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De la Vallée Poussin type approximation methods

PhD Thesis

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Budapest, 2015

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Acknowledgements

I would like to thank everyone who inspired me during all these years.

First of all, I would like to thank my project advisor, László Szili for helping me many times and introducing me a fascinating branch of mathematics.

Special thanks to my family and friends for their support and love.

Finally, I thank all my co-workers at the Department of Numerical Analysis for accepting me and guiding me during this time.

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Chapter 1

Introduction

The focus of our work is the uniform convergence of different de la Vallée Poussin type summations. We encounter this topic in theories of classical and multivariate trigonometric Fourier series, discrete Fourier series and trigonometric interpolation, and finally algebraic interpolation. There are many similarities but also some differences in our methods when dealing with these problems.

In this chapter we discuss the historical background of our study, establish the most important notations and definitions and recall some fundamental results on which the later chapters (presenting our results) are based upon.

1.1 Summations of trigonometric Fourier series

1.1.1 Preliminaries

Let $C_{2\pi}$ denote the linear space of complex valued 2π-periodic continuous functions defined on the real numbers $\mathbb{R}$. It is well known that $C_{2\pi}$ endowed with the maximum norm

$$\|f\|_\infty := \max_{x\in\mathbb{R}} |f(x)| \qquad (f \in C_{2\pi}) \tag{1.1}$$

is a complete normed space, i.e. Banach space.

The complex trigonometric system

$$\varepsilon_j(x) := e^{ijx} \qquad (x \in \mathbb{R},\ j \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}.) \tag{1.2}$$

is an orthonormal system with respect to the scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f\overline{g}\, dt \qquad (f, g \in C_{2\pi}),$$

i.e.

$$\langle \varepsilon_k, \varepsilon_l \rangle = \delta_{k,l} \qquad (k, l \in \mathbb{Z}).$$

Denote by $\mathcal{T}_n$ ($n \in \mathbb{N} := \{0, 1, \ldots\}$) the linear space of all complex valued trigonometric polynomials of degree not exceeding n:

$$\mathcal{T}_n := \text{span}\,\{\varepsilon_j \;:\; -n \le j \le n\,\}.$$

We remark that the set of all trigonometric polynomials $\mathcal{T} := \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$ form a closed system in the space $(C_{2\pi}, \|\cdot\|_\infty)$, i.e. the set is closed under linear combinations of its elements and the closure of the set is $C_{2\pi}$.

For a function $f \in C_{2\pi}$ denote the trigonometric Fourier coefficients by

$$\hat{f}(j) := \langle f, \varepsilon_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} dt \qquad (j \in \mathbb{Z}).$$

The trigonometric Fourier series of f is given by

$$S[f] := \sum_{j\in\mathbb{Z}} \hat{f}(j)\varepsilon_j. \tag{1.3}$$

Denote the n-th partial sum of this series by

$$(S_n f)(x) := \sum_{j=-n}^{n} \hat{f}(j)\varepsilon_j(x) \qquad (x \in \mathbb{R}).$$

We can rearrange this expression to the form

$$(S_n f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) dt,$$

where

$$D_n(t) := 1 + 2 \sum_{j=1}^{n} \cos jt \qquad (t \in \mathbb{R}), \tag{1.4}$$

is the so-called one dimensional Dirichlet kernel.

It is clear that $S_n : \ C_{2\pi} \to \mathcal{T}_n$ is a linear operator with the projection property

$$(S_n g)(x) = g(x) \qquad (g \in \mathcal{T}_n, x \in \mathbb{R}). \tag{1.5}$$

In the study of convergence, the concept of the operator norm plays an important role. In this case, for a map $T : \ (C_{2\pi}, \|\cdot\|_\infty) \to (C_{2\pi}, \|\cdot\|_\infty)$, the norm of operator T is defined by

$$\|T\| = \max_{\substack{f \in C_{2\pi} \\ \|f\|_\infty \le 1}} \|Tf\|_\infty .$$

Usually we don't evaluate the exact value of this expression, only give estimations.

We remark that by a simple calculation, or the usage of the Riesz representation theorem (see [8, IV. 6.3] or [61]) one can establish the well known connection between the Dirichlet kernel and the norm of S_n, namely

$$\|S_n\| = \|D_n\|_1 := \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt.$$

Consequently, for the operator S_n the following result (cf. [5, p. 42]) is known.

Theorem 1.1. *For* $1 \le n \in \mathbb{N}$ *we have*

$$\|S_n\| = \frac{4}{\pi^2} \log n + O(1),$$

i.e. there exist independent constants $c_1, c_2 \in \mathbb{R}$ *such that*

$$\frac{4}{\pi^2} \log n + c_1 \le \|S_n\| \le \frac{4}{\pi^2} \log n + c_2.$$

An important consequence of this statement is that, based on the Banach–Steinhaus theorem [33, p. 101], the sequence of partial sums of Fourier series of f does not converge uniformly to f for all $f \in C_{2\pi}$ (i.e. for some functions $f \in C_{2\pi}$ we have $\lim \|f - S_n f\|_\infty \nrightarrow 0$ as $n \to +\infty$), since the set of norms of all operators S_n is not bounded. For more details see e.g. [33, Sect. 6.6] or [5, Part I]. In summary, we have

Corollary 1.2. *For some $f \in C_{2\pi}$ the sequence $(S_n f)$ does not converge uniformly, moreover* $\sup_n \|S_n f\|_\infty = +\infty$.

We present some solutions to the problem of uniform convergence in the next subsection.

The previous train of thought shows us that the operator norms and the convergence of a sequence of operators are closely related. Indeed, if we notate by $E_n(f)$ the error of the best approximating trigonometric polynomial, i.e. $E_n(f) := \min_{p_n \in \mathcal{T}_n} \|f - p_n\|_\infty$, then we have the following estimation of error [62, (13.25)] and [62, Chap. II.12].

Theorem 1.3.

$$\|f - (S_n f)\|_\infty \leq \{\|S_n\| + 1\} \cdot E_n(f).$$

Before we move on, we recall one of the most characteristic properties of the Fourier series, the so-called Faber–Marcinkiewicz–Berman theorem (see [7, p. 281] for details), namely that the operator S_n has the smallest norm among similar projection operators.

Theorem 1.4. *Let $T_n : C_{2\pi} \to \mathcal{T}_n$ denote a linear trigonometric projection operator, i.e. suppose that $(T_n g)(x) = g(x)$, $(g \in \mathcal{T}_n, x \in \mathbb{R})$. Now we have*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (T_n g_t)(x - t) dt = (S_n g)(x),$$

where $g_t := g(\cdot + t)$ is the t-translation operator. Moreover

$$\|T_n\| \geq \|S_n\|.$$

1.1.2 φ-sums and the Natanson–Zuk theorem

It is already known that the sequence of the partial sums of Fourier series of f is not uniformly convergent for all $f \in C_{2\pi}$, since the operators in question are not uniformly bounded. This problem is usually avoided by replacing $S_n f$ $(n \in \mathbb{N})$ with a bounded linear operator obtained by applying summation over the Fourier series (see e.g. Fejér summation).

The general case, i.e. when the summation method is given by a suitable matrix, was studied (among others) in [9], [19], [35], [48]. In [46] B. Szőkefalvi-Nagy showed that in the special case when the summation is generated by a continuous function φ the uniform convergence of the φ-sums of the trigonometric Fourier series may be characterized by the Fourier transform of the summation function φ (see also [5], [25], [47], [58] and [59]).

We investigate a generalization of this idea called *φ-summation*, i.e. a summation generated by a function φ as defined below.

Let us denote by Φ the set of functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying the following requirements (cf. [42]):

(i) φ is an even function supported in $[-1, 1]$,

(ii) $\lim\limits_{t->0} \varphi(t) = \varphi(0) = 1$,

(iii) the limits

$$\varphi(t_0 \pm 0) := \lim_{t \to t_0 \pm 0} \varphi(t)$$

exist and finite in every $t_0 \in \mathbb{R}$,

(iv) for all $t \in \mathbb{R}$ the function value $\varphi(t)$ lies in the closed interval determined by $\varphi(t-0)$ and $\varphi(t+0)$.

The condition (iii) ensures that every $\varphi \in \Phi$ is Riemann integrable on $[0, 1]$. Indeed, it implies the existence of a sequence of step functions which uniformly converges on the interval $[0, 1]$ to φ. Therefore φ is continuous except at most countable points of $[0, 1]$.

Now let us fix a function $\varphi \in \Phi$. The n-th *φ-sum* of the trigonometric series of $f \in C_{2\pi}$ is defined as $\big(S_n^{\varphi} f\big)(x) := \big(S_n f\big)(x)$ for $n = 0$, and otherwise by

$$\big(S_n^{\varphi} f\big)(x) := \sum_{j\in\mathbb{Z}} \varphi\left(\frac{j}{n}\right) \hat{f}(j)e^{ijx} = \sum_{j=-n}^{n} \varphi\left(\frac{j}{n}\right) \hat{f}(j)e^{ijx}, \qquad (x \in \mathbb{R},\ 1 \le n \in \mathbb{N}). \tag{1.6}$$

The Fourier series $S[f]$ is called *uniformly φ-summable* if the sequence $(S_n^{\varphi} f, n \in \mathbb{N})$ uniformly converges on $\mathbb{R}$ as $n \to +\infty$. The limit is called the *φ-sum of $S[f]$*.

It is clear that for every $f \in C_{2\pi}$ and all $n \in \mathbb{N}$ we have

$$\big(S_n^{\varphi} f\big)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n^{\varphi}(x - t)dt,$$

where

$$D_n^{\varphi}(t) := 1 + 2\sum_{j=1}^{n} \varphi\left(\frac{j}{n}\right) \cos jt \qquad (t \in \mathbb{R}), \tag{1.7}$$

i.e. $S_n^{\varphi} f$ is a trigonometric polynomial of degree not exceeding n and

$$S_n^{\varphi} : C_{2\pi} \to \mathcal{T}_n$$

is a bounded linear operator.

Next we recall another fundamental result, the so-called Natanson–Zuk theorem. (see [25, p. 168]).

Denote by $L^1(\mathbb{R})$ the usual linear space (over $\mathbb{R}$) of measurable functions $g : \mathbb{R} \to \mathbb{R}$ for which the Lebesgue integral

$$\int_{-\infty}^{+\infty} |g(x)|dx$$

is finite. The function

$$\|g\|_{L^1(\mathbb{R})} := \int_{-\infty}^{+\infty} |g(x)|dx \qquad (g \in L^1(\mathbb{R})) \tag{1.8}$$

is a norm on the space $L^1(\mathbb{R})$ and the normed space $(L^1(\mathbb{R}), \|\cdot\|_{L^1(\mathbb{R})})$ is a Banach space.

The Fourier transform of the function $g \in L^1(\mathbb{R})$ is defined by

$$\hat{g}(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(t)e^{-ixt}dt \qquad (x \in \mathbb{R}). \tag{1.9}$$

It follows immediately from the definition that the Fourier transform of every function $g \in L^1(\mathbb{R})$ exists for all $x \in \mathbb{R}$. Further, it can be proved that if $g \in L^1(\mathbb{R})$ then the Fourier transform $\hat{g}$ is a uniformly continuous function on $\mathbb{R}$ and $\hat{g}(x)$ tends to zero as $x \to \pm\infty$ (see e.g. [5], Proposition 5.1.2).

The Fourier transform of a function from $L^1(\mathbb{R})$ does not belong to the space $L^1(\mathbb{R})$, in general. For example the function

$$g(t) := \begin{cases} e^{-t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0 \end{cases}$$

is in $L^1(\mathbb{R})$, but for its Fourier transform

$$\hat{g}(x) = \frac{1}{1-ix} \qquad (x \in \mathbb{R})$$

this is not true.

Now using definition (1.9), we are in a position to formalize the aforementioned theorem of Natanson–Zuk.

Theorem 1.5. *Suppose that* $\varphi \in \Phi$. *Then* $(S_n^{\varphi} f, n \in \mathbb{N})$ *uniformly converges to* f *on* $\mathbb{R}$ *for every function* $f \in C_{2\pi}$ *if and only if the Fourier transform of* φ *is (Lebesgue) integrable on* $\mathbb{R}$.

1.1.3 The de la Vallée Poussin sums and other examples

In this subsection we recall some of the important summations of Fourier series, and investigate them with the tools described above.

1.1.3.1 Partial sums of the Fourier series

First, let φ_1 be the unique element of Φ which equals to 1 on $[-1, 1]$ (and 0 otherwise). Now by (1.6) it is clear that $S_n^{\varphi_1} = S_n$, so we obtain the partial sums of the Fourier series as a specific φ-sum. Also

$$\hat{\varphi}_1(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_L(t)e^{-ixt}dt = \frac{1}{2\pi}\int_{-1}^{1} e^{-ixt}dt = \frac{\sin x}{\pi x} \qquad (x \in \mathbb{R}),$$

therefore $\hat{\varphi}_1 \notin L^1(\mathbb{R})$ and by Theorem 1.5 we obtain the already mentioned negative result of Corollary 1.2.

