Domain decomposition preconditioners of Neumann–Neumann type for hp-approximations on boundary layer meshes in three dimensions

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We develop and analyse Neumann–Neumann methods for hp finite-element approximations of scalar elliptic problems on geometrically refined boundary layer meshes in three dimensions. These are meshes that are highly anisotropic where the aspect ratio typically grows exponentially with the polynomial degree. The condition number of our preconditioners is shown to be independent of the aspect ratio of the mesh and of potentially large jumps of the coefficients. In addition, it only grows polylogarithmically with the polynomial degree, as in the case of p approximations on shape-regular meshes. This work generalizes our previous one on two-dimensional problems in Toselli & Vasseur (2003a, submitted to Numerische Mathematik, 2003c to appear in Comput. Methods Appl. Mech. Engng.) and the estimates derived here can be employed to prove condition number bounds for certain types of FETI methods.

Keywords: domain decomposition; preconditioning; hp finite elements; spectral elements; anisotropic meshes.

1. Introduction

Solutions of elliptic boundary value problems in polyhedral domains have corner and edge singularities and, in addition, boundary layers may also arise in laminar, viscous, incompressible flows with moderate Reynolds numbers at faces, edges and corners. Suitably graded meshes, geometrically refined towards corners, edges and/or faces, can be employed in order to achieve an exponential rate of convergence of hp finite-element approximations (see e.g. Andersson et al., 1995; Babuška & Guo, 1996; Melenk & Schwab, 1998; Schwab & Suri, 1996; Schwab et al., 1998).

Neumann–Neumann (NN) and FETI algorithms are particular iterative substructuring methods and are among the most popular and heavily tested domain decomposition (DD) methods (see e.g. Le Tallec, 1994; Farhat & Roux, 1994; Mandel & Brezina, 1996; Bhardwaj et al., 2000). Unfortunately, the performance of iterative substructuring methods might be severely compromised if very thin elements and/or subdomains or general non-quasiuniform meshes are employed.

Some work has been done on domain decomposition preconditioners for higher-order approximations of three-dimensional problems. It is well-known that on shape-regular

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meshes special care must be taken in the choice of the basis functions in order to produce
preconditioners that are robust with respect to the polynomial degree (e.g. Mandel, 1989,
1990a,b; Pavarino, 1994; Bica, 1997; Sherwin & Casarin, 2001). For \( p \) approximations that
employ nodal basis functions on Gauss–Lobatto nodes (spectral element approximations),
many iterative substructuring methods can be successfully employed and studied (see
Pavarino & Widlund, 1996, 1997; Pavarino, 1997; Pavarino & Warburton, 2000 and the
references therein). Some of these ideas can be and have been generalized to \( hp \)
approximations (e.g. Ainsworth, 1996a,b; Oden et al., 1997; Guo & Cao, 1997; Le Tallec &
Patra, 1997; Ainsworth & Sherwin, 1999; Korneev et al., 2002 and the references therein
and, in particular, Guo & Cao, 1998 for three-dimensional problems). In all the above-
mentioned works, however, the finite-element mesh is assumed to be shape-regular and
robustness with respect to the aspect ratio is not in general ensured and often unlikely to
hold in practice.

In Toselli & Vasseur (2003a,c), we showed that NN and FETI methods can be
successfully devised for the particular geometrically refined boundary layer meshes
commonly used for \( hp \) finite-element approximations of two-dimensional problems.
Indeed, these meshes are highly anisotropic, but of a particular type:

1. they are obtained by refining an initial shape-regular mesh (macromesh);
2. refinement is only carried towards the boundary of the computational domain.

These properties, also shared by three-dimensional meshes, allowed us to obtain
condition number bounds for the corresponding preconditioned operators that only grow
polylogarithmically with the polynomial degree, as is the case of \( p \) approximations
on shape-regular meshes. Our understanding and analysis was confirmed by numerical
experiments. In particular, we choose the macromesh as a decomposition into substructures
in such a way that subdomains are shape-regular. Roughly speaking, the reason why
such favourable condition numbers are retained lies in the fact that upper bounds come
from stable decompositions of finite-element functions into components associated with
geometrical objects (typically vertices and edges of the subdomains in two dimensions).
Because of our particular meshes, only components associated with internal vertices need
to be considered, i.e. relative to vertices in a neighbourhood of which the mesh is shape-
regular.

Three-dimensional boundary layer meshes also share the two characteristics mentioned
above. However, stable decompositions now involve face and wirebasket components,
where the wirebasket is the union of the subdomain edges and vertices that do not lie
on the external boundary of the computational domain. By considering, for instance, an
edge \( E \) of a macroelement that shares a face with \( \Omega \) (see the face patch in Fig. 1, left, or
Fig. 2), decoupling of face and wirebasket components is now also performed close to \( \partial \Omega \),
and thus where the mesh is not shape-regular. In this work, we are however able to provide
condition number bounds that only grow polylogarithmically with the polynomial degree,
as in the two-dimensional case, and are independent of arbitrarily large aspect ratios of the
mesh.

The core of this work lies in the careful modification and derivation of certain Sobolev-
type inequalities that are independent of the aspect ratio of the mesh for wirebasket and
face components of finite-element functions; see Section 7. Provided such inequalities are
available, the definition of the algorithms and their analysis are fairly standard procedures
in DD methods and proceed as in the two-dimensional case in Toselli & Vasseur (2003a). Here, we will only consider the balancing method, which belongs to the family of Neumann–Neumann methods, but note that the estimates derived can be employed for the analysis of other Neumann–Neumann methods and one-level FETI methods in a straightforward way (see Pavarino, 1997; Klawonn & Widlund, 2001; Toselli & Vasseur, 2003a).

We limit our analysis to the case of nodal basis functions built on Gauss–Lobatto nodes. In addition, we only consider the model problem (2.1), which does not have boundary layers but only corner and edge singularities. However, our tensor-product meshes can also be employed when only singularities are present and do not require the use of hanging nodes. We recall that numerical results in Toselli & Vasseur (2003c) for two-dimensional problems showed that better performance is obtained for certain singularly perturbed problems which exhibit boundary layers. In addition, a linear dependence in $k$ for the condition number was observed for problems with geometric refinement towards interfaces that lie in the interior of the computational domain.

The remainder of this paper is organized as follows: in Sections 2 and 3, we introduce our continuous and discrete problems, respectively. Geometric boundary layer meshes are introduced in Section 4. A particular choice of basis functions is given in Section 5 and our Neumann–Neumann preconditioners are defined in Section 6. Section 7 is the core of this work and is devoted to the proof of some discrete Sobolev-type inequalities. Comparison results for certain discrete harmonic extensions are given in Section 8. Condition number bounds are then proven in Section 9. Section 10 contains some numerical results, while some concluding remarks and perspectives are presented in Section 11.

2. Problem setting

We consider a linear, elliptic problem on a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$ of unit diameter, formulated variationally as:

Find $u \in H^1_0(\Omega)$, such that

$$a(u, v) = \int_{\Omega} \rho(x) \nabla u \cdot \nabla v \, dx = f(v), \quad v \in H^1_0(\Omega).$$

(2.1)

As usual, $H^1(\Omega)$ is the space of square summable functions with square summable first derivatives, and $H^1_0(\Omega)$ its subspace of functions that vanish on $\partial \Omega$. The functional $f(\cdot)$ belongs to the dual space $H^{-1}(\Omega)$. Here $x = (x, y, z)$ denotes the position vector.

The coefficient $\rho(x) > 0$ can be discontinuous, with very different values for different subregions, but we allow it to vary only moderately within each subregion. We will in fact assume that the region is the union of elements (also called subdomains, substructures, or macroelements) $\{\Omega_i\}$. Without decreasing the generality of our results, we will only consider the piecewise constant case:

$$\rho(x) = \rho_i, \quad x \in \Omega_i.$$

In the case of a region of diameter $H_i$, such as the substructure $\Omega_i$, we use a norm with
different relative weights obtained by a simple dilation argument:
\[ \| u \|^2_{1, \Omega} = |u|^2_{1, \Omega} + \frac{1}{H^2} \| u \|^2_{0, \Omega_b}. \]  
(2.2)

Here, \( \| \cdot \|_{0, \Omega} \) and \( | \cdot |_{1, \Omega} \) denote the norm in \( L^2(\Omega) \) and the seminorm in \( H^1(\Omega) \), respectively. In the following we also employ the space \( W^{1, \infty}(\Omega) \) of bounded functions with bounded derivatives (see e.g. Nečas, 1967).

3. hp finite-element approximations

We now specify a particular choice of finite-element spaces. Given an affine quadrilateral mesh \( \mathcal{T} \) of \( \Omega \) and a polynomial degree \( k \geq 1 \), we consider the following finite-element spaces:

\[ X = X^k(\Omega; \mathcal{T}) := \{ u \in H^1_0(\Omega) \mid u|_K \in Q_k(K), \ K \in \mathcal{T} \}. \]  
(3.1)

Here \( Q_k(K) \) is the space of polynomials of maximum degree \( k \) in each variable on \( K \). In the following, we may drop the reference to \( k, \Omega, \) and/or \( \mathcal{T} \) whenever there is no confusion.

In this paper, we always assume that the meshes are regular, i.e. the intersection between neighbouring elements is either a vertex, or an edge, or a face that is common to the two elements.

A finite-element approximation of (2.1) consists of finding \( u \in X \), such that
\[ a(u, v) = f(v), \ v \in X. \]  
(3.2)

4. Geometric boundary layer meshes

In order to resolve boundary layers and/or singularities, geometrically graded meshes can be employed. They are determined by a mesh grading factor \( \sigma \in (0,1) \) and a refinement level \( n \geq 0 \). The number of layers is \( n+1 \) and the thinnest layer has a width proportional to \( \sigma^n \). Robust exponential convergence of \( hp \) finite-element approximations is achieved if \( n \) is suitably chosen. For singularity resolution, \( n \) is required to be proportional to the polynomial degree \( k \) (see Andersson et al., 1995; Babuška & Guo, 1996). For boundary layers, the width of the thinnest layer needs to be comparable to that of the boundary layer (see Melenk & Schwab, 1998; Schwab & Suri, 1996; Schwab et al., 1998).

A geometric boundary layer mesh \( \mathcal{T} = \mathcal{T}^{n, \sigma}_{bl} \) is, roughly speaking, the tensor product of meshes that are geometrically refined towards the faces. Figure 1 shows the construction of a geometric boundary layer mesh \( \mathcal{T}^{n, \sigma}_{bl} \).

The mesh \( \mathcal{T}^{n, \sigma}_{bl} \) is built by first considering an initial shape-regular macro-triangulation \( \mathcal{T}_m \), possibly consisting of just one element, which is successively refined. This process is illustrated in Fig. 1. Every macroelement can be refined isotropically (not shown) or anisotropically in order to obtain so-called face, edge or corner patches (Fig. 1, level 2). Here and in the following, we only consider patches obtained by triangulating the reference cube \( \hat{Q} := I^3 \), with \( I := (-1,1) \). A patch for an element \( K_m \in \mathcal{T}_m \) is obtained by using an affine mapping \( F_{K_m} : \hat{Q} \rightarrow K_m \). The stability properties proven for patches on the
reference cube are equally valid for an arbitrary shape-regular element $K_m \in T_m$, with a constant that is independent of the diameter of $K_m$.

A face patch is given by an anisotropic triangulation of the form

\[ T_f := \{ K_x \times I \times I \mid K_x \in T_x \}, \quad (4.1) \]

where $T_x$ is a mesh of $I$, geometrically refined towards, say, $x = 1$, with grading factor $\sigma \in (0, 1)$ and $n$ levels of refinement; see Fig. 1 (level 2, left). We note that the mesh $T_x \times \{ I \}$ of $\hat{S} := I^2$ is a two-dimensional edge patch.

An edge patch is given by a triangulation

\[ T_e = T_{ebl} := \{ K_x \times K_y \times I \mid K_x \in T_x, \ K_y \in T_y \} = \{ K_{xy} \times I \mid K_{xy} \in T_{xy} \}, \quad (4.2) \]

where $T_x$ and $T_y$ are meshes of $I$, geometrically refined towards, say, $x = 1$ and $y = 1$, respectively, with grading factor $\sigma \in (0, 1)$ and total number of layers $n$; see Fig. 1 (level 2, centre). The mesh $T_{xy}$ of $\hat{S}$ is a two-dimensional corner patch.

