,j}$  in Fig. 1).

The down-time of interest is then the segment of the maintenance interval of length  $W_i$  with left boundary point  $t_j$  and right boundary point  $t_{j+1}$  lying to the right of the failure point  $t_{A,j}$ . The mean down-time  $A_i$  is still unknown. According to Fig. 1,

$$EA_i = E(t_{j+1} - t_{A,j}) = E(t_{j+1} - t_j) - E(t_{A,j} - t_j) = EW_i - EB'_i. \quad (10)$$

So, according to eqn. (9),

$$EW_i = \int_0^{\infty} [1 - F_i(\tau)] d\tau. \quad (11)$$

In calculating  $EB'_i$  it is assumed that subsystem  $i$  is "like new" after each maintenance is performed. It is advantageous to calculate individual  $EB'_i$  values in two steps: first  $W_i$  is held constant, i.e. a conditional expectation is calculated, and then the expectation is formed from this ( $W_i = \tau$ )-dependent random variable. In this case, since  $W_i = \tau$  is equivalent to  $B'_i \leq \tau$ ,

$$EB'_i = E[E(B'_i | B'_i \leq \tau)], \quad (12)$$

whereby the extreme expectation on the right is over  $\tau$ .



Now, by definition, the conditional expectation is given by

$$E(B'_i | B'_i \leq \tau) = \int_0^{\tau} t f_i(t) dt \quad (13)$$

with a still-unknown distribution density function  $\hat{f}_i$ . According to the general formula for the conditional probability of events a and b

$$P(a|b) = \frac{P(a \cap b)}{P(b)} \quad (14)$$

we obtain the following expression for the conditional probability that  $B'_i$  lies between  $t$  and  $t + \Delta t$  solely for cases where  $W_i = \tau$  (i.e., on condition that  $B'_i \leq \tau$ ):<sup>1)</sup>

$$f_i(t) \Delta t = \begin{cases} \Delta t f_i(t) / F_i(\tau) + o(\Delta t); & t + \Delta t \leq \tau, \\ 0; & t > \tau. \end{cases} \quad (15)$$

This density apparently also satisfies the normalization rule for distribution densities of positive random variables

$$\int_0^{\infty} f_i(t) dt = 1.$$

Note: The above events a and b can, incidentally, be interpreted directly as "hits" in certain intervals on the time axis. In Fig. 2, a is a hit, i.e. a failure in the interval  $(t, t + \Delta t]$  and b is a hit in the interval  $(0, \tau]$ . Thus,

$$a = a(t) = \{t < B'_i \leq t + \Delta t\}; \quad b = \{0 < B'_i \leq \tau\}.$$

---

1) It should be noted that  $P(B'_i \leq \tau) = F_i(\tau)$ .

A direct consequence is that for  $t + \Delta t \leq \tau$ , the average

$$a(t) \cap b = a(t).$$

since  $a(t) \subset b$ . In our case, therefore, it follows from eqn. (14) that

$$P[a(t)|b] = \frac{P[a(t)]}{P(b)}; \quad t \leq \tau - \Delta t.$$

This is simply another way of writing the top line on the right-hand side of eqn. (15). That the bottom line of eqn. (15) yields zero follows from the fact that for  $t > \tau$

$$a(t) \cap b = \emptyset \quad (\text{empty set})$$

(see Fig. 2). The probability of the empty set, i.e. of an impossible event, is always 0, however.

If eqn. (15) is now substituted into eqn. (13), we have

$$E(B'_i | W_i = \tau) = \frac{1}{F_i(\tau)} \int_0^\tau t f_i(t) dt. \quad (16)$$

After calculating the expectation over  $W_i$ , eqn. (9) leads to

$$E B'_i = \int_0^\infty \frac{f_i(\tau)}{F_i(\tau)} \int_0^\tau t f_i(t) dt d\tau. \quad (17)$$

On condition that one failure occurs between any two successive maintenance actions, the mean down-time [acc. to eqn. (10)] follows from eqn. (17) as

$$E A_i = E W_i - E B'_i \\ = \int_0^\infty \left\{ [1 - F_i(\tau)] - \frac{f_i(\tau)}{F_i(\tau)} \int_0^\tau t f_i(t) dt \right\} d\tau. \quad (18)$$

Eqn. (18) is the principle result of this study!