1.1.3.2 Fejér means of Fourier series

The so-called Fejér sums of the Fourier series holds a historical importance as being one of the first methods which yields uniform convergence [12]. These sums are the arithmetic means of the partial sums, so the operator F_n can be defined as

$$F_n := \frac{1}{n+1}\sum_{j=0}^{n} S_j \qquad (n \in \mathbb{N}).$$

Now if we let

$$\varphi_0(x) := \begin{cases} 1 - |x|, & \text{if } x \in [-1, 1], \\ 0, & \text{otherwise,} \end{cases}$$

then $\varphi_0 \in \Phi$ and a simple calculation shows the relation $F_n = S_{n+1}^{\varphi_0}$.

Since we have

$$\hat{\varphi}_0(x) = \frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2 \qquad (x \in \mathbb{R})$$

and thus $\hat{\varphi}_0 \in L^1(\mathbb{R})$, by Theorem 1.5 we obtain the result of Fejér [12].

Corollary 1.6. *The sequence* $(F_n f, n \in \mathbb{N})$ *uniformly converges to* f *on* $\mathbb{R}$ *for every function* $f \in C_{2\pi}$.

Consequently, by the Banach–Steinhaus theorem [33, p. 101] the norms of operators F_n must be uniformly bounded, i.e. $\sup_n \|F_n\| < +\infty$.

Corollary 1.7. *There exists* $c \in \mathbb{R}$ *independent of* n *such that*

$$\|F_n\| < c.$$

1.1.3.3 De la Vallée Poussin sums

Originally, the de la Vallée Poussin sums [49] were similar solutions to the problem of uniform convergence as the Fejér sums, and can be defined as arithmetic means of partial sums as well, but for $n \in \mathbb{N}$, we take the average of the partial sums $S_n, S_{n+1}, \ldots, S_{2n}$, obtaining the operator

$$G_{n,n} = \frac{1}{n+1} \sum_{j=n}^{2n} S_j \qquad (n \in \mathbb{N}).$$

The idea of taking arithmetic means of partial sums can be further generalized (see e.g. [62, Chap. III.1]), namely for two parameters $n, m \in \mathbb{N}$, we may take the average of the partial sums $S_n, S_{n+1}, \ldots, S_{n+m}$. For our work, we consider this approach and define the operator $G_{n,m}$ as

$$G_{n,m} := \frac{1}{m+1} \sum_{j=n}^{n+m} S_j \qquad (n, m \in \mathbb{N}).$$

Note that now we have two important relations with the previous operators, namely $G_{n,0} = S_n$ and $G_{0,m} = F_m$, so the partial sums and the Fejér means are obtained as two extremal cases of $G_{n,m}$. This connection makes it possible for us to use the de la Vallée Poussin means as a bridge between the previous methods, and use them to describe a transition between them.

For the norm of $G_{n,m}$, $(n, m \in \mathbb{N})$, we state the following result, a direct consequence of [22, Theorem 1.2.2].

Proposition 1.8. *Suppose that* $(n, m \in \mathbb{N})$*. We have*

$$\|G_{n,m}\| = \frac{4}{\pi^2} \log \frac{n+m+1}{m+1} + O(1),$$

i.e. there exist positive constants c_1, c_2 *independent of* n, m *such that*

$$\frac{4}{\pi^2} \log \frac{n+m+1}{m+1} + c_1 \leq \|G_{n,m}\| \leq \frac{4}{\pi^2} \log \frac{n+m+1}{m+1} + c_2.$$

Compare this result with Theorem 1.1 and Corollary 1.7.

Applying the Banach–Steinhaus theorem once again, we have

Corollary 1.9. *For* $k \in \mathbb{N}$*, consider the sequences of natural pairs* (n_k, m_k) *and suppose that* $n_k \to +\infty$ *as* $k \to +\infty$*. The sequence* $(G_{n_k,m_k} f)$ *tends uniformly to* f *for every* $f \in C_{2\pi}$ *if and only if*

$$\sup_{k \in \mathbb{N}} \left\{ \log \frac{n_k + m_k}{m_k + 1} \right\} < +\infty.$$

It is also clear that $G_{n,m} : C_{2\pi} \to \mathcal{T}_{n+m}$ and $(G_{n,m} g)(x) = g(x)$ for any $g \in \mathcal{T}_n, x \in \mathbb{R}$, so the operator has some kind of projection property. In fact, we have an analogue of the Faber–Marcinkiewicz–Berman theorem due to Nikolaev [30].

Theorem 1.10. *Fix* $n, m \in \mathbb{N}$*,* $n \geq 1$ *and let* $T_n : C_{2\pi} \to \mathcal{T}_{n+m}$ *denote a de la Vallée Poussin type trigonometric projection operator, i.e. suppose that* $(T_n g)(x) = g(x)$*,* $(g \in \mathcal{T}_n, x \in \mathbb{R})$*. Now there exist a positive constant* $c \in \mathbb{R}$ *independent of* n, m *such that*

$$\|T_{n,m}\| \geq c \log \frac{n+m}{m+1}.$$

We remark that the relation $\|T_{n,m}\| \geq \|G_{n,m}\|$ does not hold generally.

As before, the operator $G_{n,m}$ can be expressed as a specific φ-sum as well.

Definition 1.11. For $\alpha = 1$ let $\varphi_\alpha := \varphi_1$ as defined before. Otherwise, for $\alpha \in [0,1)$ let φ_α be the unique function which is 1 on the interval $[-\alpha,\alpha]$, 0 on $\mathbb{R}\setminus[-1,1]$, and is linear on the nonempty intervals $[-1,-\alpha]$ and $[\alpha,1]$, i.e. if $\alpha \neq 1$ then on $x \in [0,1]$ we have

$$\varphi_\alpha(x) = \begin{cases} 1, & \text{if } 0 \le x \le \alpha \\ \dfrac{1-x}{1-\alpha}, & \text{if } \alpha < x \le 1. \end{cases}$$

The function φ_α (see Figure 1) is called the *Fejér summation function* if $\alpha = 0$ (note that it is the same function as φ_0 before), and generally a *de la Vallée Poussin type summation function* if $0 \le \alpha \le 1$.

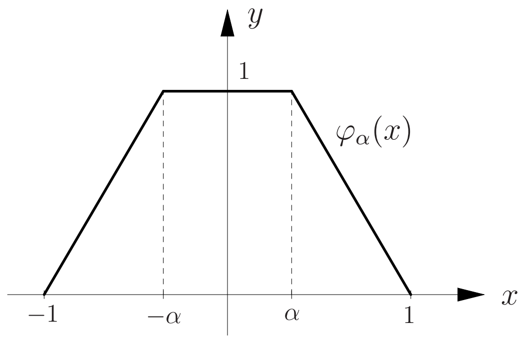

Figure 1.

Now it is clear that if $n, m \in \mathbb{N}$, $n + m \ge 1$ and the relation $\alpha = \frac{n}{n+m+1}$ holds, then by (1.6) we have $G_{n,m} = S^{\varphi_\alpha}_{n+m+1}$.

Also

$$\varphi_\alpha(t) = \frac{1}{1-\alpha}\varphi_0(t) - \frac{\alpha}{1-\alpha}\varphi_0\Big(\frac{t}{\alpha}\Big) \qquad (t \in \mathbb{R}),$$

therefore

$$\hat{\varphi}_\alpha(x) = \frac{1}{2(1-\alpha)\pi}\frac{\sin^2(x/2) - \sin^2(\alpha x/2)}{(x/2)^2} \qquad (x \in \mathbb{R}).$$

Consequently $\hat{\varphi}_\alpha \in L^1(\mathbb{R})$, and applying Theorem 1.5 we obtain the following result.

Corollary 1.12. *For any fixed* $\alpha \in [0,1)$ *the sequence* $(S_n^{\varphi_\alpha} f, n \in \mathbb{N})$ *uniformly converges to* f *on* $\mathbb{R}$ *for every function* $f \in C_{2\pi}$.

Note that this can also be obtained as a case of Corollary 1.9, since a fixed value of $\alpha \in [0,1)$ means that the ratio of n to $n+m+1$ is fixed, consequently the conditions of Corollary 1.9 hold.

A part of our work concerns the multivariate extensions of some results presented in this section. More accurately, some new results popped up lately regarding the multivariate extensions of Theorems 1.1, 1.4 and Corollary 1.7 for triangular sums of Fourier series ([41],[56]). As a sequel of these new results, we managed to obtain the corresponding variants of Proposition 1.8 and Theorem 1.10. The details are worked out in Chapter 2.

1.2 Summations of discrete trigonometric Fourier series

It is known that many theorems concerning convergence of Fourier series can be transferred to the convergence of trigonometric interpolation with equidistant points. These polynomials can be considered as partial sums of discrete Fourier series. In the paper of J. Marcinkiewicz [20] a systematic investigation of this subject is given. Also, some convergent summation processes were defined by L. Fejér and D. Jackson (see e.g. [62, X. §6]). After a paper by S. N. Bernstein [2], many authors have studied the summation of the trigonometric interpolation by the same methods which had previously been proved successful for the summation of Fourier series. S. Lozinski [19] showed that, in many cases, theorems on convergence or summability of Fourier series can be transferred to the convergence or summability of the trigonometric interpolation process with equidistant nodes.

These type of operators can be considered as special cases of discrete operators. The map $T : C_{2\pi} \to C_{2\pi}$ is called a *discrete operator* if for every $f \in C_{2\pi}$ the function $Tf \in C_{2\pi}$ is uniquely determined by the function values of f given at finitely many points of the interval $[-\pi, \pi]$.

1.2.1 Preliminaries and the discrete trigonometric φ-sums

The discrete version of the Fourier series (1.3) can be defined as follows (cf. e.g. [62], Vol. II, Chapter X). Let us fix a natural number $M \in \mathbb{N}^+ := \{1, 2, \ldots\}$ and consider the equidistant point system

$$X_M := \Big\{ x_{k,M} := k\frac{2\pi}{M} \ : \ k = 0, 1, \ldots, M-1 \ \Big\}, \tag{1.10}$$

and the discrete measure

$$\mu_M\Big(\{x_{k,M}\}\Big) := \mu_{k,M} := \frac{1}{M} \qquad (k = 0, 1, \ldots, M-1),$$

which generates the following discrete integral

$$\int_{X_M} f d\mu_M = \sum_{k=0}^{M-1} f(x_{k,M})\mu_{k,M} \qquad (f \in C_{2\pi}).$$

It is clear that

$$\langle f, g\rangle_M := \int_{X_M} f\overline{g} d\mu_M = \frac{1}{M}\sum_{k=0}^{M-1} f(x_{k,M})\overline{g}(x_{k,M}) \qquad (f, g \in C_{2\pi}) \tag{1.11}$$

is a scalar product on the space of all complex valued functions defined on X_M.

Again, consider the complex trigonometric system (1.2). For every fixed number $M \in \mathbb{N}^+$ we have

$$\langle \varepsilon_m, \varepsilon_l\rangle_M = \frac{1}{M}\sum_{k=0}^{N-1} \varepsilon_m(x_{k,M})\bar{\varepsilon}_l(x_{k,M}) =$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} e^{imx_{k,M}} e^{-ilx_{k,M}} = \frac{1}{M}\sum_{k=0}^{M-1} \Big(e^{i(m-l)\frac{2\pi}{M}}\Big)^k$$

for all $m, l \in \mathbb{Z}$. From this it follows that

$$\langle \varepsilon_m, \varepsilon_l \rangle_M = \begin{cases} 1, & \text{if } M \mid m-l, \\ 0, & \text{if } M \nmid m-l. \end{cases} \tag{1.12}$$

This means that every M consecutive terms of the sequence $(\varepsilon_j, j \in \mathbb{Z})$ are orthonormal with respect to the scalar product (1.11), i.e. for all fixed number $N \in \mathbb{Z}$ we have

$$\langle \varepsilon_m, \varepsilon_l \rangle_M = \delta_{m,l} \qquad (m, l \in \{N, N+1, \dots, N+M-1\}). \tag{1.13}$$

The discrete trigonometric Fourier coefficients with respect to the point system (1.10) of $f \in C_{2\pi}$ are defined by

$$\hat{f}_M(j) := \langle f, \varepsilon_j \rangle_M = \frac{1}{M} \sum_{k=0}^{M-1} f(x_{k,M}) e^{-ijx_{k,M}} \quad (j \in \mathbb{Z}). \tag{1.14}$$

From (1.13) it follows that the sequence $(\hat{f}_M(j), j \in \mathbb{Z})$ is periodic by M, i.e.

$$\hat{f}_M(j) = \hat{f}_M(j + lM) \qquad (l, j \in \mathbb{Z}). \tag{1.15}$$

The discrete Fourier series with respect to the point system (1.10) of the function $f \in C_{2\pi}$ is defined by

$$S_M[f] := \sum_{j \in \mathbb{Z}} \hat{f}_M(j) \varepsilon_j. \tag{1.16}$$

Now we introduce the discrete version of (1.6). Fix the summation function $\varphi \in \Phi$ and the number $M \in \mathbb{N}^+$. The *n-th discrete φ-sums with respect to the point system* (1.10) of the function $f \in C_{2\pi}$ are defined by

$$\left(S^{\varphi}_{n,M} f\right)(x) := \sum_{j \in \mathbb{Z}} \varphi\left(\frac{j}{n}\right) \hat{f}_M(j) \varepsilon_j(x) = \sum_{j=-n}^{n} \varphi\left(\frac{j}{n}\right) \hat{f}_M(j) \varepsilon_j(x)$$
$$(x \in \mathbb{R},\ f \in C_{2\pi},\ m \in \mathbb{N}). \tag{1.17}$$

Thus for every function $\varphi \in \Phi$ we have a two-parameter operator family.