In a similar way, we can define a corner patch $T_c$:

\[ T_c = T_{cbl} := \{ K_x \times K_y \times K_z \mid K_x \in T_x, \ K_y \in T_y, \ K_z \in T_z \}, \]

where $T_x$, $T_y$, and $T_z$ are meshes of $I$, geometrically refined towards, say, $x = 1$, $y = 1$, and $z = 1$, respectively; see Fig. 1 (level 2, right).

We note that every element $\hat{K}$ of $T_f$, $T_e$, and $T_c$ on the reference cube is of the form $(0, h_x) \times (0, h_y) \times (0, h_z)$ (after a possible translation and rotation) and is thus obtained from the reference element by an affine mapping $F_{\hat{K}} : \hat{Q} \to \hat{K}$ of the form

\[ \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T = \begin{bmatrix} (h_x/2)(\hat{x} + 1) \\ (h_y/2)(\hat{y} + 1) \\ (h_z/2)(\hat{z} + 1) \end{bmatrix}^T. \quad (4.3) \]

The aspect ratio of $\hat{K}$ is the maximum of all possible ratios of $h_x$, $h_y$, and $h_z$. Since the macromesh consists of affinely mapped elements $K_m$, every element $K$ of the global mesh $T = T_m^{n,\sigma}$ is obtained from the reference element by combining two affine mappings

\[ K = F_K(\hat{Q}) = F_{K_m}(F_{\hat{K}}(\hat{Q})), \quad K \subset K_m \in T_m. \quad (4.4) \]
Since $T_m$ is shape-regular, the aspect ratio is determined only by $F_{K}^{l}$; cf. (4.3). Finally, we note that the aspect ratio of the mesh is determined by $\sigma$ and $n$, and is proportional to $\sigma^{-n}$.

As in Toselli & Vasseur (2003a), our analysis will be made for a prototype mesh, obtained from a shape-regular (not necessarily quasi-uniform) macromesh, by refining elements that only touch $\partial \Omega$, either as corner, edge, or face patches. Such meshes only consist of four types of patches: unrefined, face, edge, and corner patches. We also recall that in practical applications $\sigma$ is bounded away from one and zero.

5. Basis functions on Gauss–Lobatto nodes

For the space $X^{k}(\Omega; T)$, we choose nodal basis functions on the Gauss–Lobatto nodes. We denote by $GLL(k)$ the set of Gauss–Lobatto points $[\xi_{i}; 0 \leq i \leq k]$ on $I = (−1, 1)$ in increasing order and by $[w_{l} > 0]$ the corresponding weights (see Bernardi & Maday, 1997, Section 4). We recall that the quadrature formula based on $GLL(k)$ has order $2k − 1$ and, in addition,

$$
\|u\|_{0, I}^{2} \leq \sum_{l=0}^{k} u(\xi_{l})^2 \cdot w_{l} \leq 3 \|u\|_{0, I}^{2}, \quad u \in \mathbb{Q}_{k}(I);
$$

(see Bernardi & Maday, 1997, Remark 13.3).

For the reference cube $\hat{Q} = (−1, 1)^{3}$ we set $GLL(k)^{3} = \{\xi_{ijl} = (\xi_{i}, \xi_{j}, \xi_{l}); 0 \leq i, j, l \leq k\}$. In the following, we use the same notation for the mapped Gauss–Lobatto nodes and corresponding weights for an affinely mapped element $K \in T$.

Given the nodes $GLL(k)^{3}$, our basis functions on $\mathbb{Q}_{k}(\hat{Q})$ are the tensor product of $k$th-order Lagrange interpolating polynomials on $GLL(k)$, defined by

$$
\hat{I}_{i}(\xi_{j}) = \delta_{ij},
$$

(5.2)

On the reference element we can write

$$
u(x, y, z) = \sum_{i=0}^{k} \sum_{j=0}^{k} \sum_{l=0}^{k} u(\xi_{i}, \xi_{j}, \xi_{l}) \hat{I}_{i}(x)\hat{I}_{j}(y)\hat{I}_{l}(z), \quad u \in \mathbb{Q}_{k}(\hat{Q}).
$$

(5.3)

For a general element in $T$, basis functions are obtained by mapping those on the reference element. Interior local basis functions correspond to GLL nodes inside $Q$ (all local indices differ from 0 and $k$).

Equation (5.3) defines an interpolation operator $I^{k}$ on the reference element

$$
I^{k}u(x, y, z) := \sum_{i=0}^{k} \sum_{j=0}^{k} \sum_{l=0}^{k} u(\xi_{i}, \xi_{j}, \xi_{l}) \hat{I}_{i}(x)\hat{I}_{j}(y)\hat{I}_{l}(z).
$$

The points $GLL(k)^{3}$ define a triangulation $T_{k} = T_{k}(\hat{Q})$ of $\hat{Q}$ in a natural way, consisting of $k^{3}$ parallelepipeds. Let $Y^{h} = Y^{h}(\hat{Q}) = X^{1}(\hat{Q}, T_{k})$ be the space of piecewise trilinear functions on this mesh. We also denote $Y^{k} = Y^{k}(\hat{Q}) = \mathbb{Q}_{k}(\hat{Q})$. The aspect ratio of $T_{k}$ is of the order of $k$ (see Casarin, 1996, p. 27 for details). In a similar way we can consider a Gauss–Lobatto mesh on an affinely mapped element $K$ by simply mapping the
GLL mesh on \( \hat{Q} \). In the following, we will use the notation \( T_k = T_k(K) \), \( Y^h = Y^h(K) \) and \( Y^k = Y^k(K) \) to denote the GLL mesh, the piecewise trilinear finite-element space and \( \mathbb{Q}_k \), respectively, for a mapped element. If the aspect ratio of \( K \) is e.g. \( h_x/h_y \) (cf. (4.3) and (4.4)), then that of the corresponding \( T_k \) is \((h_x/h_y)k\).

There is a one-to-one correspondence between \( Y^h \) and \( Y^k \) given by

\[
I^k : Y^h \rightarrow Y^k, \quad I^h : Y^k \rightarrow Y^h,
\]

where \( I^h \) is the nodal interpolation operator on \( Y^h \). We use the notation \( u_h \in Y^h \) and \( u_k \in Y^k \) in order to denote two corresponding functions.

**Lemma 5.1** Let \( \hat{K} = (0, h_x) \times (0, h_y) \times (0, h_z) \). Then there exist positive constants \( c \) and \( C \), such that, for \( u_h \in Y^h(\hat{K}) \),

\[
c \| u_h \|_{0, \hat{K}} \leq \| u_k \|_{0, \hat{K}} \leq C \| u_h \|_{0, \hat{K}},
\]

\[
c \| \partial_k (u_h) \|_{0, \hat{K}} \leq \| \partial_k (u_k) \|_{0, \hat{K}} \leq C \| \partial_k (u_h) \|_{0, \hat{K}},
\]

with, in particular, \( c \) and \( C \) independent of \( h_x, h_y, h_z \), and \( k \). Similar bounds hold for the \( y \) and \( z \) derivatives. If \( K \in T \) is given by (4.4), then, for \( u_h \in Y^h(K) \),

\[
c \| u_h \|_{0, K} \leq \| u_k \|_{0, K} \leq C \| u_h \|_{0, K},
\]

\[
c \| u_h \|_{1, K} \leq \| u_k \|_{1, K} \leq C \| u_h \|_{1, K},
\]

where the constants are independent of the diameter and the aspect ratio of \( K \), and \( k \).

The proof of the above result can be found in Canuto (1994, Section 2) for \( K = \hat{Q} \). For an affinely mapped element a scaling argument can be used. We note that thanks to Lemma 5.1 we can equivalently work with functions in \( Y^k \) or \( Y^h \).

The following result can be found in Casarin (1996, Lemma 3.3.3).

**Lemma 5.2** Let \( \hat{K} = (0, h_x) \times (0, h_y) \times (0, h_z) \) and \( u_h \in Y^h(\hat{K}) \). Given \( \theta \in W^{1, \infty}(\hat{K}) \), with

\[
\| \theta \|_{\infty, \hat{K}} \leq C, \quad \| \nabla \theta \|_{\infty, \hat{K}} \leq C/r,
\]

then

\[
\| I^h(\theta u_h) \|_{0, \hat{K}}^2 \leq C \| u_h \|_{0, \hat{K}}^2, \\
\| \partial_k I^h(\theta u_h) \|_{0, \hat{K}}^2 \leq C(\| u_h \|_{1, \hat{K}}^2 + r^{-2} \| u_h \|_{0, \hat{K}}^2),
\]

where \( C \) is independent of \( h_x, h_y, h_z \), and \( k \). Similar bounds hold for the \( y \) and \( z \) derivatives. If \( K \in T \) is given by (4.4), then, for \( u_h \in Y^h(K) \),

\[
\| I^h(\theta u_h) \|_{0, K}^2 \leq C \| u_h \|_{0, K}^2, \\
\| I^h(\theta u_h) \|_{1, K}^2 \leq C(\| u_h \|_{1, K}^2 + r^{-2} \| u_h \|_{0, K}^2),
\]

where \( C \) is independent of the diameter and the aspect ratio of \( K \), and \( k \).
Given an element \( \hat{K} = (0, h_x) \times (0, h_y) \times (0, h_z) \) and a coordinate direction, say \( x \), let \( a, b, c \) and \( d \) be the vertices of a face of \( \hat{K} \) perpendicular to this direction, and let \( a', b', c' \) and \( d' \) be the corresponding points on the parallel face. The following lemma relies on trivial properties of trilinear functions (cf. Casarin, 1996, Lemma 3.3.1).

**Lemma 5.3** Let \( \hat{K} = (0, h_x) \times (0, h_y) \times (0, h_z) \) and \( a, b, c \) and \( d \) be the vertices of a face of \( \hat{K} \) perpendicular to the \( x \) direction. Then there are constants independent of \( h_x, h_y \) and \( h_z \), such that, if \( u \) is trilinear on \( \hat{K} \),

\[
\begin{align*}
&c \| u \|^2_{0, \hat{K}} \leq h_x h_y h_z \sum_{x=a,b,c,d} (u(x))^2 + (u(x'))^2 \leq C \| u \|^2_{0, \hat{K}}, \\
&c \| \partial_x u \|^2_{0, \hat{K}} \leq (h_x h_y h_z/h^2) \sum_{x=a,b,c,d} (u(x) - u(x'))^2 \leq C \| \partial_x u \|^2_{0, \hat{K}}, \\
&c \| \partial_x u \|^2_{\infty, \hat{K}} \leq h_x^{-2} \sum_{x=a,b,c,d} (u(x) - u(x'))^2 \leq C \| \partial_x u \|^2_{\infty, \hat{K}}.
\end{align*}
\]

Similar bounds hold for the \( y \) and \( z \) derivatives.

### 6. Neumann–Neumann methods

Iterative substructuring methods rely on a non-overlapping partition into substructures. We mention Smith et al. (1996, Chapter 4) as a general reference to this section. In our algorithms the substructures are chosen as the macroelements in \( T_m = \{ \Omega_i | 1 \leq i \leq N \} \). We recall that the macroelements are shape-regular. This appears to be essential for the analysis and good performance.

We define the boundaries \( \Gamma_i = \partial \Omega_i \setminus \partial \Omega \) and the interface \( \Gamma' \) as their union. We remark that \( \Gamma' \) is the union of the interior subdomain faces, regarded as open sets, which are shared by two subregions, and subdomain edges and vertices, which are shared by more than two subregions. Vertices can only be endpoints of edges. In the following, we tacitly assume that points on \( \partial \Omega \) are excluded from the geometrical objects that we consider, or, in other words, we will only deal with geometrical objects (faces, edges, vertices, . . . ) that belong to \( \Gamma' \). We denote the faces of \( \Omega_i \) by \( F^i \), its edges by \( E^i \), its vertices by \( V^i \), and its wirebasket, defined as the union of its edges and vertices, by \( W^i \). Occasionally, we will also use faces, edges and vertices with one or no superscript. If a vertex (edge) lies on \( \partial \Omega \) we regard it as part of the internal edge (resp., face) that shares it with \( \partial \Omega \).