In the special case of a Poisson-type maintenance routine, i.e. if

$$f_i(t) = \gamma_i \exp(-\gamma_i t), \quad F_i(t) = 1 - \exp(-\gamma_i t), \quad (19)$$

and in the case of an exponential time-before-failure distribution, i.e. if

$$f_i(t) = \gamma_i \exp(-\gamma_i t), \quad F_i(t) = 1 - \exp(-\gamma_i t), \quad (20)$$

we obtain the following formula for calculating  $EA_1$ :

$$\begin{aligned} \int_0^{\infty} t f_i(t) dt &= \gamma_i \int_0^{\infty} t \exp(-\gamma_i t) dt = \\ &= \frac{1}{\gamma_i} [1 - \exp(-\gamma_i \tau)] - \tau \exp(-\gamma_i \tau). \end{aligned}$$

Because

$$\frac{f_i(\tau)}{F_i(\tau)} = \frac{\gamma_i \exp(-\gamma_i \tau)}{1 - \exp(-\gamma_i \tau)}$$

and

$$\int_0^{\infty} [1 - F_i(\tau)] d\tau = \frac{1}{\gamma_i}$$

we can substitute in eqn. (18) to obtain

$$\begin{aligned} EA_1 &= \frac{1}{\gamma_i} - \int_0^{\infty} \frac{\gamma_i}{\gamma_i} \left\{ \exp(-\gamma_i \tau) - \tau \gamma_i \frac{\exp[-(\gamma_i + \gamma_i) \tau]}{1 - \exp(-\gamma_i \tau)} \right\} d\tau \\ &= \frac{1}{\gamma_i} - \frac{1}{\gamma_i} + \gamma_i \sum_{k=1}^{\infty} \frac{1}{(\gamma_i + k\gamma_i)^2}. \end{aligned} \quad (21)$$

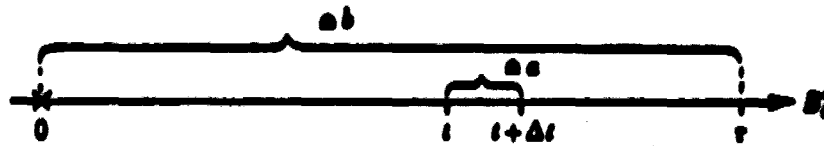


Fig. 2. The average of events a and b.

It should be noted that the sum on the right is a special case of the Riemann  $\zeta$ -function, since

$$\zeta(v, c) = \sum_{k=0}^{\infty} (k+c)^{-v}; \quad v > 1. \quad (22)$$

Several values of  $\zeta(2, c)$  are calculated in the Appendix.

More precise is

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(\tilde{\gamma}_1 + k\gamma_1)^2} &= \frac{1}{\tilde{\gamma}_1^2} \sum_{k=0}^{\infty} \left(k + \frac{\tilde{\gamma}_1}{\gamma_1}\right)^{-2} - \frac{1}{\tilde{\gamma}_1^2} \\ &= \frac{1}{\tilde{\gamma}_1^2} \zeta\left(2, \frac{\tilde{\gamma}_1}{\gamma_1}\right) - \frac{1}{\tilde{\gamma}_1^2}. \end{aligned}$$

so that

$$E A_1 = \frac{1}{\tilde{\gamma}_1} \left[ \frac{\tilde{\gamma}_1}{\gamma_1} \zeta\left(2, \frac{\tilde{\gamma}_1}{\gamma_1}\right) - 1 \right]. \quad (23)$$

If  $\tilde{\gamma}_1 \gg \gamma_1$ , i.e. if the MTBF is considerably larger than the mean interval between two maintenances, then according to the Appendix

$$\zeta\left(2, \frac{\tilde{\gamma}_1}{\gamma_1}\right) = \frac{\tilde{\gamma}_1}{\gamma_1} \left(1 + \frac{\gamma_1}{2\tilde{\gamma}_1}\right). \quad (24)$$

is usually an adequate approximation. This leads to

$$EA_i = \frac{1}{2\lambda_i}; \lambda_i \gg \gamma_i; \quad (25)$$

i.e., "on the mean" the failure occurs mid-way between two consecutive maintenance points.