Since φ is an even function thus for every $f \in C_{2\pi}$ and all $n, M \in \mathbb{N}$, $N \geq 1$ we have

$$\big(S^{\varphi}_{n,M} f\big)(x) = \frac{1}{M} \sum_{k=0}^{M-1} f(x_{k,M}) D^{\varphi}_{n}(x - x_{k,M}) \tag{1.18}$$

where the function D^{φ}_{n} is defined by (1.4).

It is clear that $S^{\varphi}_{n,M} f$ is a trigonometric polynomial of degree not exceeding n and

$$S^{\varphi}_{n,M} : C_{2\pi} \to \mathcal{T}_n$$

is a bounded linear operator.

Conditions of uniform convergence and other properties for these general operators are investigated in [39, 42], while a summary on many discrete linear interpolatory operators was given in [36]. In the following, we recall two important results concerning these operators.

1.2.2 The discrete trigonometric Natanson–Zuk theorem

In order to investigate the uniform convergence for the discrete case, first we have to choose a sequence of operators $S^{\varphi}_{n,M}$, since now we have a two-parameter operator family. This shall be done as explained below.

From the two-parameter operator family $(S^{\varphi}_{n,M},\ n \in \mathbb{N}, M \in \mathbb{N}^{+})$ we can choose a one-parameter family using two arbitrary index sequences $(n_k, k \in \mathbb{N})$ and $(M_k, k \in \mathbb{N})$. Thus we obtain a sequence of bounded linear operators:

$$S^{\varphi}_{n_k,M_k} : C_{2\pi} \to \mathcal{T}_{n_k} \qquad (k \in \mathbb{N}). \tag{1.19}$$

In this Section we investigate the uniform convergence of the operator sequence (1.19).

In 1997, F. Schipp and J. Bokor [31] published some results with respect to the discrete version of general φ-summation processes in the case when the summation function φ is a continuous function. Few years later, L. Szili and P. Vértesi [42] gave the following discrete version of the Natanson–Zuk theorem (see Theorem 1.5), which is also a generalization of a result in [31].

Theorem 1.13. *Suppose that one of the following two conditions holds:*

1^o $\varphi \in \Phi$ *and for the index sequences* $(n_k, k \in \mathbb{N})$, $(M_k, k \in \mathbb{N})$ *we have*

$$\lim_{k\to+\infty} n_k = +\infty \quad \textit{and} \quad \lim_{k\to+\infty}(M_k - n_k) = +\infty, \tag{1.20}$$

2^o $\varphi \in \Phi$, φ *is continuous at the point* 1 *(so it is continuous at* -1*, too) and the index sequences* $(n_k, k \in \mathbb{N})$ *and* $(M_k, k \in \mathbb{N})$ *satisfy the relations*

$$\lim_{k\to+\infty} n_k = +\infty \qquad \textit{and} \qquad M_k \geq n_k(1 + o(1))\ (k \to +\infty).$$[1]

Then the sequence $(S^{\varphi}_{n_k,M_k} f, k \in \mathbb{N})$ *uniformly converges on* $\mathbb{R}$ *to* f *for every* $f \in C_{2\pi}$ *if and only if the Fourier transform of* φ *belongs to* $L^1(\mathbb{R})$*.*

1.2.3 Interpolatory properties

As we stated in the beginning of the section, our aim is to investigate problems of trigonometric interpolation using the summation methods of discrete Fourier series. Therefore, it is natural to ask under what conditions is the operator $S^{\varphi}_{n,M}$ *interpolatory*, i.e. for a function $f \in C_{2\pi}$, when do the equations

$$f(x_{k,M}) = \big(S^{\varphi}_{n,M} f\big)(x_{k,M}) \qquad (x_{k,M} \in X_M)$$

hold for $k = 0, 1, \ldots, M-1$. When this happens, we also say that $S^{\varphi}_{n,M} f$ interpolates the function f at the points X_M.

Surprisingly, the interpolatory property of the operator $S^{\varphi}_{n,M}$ can be characterized by some symmetrical property of the summation function φ. We recall the following statement (cf. [39, Lemma A, p. 137]).

[1] $a_k = o(1)$ $(k \to +\infty)$ means that $a_k \to 0$ $(k \to +\infty)$.

Theorem 1.14. *Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is an even function supported in $[-1, 1]$, $\varphi(0) = 1$ and $M \geq n$. The polynomial $S^{\varphi}_{n,M} f$ interpolates the function f at the points X_M if and only if*

$$\varphi\left(\frac{j}{n}\right) + \varphi\left(\frac{M-j}{n}\right) = 1 \qquad (j = 1, 2, \ldots, M-1). \tag{1.21}$$

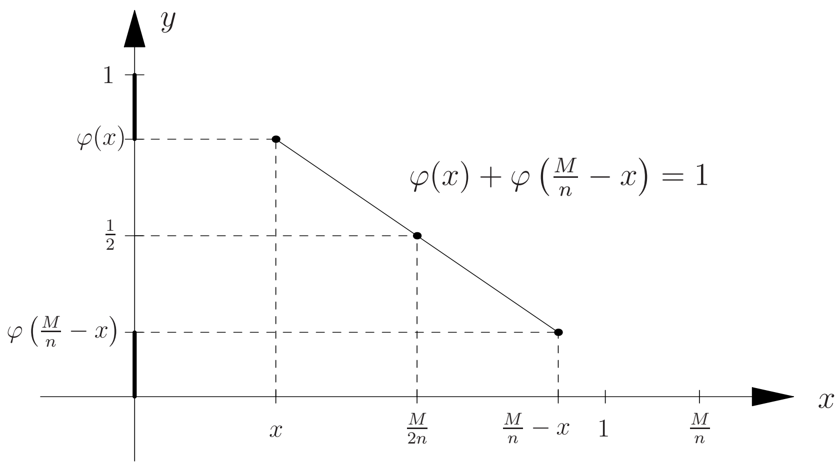

Figure 2.

This lemma visually states that for $j = 1, 2, \ldots, M-1$ the

$$\left(\frac{j}{n}, \varphi\left(\frac{j}{n}\right)\right)$$

points from the graph of φ are positioned symmetrically to the point $\left(\frac{M}{2n}, \frac{1}{2}\right)$ on the interval $\left[0, \frac{M}{n}\right]$. This property is demonstrated on Figure 2.

In Chapter 3, we discuss specific types of trigonometric interpolations, with main focus on the Lagrange and Hermite–Fejér interpolations. It turns out that the de la Vallée Poussin sums provide an excellent tool describing the transition and the connections between these two classic interpolation methods.

Chapter 2

Multivariate de la Vallée Poussin type projection operators

In this chapter we deal with the de la Vallée Poussin means of the triangular partial sums of multivariate Fourier series. We determine the exact order of the corresponding operator norms. The lower estimation of these norms will be extended to a class of projection operators having similar projection properties as the de la Vallée Poussin mean. The presented results are from our own work [26].

2.1 Introduction

Multivariate Fourier series has been the subject of intensive study. We may refer to the classical works of Zygmund [62, Ch. XVII] and Stein, Weiss [34, Ch. VII]. First, we introduce some notations.

Let $d > 1,\ d \in \mathbb{N}$ be fixed and $\mathbb{R}^d$ be the Euclidian d-dimensional space, and let $\mathbb{T}^d := \mathbb{R}^d \pmod{2\pi\mathbb{Z}^d}$ denote the d-dimensional torus.

Further, let $C(\mathbb{T}^d)$ denote the space of (complex valued) continuous functions on $\mathbb{T}^d$. By definition they are 2π-periodic in each variable.

For $g \in C(\mathbb{T}^d)$, we define its (multivariate) Fourier series by

$$g(\boldsymbol{\vartheta}) \sim \sum_{\mathbf{k}} \hat{g}(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\vartheta}}, \qquad \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\mathbf{t})\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{t}}\, d\mathbf{t}, \tag{2.1}$$

where in the above vector notation $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_d) \in \mathbb{T}^d$, $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$ and $\mathbf{k} \cdot \boldsymbol{\vartheta} = \sum_{l=1}^{d} k_l \vartheta_l$ (scalar product).

In the multivariate case, the partial sums of the Fourier series could be defined multiple ways. Our results in this chapter concern with the so–called triangular partial sums.

The *triangular* n-th partial sum of the Fourier series is defined by

$$S_{n,d}(g, \boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_1 \le n} \hat{g}(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\vartheta}} \qquad (n \in \mathbb{N}), \tag{2.2}$$

where $|\mathbf{k}|_1 = \sum_{l=1}^{d} |k_l|$ (the l_1 norm of the multiindex $\mathbf{k}$).

We remark that the *rectangular* n-th partial sum of the Fourier series is usually defined by

$$S_{n,d}^{[r]}(g, \boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_\infty \le n} \hat{g}(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\vartheta}} \qquad (n \in \mathbb{N}),$$

where $|\mathbf{k}|_\infty = \max_{l=1\ldots d} |k_l|$ (the l_∞ norm of the multiindex $\mathbf{k}$). In a way, the investigations regarding $S_{n,d}^{[r]}$ are apparent: in many cases they are essentially one variable problems (see the already mentioned works [34, 62]). Note that our results in this chapter are also true when the operator $S_{n,d}$ is replaced by $S_{n,d}^{[r]}$.

However, there are relatively few works dealing with the triangular (or l_1) sums (see Herriot [16]). In a recent paper [41] the authors gave the exact order of the norms of the operators $S_{n,d}$ together with some similar types of projection operators. Others were dealing with the so-called Fejér–summability (among some other summation methods) of the triangular partial sums [1, 56]. We recall the details of some of these results later.

Our aim is to investigate the de la Vallée Poussin means of the partial sums of Fourier series.

Definition 2.1. For $n, m \in \mathbb{N}$, let the de la Vallée Poussin mean $G_{n,m,d}$ of the Fourier series of the function g be defined by

$$G_{n,m,d}(g, \boldsymbol{\vartheta}) := \frac{1}{m+1} \sum_{j=0}^{m} S_{n+j,d}(g, \boldsymbol{\vartheta}). \tag{2.3}$$

Note that for $d = 1$ we obtain $G_{n,m,0} = G_{n,m}$ from Subsection 1.1.3, i.e. the one-dimensional de la Vallée Poussin mean.

Further, let $\mathcal{T}_{n,d}$ be the space of trigonometric polynomials of form

$$\sum_{|\mathbf{k}|_1 \le n} a_{\mathbf{k}} \, \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\vartheta}},$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$.

Definition 2.2. In the following, let $T_{n,m,d}$ denote a de la Vallée Poussin type projection operator, i.e. an arbitrary linear projection operator such that $T_{n,m,d} : \; C(\mathbb{T}^d) \to \mathcal{T}_{n+m,d}$ and $T_{n,m,d}(g, \boldsymbol{\vartheta}) = g(\boldsymbol{\vartheta})$ for every $g \in \mathcal{T}_{n,d}$.

Note that $G_{n,m,d}$ is an operator of this type, and that $S_{n,d} = G_{n,0,d}$.

In the next section we derive a formula for the kernel function of $G_{n,m,d}$.

2.2 Kernel function of the de la Vallée Poussin operator

Introducing the notation

$$D_{n,d}(\boldsymbol{\vartheta}) = \sum_{|\mathbf{k}|_1 \le n} \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\vartheta}} \qquad (n \ge 0), \tag{2.4}$$

where $\mathbf{k} \in \mathbb{Z}^d$, one can see that

$$S_{n,d}(g, \boldsymbol{\vartheta}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\boldsymbol{\vartheta} - \mathbf{t}) D_{n,d}(\mathbf{t}) d\mathbf{t},$$

where $g \in C(\mathbb{T}^d), \boldsymbol{\vartheta}, \mathbf{t} \in \mathbb{T}^d$ (cf. [34, Chs I, VII], [1]).