When restricted to the subdomain \( \Omega_i \), the global triangulation \( T \) determines a local mesh \( T_i \). This mesh can be of four types: face, edge, corner or consisting of just one element. We define the local spaces \( X_i = X^k(\Omega_i; T_i) \), of local finite-element functions that vanish on \( \partial \Omega \cap \partial \Omega_i \).

In our analysis, we will also employ the GLL mesh \( T_h(\Omega_i) \) on \( \Omega_i \), generated by the local GLL meshes \( T_h(K) \) for \( K \in T_i \). The corresponding space of piecewise trilinear functions on \( T_h(\Omega_i) \) that vanish on \( \partial \Omega \cap \partial \Omega_i \) is denoted by \( Y_h(\Omega_i) \). We set \( Y_h(\Omega_i) = X_h(\Omega_i; T_i) \).

We next define the local bilinear forms

\[
a_i(u, v) = \int_{\Omega_h} \rho_i \nabla u \cdot \nabla v \, dx, \quad u, v \in X_i.
\]

We note that if \( \Omega_i \) is a floating subdomain (i.e. its boundary does not touch \( \partial \Omega \)), \( a_i(\cdot, \cdot) \) is
only positive semi-definite and for \( u \in X \) we have
\[
a_i(u, u) = 0 \quad \text{iff} \quad u \text{ constant in } \Omega_i.
\]

The sets of nodal points on \( \Gamma_i, \Gamma', F^{ij}, E^{ij} \) and \( W^i \) are denoted by \( \Gamma_{i,h}, \Gamma_h, F^{ij}_h, E^{ij}_h \) and \( W^i_h \), respectively. We will identify these sets with the corresponding sets of degrees of freedom. As for the corresponding regions, we will also use notation with one or no superscript.

We introduce some spaces defined on the interfaces: \( U_i \) is the space of restrictions to \( \Gamma_i \) of functions in \( X_k(\Omega_i; T_i) \) and \( U \) of restrictions to \( \Gamma \) of functions in \( X_k(\Omega; T) \). We note that functions in \( U_i \) and \( U \) are uniquely determined by the nodal values in \( \Gamma_i, h \) and \( \Gamma_h \), respectively. In the following we will identify these spaces with those of the corresponding harmonic extensions; see in particular Lemma 6.1 below. For every substructure \( \Omega_i \), there is a natural interpolation operator
\[
R_i^T : U_i \longrightarrow U
\]
that extends a function on \( \Gamma_i \) to a global function on \( \Gamma \) with vanishing degrees of freedom in \( \Gamma_h \backslash \Gamma_i, h \). Its transpose with respect to the Euclidean scalar product \( R_i : U \rightarrow U_i \) extracts the degrees of freedom in \( \Gamma_i, h \).

Once a vector \( u \in X_k(\Omega; T) \) is expanded using the basis functions introduced in Section 5, problem (3.2) can be written as a linear system
\[
Au = f.
\]
We recall that the condition number of \( A \) is expected to grow at least as \( k^3/(h_{\min})^2 \sim k^3 \sigma^{-2n} \sim k^3 \sigma^{-2k} \) (see Melenk, 2002 for a result in two dimensions) and may thus be extremely large for large values of \( k \).

The contributions to the stiffness matrix and the right-hand side can be formed one subdomain at a time. The stiffness matrix is then obtained by subassembly of these parts. We will order the nodal points interior to the subdomains first, followed by those on the interface \( \Gamma' \). Similarly, for the stiffness matrix relative to a substructure \( \Omega_i \), we have
\[
A^{(i)} = \begin{pmatrix}
A^{(i)}_{II} & A^{(i)}_{I \Gamma} \\
A^{(i)}_{I I} & A^{(i)}_{\Gamma \Gamma}
\end{pmatrix}.
\]  
(6.1)

In a first step of many iterative substructuring algorithms, the unknowns in the interior of the subdomains are eliminated by block Gaussian elimination. In this step, the Schur complements, with respect to the variables associated with the boundaries of the individual substructures, are calculated. The resulting linear system can be written as
\[
Su_{\Gamma} = g_{\Gamma}.
\]  
(6.2)

Given the local Schur complements
\[
S_i = A^{(i)}_{I \Gamma} - A^{(i)}_{I I} A^{(i)}_{I I}^{-1} A^{(i)}_{I \Gamma} : U_i \longrightarrow U_i.
\]
we have

\[ S = \sum_{i=1}^{N} R_i^T S_i R_i : U \rightarrow U \]

and an analogous formula can be found for \( g \) (see Smith et al., 1996, Chapter 4).

A function \( u^{(i)} \) defined on \( \Omega_i \) is said to be discrete harmonic on \( \Omega_i \) if

\[ A^{(i)} u^{(i)} + A^{(i)}_T u^{(i)} = 0. \]

In this case, it is easy to see that \( \mathcal{H}_i(u^{(i)}) := u^{(i)} \) is completely defined by its value on \( \Gamma_i \).

The space of piecewise discrete harmonic functions \( u \) consists of functions in \( X \) that are discrete harmonic on each substructure. In this case, \( u =: \mathcal{H}(u\Gamma) \) is completely defined by its value on \( \Gamma \).

Our preconditioners will be defined with respect to the inner product

\[ s(u, v) = u^T S v, \quad u, v \in U. \]

It follows immediately from the definition of \( S \) that \( s(\cdot, \cdot) \) is symmetric and coercive.

The following lemma results from elementary variational arguments.

**Lemma 6.1** Let \( u^{(i)}_\Gamma \) be the restriction of a finite-element function to \( \Gamma_i \). Then the discrete harmonic extension \( u^{(i)} = \mathcal{H}_i(u^{(i)}_\Gamma) \) of \( u^{(i)}_\Gamma \) into \( \Omega_i \) satisfies

\[ a_i(u^{(i)}, u^{(i)}) = \min_{v^{(i)}|_{\partial \Omega_i} = u^{(i)}_\Gamma} a_i(v^{(i)}, v^{(i)}) = u^{(i)\Gamma}_T S^{(i)} u^{(i)}_\Gamma. \]

Analogously, if \( u\Gamma \) is the restriction of a finite-element function to \( \Gamma \), the piecewise discrete harmonic extension \( u = \mathcal{H}(u\Gamma) \) of \( u\Gamma \) into the interior of the subdomains satisfies

\[ a(u, u) = \min_{v|_{\Gamma} = u\Gamma} a(v, v) = s(u, u) = u^T S u\Gamma. \]

This lemma ensures that instead of working with functions defined on the interface \( \Gamma \), we can equivalently work with the corresponding discrete harmonic extensions. For this reason, in the following we will identify spaces of traces on the interfaces, \( U_i \) and \( U \), with spaces of discrete harmonic extensions. We point out, however, that due to the particular meshes considered, we cannot equivalently work with norms of local discrete harmonic extensions and traces on the subdomain boundaries since our local meshes are not in general quasi-uniform or shape-regular, and stable discrete harmonic extensions cannot be found in general; see Section 8.

Neumann–Neumann methods provide preconditioners for the Schur complement system: instead of solving (6.2) using, e.g., the conjugate gradient method, they employ an equivalent system involving a preconditioned operator of the form

\[ \tilde{S}^{-1} S = P_{NN} = P_0 + \left( I - P_0 \left( \sum_{i=1}^{N} P_i \right) (I - P_0) \right). \]
We refer to Dryja & Widlund (1995), Mandel & Brezina (1996), Pavarino (1997) and Klawonn & Widlund (2001) for some NN methods for the $h$ and $p$ finite-element approximations. We are unaware on any such method for $hp$-approximations.

The operators $P_i$ are projection-like operators associated to a family of subspaces $U_i$ and determined by a set of local bilinear forms defined on them:

$$\tilde{s}_i(u, v), \quad u, v \in U_i.$$ 

Given the interpolation operators $R^T_i : U_i \to U$, we have

$$P_i = R^T_i \tilde{P}_i, \quad \tilde{P}_i : U \to U_i, \quad (6.3)$$

with

$$\tilde{s}_i(\tilde{P}_i u, v_i) = s(u, R^T_i v_i), \quad v_i \in U_i. \quad (6.4)$$

While $P_0$ is associated with a low-dimensional global problem, the others are associated with the single substructures. The remainder of this section is devoted to the definition of the various components of $P_{NN}$.

An important role is played by a family of weighted counting functions $\delta_i$, which are associated with and defined on the individual $\Gamma_i$ (cf. Dryja et al., 1996; Dryja & Widlund, 1995; Mandel & Brezina, 1996; Sarkis, 1994; Pavarino, 1997) and are defined for $\gamma \in [1/2, \infty)$. Given $\Omega_i$ and $x \in \Gamma_i$, $\delta_i(x)$ is determined by a sum of contributions from $\Omega_i$ and its relevant next neighbours,

$$\delta_i(x) = \sum_{j \in N_x} \rho^\gamma_j(x)/\rho^\gamma_i(x), \quad x \in \Gamma_i, \quad (6.5)$$

Here $N_x$, $x \in \Gamma_h$, is the set of indices $j$ of the subregions such that $x \in \Gamma_j$. These nodal values on $\Gamma_i$ are then interpolated in order to obtain a function of $\delta_i \in U_i$. The pseudoinverses $\delta^\dagger_i \in U_i$ are defined, for $x \in \Gamma_i$, by

$$\delta^\dagger_i(x) = \delta_i^{-1}(x), \quad x \in \Gamma_i. \quad (6.6)$$

We note that these functions provide a partition of unity:

$$\sum_{i=1}^N R^T_i \delta^\dagger_i(x) \equiv 1. \quad (6.7)$$

In particular, for $u \in U$ we can use the formula

$$u = \sum_{i=1}^N R^T_i u_i, \quad \text{with } u_i = H_i(\delta^\dagger_i u). \quad (6.8)$$

Here and from now on, we will tacitly assume that whenever we write $H_i(uv)$ or $H(uv)$ we first form $I^h(uv)$, i.e. map the product of the two functions $u$ and $v$ into the $hp$ finite-element space by interpolation, and then extend the result as a discrete harmonic function.
If there is no confusion, we will sometimes use the notation $u\, v$ in order to denote $I^k(\mathcal{H}_i(\mathcal{H}_i(u), \mathcal{H}_i(\mathcal{H}_i(v))))$ or $\mathcal{H}_i(\mathcal{H}_i(u), \mathcal{H}_i(\mathcal{H}_i(v)))$.

A coarse space $U_0$ of minimal dimension is defined as

$$U_0 = \text{span}\{R_T^i \delta_i^i\} \subset U,$$

where the span is taken over the floating subdomains. We note that $U_0$ consists of piecewise discrete harmonic functions and $R_T^i$ is the natural injection $U_0 \subset U$. We consider an exact solver on $U_0$

$$\tilde{s}_0(u, v) := a(\mathcal{H}u, \mathcal{H}v) = a(u, v).$$

For each substructure $\Omega_i$, the local bilinear form is

$$\tilde{s}_i(u, v) := a_i(\mathcal{H}_i(\delta_i u), \mathcal{H}_i(\delta_i v)), \quad u, v \in U_i.$$

For a floating subdomain $\tilde{P}_i$ is defined only for those $u \in U$ for which $s(u, v) = 0$ for all $v = R_T^i v_i$ such that $\mathcal{H}_i(\delta_i v_i)$ is constant on $\Omega_i$. This condition is satisfied if $a(u, R_T^i \delta_i^i) = 0$; we note that $R_T^i \delta_i^i$ is a basis function for $U_0$. For such subdomains, we make the solution $\tilde{P}_i u$ of (6.4) unique by imposing the constraint

$$\int_{\Omega_i} \mathcal{H}_i(\delta_i \tilde{P}_i u) \, dx = 0,$$

(6.9)

which just means that we select the solution orthogonal to the null space of the Neumann operator. Thus, Range$(\tilde{P}_i)$ has codimension 1 with respect to the space $U_i$.

We can equally well use matrix notations. Let $D_i$ be the diagonal matrix with the elements $\delta_i^i(x)$ corresponding to the point $x \in \Gamma_{i,h}$. Then

$$\tilde{s}_i(u, v) = u^T D_i^{-1} S_i D_i^{-1} v.$$

We also have

$$P_i = R_T^i D_i S_i^1 D_i R_i S,$$

where $S_i^1$ is a pseudoinverse of $S_i$. Analogously for the coarse projection,

$$P_0 = R_T^0 S_0^{-1} R_0 S,$$

where $S_0 = R_0 S R_T^0$ the restriction of $S$ to $U_0$.