### 5. Comparison with Periodic Maintenance

The above-mentioned special interpretation of the problem, i.e. that the arbitrariness of the maintenance points is determined by "self-indicated" failures, is obviously unsuitable in the case of periodic maintenance. In practice, periodic maintenance should be practiced as a supplement to stochastic maintenance.

In other respects the solution for the case of periodic maintenance at time intervals  $T$  has already been presented in the preceding Section. It is necessary merely to set  $\tau = T$  in eqn. (16) to obtain

$$EB_i = \frac{1}{F_i(T)} \int_0^T t f_i(t) dt.$$

[This result also follows formally from eqn. (17) with  $\hat{f}_i(\tau) = \delta(\tau - T)$ .]

Moreover, since in the trivial case

$$EW_i = T,$$

the mean down-time according to eqn. (10) becomes

$$EA_i = T - \frac{1}{F_i(T)} \int_0^T t f_i(t) dt. \quad (26)$$

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In the special case of an exponential distribution for  $B_1$ , therefore, it follows from eqn. (20) that

$$\begin{aligned}
 EA_1 &= \\
 &= T - \frac{1}{1 - (\exp - \gamma_1 T)} \left\{ \frac{1}{\gamma_1} [1 - \exp(-\gamma_1 T)] - T \exp(-\gamma_1 T) \right\} \\
 &\approx T - \frac{1}{\gamma_1} + \frac{T}{\exp(\gamma_1 T) - 1}.
 \end{aligned} \tag{27}$$

Of interest here is an approximation for the case of frequent maintenance

$$\gamma_1 T \ll 1, \text{ i.e. } T \ll \frac{1}{\gamma_1}.$$

Then, because

$$\exp(\gamma_1 T) = 1 + \gamma_1 T + \frac{\gamma_1^2 T^2}{2} + \frac{\gamma_1^3 T^3}{6} + \dots$$

the quotient

$$\frac{T}{\exp(\gamma_1 T) - 1} \approx \frac{1}{\gamma_1} \left( 1 - \frac{\gamma_1 T}{2} + \dots \right). \tag{28}$$

But this yields the approximation

$$\begin{aligned}
 EA_1 &= T - \frac{1}{\gamma_1} + \frac{1}{\gamma_1} \left( 1 - \frac{\gamma_1 T}{2} + \dots \right) \\
 &\approx \frac{T}{2}.
 \end{aligned} \tag{29}$$

which is quite plausible, because the failure probability density will be constant for a  $T$  which is small compared to  $1/\gamma_1$ .

## 6. Example

### The Two-Out-Of-Four Selection System

If the entire system is functional when at least two of its four subsystems are intact, the system function is as follows:

$$X_s = X_1 \& X_2 \vee X_1 \& X_3 \vee X_1 \& X_4 \vee X_2 \& X_3 \vee X_2 \& X_4 \vee X_3 \& X_4. \quad (30)$$

The expressions

$$X_i \& X_k = X_i X_k \quad \text{and} \quad X_i \vee X_k = X_i + X_k - X_i X_k \quad (31)$$

also lead to

$$X_s = 1 - (1 - X_1 X_2)(1 - X_1 X_3)(1 - X_1 X_4) \cdot (1 - X_2 X_3)(1 - X_2 X_4)(1 - X_3 X_4). \quad (32)$$

Because of the so-called idempotent relation  $X^N = X$  for Boolean variables we now have, after some elementary intermediate operations,

$$X_s = X_1 X_2 + X_1 X_3 + X_1 X_4 + X_2 X_3 + X_2 X_4 + X_3 X_4 - 2(X_1 X_2 X_3 + X_1 X_2 X_4 + X_1 X_3 X_4 + X_2 X_3 X_4) + 3X_1 X_2 X_3 X_4. \quad (33)$$