For $D_{n,d}(\boldsymbol{\vartheta})$, Xu proved the following relation (cf. [60, Lemma 1])

$$D_{n,d}(\boldsymbol{\vartheta}) = (-1)^{\left[\frac{d-1}{2}\right]} \sum_{l=1}^{d} \frac{2\cos\frac{\vartheta_l}{2}(\sin\vartheta_l)^{d-2}\operatorname{soc}\frac{2n+1}{2}\vartheta_l}{\prod_{j=1,j\neq l}^{d}(\cos\vartheta_l - \cos\vartheta_j)}, \tag{2.5}$$

where the function soc (sin or cos) is defined by

$$\operatorname{soc}\vartheta = \begin{cases} \sin\vartheta, & \text{if } d \text{ is odd;} \\ \cos\vartheta, & \text{if } d \text{ is even.} \end{cases}$$

Similarly, for $G_{n,m,d}$, from (2.3) we have

$$\begin{aligned} G_{n,m,d}(g,\boldsymbol{\vartheta}) =& \frac{1}{m+1}\sum_{j=0}^{m}\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(\boldsymbol{\vartheta}-\mathbf{t})D_{n+j,d}(\mathbf{t})d\mathbf{t} \\ =& \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(\boldsymbol{\vartheta}-\mathbf{t})\cdot\Big(\frac{1}{m+1}\sum_{j=0}^{m}D_{n+j,d}(\mathbf{t})\Big)d\mathbf{t}, \end{aligned}$$

so let us introduce the notation

$$V_{n,m,d}(\boldsymbol{\vartheta}) = \frac{1}{m+1}\sum_{j=0}^{m} D_{n+j,d}(\boldsymbol{\vartheta}). \tag{2.6}$$

This kernel function of the de la Vallée Poussin mean has an explicit form similar to the aforementioned result of Xu.

Theorem 2.3. *The kernel $V_{n,m,d}(\boldsymbol{\vartheta})$ takes the form*

$$V_{n,m,d}(\boldsymbol{\vartheta}) = (-1)^{\left[\frac{d-1}{2}\right]} \sum_{l=1}^{d} \frac{(\sin\vartheta_l)^{d-1}\sin\frac{m+1}{2}\vartheta_l\operatorname{soc}\frac{2n+m+1}{2}\vartheta_l}{\prod_{j=1,j\neq l}^{d}(\cos\vartheta_l-\cos\vartheta_j)\cdot(m+1)\sin^2\frac{\vartheta_l}{2}}. \tag{2.7}$$

Proof. From (2.5) and (2.6) we have, by changing the order of the sums,

$$V_{n,m,d}(\boldsymbol{\vartheta}) = (-1)^{\left[\frac{d-1}{2}\right]} \sum_{l=1}^{d} \frac{2\cos\frac{\vartheta_l}{2}(\sin\vartheta_l)^{d-2}\left(\sum_{k=0}^{m}\operatorname{soc}\frac{2(n+k)+1}{2}\vartheta_l\right)}{\prod_{j=1,j\neq l}^{d}(\cos\vartheta_l-\cos\vartheta_j)\cdot(m+1)}.$$

If d is odd, then

$$\begin{aligned}\sum_{k=0}^{m} \operatorname{soc} \frac{2(n+k)+1}{2}\vartheta_l =& \sum_{k=0}^{m} \sin\left(n+k+\frac{1}{2}\right)\vartheta_l \\ =& \frac{1}{2\sin\frac{\vartheta_l}{2}} \sum_{k=0}^{m} (\cos(n+k)\vartheta_l - \cos(n+k+1)\vartheta_l) \\ =& \frac{\cos n\vartheta_l - \cos(n+m+1)\vartheta_l}{2\sin\frac{\vartheta_l}{2}} \\ =& \frac{\sin\left(n+\frac{m+1}{2}\right)\vartheta_l \cdot \sin\frac{m+1}{2}\vartheta_l}{\sin\frac{\vartheta_l}{2}}.\end{aligned}$$

If d is even, then

$$\begin{aligned}\sum_{k=0}^{m} \operatorname{soc} \frac{2(n+k)+1}{2}\vartheta_l =& \sum_{k=0}^{m} \cos\left(n+k+\frac{1}{2}\right)\vartheta_l \\ =& \frac{1}{2\sin\frac{\vartheta_l}{2}} \sum_{k=0}^{m} (\sin(n+k+1)\vartheta_l - \sin(n+k)\vartheta_l) \\ =& \frac{\sin(n+m+1)\vartheta_l - \sin n\vartheta_l}{2\sin\frac{\vartheta_l}{2}} \\ =& \frac{\cos\left(n+\frac{m+1}{2}\right)\vartheta_l \cdot \sin\frac{m+1}{2}\vartheta_l}{\sin\frac{\vartheta_l}{2}}.\end{aligned}$$

Consequently, for any $d \geq 1$ we have

$$\sum_{k=0}^{m} \operatorname{soc} \frac{2(n+k)+1}{2}\vartheta_l = \frac{\operatorname{soc}\left(n+\frac{m+1}{2}\right)\vartheta_l \cdot \sin\frac{m+1}{2}\vartheta_l}{\sin\frac{\vartheta_l}{2}},$$

thus the proof of (2.7) is complete. □

2.3 Characterization of the operator norms

The concepts of maximum norm and operator norm are defined similarly to the one-dimensional case. For a function $g \in C(\mathbb{T}^d)$ let

$$\|g\| := \max_{\boldsymbol{\vartheta} \in \mathbb{T}^d} |g(\boldsymbol{\vartheta})|$$

and for the norm of $T_{n,m,d}$ let

$$\|T_{n,m,d}\| := \max_{\substack{g \in C(\mathbb{T}^d) \\ \|g\| \leq 1}} \|T_{n,m,d}(g, \boldsymbol{\vartheta})\| \qquad (n, m \in \mathbb{N}).$$

In a recent paper [41] the authors evaluate the exact order of the operator norm $\|S_{n,d}\| = \|G_{n,0,d}\|$, and give a lower bound for the norm of an arbitrary projection operator of type $T_{n,0,d}$, namely for $n \geq 2$ and $d \geq 1$ we have

$$\|T_{n,0,d}\| \geq \|S_{n,d}\| \sim (\log n)^d .$$

Note that the above inequality is a multivariate extension of the Faber–Marcinkiewicz–Berman theorem introduced in Chapter 1 (see Theorem 1.4), namely the operator $S_{n,d}$ has the smallest norm among all projection operators of the type $T_{n,0,d}$. The second half of the statement is also interesting, stating that the exact order of the norm of $S_{n,d}$ is $(\log n)^d$. This is the d-dimensional version of a weaker variant of Theorem 1.1.

A specific type of the de la Vallée Poussin means are the so-called $F_{m,d}$ Fejér means of the (triangular) partial sums of Fourier series, defined as

$$F_{m,d} := G_{0,m,d} = \frac{1}{m+1} \sum_{j=0}^{m} S_{j,d},$$

where $d \geq 1,\ m \in \mathbb{N}$. Note that for $d = 1$ these are the classical F_m Fejér means of Fourier series defined in Subsection 1.1.3.

Regarding the norms of these operators we recall the result of Weisz [56], i.e. there exists a c positive constant independent of m such that

$$\|F_{m,d}\| \leq c.$$

This is a direct multivariate extension of Corollary 1.7.

With the next theorem we try to establish a connection between the aforementioned results by evaluating the exact order of $\|G_{n,m,d}\|$ and giving a lower estimation for the norms of de la Vallée Poussin type projection operators in general.

Theorem 2.4. *Fix $d \geq 1$ and suppose that $n, m \in \mathbb{N}$ and $n \geq 1$. For any de la Vallée Poussin type projection operator $T_{n,m,d}$, we have*

$$\|T_{n,m,d}\| \geq c \left(\log \frac{n+m}{m+1} \right)^d, \tag{2.8}$$

where $c > 0$ is a positive constant independent of n, m.

Further, for the operator $G_{n,m,d}$ we have

$$\|G_{n,m,d}\| \leq c \left\{ \left(\log \frac{n+m}{m+1} \right)^d + 1 \right\}. \tag{2.9}$$

The first inequality is a multivariate extension of Nikolaev's result (see Theorem 1.10) for de la Vallée Poussin type projection operators in one dimension. The second inequality is a weaker d-dimensional variant of Proposition 1.8.

We conjecture that, similarly to the one-dimensional case (see [6]), the relation $\|T_{n,m,d}\| \geq \|G_{n,m,d}\|$ does not hold generally. Giving necessary and sufficient conditions for this inequality, e.g. generalizing the results of [6] to the multivariate case, may be a subject of further study.

Since the set of all trigonometric polynomials form a closed system in the Banach space $\left(C(\mathbb{T}^d), \|\cdot\|\right)$, the Banach–Steinhaus theorem may be applied, and we have the analogue of Corollary 1.9.

Corollary 2.5. *For $k \in \mathbb{N}$, consider the sequences of natural pairs (n_k, m_k) and suppose that $n_k \to +\infty$ as $k \to +\infty$. The sequence $(G_{n_k,m_k,d}(g, \cdot))$ tends uniformly to g for every $g \in C(\mathbb{T}^d)$ if and only if*

$$\sup_{k \in \mathbb{N}} \left\{ \left(\log \frac{n_k + m_k}{m_k + 1} \right)^d \right\} < +\infty.$$

Before we prove Theorem 2.4, we remark that the concept of φ-summations can be introduced in the multivariate case the same way as in the one-dimensional case. For example, for $\varphi \in \Phi$ one could consider the formula

$$S_{n,d}^{\varphi}(g, \boldsymbol{\vartheta}) = \sum_{j=0}^{n} \varphi\left(\frac{j}{n}\right) \sum_{|\mathbf{k}|_1 = j} \hat{g}(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\boldsymbol{\vartheta}}\,.$$

This way we can obtain the de la Vallée Poussin means by applying the same φ_α function as in Chapter 1 (see Definition 1.11). The characterization of the uniform convergence by the Fourier transform of φ (if possible) in this case is an open problem.

In the last section of this chapter, we prove Theorem 2.4.

2.4 Proof of Theorem 2.4

In order to show

$$\|T_{n,m,d}\| \geq c \left(\log \frac{n+m}{m+1}\right)^d,$$

the case $n = 1$ is obvious, so for every $n, m \in \mathbb{N}$, $n \geq 2$ we construct a trigonometric polynomial $f_{n,m}(\mathbf{t}) = \sum_{\mathbf{j}} a_{\mathbf{j}}\, \mathrm{e}^{i\mathbf{j}\mathbf{t}}$ with

$$\|f_{n,m}\| \leq 1 \quad \text{and} \quad |T_{n,m,d}\left(f_{n,m}(\cdot - \boldsymbol{\gamma}'), \boldsymbol{\gamma}'\right)| \geq c \left(\log \frac{n+m}{m+1}\right)^d \tag{2.10}$$

for an appropriate $\boldsymbol{\gamma}' \in \mathbb{T}^d$.

With these polynomials we have

$$\max_{\substack{g \in C(\mathbb{T}^d) \\ \|g\| \leq 1}} \|T_{n,m,d}(g, \cdot)\| \geq |T_{n,m,d}\left(f_{n,m}(\cdot - \boldsymbol{\gamma}'), \boldsymbol{\gamma}'\right)|$$

which proves (2.8).

Our proof is based on two ingredients. The first one is Fejér's classical example (see [24, Vol. II,Ch. 2/1]) and its application for the multivariate case [41]. We remark that the latter work also relies on Fejér's example and ideas communicated by Gábor Halász. The second one is Nikolaev's argument for one dimension (see [30]).