The main result of this paper is a bound for the condition number of $P_{NN}$. Such bound can be found using the abstract Schwarz theory (see e.g. Smith et al., 1996, Chapter. 6). We refer to Mandel & Brezina (1996), Dryja & Widlund (1995), Pavarino (1997) and Klawonn & Widlund (2001) for similar proofs.

A uniform bound for the smallest eigenvalue can be found using the decomposition (6.8) and the fact that $P_0$ is an orthogonal projection.

**Lemma 6.2** We have

$$s(P_{NN} u, u) \geq s(u, u), \quad u \in U.$$
In order to find a bound for the largest eigenvalue, we need a stability property for the local bilinear forms (see e.g. Smith et al., 1996).

**Assumption 6.1** We have

\[ s(R_i^T u_i, R_i^T u_i) \leq \omega \tilde{s}_i(u_i, u_i), \quad u_i \in \text{Range}(\tilde{P}_i), \quad i = 1, \ldots, N, \]

with

\[ \omega = C (1 - \sigma)^{-6} \left( 1 + \log \left( \frac{k}{1 - \sigma} \right) \right)^2 \]

and \( C \) independent of \( k, n, \sigma, \gamma \), the coefficients \( \rho_i \) and the diameters \( H_i \).

The proof of Assumption 6.1 is given in Section 9. Assumption 6.1 and a colouring argument provide a bound for the largest eigenvalue (see e.g. Pavarino, 1997, Section 8).

**Lemma 6.3** Let Assumption 6.1 be satisfied. Then

\[ s(P_{NN} u, u) \leq C \omega s(u, u), \quad u \in U. \]

Consequently, the condition number of \( P_{NN} \) satisfies

\[ \kappa(P_{NN}) \leq C \omega = C (1 - \sigma)^{-6} \left( 1 + \log \left( \frac{k}{1 - \sigma} \right) \right)^2. \]

### 7. Decomposition results

A key ingredient for the proof of Assumption 6.1 and for the analysis of many iterative substructuring methods in three dimensions is a decomposition result for local functions in \( U_i \) into face and wirebasket components:

\[ u = \sum_j u_{Fij} + u_{W_i}, \quad u \in U_i. \tag{7.1} \]

The face component \( u_{Fij} \) vanishes on \( \partial \Omega_i \setminus F_i \) and is discrete harmonic. It is uniquely determined by the nodal values in \( F_i \). The wirebasket component \( u_{W_i} \) is also discrete harmonic and vanishes at all points of \( \Gamma_i \) except at those in \( W_i \).

We can further decompose a local function by also defining edge and vertex components:

\[ u = \sum_j u_{Fij} + \sum_j u_{Eij} + \sum_j u_{Vij}, \quad u \in U_i, \tag{7.2} \]

where \( u_{Eij} \) is discrete harmonic and vanishes on \( \partial \Omega_i \setminus E_i \), and \( u_{Vij} \) vanishes at all nodes in \( \Gamma_i \) except at the vertex \( V_i \). We recall that we exclude geometrical objects on \( \partial \Omega \) and that therefore the sums in (7.1) and (7.2) are taken over faces, edges and vertices that do not belong to \( \partial \Omega \). Discrete harmonic functions of type \( u_{Fij}, u_{Eij}, u_{Vij} \) and \( u_{W_i} \) are called face, edge, vertex and wirebasket functions, respectively.
Here and in the following section, we only carry out proofs for the reference cube \( \hat{Q} \): since elements in the macromesh \( T_m \) are shape-regular and affinely mapped, the corresponding bounds for a generic substructure \( \Omega_l \in T_m \), of diameter \( H_l \) can be obtained by a standard scaling argument and involve the scaled norm (2.2). We recall that we only need to consider four types of patches: face, edge, corner and unrefined ones, together with the corresponding triangulations \( T_f, T_e, T_c \) and \( \hat{Q} \), respectively; cf. Fig. 1. We recall that a generic patch is denoted by \( \Omega_i \) and its triangulation by \( T_i \).

### 7.1 Wirebasket components

Given an edge \( E = E^{ij} \subset W^i \), we define a discrete \( L^2 \) norm on \( E \). If \( E \) does not touch the boundary \( \partial \Omega \), we simply set

\[
\| u \|_{h,E} := \| u \|_{0,E}.
\]

Let now \( E \) be an edge that touches \( \partial \Omega \); see Fig. 2, left, for an example of a face patch. After a possible translation and rotation, \( E \) can always be written as

\[
E = \{(1,1,z) \mid z \in I\}.
\]

Then, the local mesh \( T_l \) gives rise to a one-dimensional triangulation on \( E \), \( T_E \), which is not quasiuniform and is geometrically refined towards one end point, say \( z = 1 \). In addition, \( E \) can be partitioned as

\[
E = E_{1-\sigma} \cup E_\sigma,
\]

\[
E_{1-\sigma} = (-1,-1+2(1-\sigma)), \quad E_\sigma = (-1+2(1-\sigma),1).
\]

We note that \( E_{1-\sigma} \) consists of exactly one element of length \( 2(1-\sigma) \) in \( T_E \), while the elements on \( E_\sigma \) are geometrically refined towards \( z = 1 \). We now consider the GLL mesh \( T_k(\Omega_l) \) and observe that all the elements that touch the edge \( E \) have the same diameters \( h_{1,x} \) and \( h_{1,y} \), along the two directions perpendicular to \( E \); cf. Fig. 2. Indeed, \( h_{1,x} \) and \( h_{1,y} \) are of order \( k^{-2} \) for a face patch, of order \( k^{-2}(1-\sigma) \) for a corner patch and of order \( k^{-2} \)
and $k^{-2}(1 - \sigma)$, respectively, for an edge patch. Moreover, thanks to our particular meshes and to the fact that local spaces of the same degree $k$ are employed on each element, we have the following property.

**PROPERTY 7.1** Let $E$ be an edge parallel to e.g. $z$, that is shared by two substructures $\Omega_i$ and $\Omega_j$. Then, the mesh sizes $h_{i,x}$ and $h_{j,x}$, and $h_{i,y}$ and $h_{j,y}$ are comparable. In particular, there exist constants, depending only on the aspect ratios of $\Omega_i$ and $\Omega_j$, such that

$$c(1 - \sigma)h_{i,x} \leq h_{j,x} \leq C(1 - \sigma)^{-1}h_{i,x}.$$  

Similar bounds hold for $h_{i,y}$ and $h_{j,y}$.

We define

$$\|u\|_{h, E}^2 := \|u\|_{0,E}^2 + \|u\|_{h,E_\sigma}^2 = \|u\|_{0,E}^2 + h_{i,x}h_{j,y}\|\partial_z u\|_{0,E_\sigma}^2.$$  

We note that in this case the discrete norm is obtained by adding to the $L^2$ norm on $E$ a weighted $L^2$ norm of $\partial_z u$ over a part of $E$ where $T_E$ is not quasiuniform. A discrete wirebasket norm is obtained by summing the contributions over all the edges:

$$\|u\|_{h, W}^2 := \sum_{E \subset W} \|u\|_{h, E}^2.$$  

**LEMMA 7.1** Let $u_{W_i} \in U_i$ be discrete harmonic and vanish at all nodal points $\Gamma_i,h$ except at those on $W_i$. Then there is a constant independence of $u_{W_i}$, $H_i$, $\sigma$ and $n$, such that

$$|u_{W_i}|_{1,\Omega_i}^2 \leq C(1 - \sigma)^{-2}\|u_{W_i}\|_{h, W_i}^2.$$  

**Proof.** The result follows by estimating the energy norm of the zero extension of the boundary values and by noting that the harmonic extension has a smaller energy (cf. Lemma 6.1). More precisely, let $u_k$ be the function that vanishes at all nodal points in $\Omega_{i,h} \cup \Gamma_{i,h}$ except at those on $W_i$, and $u = u_h = I_h u_k$ the corresponding piecewise trilinear function defined on the GLL mesh $T_k(\Omega_i)$. We will estimate the energy of $u_h$ on each element $K \in T_k(\Omega_i)$ that touch an edge $E \subset W_i$. Without loss of generality, we assume that $E$ is parallel to the $z$ axis. We only consider the worst possible case, i.e. that of a face patch and refer to Fig. 2.

Let us first suppose that $E$ does not touch $\partial \Omega_i$. For a face patch, $K$ has dimensions $h_x$, $h_y$ and $h_z$ of order

$$k^{-2} \times k^{-2}(1 - \sigma) \times k^{-2},$$  

or

$$k^{-2} \times k^{-2}(1 - \sigma) \times k^{-1},$$  

and thus

$$c(1 - \sigma)h_x \leq h_y \leq Ch_x,$$

$$h_x \leq Ch_z;$$  

(7.3)
see Fig. 2. If $a$ and $b$ are the vertices of $K$ that lie on $E$, Lemma 5.3 yields

$$\|\partial_x u\|_{0,K}^2 \leq C (h_y h_z / h_x) (u(a)^2 + u(b)^2) \leq C \int_a^b u^2 dz,$$

where for the last inequality we have used (7.3) and standard properties of linear functions. In a similar way, we find

$$\|\partial_y u\|_{0,K}^2 \leq C (1 - \sigma)^{-1} \int_a^b u^2 dz, \quad \|\partial_z u\|_{0,K}^2 \leq C \int_a^b u^2 dz.$$

Let now $E$ be an edge that touches $\partial \Omega$ and $K \in T_k(\Omega_i)$ be an element that shares an edge with $E_{1-\sigma}$. For a face patch, $K$ has dimensions of the order

$$k^{-2} \times k^{-2} \times k^{-2} (1 - \sigma),$$

or

$$k^{-2} \times k^{-2} \times k^{-1} (1 - \sigma),$$

and thus

$$ch_x \leq h_y \leq Ch_x, \quad h_x \leq C (1 - \sigma)^{-1} h_z; \quad (7.4)$$

see Fig. 2, left. As before, Lemma 5.3 yields

$$\|\partial_x u\|_{0,K}^2 \leq C \int_a^b u^2 dz, \quad \|\partial_y u\|_{0,K}^2 \leq C \int_a^b u^2 dz, \quad \|\partial_z u\|_{0,K}^2 \leq C \int_a^b u^2 dz.$$

We are now left with the case of an element $K \in T_k(\Omega_i)$ that shares an edge with $E_{1-\sigma}$. We note that the first of (7.4) remains valid in this case. We then have

$$\|\partial_x u\|_{0,K}^2 \leq C \int_a^b u^2 dz, \quad \|\partial_y u\|_{0,K}^2 \leq C \int_a^b u^2 dz.$$

For $\partial_z u$, we trivially have

$$\|\partial_z u\|_{0,K}^2 \leq C (h_x h_y / h_z) (u(a) - u(b))^2 \leq Ch_x h_y \int_a^b (\partial_z u)^2 dz.$$

The proof is concluded by summing over the elements $K \in T_k(\Omega_i)$ and using Lemma 5.1. □

We now have a bound for the wirebasket component.

**Corollary 7.2** Let $u \in U_i$ and $u_{W_i}$ be its wirebasket component. Then there is a constant independent of $u$, $H_i$, $\sigma$ and $n$ such that

$$|u_{W_i}|_{1,\Omega_i}^2 \leq C (1 - \sigma)^{-2} \|u\|_{h_{W_i}}^2.$$
A complementary result is given by the trace estimates in Lemma 7.3. We first introduce some additional notation. Let $E$ be an edge of a substructure $\Omega_i$. Without loss of generality, we assume that $\Omega_i$ coincides with the reference cube $\hat{Q}$ and that $E = \{(1, 1, z) \mid z \in I\}$. The intersection between the plane corresponding to a constant $z \in I$ and $\hat{Q}$ is the unit square $\hat{S} = (-1, 1)^2$, and the local mesh $T_i$ gives rise to a two-dimensional mesh $T(z)$ on $\hat{S}$ which is either a two-dimensional edge or corner patch, or it consists of a single element $\hat{S}$; see Fig. 2, right. Let $V = (1, 1)$ be the intersection between $E$ and the closure of $\hat{S}$. If $K_V \in T(z)$ is the two-dimensional element that contains $V$, we note that, since $E$ does not belong to $\partial \Omega_i$, $K_V$ has dimensions in $[2, 2(1 - \sigma)]$, and thus is independent of the level of refinement $n$. For a fixed $(x, y) \in \overline{K_V}$, we finally define the edge $E(x, y) = \{(x, y, z) \mid z \in I\}$.