If all subsystems have the same availability  $V_0$ , then according to eqn. (4) the system availability  $V_s$  becomes

$$V_s = 6V_0^2 - 8V_0^3 + 3V_0^4 = V_0^2(6 - 8V_0 + 3V_0^2). \quad (34)$$

---

1) The same result is obtained by elementary probability theory and combinatory analysis from

$$V_s = \binom{4}{2} V_0^2 (1 - V_0)^2 + \binom{4}{3} V_0^3 (1 - V_0) + \binom{4}{4} V_0^4.$$

and, according to eqn. (5), the defining equation for the mean time between failures (MTBF)  $EB_s$  is as follows (with the MTBF of all subsystems designated as  $EB_0$ ):

$$\begin{aligned} \frac{V_s}{EB_s} &= 6 \cdot 2 \frac{V_0^2}{EB_0} - 8 \cdot 3 \frac{V_0^3}{EB_0} + 3 \cdot 4 \frac{V_0^4}{EB_0} \\ &= \frac{12}{EB_0} V_0^2 (1 - V_0)^2. \end{aligned} \quad (35)$$

The MTBF follows explicitly from the last two equations:

$$EB_s = \frac{EB_0 (6 - 8V_0 + 3V_0^2)}{12(1 - V_0)^2}. \quad (36)$$

We must next determine  $EB_0$  and, for  $V_0$ ,  $EA_0$ : For an exponentially-distributed  $B_0$  according to

$$F_0(t) = 1 - \exp(-\gamma_0 t)$$

we have the known relation

$$EB_0 = 1/\gamma_0; \quad (37)$$

and, according to eqn. (25), for maintenance points with a Poisson distribution (exponential intervals with expectation  $1/\tilde{\gamma}_0$ , which may be equal for all subsystems) we have

$$\begin{aligned} EA_0 &= \frac{1}{\gamma_0} \left[ \frac{\tilde{\gamma}_0}{\gamma_0} \zeta \left( 2, \frac{\tilde{\gamma}_0}{\gamma_0} \right) - 1 \right]. \\ &\approx \frac{1}{2\tilde{\gamma}_0}; \quad \tilde{\gamma}_0 \gg \gamma_0. \end{aligned} \quad (38)$$

By definition, the availability is derived from  $EB_0$  and  $EA_0$ :

$$V_0 = \frac{EB_0}{EB_0 + EA_0}.$$



For the practically-important case of  $\tilde{\gamma}_0 \gg \gamma_0$ , i.e. for maintenance which is frequent relative to the failure rate, eqn. (37) and (38) lead to

$$V_0 = \frac{1/\gamma_0}{1/\gamma_0 + 1/(2\tilde{\gamma}_0)} = \frac{2\tilde{\gamma}_0}{2\tilde{\gamma}_0 + \gamma_0} = \frac{1}{1 + \gamma_0/(2\tilde{\gamma}_0)} \quad (39)$$

Tab. 1 gives the  $V_0$ ,  $EB_0$ ,  $V_S$  and  $EB_S$  for several values of  $\gamma_0$  and  $\tilde{\gamma}_0$ .

Table 1. Availability  $V_S$  and MTBF  $EB_S$  of the 2-out-of-4 System for the Case of Poisson Maintenance with Rate  $\tilde{\gamma}_0$  and a Failure Rate  $\gamma_0$  of the Individual System<sup>0</sup>