Now let us define the function

$$h(x) = (m+1)^{1-x}(n+m)^x.$$

Note that $h(0) = m+1$, $h(1) = n+m$ and for arbitrary $\alpha, \beta \in [0,1]$, $\alpha > \beta$ we have $h(\alpha) > h(\beta)$ and

$$\log \frac{h(\alpha)}{h(\beta)} = \log \left(\frac{n+m}{m+1}\right)^{\alpha-\beta} = (\alpha-\beta) \log \frac{n+m}{m+1}.$$

For the construction of $f_{n,m}$, first choose real numbers α_j, β_j $(j = 1, 2, \dots, d)$ for which

$$0 \le \beta_d < \alpha_d < \beta_{d-1} < \alpha_{d-1} < \dots < \beta_1 < \alpha_1 \le 1. \tag{2.11}$$

Let us consider the trigonometric polynomials

$$F_j(t) = \sum_{|k_j|=[h(\beta_j)]}^{[h(\alpha_j)]} \frac{1}{k_j} \mathrm{e}^{ik_jt}, \qquad (t \in [0, 2\pi), 1 \le j \le d).$$

As we know for the trigonometric polynomials

$$\mathcal{F}_k(t) = \sum_{0 \le |l| \le k} \frac{1}{l} \mathrm{e}^{ilt}, \qquad (t \in \mathbb{R}, k \in \mathbb{N})$$

we have

$$|\mathcal{F}_k(t)| = 2 \left| \sum_{l=1}^{k} \frac{\sin lt}{l} \right| \le 4\sqrt{\pi}$$

(see [13], [14], [24, Vol. I,(118)]). Therefore we get

$$|F_j(t)| = \left|\mathcal{F}_{[h(\alpha_j)]}(t) - \mathcal{F}_{[h(\beta_j)]-1}(t)\right| \le 8\sqrt{\pi} =: M. \tag{2.12}$$

Denoting the canonical unit vectors of $\mathbb{R}^d$ by $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, $(1 \le j \le d)$, we define the polynomial $g_{n,m}(\mathbf{t}) =: M^d f_{n,m}(\mathbf{t})$ as follows

$$\begin{aligned} g_{n,m}(\mathbf{t}) &= \mathrm{e}^{i(n+m)\mathbf{e}_1\cdot\mathbf{t}} \cdot F_d(-\mathbf{e}_d\cdot\mathbf{t})\cdot\prod_{j=1}^{d-1} F_j\left((\mathbf{e}_{j+1}-\mathbf{e}_j)\cdot\mathbf{t}\right) \\ &= \mathrm{e}^{i(n+m)t_1}\sum_{|k_d|=[h(\beta_d)]}^{[h(\alpha_d)]}\frac{\mathrm{e}^{-ik_dt_d}}{k_d}\prod_{j=1}^{d-1}\left(\sum_{|k_j|=[h(\beta_j)]}^{[h(\alpha_j)]}\frac{\mathrm{e}^{i(k_jt_{j+1}-k_jt_j)}}{k_j}\right), \end{aligned} \tag{2.13}$$

where $\prod_{j=1}^{0}\cdots := 1$ and $(\mathbf{e}_{j+1}-\mathbf{e}_j)\cdot\mathbf{t} = t_{j+1}-t_j$.

Using (2.12) we obtain that $|f_{n,m}(\mathbf{t})| \le 1$ $(t \in \mathbb{T}^d)$, i.e. the first requirement of (2.10) holds.

The polynomial $g_{n,m}(\mathbf{t})$ can be written as

$$g_{n,m}(\mathbf{t}) = \sum_{k_1,\ldots,k_d}\frac{1}{k_1k_2\ldots k_d}\mathrm{e}^{i\mathbf{kt}},$$

where $\mathbf{k} = (n+m-k_1, k_1-k_2, k_2-k_3, \ldots, k_{d-1}-k_d)$, and we take the summation for the indices $[h(\beta_j)] \le k_j \le [h(\alpha_j)]$, $(1 \le j \le d)$.

Now we write $f_{n,m}(\mathbf{t})$ as

$$\begin{aligned} f_{n,m}(\mathbf{t}) &= \frac{1}{M^d}\sum_{\substack{k_1,\ldots,k_d\\ |\mathbf{k}|_1\le n}}\frac{1}{k_1k_2\ldots k_d}\mathrm{e}^{i\mathbf{kt}} + \frac{1}{M^d}\sum_{\substack{k_1,\ldots,k_d\\ |\mathbf{k}|_1> n}}\frac{1}{k_1k_2\ldots k_d}\mathrm{e}^{i\mathbf{kt}} \\ &=: \psi_{n,m}(\mathbf{t}) + \chi_{n,m}(\mathbf{t}). \end{aligned} \tag{2.14}$$

We prove that in the sum $\psi_{n,m}(\mathbf{t})$ only the *positive* indices $k_1, \ldots, k_d$ appear. Observe that $|n+m-k_1| = n+m-k_1$. Using $|k_j-k_{j+1}| \ge k_j-k_{j+1}$, $(1 \le j \le d-1)$, we get

$$\begin{aligned} n \ge |\mathbf{k}|_1 &= |n+m-k_1| + |k_1-k_2| + |k_2-k_3| + \ldots + |k_{d-1}-k_d| \\ &\ge n+m-k_d, \end{aligned} \tag{2.15}$$

whence we obtain that $k_d > 0$.

Now let us suppose that for a fixed index j^*, $(1 \le j^* \le d-1)$, we have $k_{j^*} < 0$ (consequently $|k_{j^*} - k_{j^*+1}| = -k_{j^*} + k_{j^*+1}$), and $k_j > 0$ for every $j^* < j \le d$. We get

$$|\mathbf{k}|_1 \ge n + m - 2k_j + 2k_{j+1} - k_d > n + m, \tag{2.16}$$

which is a contradiction. This means that in $\psi_{n,m}(\mathbf{t})$ only positive indices $k_1, \dots, k_d$ appear indeed. On the other hand, if all of the indices $k_1, \dots, k_d$ are positive, then we have $|\mathbf{k}|_1 = n + m - k_d \le n$, consequently

$$\psi_{n,m}(\mathbf{0}) = \frac{1}{M^d} \prod_{j=1}^{d} \left(\sum_{k_j=[h(\beta_j)]}^{[h(\alpha_j)]} \frac{1}{k_j} \right) \ge c \left(\log \frac{n+m}{m+1} \right)^d . \tag{2.17}$$

Now we show that in the sum $\chi_{n,m}(\mathbf{t})$ we have $|\mathbf{k}|_1 > n + m$. The previous argument shows that in $\chi_{n,m}(\mathbf{t})$ not all of the indices $k_1, \dots, k_d$ are positive, and from inequalities (2.15) and (2.16) we immediately obtain that if for any index j^*, $(1 \le j^* \le d)$, we have $k_{j^*} < 0$, then $|\mathbf{k}|_1 > n + m$.

So we have

$$\chi_{n,m}(\mathbf{t}) = \frac{1}{M^d} \sum_{\substack{k_1,\dots,k_d \\ |\mathbf{k}|_1 > n+m}} \frac{1}{k_1 k_2 \dots k_d} \, \mathrm{e}^{i\mathbf{k}\mathbf{t}} \, .$$

For the function $f_{n,m}(\mathbf{t} - \boldsymbol{\gamma})$ ($\boldsymbol{\gamma} \in \mathbb{T}^d$ arbitrary) we have

$$\begin{aligned} f_{n,m}(\mathbf{t} - \boldsymbol{\gamma}) &= \psi_{n,m}(\mathbf{t} - \boldsymbol{\gamma}) + \frac{1}{M^d} \sum_{\substack{k_1,\dots,k_d \\ |\mathbf{k}|_1 > n+m}} \frac{1}{k_1 k_2 \dots k_d} \, \mathrm{e}^{i\mathbf{k}(\mathbf{t} - \boldsymbol{\gamma})} \\ &= \psi_{n,m}(\mathbf{t} - \boldsymbol{\gamma}) + \sum_{\substack{k_1,\dots,k_d \\ |\mathbf{k}|_1 > n+m}} p_{\mathbf{k}}(\mathbf{t}) \, \mathrm{e}^{-i\mathbf{k}\boldsymbol{\gamma}}, \end{aligned}$$

where $p_{\mathbf{k}}(\mathbf{t})$ are trigonometric polynomials with degree $|\mathbf{k}|_1 > n + m$.

Applying the linear projection operator $T_{n,m,d}$ we have $T_{n,m,d}(\psi_{n,m}(\cdot - \boldsymbol{\gamma}), \mathbf{t}) = \psi_{n,m}(\mathbf{t} - \boldsymbol{\gamma})$, so

$$T_{n,m,d}\left(f_{n,m}(\cdot - \boldsymbol{\gamma}), \mathbf{t}\right) = \psi_{n,m}(\mathbf{t} - \boldsymbol{\gamma}) + \sum_{\substack{k_1,\ldots,k_d \\ |\mathbf{k}|_1 > n+m}} P_{\mathbf{k}}(\mathbf{t})\, \mathrm{e}^{-i\mathbf{k}\boldsymbol{\gamma}},$$

where $P_{\mathbf{k}}(\mathbf{t}) \in \mathcal{T}_{n+m,d}$ for every multiindex $\mathbf{k}$.

Now by letting $\mathbf{t} = \boldsymbol{\gamma}$ we get

$$\begin{aligned} T_{n,m,d}\left(f_{n,m}(\cdot - \boldsymbol{\gamma}), \boldsymbol{\gamma}\right) &= \psi_{n,m}(\mathbf{0}) + \sum_{\substack{k_1,\ldots,k_d \\ |\mathbf{k}|_1 > n+m}} P_{\mathbf{k}}(\boldsymbol{\gamma})\, \mathrm{e}^{-i\mathbf{k}\boldsymbol{\gamma}} \\ &= \psi_{n,m}(\mathbf{0}) + Q_{n,m}(\boldsymbol{\gamma}), \end{aligned}$$

where $Q_{n,m}(\boldsymbol{\gamma})$ is a trigonometric polynomial without a constant term, consequently $\int_{\mathbb{T}^d} Q_{n,m}(\mathbf{t})\, d\mathbf{t} = 0$, so that $Q_{n,m}$ has to change sign on $\mathbb{T}^d$. By continuity, $Q_{n,m}$ has to be zero somewhere, i.e. there exists $\boldsymbol{\gamma}' \in \mathbb{T}^d$ so that $Q_{n,m}(\boldsymbol{\gamma}') = 0$, and

$$T_{n,m,d}\left(f_{n,m}(\cdot - \boldsymbol{\gamma}'), \boldsymbol{\gamma}'\right) = \psi_{n,m}(\mathbf{0}).$$

Using (2.17) we get that the second requirement of (2.10) holds, thus the proof of (2.8) is complete.

In order to prove (2.9), we only need to show that there exists a positive constant $c > 0$ independent of n, m such that

$$\|G_{n,m,d}\| \leq c \left\{ \left(\log \frac{n+m}{m+1} \right)^d + 1 \right\}$$

holds. As a consequence of the Riesz representation theorem (see [8, IV. 6.3] or [61]), we have

$$\|G_{n,m,d}\| = \|V_{n,m,d}\|_1 = \int_{\mathbb{T}^d} |V_{n,m,d}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta}. \tag{2.18}$$

For the proof we use a generalized version of the argument of Weisz [56] for the $n = 0$ case.

In order to estimate the integral in (2.18), we introduce an inductive form for the kernel function $V_{n,m,d}$. The n-th divided difference of a function f at the (pairwise distinct) knots $x_1, \ldots, x_n \in \mathbb{R}$ is introduced inductively as

$$[x_1]f := f(x_1), \qquad [x_1, \ldots, x_n]f := \frac{[x_1, \ldots, x_{n-1}]f - [x_2, \ldots, x_n]f}{x_1 - x_n}. \tag{2.19}$$

Note that the difference is a symmetric function of the knots.

Berens and Xu proved [1] [60] that

$$D_{n,d}(\boldsymbol{\vartheta}) = [\cos\vartheta_1, \ldots, \cos\vartheta_d]s_{n,d}, \qquad (\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n) \in \mathbb{T}^d),$$

where

$$s_{n,d}(\cos t) = (-1)^{[(d-1)/2]}\ 2\cos\frac{t}{2}\ (\sin t)^{d-2}\operatorname{soc}\frac{2n+1}{2}t.$$

Similarly, by (2.7) one can see that for $V_{n,m,d}$ we have

$$V_{n,m,d}(\boldsymbol{\vartheta}) = [\cos\vartheta_1, \ldots, \cos\vartheta_d]v_{n,d}, \qquad (\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n) \in \mathbb{T}^d),$$

where

$$v_{n,m,d}(\cos t) = (-1)^{[(d-1)/2]}\ \frac{(\sin t)^{d-1}\sin\frac{m+1}{2}t\operatorname{soc}\frac{2n+m+1}{2}t}{(m+1)\sin^2\frac{t}{2}}.$$

If we apply the inductive definition (2.19) to $V_{n,m,d}$, then in the denominator we have factors of elements of the following table:

$$\begin{array}{llll}
\cos\vartheta_1 - \cos\vartheta_d & & & \\
\cos\vartheta_1 - \cos\vartheta_{d-1} & \cos\vartheta_2 - \cos\vartheta_d & & \\
\ldots & & & \\
\cos\vartheta_1 - \cos\vartheta_{d-k+1} & \cos\vartheta_2 - \cos\vartheta_{d-k+2} & \ldots & \cos\vartheta_k - \cos\vartheta_d \\
\ldots & & & \\
\cos\vartheta_1 - \cos\vartheta_2 & \cos\vartheta_2 - \cos\vartheta_3 & \ldots & \cos\vartheta_{d-1} - \cos\vartheta_d
\end{array}$$

We have to choose exactly one factor from each row in the following manner: let $\mathcal{I}$ denote the set of sequences of integer pairs $((i_n, j_n),\ n = 1, \ldots, d-1)$ for which $i_1 = 1$, $j_1 = d$, (i_n) is non-decreasing, (j_n) is non-increasing and if (i_n, j_n) is given then let $(i_{n+1}, j_{n+1}) = (i_n, j_n - 1)$ or $(i_{n+1}, j_{n+1}) = (i_n + 1, j_n)$. Observe that the difference $\cos\vartheta_{i_k} - \cos\vartheta_{j_k}$ is in the k-th row of the table ($k = 1, \ldots, d-1$). So the factors we have chosen can be written as $\prod_{l=1}^{d-1} \cos\vartheta_{i_l} - \cos\vartheta_{j_l}$, and with these we have

$$\begin{aligned} V_{n,m,d}(\boldsymbol{\vartheta}) &= \sum_{(i_l,j_l)\in\mathcal{I}} (-1)^{i_{d-1}-1} \frac{[\cos\vartheta_{i_{d-1}}, \cos\vartheta_{j_{d-1}}]v_{n,m,d}}{\prod_{l=1}^{d-2}(\cos\vartheta_{i_l} - \cos\vartheta_{j_l})} \\ &= \sum_{(i_l,j_l)\in\mathcal{I}} (-1)^{i_{d-1}-1} \frac{v_{n,m,d}(\cos\vartheta_{i_{d-1}}) - v_{n,m,d}(\cos\vartheta_{j_{d-1}})}{\prod_{l=1}^{d-1}(\cos\vartheta_{i_l} - \cos\vartheta_{j_l})} \\ &=: \sum_{(i_l,j_l)\in\mathcal{I}} V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta}). \end{aligned} \tag{2.20}$$

A similar argument was used by Weisz [56, page 102] to represent the Dirichlet kernel $D_{n,d}$, see more details there.