**Lemma 7.3** Let $u_k \in X_i$ and $E$ and edge of $\Omega_i$. Then there is a constant independent of $u_k, H_i, \sigma$ and $n$ such that

$$
\|u_k\|^2_{0, E} \leq C (1 - \sigma)^{-2} (1 + \log k) \|u_k\|^2_{1, \Omega_i},
$$

$$
\|u_k\|^2_{\infty, E} \leq C (1 - \sigma)^{-2} (1 + \log k) \|u_k\|^2_{1, \Omega_i}.
$$

**Proof.** As before, it is enough to find bounds for $u = I^h u_k$. Without loss of generality, we assume $E = \{(1, 1, z) \mid z \in I\}$. We consider the two-dimensional mesh $T(z)$ on the intersection between the plane corresponding to a constant $z$ and the substructure; cf. Fig. 2, right. Since geometric refinement on $T(z)$ takes place far from the vertex $(1, 1)$, we can apply the two-dimensional result in Toselli & Vasseur (2003a, Lemma 7.6) and write

$$
|u(1, 1, z)|^2 \leq C (1 - \sigma)^{-2} (1 + \log k) \|u(\cdot, \cdot, z)\|^2_{1, \hat{S}}, \quad z \in (-1, 1),
$$

with a constant that is independent of $n, \sigma$ and $z$. Integrating over $z$ then gives

$$
\|u\|^2_{0, E} \leq C (1 - \sigma)^{-2} (1 + \log k) \|u\|^2_{1, \Omega_i},
$$

which proves the first inequality and the second one for edges that do not touch $\partial \Omega_i$.

We now bound $\|u\|_{h, E_\sigma}$ for an edge that touches the boundary $\partial \Omega_i$. We consider the one-dimensional GLL meshes for each one of the elements in $T_E$ and estimate the single contributions from the elements of these meshes. Let $e$ be one of these elements of length $h_e$ and end points $a$ and $b$. The edge $e$ belongs to a parallelepiped $K_e \in T_e(\Omega_i)$. We note that $K_e$ has dimensions $h_e = h_{i,x}, h_y = h_{i,y}$, and $h_z$. Since $u$ is linear on $e$ and trilinear on $K_e$, we have

$$
h_x h_y \int_e \partial_z u^2 dz \leq C \frac{h_x h_y}{h_z} (u(a) - u(b))^2 \leq C \|\partial_z u\|^2_{0, K_e},
$$

where, for the last inequality, we have used Lemma 5.3. Summing over the edges $e$ in $E_\sigma$ yields

$$
\|u\|^2_{h, E_\sigma} \leq C \|\partial_z u\|^2_{0, \Omega_i},
$$

which, combined with the first inequality, proves the second bound. \qed
The next lemma can be proved using the two-dimensional bound in Toselli & Vasseur (2003a, Lemma 7.6) and similar arguments as before. We note that it is only valid for edges \( E(x, y) \) that are not too far from \( E \) and thus not too close to the part of \( \Omega_i \) where anisotropic refinement takes place.

**Lemma 7.4** Let \( E \) be an edge of a substructure \( \Omega_i \) which is parallel, say, to \( z \) and intersects the plane corresponding to a constant \( z \) in \( V \). Let in addition \( K_V \) be the element in the two-dimensional mesh \( T(z) \) that contains \( V \). Then, for every \((x, y) \in K_V \) and \( u_k \in X_i \),

\[
\|u_k\|_{h,E(x,y)}^2 \leq C (1 - \sigma)^{-2} (1 + \log k) \|u_k\|_{1,\Omega_i}^2,
\]

where \( C \) is independent of \( u_k, \sigma, n, k \), but depends only on the aspect ratio of \( \Omega_i \).

**Proof.** The proof can be carried out as in the previous lemma by using the two-dimensional result in Toselli & Vasseur (2003a, Lemma 7.6). Indeed, since the point \((x, y) \in K_V \) and is thus far from the region where anisotropic refinement takes place, we have

\[
|u(x,y,z)|^2 \leq C (1 - \sigma)^{-2} (1 + \log k) \|u(\cdot, \cdot, z)\|_{1,\hat{\Omega}}^2, \quad z \in (-1, 1).
\]

Integration along \( z \) concludes the proof. \( \square \)

We end this section with a stability result for vertex and edge components. It is a direct consequence of (5.1) and of the fact that for a vertex function the modified norm \( \|\cdot\|_{h,E} \) coincides with \( \|\cdot\|_{0,E} \).

**Lemma 7.5** Let \( E \) be an edge of a substructure \( \Omega_i \) and \( V \) one of its end points. Then, for every \( u \in X_i \),

\[
\|u_V\|_{h, W^i}^2 \leq C \|u\|_{h,E}^2, \quad \|u_E\|_{h, W^i}^2 \leq C \|u\|_{h,E}^2,
\]

where \( C \) is independent of \( u, \sigma, n, k \).

### 7.2 Face components

We next consider the face contributions of the decomposition (7.1). Bounds for face contributions on the unrefined patch follow from standard results for spectral elements. For face, edge and corner patches, we employ cut-off functions \( \theta_F \) for each face and Lemma 5.2. We note that we need to consider one possible case for faces of the corner patch, and two for the edge and face patches; cf. Fig. 1. In this section we only consider the case of an edge patch \( \Omega_i \) in full detail, with the edge \((1, y, -1), y \in I \), and the two adjacent faces in common with \( \partial \Omega \); see Fig. 3. The other patches can be dealt with in a similar way.

As shown in Fig. 3 for the reference cube, the edges that do not lie on \( \partial \Omega \) are denoted by \( E^l, l = 1, \ldots, 5 \), with \( E^5 \) the edge that does not touch the boundary \( \partial \Omega \). An edge patch is further partitioned into three regions. The first step of geometric refinement partitions \( \hat{Q} \) into four parallelepipeds with dimensions in \([2, 2(1 - \sigma), 2\sigma]\). Let \( K_{\Omega} \) be the one that
Fig. 3. Edge patch on the reference cube \((-1, 1)^3\) employed in the proofs of Lemmas 7.6 and 7.7.

contains the boundary edge and \(K_{int}\), the one that does not touch \(\partial \Omega\) and contains the inner edge \(E^5\). The two remaining parallelepipeds are denoted by \(K^{12}\) and \(K^{34}\) and they touch the edges \(E^1\) and \(E^2\), and \(E^3\) and \(E^4\), respectively. The region \(K_{edge}\) is the union of \(K^{12}\) and \(K^{34}\), cf. Fig. 3.

The proof of the following lemma is a modification of those of Casarin (1996, Lemma 3.3.6) and Toselli & Vasseur (2003a, Lemma 7.7).

**Lemma 7.6** Given a face \(F_j\) of \(\Omega_i\) that does not lie on \(\partial \Omega\), there exists a continuous function \(\theta_{F_j}\), defined on \(\Omega_i\), that is equal to one at the nodal points of \(F_j^h\) and zero on \(\Gamma_i^h \setminus F_j^h\), such that

\[
\sum_{F_j^h \subset \Gamma_i} \theta_{F_j}(x) = 1, \quad x \in (\Omega_{i,h} \cup \Gamma_{i,h}) \setminus W_j^h, \\
0 \leq \theta_{F_j} \leq 1, \\
|\nabla \theta_{F_j}| \leq C/r, \quad \text{in} \quad \Omega_i \setminus K_{\Omega}, \\
|\nabla \theta_{F_j}| \leq C/H_i, \quad \text{in} \quad K_{\Omega},
\]

(7.7)

where \(r = r(x)\) is the distance to the closest edge of \(\Omega_i\) that does not lie on \(\partial \Omega\).

**Proof.** We only need to construct four functions and we will do that by constructing them in the three regions \(K_{int}\), \(K_{edge}\), and \(K_{\Omega}\) separately.

We start with the inner region \(K_{int}\) and employ a similar construction as in Casarin (1996, Lemma 3.3.6). We further partition \(\Omega_i\) into eight parallelepipeds by bisecting \(\{K_{int}, K^{12}, K^{34}, K_{\Omega}\}\) with the plane \(y = 0\); see Fig. 3, left. Let the centre \(C\) be the common vertex to these parallelepipeds and \(\{C^j, j = 1, \ldots, 6\}\) be their vertices that belong to the six faces of \(\Omega_i\); see Fig. 3, right. By connecting the centre \(C\) with the centres \(C^j\) and with the eight vertices of \(\Omega_i\), and, for each face, by connecting the point \(C^j\) with the four vertices of this face, we can partition \(\Omega_i\) into 24 tetrahedra; see Fig. 3, right. By intersecting them with \(K_{int}\), we obtain a partition of \(K_{int}\) into eight tetrahedra. We first define a function \(\theta_{F_j}\) associated with the face \(F_j\), defined to be 1/4 at the centre \(C\)
and \( \vartheta_{F_j} (C^l) = \delta_{jl} \) at the centres of the faces. On the segments \( CC^j \), these functions are obtained by linear interpolation of the values at \( C \) and \( C^l \); see Fig. 3, right. The values inside each subtetrahedron formed by the segment \( CC^j \) and one edge of \( F^j \) are defined to be constant on the intersection of any plane through that edge, and are given by the value on the segment \( CC^j \). We note that this procedure determines \( \vartheta_{F_j} \) at all points in \( \Omega \) except on the wirebasket \( W^j \).

We next consider the GLL triangulation \( T_k(\Omega_i) \) and interpolate \( \vartheta_{F_j} \) at the GLL nodes in \( K_{\text{int}} \setminus W^j \):

\[
\theta_{F_j}(x) = (l^h \vartheta_{F_j})(x), \quad x \in K_{\text{int}} \setminus W^j.
\]

The function \( \theta_{F_j} \) is set to zero on the nodes in \( W^j \). The functions \( \theta_{F_j} \) are non-negative and bounded by one: this proves the second of (7.7) for points in \( K_{\text{int}} \). By construction, also the first of (7.7) holds for every node in \( K_{\text{int}} \setminus W^j \). The third of (7.7) can be proven by proceeding in the same way as for Casarin (1996, Lemma 3.3.6).

We next construct the functions \( \theta_{F_j} \) in \( K_{\text{edge}} \). We start with \( K \in T_k(\Omega_i) \) and extend them as constants into \( K \cap K_{\text{int}} \) along the segments parallel to \( E^1 \) and \( E^2 \); see Fig. 3, left. The inequalities in (7.7) remain valid. We note that the function obtained is independent of \( x \) in \( K \). A similar construction is carried out in \( K \cap K_{\text{int}} \).

Finally, we construct \( \theta_{F_j} \) in \( K_{\Omega} \). We note that \( K_{\Omega} \) is divided into two parallelepipeds and that on their internal faces the function \( \theta_{F_j} \) has already been defined. In addition, \( \theta_{F_j} \) is bilinear on these faces. It is then enough to assign the value 1/4 at the end points and mid-point of the boundary edge and interpolate these values in \( K_{\Omega} \) in order to obtain a piecewise trilinear function. The first, second and fourth of (7.7) follow from standard properties of trilinear functions.

By examining the proof of the previous lemma, we see that, for an edge \( E \) that touches \( \partial \Omega \), the value of the functions \( \theta_{F_j} \) is independent of the coordinate along the direction of \( E \) in all the elements of the GLL meshes that touch \( E \); cf. Fig. 3, left.

**Property 7.2** Let \( F \) be a face of \( \Omega \) and \( E \) be an edge, parallel to say \( z \), that touches \( \partial \Omega \). In any element \( K \in T_k(\Omega_i) \) that shares an edge with \( E \) the function \( \theta_F \) is independent of \( z \).

We are now able to bound the face components in the decomposition (7.1).