$\gamma_0$	$\tilde{\gamma}_0$	$V_0$	$EB_0$	$V_S$	$EB_S$
$10^{-6}$	$10^{-3}$	9.995003 <sub>10</sub> -001	$10^{-6}$	9.999997 <sub>10</sub> -001	3.340334 <sub>10</sub> -011
	$5 \cdot 10^{-4}$	9.990011 <sub>10</sub> -001		9.999998 <sub>10</sub> -001	8.367166 <sub>10</sub> -010
	$10^{-4}$	9.950250 <sub>10</sub> -001		9.999991 <sub>10</sub> -001	3.400558 <sub>10</sub> -009
$5 \cdot 10^{-7}$	$5 \cdot 10^{-4}$	9.995003 <sub>10</sub> -001	$2 \cdot 10^{-6}$	9.999997 <sub>10</sub> -001	6.680669 <sub>10</sub> -011
	$2.5 \cdot 10^{-4}$	9.990011 <sub>10</sub> -001		9.999998 <sub>10</sub> -001	1.673433 <sub>10</sub> -011
	$8 \cdot 10^{-5}$	9.950250 <sub>10</sub> -001		9.999991 <sub>10</sub> -001	6.801116 <sub>10</sub> -009
$10^{-7}$	$10^{-4}$	9.995003 <sub>10</sub> -001	$10^{-7}$	9.999997 <sub>10</sub> -001	3.340333 <sub>10</sub> -012
	$5 \cdot 10^{-5}$	9.990011 <sub>10</sub> -001		9.999998 <sub>10</sub> -001	8.367168 <sub>10</sub> -011
	$10^{-5}$	9.950250 <sub>10</sub> -001		9.999991 <sub>10</sub> -001	3.400557 <sub>10</sub> -010

## 7. Appendix

### Calculating Values of the Two-Parameter Zeta Function

We shall attempt to calculate numerical values for the two-parameter zeta function

$$\zeta(2, c) = \sum_{k=0}^{\infty} \frac{1}{(k+c)^2} \quad (40)$$

$\zeta(2, c)$  defines the surface which lies to the right of the ordinate below the step curve in Fig. 3 (schematic).

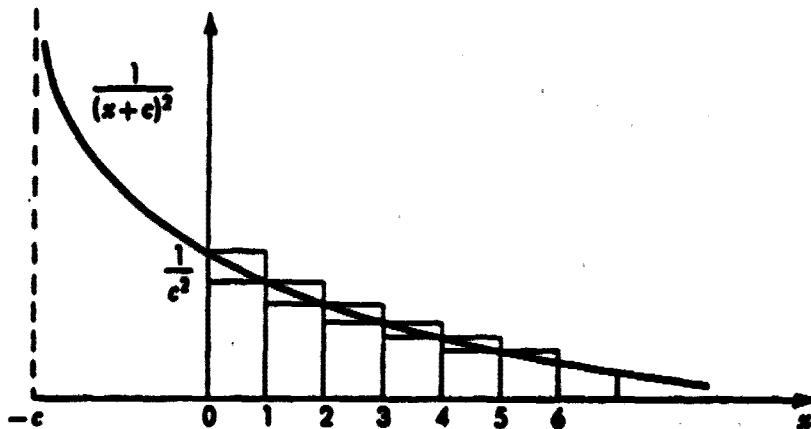


Fig. 3. The 2-parameter zeta function.

A lower approximation is

$$\int_0^{\infty} \frac{dx}{(x+c)^2} = -\left. \frac{1}{x+c} \right|_0^{\infty} = \frac{1}{c}$$

An upper approximation is, according to Fig. 3,

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$$\frac{1}{c} + \frac{1}{c^2}$$

If accuracy requirements are not too high, then

$$\zeta(2, c) \approx \frac{1}{c} \left( 1 + \frac{1}{2c} \right)$$

will probably be a satisfactory approximation for  $c \gg 1$ .

If requirements are high,  $N$  summands of series (40) must be computed and the rest estimated by

$$\int_N^{\infty} \frac{dx}{(x+c)^2} = \frac{1}{N+c}$$

Then the relation

$$\frac{1}{N+c} < \zeta(2, c) - \sum_{k=0}^{N-1} \frac{1}{(k+c)^2} < \frac{1}{N+c} \left( 1 + \frac{1}{N+c} \right). \quad (41)$$

is more precise.

The accuracy is thus dependent on the sum  $N + c$  ab, i.e. a smaller  $N$  is sufficient for large values of  $c$ . Further results have been published by Jahnke/Emde [5].

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