Due to symmetrical properties, it is enough to estimate the integral (2.18) on $\left[0, \frac{\pi}{2}\right]^d$. We may also suppose that $\frac{\pi}{2} > \vartheta_1 > \vartheta_2 > \ldots > \vartheta_d > 0$.

We will need the following estimations of the kernel functions. Note that these are generalized versions of [56, Lemmas 1,2].

Lemma 2.6. *Let k_1 and k_2 be positive integers satisfying $1 \le k_1 \le k_2 \le d$ and $\alpha = 0$ or 1. For all $0 \le \beta, \gamma, \delta \in \mathbb{R}$, $\beta(k_1 - 1) + \delta(k_2 - k_1) + \gamma(d - k_2) < 1 + \alpha$ we have*

$$\begin{aligned} &|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| \le \\ &c\frac{\vartheta_{j_{d-1}}^{\beta(k_1-1)+\delta(k_2-k_1)+\gamma(d-k_2)-1-\alpha}}{(m+1)^\alpha \prod_{l=1}^{k_1-1}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\beta} \prod_{l=k_1}^{k_2-1}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\delta} \prod_{l=k_2}^{d-1}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\gamma}}. \end{aligned} \tag{2.21}$$

Proof. First, let $\alpha = 1$. Since

$$|v_{n,m,d}(\cos t)| \le \left| \frac{2(\sin t)^{d-2}}{(m+1)\sin\frac{t}{2}} \right|$$

and

$$\cos\vartheta_{i_l} - \cos\vartheta_{j_l} = 2\sin\frac{\vartheta_{i_l} - \vartheta_{j_l}}{2}\sin\frac{\vartheta_{i_l} + \vartheta_{j_l}}{2},$$

we have

$$|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| \le \frac{(\sin\vartheta_{i_{d-1}})^{d-2}/\sin\frac{\vartheta_{i_{d-1}}}{2} + (\sin\vartheta_{j_{d-1}})^{d-2}/\sin\frac{\vartheta_{j_{d-1}}}{2}}{(m+1)\prod_{l=1}^{d-1}\sin\frac{\vartheta_{i_l}-\vartheta_{j_l}}{2}\sin\frac{\vartheta_{i_l}+\vartheta_{j_l}}{2}},$$

and so we can conclude

$$|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| \le c\frac{\vartheta_{i_{d-1}}^{d-3} + \vartheta_{j_{d-1}}^{d-3}}{(m+1)\prod_{l=1}^{d-1}(\vartheta_{i_l}-\vartheta_{j_l})(\vartheta_{i_l}+\vartheta_{j_l})}.$$

Now, using $\vartheta_{i_l}+\vartheta_{j_l} > \vartheta_{i_{d-1}} > \vartheta_{j_{d-1}}$ and then $\vartheta_{i_l}+\vartheta_{j_l} > \vartheta_{i_l}-\vartheta_{j_l}$, we obtain

$$\begin{aligned}
&|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| \le \\
&c\frac{\vartheta_{i_{d-1}}^{d-3+(\beta-1)(k_1-1)+(\delta-1)(k_2-k_1)+(\gamma-1)(d-k_2)} + \vartheta_{j_{d-1}}^{d-3+(\beta-1)(k_1-1)+(\delta-1)(k_2-k_1)+(\gamma-1)(d-k_2)}}{(m+1)\prod_{l=1}^{k_1-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\beta}\prod_{l=k_1}^{k_2-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\delta}\prod_{l=k_2}^{d-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\gamma}} \\
&\le c\frac{\vartheta_{j_{d-1}}^{\beta(k_1-1)+\delta(k_2-k_1)+\gamma(d-k_2)-2}}{(m+1)\prod_{l=1}^{k_1-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\beta}\prod_{l=k_1}^{k_2-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\delta}\prod_{l=k_2}^{d-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\gamma}}
\end{aligned}$$

for all $0 \le \beta,\delta,\gamma \in \mathbb{R}$, $\beta(k_1-1)+\delta(k_2-k_1)+\gamma(d-k_2) < 1+\alpha$.

If $\alpha = 0$, using

$$\left|\frac{\sin\frac{m+1}{2}t}{(m+1)\sin\frac{t}{2}}\right| \le 1,$$

we have

$$|v_{n,m,d}(\cos t)| \le \left|2(\sin t)^{d-2}\right|,$$

and the statement can be proved similarly. □

Lemma 2.7. *Let k_1 and k_2 be positive integers satisfying $1 \le k_1 \le k_2 \le d$ and $\alpha = 0$ or 1. For all $0 \le \beta,\delta,\gamma \in \mathbb{R}$, $\beta(k_1-1)+\delta(k_2-k_1)+\gamma(d-k_2-1) < 1+\alpha$ we have*

$$\begin{aligned}
&|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| \le \\
&c\frac{(n+m)\vartheta_{j_{d-1}}^{\beta(k_1-1)+\delta(k_2-k_1)+\gamma(d-k_2-1)-1-\alpha}}{(m+1)^{\alpha}\prod_{l=1}^{k_1-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\beta}\prod_{l=k_1}^{k_2-1}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\delta}\prod_{l=k_2}^{d-2}(\vartheta_{i_l}-\vartheta_{j_l})^{1+\gamma}}.
\end{aligned} \tag{2.22}$$

Proof. Similarly to (2.20) we can write

$$
\begin{aligned}
D_{n,d}(\boldsymbol{\vartheta}) &= \sum_{(i_l,j_l)\in\mathcal{I}} (-1)^{i_{d-1}-1} \frac{d_{n,d}(\cos\vartheta_{i_{d-1}}) - d_{n,d}(\cos\vartheta_{j_{d-1}})}{\prod_{l=1}^{d-1}(\cos\vartheta_{i_l} - \cos\vartheta_{j_l})} \\
&=: \sum_{(i_l,j_l)\in\mathcal{I}} D_{n,d,(i_l,j_l)}(\boldsymbol{\vartheta}),
\end{aligned} \tag{2.23}
$$

and then we have

$$
V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta}) = \frac{(-1)^{i_{d-1}-1}}{m+1} \frac{\sum_{k=n}^{n+m} d_{k,d}(\cos\vartheta_{i_{d-1}}) - d_{k,d}(\cos\vartheta_{j_{d-1}})}{\prod_{l=1}^{d-1}(\cos\vartheta_{i_l} - \cos\vartheta_{j_l})}.
$$

The Lagrange mean value theorem imply that there exists $\vartheta_{i_{d-1}} > \xi > \vartheta_{j_{d-1}}$ such that

$$
V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta}) = \frac{(-1)^{i_{d-1}-1}}{m+1} \frac{\sum_{k=n}^{n+m} H'_{k,d}(\xi)(\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}})}{\prod_{l=1}^{d-1}(\cos\vartheta_{i_l} - \cos\vartheta_{j_l})},
$$

where

$$
H_{k,d}(t) = (-1)^{[(d-1)/2]}\ 2\cos\frac{t}{2}\ (\sin t)^{d-2} \operatorname{soc} \frac{2k+1}{2} t.
$$

Let $\alpha = 1$. We obtain

$$
\begin{aligned}
|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| \le & c \frac{(\sin\xi)^{d-2} + (n+m)(\sin\xi)^{d-2}}{(m+1)\sin\frac{\xi}{2}\prod_{l=1}^{d-1}\sin\frac{\vartheta_{i_l}-\vartheta_{j_l}}{2}\sin\frac{\vartheta_{i_l}+\vartheta_{j_l}}{2}}(\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}}) \\
& + c\frac{(\sin\xi)^{d-3}}{\prod_{l=1}^{d-1}\sin\frac{\vartheta_{i_l}-\vartheta_{j_l}}{2}\sin\frac{\vartheta_{i_l}+\vartheta_{j_l}}{2}}(\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}}).
\end{aligned}
$$

We have used the formulas from the proof of (2.7) and that $|\sum_{k=n}^{n+m} \operatorname{soc}(k+1/2)t| \le m+1$.

Now

$$
\begin{aligned}
|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| &\le c\frac{n+m}{m+1}\frac{(\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}})\xi^{d-3}}{\prod_{l=1}^{d-1}(\vartheta_{i_l} - \vartheta_{j_l})(\vartheta_{i_l} + \vartheta_{j_l})} \\
\le c\frac{n+m}{m+1} & \frac{\xi^{d-4+(\beta-1)(k_1-1)+(\delta-1)(k_2-k_1)+(\gamma-1)(d-k_2-1)}}{\prod_{l=1}^{k_1-1}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\beta}\prod_{l=k_1}^{k_2-1}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\delta}\prod_{l=k_2}^{d-2}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\gamma}} \\
\le c\frac{n+m}{m+1} & \frac{\vartheta_{j_{d-1}}^{\beta(k_1-1)+\delta(k_2-k_1)+\gamma(d-k_2-1)-2}}{\prod_{l=1}^{k_1-1}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\beta}\prod_{l=k_1}^{k_2-1}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\delta}\prod_{l=k_2}^{d-2}(\vartheta_{i_l} - \vartheta_{j_l})^{1+\gamma}},
\end{aligned}
$$

for all $0 \leq \beta, \delta, \gamma \in \mathbb{R}$, $\beta(k_1-1)+\delta(k_2-k_1)+\gamma(d-k_2-1)<2$.

The proof is similar for $\alpha = 0$. □

Now we proceed with the estimation of the integral (2.18). Since in (2.20) the number of sequences in $\mathcal{I}$ only depends on d, it is enough to show that for any $(i_l, j_l) \in \mathcal{I}$ the inequality

$$\int_{\{\frac{\pi}{2}>\vartheta_1>\vartheta_2>\ldots>\vartheta_d>0\}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq c\left\{\left(\log\frac{n+m}{m+1}\right)^d + 1\right\}$$

holds.

First let us divide the domain $\left[0, \frac{\pi}{2}\right]^d$ into the following parts.

$$\begin{aligned} \mathcal{S}' &:= \left\{\boldsymbol{\vartheta} \in \mathbb{T}^d : \ \frac{1}{n+m} \geq \vartheta_1 > \vartheta_2 > \ldots > \vartheta_d > 0\right\} \\ \mathcal{S} &:= \left[0, \frac{\pi}{2}\right]^d \setminus \mathcal{S}' \end{aligned}$$

Since $|\mathcal{S}'| := \int_{\mathcal{S}'} 1 \ d\boldsymbol{\vartheta} \leq c\frac{1}{(n+m)^d}$ and $|V_{n,m,d}(\boldsymbol{\vartheta})| \leq c(n+m)^d$, we get

$$\int_{\mathcal{S}'} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq c.$$

Now we consider $\mathcal{S}$. For an $(i_l, j_l) \in \mathcal{I}$, let us define

$$\begin{aligned} \mathcal{S}_{(i_l,j_l),k_1,k_2} := \Big\{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{i_l} - \vartheta_{j_l} > \frac{1}{m+1}, l = 1, \ldots, k_1 - 1, \\ \frac{1}{n+m} < \vartheta_{i_l} - \vartheta_{j_l} \leq \frac{1}{m+1}, l = k_1, \ldots, k_2 - 1, \\ \frac{1}{n+m} \geq \vartheta_{i_l} - \vartheta_{j_l}, l = k_2, \ldots, d-1\Big\}, \end{aligned}$$

where $1 \leq k_1 \leq k_2 \leq d$, $k_1, k_2 \in \mathbb{Z}$. Note that for $n = 1$ the middle interval is empty so we only have two intervals for the values of $\vartheta_{i_l} - \vartheta_{j_l}$, consequently one index k_1 is enough to complete the proof, similarly to [56].