**Lemma 7.7** Let \( \theta_{F_j} \) be the functions in Lemma 7.6, where \( F^j \) is a face of the substructure \( \Omega_i \), then for every \( x \in \Omega_{h,h} \cup \Gamma_{i,h} \) that is not on the wirebasket of \( \Omega_h \),

\[
\sum_j t^k(\theta_{F_j} u)(x) = \sum_j t^h(\theta_{F_j} u)(x) = u(x), \quad u \in X_i
\]

and

\[
|t^k(\theta_{F_j} u)|_{1,\Omega_i}^2 \leq C (1 - \sigma)^{-4} \left( 1 + \log \left( \frac{k}{1 - \sigma} \right) \right)^2 ||u||_{1,\Omega_i}^2.
\]
Proof. We only consider the case of an edge patch $\Omega_g$ in full detail; see Fig. 3. The proof is similar to that in Toselli & Vasseur (2003a, Lemma 7.8) and Casarin (1996, Lemma 3.3.7) but particular care is required close to the edges that touch $\partial \Omega$. Indeed, thanks to Lemma 5.1, it is enough to find a bound for the piecewise trilinear function $I_h^k(\theta_{F_j} u)$. The first equality follows directly from the first of $(7.7)$. For the second inequality, we consider an element $K$, of dimensions $h_x, h_y, h_z$, in the GLL mesh $T_k(\Omega_k)$. We consider three cases (as opposed to Casarin, 1996, Lemma 3.3.7 where only two cases are considered): $K$ may belong to the region $K_{\Omega}$ containing the boundary edge, touch the wirebasket, or may not touch it; see Fig. 3.

Case 1. We start with an element that touches an edge $E$ and does not belong to $K_{\Omega}$. We can proceed as in Casarin (1996, Lemma 3.3.7) if $E$ does not touch $\partial \Omega (E = E^3)$ or, in case it does ($E = E^l, l = 1, \ldots, 4$), if $K$ does not touch $E_\sigma$. We only consider the case of $E = E^3$ in full detail; cf. Fig. 3, left. The nodal values of $I_h^k(\theta_{F_j} u)$ on $K$ are 0, 0, 0, $u(a)$, $u(b)$, $\theta_{F_j}(c)u(c)$ and $\theta_{F_j}(d)u(d)$, with $a$ and $b$ vertices on a face and $c$ and $d$ vertices inside $\Omega_k$. It is immediate to see that
\begin{align}
c(1-\sigma)h_x & \leq h_y \leq C(1-\sigma)^{-1}h_x, \\
h_z & \leq C(1-\sigma)^{-1}h_x.
\end{align}
Using Lemma 5.3 and (7.8), we can easily find
\begin{align}
|I_h^k(\theta_{F_j} u)|_{1,K}^2 & \leq C(1-\sigma)^{-2h_z} (u(a)^2 + u(b)^2 + u(c)^2 + u(d)^2) \\
& \leq C(1-\sigma)^{-2} \left( \int_a^b u^2 dz + \int_c^d u^2 dz \right),
\end{align}
where we have also used the fact that $\theta_{F_j}$ has values between zero and one. Summing over the element $K$ and using in Lemma 7.4 for segments that are parallel to $E$ gives
\begin{align}
\sum_K |I_h^k(\theta_{F_j} u)|_{1,K}^2 & \leq C (1-\sigma)^{-4} (1 + \log k) \|u\|^2_{1,\Omega},
\end{align}
where the sum is taken over the elements in $T_k(\Omega_k)$ that touch an edge $E$, such that $E$ does not touch $\partial \Omega$ or, if it does, $K$ does not touch $E_\sigma$.

We next consider the case where $K$ shares an edge with $E_\sigma$. The terms involving the $x$ and $y$ derivatives can be bounded as before: indeed, the first of (7.8) still holds in this case. However, the second of (7.8), needed to bound the $z$ derivative, does not hold. Using Lemma 5.3 we find
\begin{align}
\|\partial_z I_h^k(\theta_{F_j} u)\|_{0,K}^2 & \leq C(h_x h_y / h_z) \left( (u(a) - u(b))^2 + (\theta_{F_j}(c)u(c) - \theta_{F_j}(d)u(d))^2 \right).
\end{align}
Property 7.1 ensures that $\theta_{F_j}(c) = \theta_{F_j}(d)$ and thus
\begin{align}
\|\partial_z I_h^k(\theta_{F_j} u)\|_{0,K}^2 & \leq C\|\partial_z(\theta_{F_j} u)\|_{0,K}^2.
\end{align}
Summing over the elements $K$ that touch $E_\sigma$ gives
\begin{align}
\sum_K \|\partial_z(I_h^k(\theta_{F_j} u))\|_{0,K}^2 & \leq C \|\partial_z(\theta_{F_j} u)\|_{0,\Omega}^2.
\end{align}
and thus
\[
\sum_{K \cap W_i \neq \emptyset} |I^h(\theta_F u)|^2_{1,K} \leq C (1 - \sigma)^{-4} (1 + \log k) \|u\|^2_{1,\Omega_i}, \tag{7.9}
\]

**Case 2.** We now consider an element \( K \in T_k(\Omega_i) \) that does not touch the wirebasket and does not belong to \( K_{\Omega} \). The proof for this case is similar to that of Casarin (1996, Lemma 3.3.7). Using Lemma 5.2 and the second of (7.7), we have
\[
\sum_{K \subset \Omega_i \setminus K_{\Omega}} |I^h(\theta_F u)|^2_{1,K} \leq C \sum_{K} (|u|^2_{1,K} + r_K^{-2} \|u\|^2_{0,K}),
\]
where \( r_K \) is the distance of the baricentre of \( K \) from the wirebasket. We have
\[
\sum_{K} r_K^{-2} \|u\|^2_{0,K} \leq C \int_{K_{\text{int}}} r^{-2} u^2 \, dx + C \sum_{j=1}^{2} \int_{K_{3j}\cup K_{\text{int}}} r_j^{-2} u^2 \, dx + C \sum_{j=3}^{4} \int_{K_{3j}\cup K_{\text{int}}} r_j^{-2} u^2 \, dx,
\]
where \( r_j \) denotes the distance of a point from the edge \( E_j \), and the region consisting of the elements in the GLL mesh \( T_k(\Omega_i) \) that touch the wirebasket is assumed to be excluded from the domains of integration; cf. Fig. 3, left. Each of the integrals on the right, associated with an edge \( E = E_j \), can be estimated using cylindrical coordinates with the \( \zeta \) axis coinciding with \( E_j \) and the radial direction \( r_j \) normal to \( E_j \). We only consider \( E_5 \) in detail; cf. Fig. 3. The other integrals can be estimated in the same way. If the point \( V \) is the intersection between \( E_5 \) and the section corresponding to a fixed \( \zeta \), and \( K_V \) is the element of the two-dimensional mesh \( T(\zeta) \) that contains \( V \), we can write
\[
\int_{K_{\text{int}}} r_5^{-2} u^2 \, dx \leq C \int_{K_V} r_5^{-2} \, dx \int_{-1}^{1} u^2 \, d\zeta \leq C (1 - \sigma)^{-2} (1 + \log k) \|u\|^2_{1,\Omega_i} \int_{K_V} r_5^{-2} \, dx \, dy,
\]
where we have used Lemma 7.4 for the last inequality; cf. Fig. 2, right. The last integral can be estimated by
\[
\int_{K_V} r_5^{-2} \, dx \, dy \leq C \int_{k^{-2} \zeta_{1-\sigma}}^{2} r_5^{-1} \, dr \int_{0}^{2\pi} d\phi \leq C \left( 1 + \log \left( \frac{k}{1-\sigma} \right) \right)^2.
\]
Considering similar contributions for the other edges, we then find
\[
\sum_{K \subset \Omega_i \setminus K_{\Omega}} |I^h(\theta_F u)|^2_{1,K} \leq C |u|^2_{1,\Omega_i} + C (1 - \sigma)^{-2} \left( 1 + \log \left( \frac{k}{1-\sigma} \right) \right)^2 \|u\|^2_{1,\Omega_i}, \tag{7.10}
\]
Case 3. We are now left with the case $K \subset K_\Omega$. Since, in this case, $|\nabla \theta_{F_i}|$ is bounded by a constant, Lemma 5.2 ensures

$$\sum_{K \subset K_\Omega} |I^h(\theta_{F_j} u)|^2_{1,K} \leq C \|u\|_{1,\Omega_\iota}^2.$$ 

The proof is concluded by combining this inequality with (7.9) and (7.10), and applying Lemma 5.1.

8. Comparison results

In the analysis of many iterative substructuring methods, it is necessary to compare certain norms of discrete harmonic functions on different substructures that have the same trace on a common face, edge or vertex.

As already pointed out in Toselli & Vasseur (2003a), if the local meshes are shape-regular and quasi-uniform, the comparison for functions on adjacent substructures that have the same value on a common face can be made using a trace theorem (which is valid for general functions in $H^1$) and a stable extension from the face. However, the existence of stable extensions for meshes that are not quasi-uniform or shape-regular is far from trivial. For this reason, here we will adopt the same strategy as in Toselli & Vasseur (2003a), since the meshes considered are highly anisotropic but of a particular type.

We note that we only need to consider three cases: that of a face shared by an unrefined and a face patch, by a face and an edge patch, and by an edge and a corner patch. We only consider the last two cases in full detail, since the former can be treated in exactly the same way. We consider the two substructures $\Omega_i$ and $\Omega_j$ in Fig. 4, which share the face $F$. Since we proceed in exactly the same way as in Toselli & Vasseur (2003a, Section 7.3), we do not present any proof here. We first consider $\Omega_i$ and suppose that it coincides with the reference cube $\hat{Q}$. The face $F$ corresponds to $x = 1$. Let $\Omega_F$ be the layer of points in $\Omega_i$ within a distance $2(1 - \sigma)$ from $F$.

The following lemma can be proven in the same way as Toselli & Vasseur (2003a, Lemma 7.9).

**Lemma 8.1** Let $u_F \in U_i$ be a face function on $\Omega_i$, i.e. a discrete harmonic function that vanishes on $\partial \Omega_i \setminus F$, and $\bar{u}_F \in X_i$, such that
1. \( \tilde{u}_F \) is equal to \( u_F \) on \( F \) and vanishes on \( \partial \Omega_F \setminus F \);
2. \( \tilde{u}_F \) is discrete harmonic in \( \Omega_F \);
3. \( \tilde{u}_F \) vanishes in \( \Omega_i \setminus \Omega_F \).

Then

\[
|u_F|^2_{1, \Omega_i} \leq |\tilde{u}_F|^2_{1, \Omega_i} \leq \|\nabla \theta_{\sigma, F}\|_{\infty}^2 |u_F|^2_{1, \Omega_i},
\]

where \( \theta_{\sigma, F} \in W^{1, \infty}(\Omega) \) is any function that is equal to one on \( F \), vanishes in \( \Omega_i \setminus \Omega_F \), and has values in \((0, 1)\) in the rest of \( \Omega_i \). In particular, we can find a function such that

\[
\|\nabla \theta_{\sigma, F}\|_{\infty} \leq C(1 - \sigma)^{-1}.
\]

The comparison result for face functions can be then found by noting that we can map \( \Omega_j \) and its mesh into \( \Omega_F \) and the corresponding local mesh, by a simple dilation in the horizontal direction.