First let us consider the case $k_2 < d$ and $k_1 > 1$. For the domain $\mathcal{S}_{(i_l,j_l),k_1,k_2,1} := \mathcal{S}_{(i_l,j_l),k_1,k_2} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} > \frac{1}{m+1}\}$, using (2.22) with $\alpha = 1$ and $\gamma = \delta = 0$ gives

$$\int_{\mathcal{S}_{(i_l,j_l),k_1,k_2,1}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq$$
$$c \int_{\mathcal{S}_{(i_l,j_l),k_1,k_2,1}} \frac{n+m}{m+1} \prod_{l=1}^{k_1-1} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})^{\beta+1}} \prod_{l=k_1}^{k_2-1} \frac{1}{\vartheta_{i_l} - \vartheta_{j_l}}$$
$$\cdot \prod_{l=k_2}^{d-2} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})^{1-\frac{1}{d-k_2}}} \cdot \frac{1}{(\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}})^{1-\frac{1}{d-k_2}}} \vartheta_{j_{d-1}}^{\beta(k_1-1)-2} d\boldsymbol{\vartheta},$$

since $\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}} \leq \vartheta_{i_l} - \vartheta_{j_l}$. Now we choose the indices $j_{d-1}(= i'_d)$, $i_{d-1}(= i'_{d-1})$ and then i_{d-2} if $i_{d-2} \neq i_{d-1}$ or j_{d-2} if $j_{d-2} \neq j_{d-1}$. (Exactly one of these two cases is satisfied.) If we repeat this process we get an injective sequence $(i'_l, l = 1, \dots, d)$. We integrate the term $\vartheta_{i_1} - \vartheta_{j_1}$ in $\vartheta_{i'_1}$, the term $\vartheta_{i_2} - \vartheta_{j_2}$ in $\vartheta_{i'_2}$, ..., and finally the term $\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}}$ in $\vartheta_{i'_{d-1}}$ and $\vartheta_{j_{d-1}}$ in $\vartheta_{i'_d}$. By the definition of the domain $\mathcal{S}_{(i_l,j_l),k_1,k_2,1}$ we have

$$\int_{\mathcal{S}_{(i_l,j_l),k_1,k_2,1}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq c \frac{n+m}{m+1} \prod_{l=1}^{k_1-1} \left(\frac{1}{m+1}\right)^{-\beta} \cdot \left(\log \frac{n+m}{m+1}\right)^{k_2-k_1}$$
$$\cdot \prod_{l=k_2}^{d-2} \left(\frac{1}{n+m}\right)^{\frac{1}{d-k_2}} \cdot \left(\frac{1}{n+m}\right)^{\frac{1}{d-k_2}} \cdot \left(\frac{1}{m+1}\right)^{\beta(k_1-1)-1}$$
$$\leq c \left(\log \frac{n+m}{m+1}\right)^{k_2-k_1}$$

for any $0 < \beta < \frac{1}{k_1-1}$.

The same argument holds for the domain $\mathcal{S}_{(i_l,j_l),k_1,k_2,2} := \mathcal{S}_{(i_l,j_l),k_1,k_2} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} \leq \frac{1}{m+1}\}$ after using (2.22) with $\alpha = 0$ and $\gamma = \delta = 0$.

Next consider the case $k_2 < d-1$ and $k_1 = 1$. Now for the domain $\mathcal{S}_{(i_l,j_l),1,k_2,1} := \mathcal{S}_{(i_l,j_l),1,k_2} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} > \frac{1}{m+1}\}$, using (2.22) with $\alpha = 1$ and $\gamma = \delta = 0$ gives

$$\begin{aligned}
\int_{\mathcal{S}_{(i_l,j_l),1,k_2,1}} &|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq \\
c \int_{\mathcal{S}_{(i_l,j_l),1,k_2,1}} &\frac{n+m}{m+1} \prod_{l=1}^{k_2-1} \frac{1}{\vartheta_{i_l} - \vartheta_{j_l}} \prod_{l=k_2}^{d-2} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})^{1-\frac{1}{d-k_2}}} \\
&\cdot \frac{1}{(\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}})^{1-\frac{1}{d-k_2}}} \vartheta_{j_{d-1}}^{-2} d\boldsymbol{\vartheta} \\
\leq c \frac{n+m}{m+1} \left(\log \frac{n+m}{m+1}\right)^{k_2-k_1} &\cdot \prod_{l=k_2}^{d-2} \left(\frac{1}{n+m}\right)^{\frac{1}{d-k_2}} \cdot \left(\frac{1}{n+m}\right)^{\frac{1}{d-k_2}} \cdot \left(\frac{1}{m+1}\right)^{-1} \\
&\leq c \left(\log \frac{n+m}{m+1}\right)^{k_2-k_1}.
\end{aligned}$$

The same argument for the domain $\mathcal{S}_{(i_l,j_l),1,k_2,2} := \mathcal{S}_{(i_l,j_l),1,k_2} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \frac{1}{n+m} < \vartheta_{j_{d-1}} \leq \frac{1}{m+1}\}$, using (2.22) with $\alpha = 0$ and $\gamma = \delta = 0$ gives

$$\int_{\mathcal{S}_{(i_l,j_l),1,k_2,2}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq c \left(\log \frac{n+m}{m+1}\right)^{k_2-k_1+1}.$$

Finally for $\mathcal{S}_{(i_l,j_l),1,k_2,3} := \mathcal{S}_{(i_l,j_l),1,k_2} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} \leq \frac{1}{n+m}\}$, using (2.22) with $\alpha = 0$ and $\delta = 0$ we get

$$\begin{aligned}
\int_{\mathcal{S}_{(i_l,j_l),1,k_2,3}} &|V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq \\
c \int_{\mathcal{S}_{(i_l,j_l),1,k_2,3}} &(n+m) \prod_{l=1}^{k_2-1} \frac{1}{\vartheta_{i_l} - \vartheta_{j_l}} \prod_{l=k_2}^{d-2} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})^{1+\gamma-\frac{1}{d-k_2}}} \\
&\cdot \frac{1}{(\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}})^{1-\frac{1}{d-k_2}}} \vartheta_{j_{d-1}}^{\gamma(d-k_2-1)-1} d\boldsymbol{\vartheta} \\
\leq c(n+m) \left(\log \frac{n+m}{m+1}\right)^{k_2-k_1} &\cdot \prod_{l=k_2}^{d-2} \left(\frac{1}{n+m}\right)^{\frac{1}{d-k_2}-\gamma} \cdot \left(\frac{1}{n+m}\right)^{\frac{1}{d-k_2}} \\
\cdot \left(\frac{1}{n+m}\right)^{\gamma(d-k_2-1)} &\leq c \left(\log \frac{n+m}{m+1}\right)^{k_2-k_1}
\end{aligned}$$

for any $0 < \gamma < \frac{1}{d-k_2}$.

Now let $k_2 = d-1$ and $k_1 = 1$. For $\mathcal{S}_{(i_l,j_l),1,d-1,1} := \mathcal{S}_{(i_l,j_l),1,d-1} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} > \frac{1}{m+1}\}$, using (2.22) with $\alpha = 1$ and $\delta = 0$ we get

$$\int_{\mathcal{S}_{(i_l,j_l),1,d-1,1}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq$$
$$c \int_{\mathcal{S}_{(i_l,j_l),1,d-1,1}} \frac{n+m}{m+1} \prod_{l=1}^{d-2} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})} \cdot (\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}})^0 \cdot \vartheta_{j_{d-1}}^{-2} d\boldsymbol{\vartheta}$$
$$\leq c \frac{n+m}{m+1} \cdot \left(\log \frac{n+m}{m+1}\right)^{d-2} \cdot \frac{1}{n+m} \cdot \left(\frac{1}{m+1}\right)^{-1} \leq c \left(\log \frac{n+m}{m+1}\right)^{d-2}.$$

For $\mathcal{S}_{(i_l,j_l),1,d-1,2} := \mathcal{S}_{(i_l,j_l),1,d-1} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \frac{1}{n+m} < \vartheta_{j_{d-1}} \leq \frac{1}{m+1}\}$, using (2.22) with $\alpha = 0$ and $\delta = 0$ in a similar way gives

$$\int_{\mathcal{S}_{(i_l,j_l),1,d-1,2}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq c \left(\log \frac{n+m}{m+1}\right)^{d-1}.$$

Finally for $\mathcal{S}_{(i_l,j_l),1,d-1,3} := \mathcal{S}_{(i_l,j_l),1,d-1} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} \leq \frac{1}{n+m}\}$, using (2.22) with $\alpha = 0$ yields

$$\int_{\mathcal{S}_{(i_l,j_l),1,d-1,3}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq$$
$$c \int_{\mathcal{S}_{(i_l,j_l),1,d-1,3}} (n+m) \prod_{l=1}^{d-2} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})^{1+\delta}} \cdot (\vartheta_{i_{d-1}} - \vartheta_{j_{d-1}})^0 \cdot \vartheta_{j_{d-1}}^{\delta(d-2)-1} d\boldsymbol{\vartheta}$$
$$\leq c(n+m) \prod_{l=1}^{d-2} \left(\frac{1}{n+m}\right)^{-\delta} \cdot \frac{1}{n+m} \cdot \left(\frac{1}{n+m}\right)^{\delta(d-2)} \leq c$$

for any $0 < \delta < \frac{1}{d-2}$.

Now we only need to investigate the case $k_2 = d$. First, let $k_1 > 1$. For the domain $\mathcal{S}_{(i_l,j_l),k_1,d,1} := \mathcal{S}_{(i_l,j_l),k_1,d} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} > \frac{1}{m+1}\}$, using (2.21) with $\alpha = 1$ and

$\delta = 0$ we get

$$\begin{aligned}
&\int_{\mathcal{S}_{(i_l,j_l),k_1,d,1}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq \\
c &\int_{\mathcal{S}_{(i_l,j_l),k_1,d,1}} \frac{1}{m+1} \prod_{l=1}^{k_1-1} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})^{1+\beta}} \prod_{l=k_1}^{d-1} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})} \cdot \vartheta_{j_{d-1}}^{\beta(k_1-1)-2} d\boldsymbol{\vartheta} \\
&\leq c \frac{1}{m+1} \left(\log \frac{n+m}{m+1} \right)^{d-k_1} \cdot \prod_{l=1}^{k_1-1} \left(\frac{1}{m+1} \right)^{-\beta} \cdot \left(\frac{1}{m+1} \right)^{\beta(k_1-1)-1} \\
&\leq c \left(\log \frac{n+m}{m+1} \right)^{d-k_1}
\end{aligned}$$

for any $0 < \beta < \frac{2}{k_1-1}$.

We obtain the same result for $\mathcal{S}_{(i_l,j_l),k_1,d,2} := \mathcal{S}_{(i_l,j_l),k_1,d} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} \leq \frac{1}{m+1}\}$, using (2.21) with $\alpha = 0$ and $\delta = 0$.

Finally let $k_1 = 1$. For $\mathcal{S}_{(i_l,j_l),1,d,1} := \mathcal{S}_{(i_l,j_l),k_1,d} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} > \frac{1}{m+1}\}$, we use (2.21) with $\alpha = 1$ and $\delta = 0$ to obtain

$$\begin{aligned}
\int_{\mathcal{S}_{(i_l,j_l),1,d,1}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq & c \int_{\mathcal{S}_{(i_l,j_l),1,d,1}} \frac{1}{m+1} \prod_{l=1}^{d-1} \frac{1}{\vartheta_{i_l} - \vartheta_{j_l}} \cdot \vartheta_{j_{d-1}}^{-2} d\boldsymbol{\vartheta} \\
\leq & c \frac{1}{m+1} \left(\log \frac{n+m}{m+1} \right)^{d-1} \cdot \left(\frac{1}{m+1} \right)^{-1} \\
\leq & c \left(\log \frac{n+m}{m+1} \right)^{d-1} .
\end{aligned}$$

For the domain $\mathcal{S}_{(i_l,j_l),1,d,2} := \mathcal{S}_{(i_l,j_l),1,d} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \frac{1}{n+m} < \vartheta_{j_{d-1}} \leq \frac{1}{m+1}\}$, the same argument using (2.21) with $\alpha = 0$ and $\delta = 0$ gives

$$\int_{\mathcal{S}_{(i_l,j_l),k_1,d,2}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq c \left(\log \frac{n+m}{m+1} \right)^{d} .$$

For the final domain $\mathcal{S}_{(i_l,j_l),1,d,3} := \mathcal{S}_{(i_l,j_l),1,d} \cap \{\boldsymbol{\vartheta} \in \mathcal{S} : \vartheta_{j_{d-1}} \leq \frac{1}{n+m}\}$, let us apply (2.21) with $\alpha = 0$.

$$\begin{aligned}\int_{\mathcal{S}_{(i_l,j_l),1,d,3}} |V_{n,m,d,(i_l,j_l)}(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq & c \int_{\mathcal{S}_{(i_l,j_l),1,d,3}} \prod_{l=1}^{d-1} \frac{1}{(\vartheta_{i_l} - \vartheta_{j_l})^{1+\delta}} \cdot \vartheta_{j_{d-1}}^{\delta(d-1)-1} d\boldsymbol{\vartheta} \\ \leq & c \prod_{l=1}^{d-1} \left(\frac{1}{n+m}\right)^{-\delta} \cdot \left(\frac{1}{n+m}\right)^{\delta(d-1)} \leq c,\end{aligned}$$

for any $0 < \delta < \frac{1}{d-1}$. Thus we proved our statement. □

Chapter 3

De la Vallée Poussin sums in trigonometric interpolation

In this chapter we discuss some classic methods of trigonometric interpolation, mainly the well known (trigonometric) Lagrange and Hermite–Fejér interpolations, using the tools of discrete φ-summations introduced in Section 1.2. We also define the de la Vallée Poussin sums, which will be used as a tool to describe the transition between these two classic methods. We give general properties, the precise operator norm and (uniform) convergence order for these cases. The results of this chapter are from our own work [27].