**Corollary 8.2** Let \( F \) be a face that is common to \( \Omega_i \) and \( \Omega_j \) and \( u_F \in U \) be a piecewise discrete harmonic function that is identically zero at all nodal points in \( \Gamma_h \setminus F_h \). Then,

\[
(1 - \sigma) |u_F|^2_{1, \Omega_i} \leq |u_F|^2_{1, \Omega_j} \leq C(1 - \sigma)^{-1} |u_F|^2_{1, \Omega_i}.
\]

**Lemma 8.3** Let \( \Omega_i \) and \( \Omega_j \) be two substructures and \( u \in X \). If \( V = V^i = V^j \) is a common vertex, then the vertex components of \( u \) satisfy

\[
\|u_{V^i}\|_{h, V^j}^2 \leq C(1 - \sigma)^{-1} \|u_{V^i}\|_{h, V^j}^2.
\]

If \( E = E^i = E^j \) is a common edge, then the edge components of \( u \) satisfy

\[
\|u_{E^i}\|_{h, E^j}^2 \leq C(1 - \sigma)^{-2} \|u_{E^i}\|_{h, E^j}^2.
\]

**Proof.** For the first inequality, we note that the modified norms \( \| \cdot \|_{h, W^i} \) and \( \| \cdot \|_{h, W^j} \) coincide with the \( L^2 \) norms, since a vertex function vanishes at all nodal points in \( \Gamma_h \) except at that vertex and we only consider internal vertices. It is enough to compare a contribution from an edge \( E^j \) of \( \Omega_j \) with that of an edge \( E^i \) of \( \Omega_i \). The worst possible case occurs when \( E^j \) does not touch \( \partial \Omega \) but \( E^i \) does; cf. Fig. 4. Let \( \phi(\hat{z}) \) be the function in \( Q_h \) that vanishes at all the GLL nodes in \( I_h \), except at \( -1 \) where it is equal to \( u(V) \). Using the change of variables \( z = (1 - \sigma)(\hat{z} + 1) - 1 \) and the fact that \( u_{V^i} \) vanishes in \( E^i_{\sigma} \), we have

\[
\int_{E^i_{\sigma}} u_{V^i}(\hat{z})^2 d\hat{z} = \int_{-1}^1 \phi(\hat{z})^2 d\hat{z} = (1 - \sigma)^{-1} \int_{-1}^{1+2(1-\sigma)} \phi(z)^2 dz = (1 - \sigma)^{-1} \int_{E^i_{\sigma}} u_{V^i}(z)^2 dz.
\]

For the second inequality, it is enough to use the definition of the modified norms \( \| \cdot \|_{h, W^i} \) and \( \| \cdot \|_{h, W^j} \) and Property 7.1.
9. Proof of Assumption 6.1

We are now ready to give an upper bound for $\omega$ in Assumption 6.1. Our proof is similar to that in Pavarino (1998, Lemma 9.1). We note that if $u_i \in U_i$, its extension $u = R_i^T u_i$ vanishes on $\Gamma_i$ except at the nodal points in $\Gamma_{i,h}$ and its support is thus contained in the union of $\Omega_i$ and its neighbouring substructures. In order to estimate $\omega$ we thus have to estimate the energy of $u$ in these substructures in terms of the energy of $\mathcal{H}(\delta_i u_i)$ in $\Omega_i$ alone.

We first note that, by simple calculation, we have

$$\rho_j(\delta_i(x))^2 = \rho_j(\delta_i(x))^{-2} \leq \min\{\rho_i, \rho_j\}, \quad x \in \Gamma_{i,h}, \quad j \in N_x.$$  \hfill (9.1)

Let $u_i \in \text{Range}(\tilde{P}_i)$. We start with a substructure $\Omega_j$ that only has a vertex $V = V^i = V^j$ in common with $\Omega_i$. We note that, according to the decomposition (7.2), $u$ has only a wirebasket component $u_{V^j} = u$ on $\Omega_j$, which vanishes at all nodes in $\Gamma_{j,h}$ except at $V$. Using Lemma 7.1, we find

$$a_j(u, u) = \rho_j |u_{V^j}|_{1, \Omega_j}^2 \leq C \rho_j (1 - \sigma)^{-2} \|u_{V^j}\|_{h, \Omega_j}^2 = C \rho_j \delta_i(u_{V^j}) (1 - \sigma)^{-2} \|\delta_i u_{V^j}\|_{h, \Omega_j}^2,$$

where $\delta_i u_{V^j} = \delta_i(V)$. We next note that, thanks to Lemma 8.3, the norm $\|\cdot\|_{h, \Omega_j}$ associated with $\Omega_j$ can be bounded by $\|\cdot\|_{h, \Omega^j}$. In addition, we can apply Lemmas 7.5 and 7.3 and find

$$\rho_i \|\delta_i u_{V^j}\|_{h, \Omega_j}^2 \leq C (1 - \sigma)^{-3} \rho_i \|\delta_i u_i\|_{h, \Omega^j}^2 \leq C (1 - \sigma)^{-1} \rho_i \|\mathcal{H}_i(\delta_i u_i)\|_{h, \Omega^j}^2 \leq C (1 - \sigma)^{-3} (1 + \log k) \rho_i \|\mathcal{H}_i(\delta_i u_i)\|_{h, \Omega^j}^2 \leq C (1 - \sigma)^{-3} (1 + \log k) \rho_i \|\mathcal{H}_i(\delta_i u_i)\|_{h, \Omega^j}^2 \leq \rho_i \|\mathcal{H}_i(\delta_i u_i)\|_{h, \Omega^j}^2.$$  \hfill (9.2)

The $L^2$ component in the last term can be bounded by the local bilinear form $a_i(\cdot, \cdot)$, thanks to a Poincaré inequality for floating subdomains (cf. (6.9)), or thanks to a Friedrichs inequality for substructures that touch $\partial \Omega$. Combining these two estimates and using (9.1), we find

$$a_j(u, u) = a_j(u_{V^j}, u_{V^j}) \leq C (1 - \sigma)^{-5} (1 + \log k) a_i(\mathcal{H}_i(\delta_i u_i), \mathcal{H}_i(\delta_i u_i)).$$  \hfill (9.3)

We next consider a substructure $\Omega_j$ that only has an edge $E = E^j = E^j$ in common with $\Omega_i$, with vertices $V^j_1 = V^j_1$ and $V^j_2 = V^j_2$. We note that, according to the decompositions (7.1) and (7.2), $u$ has only a wirebasket component on $\Omega_j$,

$$u = u_{W^j} = u_{V^j_1} + u_{V^j_2} + u_{E^j},$$

which vanishes at all nodes in $\Gamma_{j,h}$ except at those on the closure $\overline{E^j}$. We then have

$$a_j(u, u) \leq 3 a_j(u_{V^j_1}, u_{V^j_1}) + 3 a_j(u_{V^j_2}, u_{V^j_2}) + 3 a_j(u_{E^j}, u_{E^j}).$$

For the two vertex components, we can proceed as before and find similar bounds to (9.2). For the edge component, we use Lemma 7.1, the definition of $\|\cdot\|_{h, E^j}$ and the fact that $\delta_i$
is constant at all the nodal points in $E_h$. We find

$$a_j(u_{Ej}, u_{Ej}) = \rho_j|u_{Ej}|_{1,\Omega_j}^2 \leq C \frac{\rho_j}{(1-\sigma)^2} |u_{Ej}|_{h,\Omega_j}^2 \leq C \frac{\rho_j \delta_{i,E}^{-2}}{(1-\sigma)^2} \|\delta_i u_{Ej}\|_{h,\Omega_j}^2,$$

where $\delta_{i,E}$ is the constant value of $\delta_i$ on $E$. Thanks to Lemma 8.3, the norm $\|\cdot\|_{h,\Omega_j}$ associated with $\Omega_j$ can be bounded by $\|\cdot\|_{h,E_j}$. In addition, we can apply Lemmas 7.5 and 7.3 and find

$$\rho_i |\delta_i u_{Ej}|_{h,\Omega_j}^2 \leq C (1-\sigma)^{-2} \rho_i \|\delta_i u_{Ej}\|_{h,\Omega_j}^2 \leq C (1-\sigma)^{-2} \rho_i \|\mathcal{H}_i(\delta_i u_i)\|_{h,E_i}^2 \leq C (1-\sigma)^{-4} (1 + \log k) \rho_i \|\mathcal{H}_i(\delta_i u_i)\|_{1,\Omega_i}^2 \leq C (1-\sigma)^{-4} (1 + \log k) \rho_i \|\mathcal{H}_i(\delta_i u_i)\|_{1,\Omega_i}^2.$$ 

As before, the $L^2$ component in the last term can be bounded by the local bilinear form $a_i(\cdot, \cdot)$, thanks to a Poincaré or a Friedrichs inequality. Combining these two estimates and using (9.1), we find

$$a_j(u_{Ej}, u_{Ej}) \leq C (1-\sigma)^{-6} (1 + \log k) a_i(\mathcal{H}_i(\delta_i u_i), \mathcal{H}_i(\delta_i u_i)).$$

(9.3)

We next consider a substructure $\Omega_j$ that shares a face $F$ and thus also the edges and vertices that lie on $\partial F$. We note that on $\Omega_j$, $u$ can be decomposed as

$$u = u_{\omega_j} + u_F.$$ 

We have

$$a_j(u, u) = \rho_j |u|_{1,\Omega_j}^2 \leq 2 \rho_j (|u_{\omega_j}|_{1,\Omega_j}^2 + |u_F|_{1,\Omega_j}^2).$$

The wirebasket component can be bounded as before; cf. (9.2) and (9.3). For the face component, we first note that the function $\delta_i$ is equal to a constant value $\delta_{i,F}$ at all nodal points inside $F$. Using (9.1), we can then write

$$\rho_j |u_F|_{1,\Omega_j}^2 = \rho_j \delta_{i,F}^{-2} |\mathcal{H}_j(\delta_i u_F)|_{1,\Omega_j}^2 \leq \rho_i |\mathcal{H}_j(\delta_i u_F)|_{1,\Omega_j}^2.$$ 

Using Corollary 8.2 and Lemma 7.7 yields

$$|\mathcal{H}_j(\delta_i u_F)|_{1,\Omega_j}^2 \leq C (1-\sigma)^{-1} |\mathcal{H}_j(\delta_i u_F)|_{1,\Omega_j}^2 \leq C (1-\sigma)^{-1} \left(1 + \log \left(\frac{k}{1-\sigma}\right)\right)^2 \|u\|^2_{1,\Omega_j}.$$ 

Combining the last two estimates and using a Poincaré or a Friedrichs inequality, we find

$$a_j(u_F, u_F) \leq C (1-\sigma)^{-5} \left(1 + \log \left(\frac{k}{1-\sigma}\right)\right)^2 a_i(\mathcal{H}_i(\delta_i u), \mathcal{H}_i(\delta_i u)).$$

(9.4)

We finally need to consider the energy of $u$ in $\Omega_j, a_i(u, u)$. We note that we can decompose $u$ on $\Omega_j$ according to (7.1). The wirebasket and the face components can be bounded as before. Summing over $j$ and the neighbouring subdomains, we then find

$$a(u, u) \leq \frac{C}{(1-\sigma)^6} \left(1 + \log \left(\frac{k}{1-\sigma}\right)\right)^2 \left(\sum_{V} + \sum_{E} + \sum_{F} \right) a_i(\mathcal{H}_i(\delta_i u), \mathcal{H}_i(\delta_i u)).$$
Table 1  Balancing Neumann–Neumann algorithm

1. Initialize
\begin{align*}
u_0 &= R_0^T S_0^{-1} R_0 g \Gamma + \tilde{w}, \quad \tilde{w} \in \text{Range}(I - P_0) \\
q_0 &= g \Gamma - S u_0
\end{align*}

2. Iterate $j = 1, 2, \ldots$ until convergence

Project:
\begin{align*}
w_{j-1} &= (I - P_0^T) q_{j-1}
\end{align*}

Precondition:
\begin{align*}
\tilde{z}_{j-1} &= \sum_{i=1}^{N} R_i^T D_i S_i^\dagger D_i R_i \tilde{w}_{j-1}
\end{align*}

Project:
\begin{align*}
y_{j-1} &= (I - P_0) \tilde{z}_{j-1} \\
\beta_j &= \langle y_{j-2}, w_{j-2} \rangle \\
p_j &= y_{j-1} + \beta_j p_{j-1} \quad [p_1 = y_0] \\
\alpha_j &= \langle y_{j-1}, w_{j-1} \rangle / \langle p_j, S p_j \rangle \\
u_j &= u_{j-1} - \alpha_j S p_j \\
q_j &= q_{j-1} - \alpha_j p_j
\end{align*}

Since the partition $T_m$ is shape-regular, the number of subdomains to which an edge or a vertex may belong is bounded. We finally obtain

$$\omega \leq C (1 - \sigma)^{-6} \left( 1 + \log \left( \frac{k}{1 - \sigma} \right) \right)^2.$$ 

Since in practice $\sigma$ is bounded away from one, we obtain the same bound as for Neumann–Neumann methods for $p$ finite-element approximations on shape-regular meshes

$$\kappa(P_{NN}) \leq C (1 + \log k)^2;$$

(see e.g. Pavarino, 1997). We stress the fact that the constants in the last two estimates are independent of the coefficients $\rho_i$ and the refinement level $n$ (and thus of the aspect ratio of the mesh $T_{bl}^n$).

10. Numerical results

The purpose of this section is to present two numerical experiments in order to validate our analysis on some medium-size problems. A more detailed and thorough study will be presented in Toselli & Vasseur (2003b).

The balancing Neumann–Neumann method of Section 6 can be implemented as a projected preconditioned conjugate gradient algorithm and is shown in Table 1 (see Toselli & Vasseur, 2003c for more details). In this table $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

It is easy to show that $w_j = q_j$ thanks to the choice of the initial guess, and the first projection step can therefore be omitted. In addition, the application of the pseudoinverses
$S^\dagger_i$ can be carried out by applying the pseudoinverses of the original matrices $A^{(i)}$, cf. (6.1), which amounts to solving local Neumann problems on the substructures (see Smith et al., 1996, Section 4.2.1 for details). The total amount of work for each step consists of the solution of one coarse problem (application of $S_0^{-1}$), one Neumann problem (application of $S^\dagger_i$) and two Dirichlet problems (application of $S$ for $P_0$ and for the calculation of the new search direction) on each subdomain. The most expensive parts of the methods are the factorizations of the local matrices $A^{(i)}$ and $A^{(i)}_{II}$, and of the global $S_0$. The matrices $A^{(i)}$ and $A^{(i)}_{II}$ have roughly the same size.

We remark that the amount of work per step of the unpreconditioned conjugate gradient algorithm for the Schur complement system (6.2) amounts to solving one Dirichlet problem on each substructure (one application of $S$ for the calculation of the new search direction). The rate of convergence however deteriorates very fast with the problem size. A more detailed numerical study on the performance and cost of our algorithm will be performed in Toselli & Vasseur (2003b).

Our first numerical experiment targets the efficiency of the Neumann–Neumann preconditioner for a Laplace problem defined on a boundary layer mesh (corner refinement), whereas the second one is a standard domain decomposition test case defined on a uniform mesh. In both experiments, the conjugate gradient iteration is stopped after a reduction of the Euclidean norm of the initial residual of $10^{-14}$ and homogeneous Dirichlet boundary conditions have been used.

### 10.1 Laplace problem on a boundary layer mesh

We consider approximations on the unit cube $\Omega = (0, 1)^3$. We choose $\rho \equiv 1$ and the right-hand side $f \equiv 1$. The macromesh $T_m$ consists of $N \times N \times N$ cubic substructures. Geometric refinement is performed towards the three edges $x = 0, y = 0, z = 0$, with $\sigma = 0.5$; see Fig. 5, left. Given a polynomial degree $k$, we choose $n = k$ as is required for robust exponential convergence (see e.g. Andersson et al., 1995; Babuška & Guo, 1996).

We note that even for moderate values of $k$ and $N$, extremely large linear systems are obtained; cf. Tables 2 and 3. Huge local blocks need to be inverted, both for the application of $S$ (solution of local Dirichlet problems) and the preconditioner (solution of local Neumann problems). Due to memory limitations in our Matlab implementation, direct solvers could not always be employed and thus we have employed approximate solvers for local Dirichlet and Neumann problems. We refer to Smith et al. (1996, Section 4.4) for details on the implementation. In particular, we have used a conjugate gradient iteration with an incomplete Cholesky factorization with drop tolerance $10^{-3}$ for all local problems. The iteration is stopped after a reduction of the initial residual of a factor $10^{-3}$ or after 20 iteration steps. In the sequel, we denote by NN (inexact) the resulting balancing Neumann–Neumann method with this strategy for the approximate solvers. An exact variant denoted by NN (exact) is derived, when solving all the local subproblems now up to machine precision with the same iterative solver as in the inexact case. Our numerical results show that the theoretical bounds for the case of exact solvers in Lemma 6.2 remain valid in this case; cf. Tables 2 and 3.

For a fixed partition into substructures with $N = 3$, Table 2 shows the size of the original problem, the iteration count, the estimated maximum and minimum eigenvalues,
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![Geometric refinement towards one corner (N = 3, σ = 0.5, and n = 6), left, and estimated condition numbers (circles) from Table 2 (inexact variant) and least-square second-order logarithmic polynomial fit (solid line) versus k, right.](image)

**TABLE 2** Conjugate gradient method for the global system with Neumann–Neumann preconditioner with inexact and exact solvers: iteration counts, maximum and minimum eigenvalues, and condition numbers, versus the polynomial degree, for the case of a fixed partition. The size of the original problem is also reported. Fixed number of subdomains (N = 3)

<table>
<thead>
<tr>
<th>k</th>
<th>Size</th>
<th>It</th>
<th>(\lambda_{\text{max}})</th>
<th>(\lambda_{\text{min}})</th>
<th>(\kappa)</th>
<th>(\lambda_{\text{max}})</th>
<th>(\lambda_{\text{min}})</th>
<th>(\kappa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1331</td>
<td>15</td>
<td>1.8379</td>
<td>1</td>
<td>13</td>
<td>1.6255</td>
<td>0.00002</td>
<td>1.6255</td>
</tr>
<tr>
<td>3</td>
<td>6859</td>
<td>20</td>
<td>2.8165</td>
<td>0.99997</td>
<td>23</td>
<td>2.8165</td>
<td>0.00001</td>
<td>2.8161</td>
</tr>
<tr>
<td>4</td>
<td>24389</td>
<td>25</td>
<td>3.9507</td>
<td>0.99947</td>
<td>23</td>
<td>3.9506</td>
<td>0.00002</td>
<td>3.9498</td>
</tr>
<tr>
<td>5</td>
<td>68921</td>
<td>29</td>
<td>5.1507</td>
<td>0.99799</td>
<td>25</td>
<td>5.1507</td>
<td>0.00002</td>
<td>5.1493</td>
</tr>
<tr>
<td>6</td>
<td>106375</td>
<td>34</td>
<td>6.3675</td>
<td>0.99801</td>
<td>28</td>
<td>6.3675</td>
<td>0.00002</td>
<td>6.3658</td>
</tr>
<tr>
<td>7</td>
<td>357911</td>
<td>38</td>
<td>7.5082</td>
<td>0.99395</td>
<td>32</td>
<td>7.5067</td>
<td>0.00002</td>
<td>7.5065</td>
</tr>
<tr>
<td>8</td>
<td>704969</td>
<td>40</td>
<td>8.5298</td>
<td>0.99574</td>
<td>34</td>
<td>8.5064</td>
<td>0.00002</td>
<td>8.5062</td>
</tr>
</tbody>
</table>

and the condition number for different values of \(k\) for both inexact and exact variants. We note that the minimum eigenvalue is close to one when using inexact solvers; see Lemma 6.2. In addition, a moderate growth of the maximum eigenvalue is observed with \(k\); such growth is consistent with the quadratic bound in Lemma 6.3; see Fig. 5, right. Using inexact solvers for the local subproblems induces a moderate increase of number of iterations. Nevertheless, quite satisfactory condition numbers are still obtained, see Table 2.

We next consider the same problem, and fix the polynomial degree \(k = 4\). Table 3 shows the results for different values of \(N\). In both variants, the iteration counts, and the smallest and largest eigenvalues appear to be bounded independently of the number of subdomains. We note that when the number of subdomains increases, the number of iterations to reach the convergence criterion for both variants are nearly identical.
Nevertheless, the difference on the condition number estimates is more pronounced than in Table 2.

10.2 Laplace problem with jump coefficients

The theoretical bound for the condition number in Lemma 6.3 is independent of arbitrary jumps on the coefficients between the substructures. The purpose of this numerical experiment is to check this property. In consequence, the coefficient \( \rho \) possibly changes between the substructures by orders of magnitudes. The right-hand side is \( f \equiv 1 \). Given a partition of \( \Omega = (0, 1)^3 \) into \( N \times N \times N \) cubic substructures \( (T = T_m = N \times N \times N) \), a checkerboard distribution on this partition is considered for \( \rho \) which is equal to either \( \rho_1 \) or \( \rho_2 \) as in Mandel & Brezina (1996). Inexact solvers for the Dirichlet and Neumann problems have been considered.

For a fixed partition into substructures with \( N = 3 \) and for fixed jumps between the substructures with \( \rho_1 = 10^{-3} \) and \( \rho_2 = 10^3 \), we have investigated the behaviour of the condition number of the preconditioned operator versus the polynomial degree \( k \). This behaviour is shown in Fig. 6 and is consistent with the quadratic bound in Lemma 6.3.

For a fixed partition into substructures with \( N = 3 \) and for a fixed polynomial degree \( k = 4 \), we have investigated the influence of the jump \( \rho_2/\rho_1 \) on the convergence behaviour of the balancing Neumann–Neumann method. \( \rho_1 \) is fixed to 1, whereas \( \rho_2 \) is varying from 1 to \( 10^6 \). A checkerboard distribution has also been used. The results are presented in Table 4. The number of preconditioned CG iterations in order to satisfy the stopping criterion is bounded independently of the ratio \( \rho_2/\rho_1 \), in agreement with the bound for the case of exact solvers in Lemma 6.3.
Fig. 6. Laplace problem with jump coefficients. Case of $\rho_1 = 10^{-3}$ and $\rho_2 = 10^3$. Fixed partition $3 \times 3 \times 3$. Estimated condition numbers (circles) and least-square second order logarithmic polynomial (solid line) versus the spectral degree for the balancing Neumann–Neumann method (inexact variant).

Table 4 Laplace problem with jump coefficients. Case of $k = 4$ and $\rho_1 = 1$. Conjugate gradient method for the global system with balancing Neumann–Neumann method (inexact solvers): iteration counts, maximum and minimum eigenvalues, and condition numbers versus $\rho_2$. Fixed number of subdomains ($N = 3$)

<table>
<thead>
<tr>
<th>$\rho_2$</th>
<th>$\text{It}$</th>
<th>$\lambda_{\text{max}}$</th>
<th>$\lambda_{\text{min}}$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$</td>
<td>15</td>
<td>2.1153</td>
<td>0.99999</td>
<td>2.1153</td>
</tr>
<tr>
<td>$10^2$</td>
<td>15</td>
<td>2.1185</td>
<td>0.99999</td>
<td>2.1186</td>
</tr>
<tr>
<td>$10^3$</td>
<td>15</td>
<td>2.0370</td>
<td>1</td>
<td>2.0370</td>
</tr>
<tr>
<td>$10^4$</td>
<td>14</td>
<td>2.0262</td>
<td>1</td>
<td>2.0262</td>
</tr>
<tr>
<td>$10^5$</td>
<td>14</td>
<td>2.0251</td>
<td>0.99999</td>
<td>2.0253</td>
</tr>
<tr>
<td>$10^6$</td>
<td>17</td>
<td>2.0275</td>
<td>0.96406</td>
<td>2.1031</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>2.0266</td>
<td>0.98234</td>
<td>2.0630</td>
</tr>
</tbody>
</table>

11. Concluding remarks
As for the analysis in Toselli & Vasseur (2003a), some important issues still need to be addressed. We refer to our previous work for a full discussion of these issues.

Our analysis is restricted to approximations that employ nodal basis functions on the Gauss–Lobatto nodes. Indeed, for three-dimensional shape-regular meshes good performance of iterative substructuring methods is in general ensured only if these basis functions are employed and for more general $p$ or $hp$ version finite-element
approximations many important issues remain to be solved even for shape-regular meshes (see e.g. Sherwin & Casarin, 2001 and the references therein).

The Dirichlet and Neumann problems that we need to solve ($S_i$ and $S_i^\dagger$) can be potentially very large. Approximate local solvers can be employed for iterative substructuring methods (see e.g. Smith et al., 1996; Klawonn & Widlund, 2000) and some have been proposed in Korneev et al. (2002) for $hp$-approximations. In our numerical experiments, we have employed a conjugate gradient iteration with an incomplete Choleski preconditioner. However, we believe that the tensor product structure of corner, edge and face patches can be exploited. This is left to a future work.

We believe that the analysis and/or the development of iterative substructuring methods for general meshes with hanging nodes still need to be fully addressed. These meshes are widely used in practice. There is no straightforward way of defining Neumann–Neumann or FETI algorithms when hanging nodes lie on the interface $\Gamma$ (see Toselli & Vasseur, 2003a, Remark 6.1 for more details).

Finally, our analysis has been carried out for the model problem (2.1), which indeed does not exhibit boundary layers. As for the two-dimensional problems in Toselli & Vasseur (2003a,c), numerical results show that our algorithms are robust when applied to certain singularly perturbed problems. Extensive numerical results will be presented in Toselli & Vasseur (2003b).

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