3.1 Some specific types of interpolation

We show that using the specific $\varphi \in \Phi$ functions defined in Subsection 1.1.3 for a discrete operator $S^{\varphi}_{n,M}f$, $(n \in \mathbb{N}, M \in \mathbb{N}^{+})$ defined by (1.17), we may obtain some well known interpolation methods. The point system X_M, $(M \in \mathbb{N}^{+})$ is defined by (1.10).

3.1.1 Lagrange interpolation

For a fixed $M \in \mathbb{N}^+$, a trigonometric Lagrange interpolation polynomial of a function $f \in C_{2\pi}$ is a trigonometric polynomial of degree $\leq \left[\frac{M}{2}\right]$ which interpolates f at the points of X_M, i.e. the equations

$$f(x_{k,M}) = \left(S^{\varphi_\alpha}_{n,M} f\right)(x_{k,M}) \qquad (x_{k,M} \in X_M)$$

hold for $k = 0, 1, \ldots, M-1$.

When $M = 2m+1$ is odd, letting $n = m$ and using φ_1 of Subsection 1.1.3, it is clear that $S^{\varphi_1}_{m,M} f$ has a degree $\leq m = \left[\frac{M}{2}\right]$ and, by Theorem 1.14, interpolates $f \in C_{2\pi}$. We denote the operator $S^{\varphi_1}_{m,M}$ by L_M for odd M. It is known that in the odd case, the Lagrange interpolation polynomial is unique [39].

When $M = 2m$ is even, letting $n = m$ and defining $\varphi_1^* \in \Phi$ as the element of Φ which equals to 1 on $(-1,1)$ and $\varphi_1^*(1) = \frac{1}{2}$, it is clear that $S^{\varphi_1^*}_{m,M} f$ has a degree $\leq m = \left[\frac{M}{2}\right]$ and, by Theorem 1.14, interpolates $f \in C_{2\pi}$. We denote the operator $S^{\varphi_1^*}_{m,M}$ by L_M for even M. It is also known that in the even case, the Lagrange interpolation polynomial is *not* uniquely determined [39].

We already know that $\hat{\varphi}_1^* = \hat{\varphi}_1 \notin L^1(\mathbb{R})$, so considering Theorem 1.13, we obtain the result of Faber [11] for the trigonometric Lagrange interpolation:

Corollary 3.1. *There exists some $f \in C_{2\pi}$ for which*

$$\|f - (L_M f)\|_\infty \nrightarrow 0 \quad \text{as} \quad M \to +\infty.$$

For the trigonometric Lagrange interpolation, the following results are known (see e.g. [22]). Later we are going to obtain this as a corollary of our results.

Proposition 3.2. *The exact norm of operator L_M is given by*

$$\|L_M\| = \frac{2}{\pi} \log M + O(1).$$

If $E_n(f)$ notates the error of the best approximating trigonometric polynomial of degree $\leq n$, we have

$$\|L_M f - f\|_\infty \leq \left\{\frac{2}{\pi}\log M + O(1)\right\} \cdot E_m(f),$$

where $m = n - 1 = \left[\frac{M}{2}\right]$.

Compare these results to Theorems 1.1, 1.3 and Corollary 1.2 for the partial sums of trigonometric Fourier series.

3.1.2 Hermite–Fejér interpolation

For a fixed $M \in \mathbb{N}^+$, the trigonometric Hermite–Fejér interpolation polynomial of a function $f \in C_{2\pi}$ is the trigonometric polynomial $H_M f$ of degree $\leq M-1$ which satisfies the following Hermite–Fejér interpolation properties:

$$(H_M f)(x_{k,M}) = f(x_{k,M}), \quad (H_M f)'(x_{k,M}) = 0 \quad (k = 0, 1, \dots, M-1).$$

It was shown by Szili [39, Theorem 1, p. 142] that the operator $S^{\varphi_0}_{M,M}$ satisfies the conditions above, and it is known that the trigonometric polynomial $H_M f$ uniquely exists for $f \in C_{2\pi}$ [22], so we have $H_M = S^{\varphi_0}_{M,M}$.

We already know that $\hat{\varphi}_0 \in L^1(\mathbb{R})$, so applying Theorem 1.13, we obtain the trigonometric version of Fejér's classical result for first kind Chebyshev roots in the unweighted case. (See e.g. [37, p. 165], [53].), and also the order of the norm of H_M by the Banach–Steinhaus theorem.

Corollary 3.3. *The sequence $(H_M f, n \in \mathbb{N})$ uniformly converges to f on $\mathbb{R}$ for every function $f \in C_{2\pi}$. Moreover, there exists a positive constant $c \in \mathbb{R}$ independent of M such that*

$$\|H_M\| \leq c.$$

Compare these results to Corollaries 1.6 and 1.7 regarding the Fejér means of trigonometric Fourier series.

3.1.3 De la Vallée Poussin type interpolation

Let us consider the de la Vallée Poussin type summation functions $\varphi_\alpha \in \Phi$, ($\alpha \in [0,1)$) (see Definition 1.11). The case $\alpha = 1$ is left out because of practical reasons, as we will see at the generation of Lagrange interpolation by these summation functions.

Note that now the $j = n$ term of the sum (1.4) is zero for every φ_α-summation, so *the degree of the polynomial* $S_{n,M}^{\varphi_\alpha} f$ *(*$f \in C_{2\pi}$*) is at most* $n-1$.

It is clear that the operator $S_{n,M}^{\varphi_\alpha}$ is not interpolatory in general, i.e. the polynomial $S_{n,M}^{\varphi_\alpha} f$ does not interpolate f for some $f \in C_{2\pi}$. Considering Theorem 1.14, we give the following necessary and sufficient condition for $S_{n,M}^{\varphi_\alpha} f$ interpolating the function $f \in C_{2\pi}$.

Theorem 3.4. *Let* $\alpha \in [0,1)$*.* $S_{n,M}^{\varphi_\alpha} f$ *interpolates* $f \in C_{2\pi}$ *on the point system* X_M *if and only if one of these conditions holds:*
a) $\alpha = \dfrac{M}{n} - 1$;
b) $M = 2n-1$ *and* $\alpha > 1 - \dfrac{1}{n}$.

Remark that for arbitrary parameters M, n, α_1 satisfying condition b), there exists $\alpha_2 \in [0,1)$ such that

$$S_{n,M}^{\varphi_{\alpha_1}} f = S_{n,M}^{\varphi_{\alpha_2}} f \qquad (f \in C_{2\pi})$$

and condition a) holds for parameters M, n, α_2.

Indeed, suppose that $M = 2n-1$ and $\alpha_1 > 1 - \frac{1}{n}$, and let $\alpha_2 = \frac{M}{n} - 1 = 1 - \frac{1}{n}$. It is clear that condition a) holds for the values M, n, α_2, and one can easily see that in (1.17) we have

$$\varphi_{\alpha_1}\left(\frac{j}{n}\right) = \varphi_{\alpha_2}\left(\frac{j}{n}\right) = \begin{cases} 1, & \text{if } j = -n+1, -n+2, \ldots, n-1, \\ 0, & \text{if } j = -n, n. \end{cases}$$

Consequently $S_{n,M}^{\varphi_{\alpha_1}} f = S_{n,M}^{\varphi_{\alpha_2}} f$ for every $f \in C_{2\pi}$.

Proof of Theorem 3.4. First, let us assume that *a)* $\alpha = \frac{M}{n} - 1$. In this case it is enough to show that the graph of φ_α on the interval $\left(0, \frac{M}{n}\right)$ is symmetrical to the point $\left(\frac{M}{2n}, \frac{1}{2}\right)$. Namely when it is true, then

$$\varphi_\alpha(x) + \varphi_\alpha\left(\frac{M}{n} - x\right) = 1$$

holds and considering Theorem 1.14 the polynomial $S^{\varphi_\alpha}_{n,M} f$ interpolates f.

The graph of φ_α is symmetrical to the point $\left(\frac{M}{2n}, \frac{1}{2}\right)$ because the center points of intervals $\left(0, \frac{M}{n}\right)$ and $(\alpha, 1)$ are the same, and

$$\varphi_\alpha\left(\frac{\alpha+1}{2}\right) = \frac{1 - \frac{\alpha+1}{2}}{1-\alpha} = \frac{1}{2}.$$

Now let us assume that condition *b)* holds. From $\alpha \geq 1 - \frac{1}{n}$ we have that $\varphi_\alpha\left(\frac{n-1}{n}\right) = 1$, and also we can see that

$$\varphi_\alpha\left(\frac{M-(n-1)}{n}\right) = \varphi_\alpha\left(\frac{(2n-1)-(n-1)}{n}\right) = \varphi_\alpha(1) = 0.$$

This means that (1.21) holds for $j = n-1$ and $j = n$, as well. It also implies that for every $j < n-1$ we have $\varphi_\alpha\left(\frac{j}{n}\right) = 1$, and $\varphi_\alpha\left(\frac{M-j}{n}\right) = 0$, so (1.21) holds for every $j \leq n-1$. From this we get that it is true for every $j \geq n$, and by using Theorem 1.14 we proved that $S^{\varphi_\alpha}_{n,M} f$ interpolates f.

Conversely let us assume that $S^{\varphi_\alpha}_{n,M} f$ interpolates $f \in C_{2\pi}$, that means condition (1.21) holds. We consider two cases:

i) If $\alpha < 1 - \frac{1}{n}$, then we have $\frac{n-1}{n} \in (\alpha, 1)$. Using (1.21) for $j = n$, and considering $\varphi_\alpha(1) = 0$, we get $\varphi_\alpha\left(\frac{M-n}{n}\right) = 1$, whence $\frac{M-n}{n} \leq \alpha$.

Now using (1.21) for $j = n-1$ we can write

$$\varphi_\alpha\left(\frac{M-n}{n}\right) - \varphi_\alpha\left(\frac{M-n+1}{n}\right) = 1 - \varphi_\alpha\left(\frac{M-n+1}{n}\right) =$$
$$= \varphi_\alpha\left(\frac{n-1}{n}\right) = \varphi_\alpha\left(\frac{n-1}{n}\right) - \varphi_\alpha(1),$$

and considering that φ_α is linear on $[\alpha, 1]$, this can only happen if $\alpha = \frac{M-n}{n}$, that means condition *a)* holds.

ii) If $\frac{n-1}{n} \leq \alpha$, then we only have to show that $M = 2n-1$. Consider (1.21) for $j = n-1$. We get

$$\varphi_\alpha\left(\frac{M-(n-1)}{n}\right) = 0,$$

because in this case $\varphi_\alpha\left(\frac{n-1}{n}\right) = 1$. It can only happen if $\frac{M-(n-1)}{n} \geq 1$, so $M \geq 2n-1$.

Observe that we have

$$\varphi_\alpha\left(\frac{M-n}{n}\right) = 1$$

as well, because $\varphi_\alpha\left(\frac{n}{n}\right) = 0$. This means that $\frac{M-n}{n} < \frac{n}{n}$, so $M < 2n$. Putting these together we have $M = 2n-1$. □

With this result at hand, we are in the position to prove the following statement, which shows that the classical cases of Lagrange and Hermite–Fejér interpolations can be obtained as specific de la Vallée Poussin type interpolations as well.

Theorem 3.5. a) *The (trigonometric) Lagrange interpolation, odd case: If*

$$M = 2m+1, \quad n = m+1 = \left[\frac{M}{2}\right] + 1 \quad \text{and} \quad \alpha = \frac{M-n}{n} = 1 - \frac{1}{n}$$

then $S_{n,M}^{\varphi_\alpha} = L_M$*, i.e.* $S_{n,M}^{\varphi_\alpha} f$ *is the uniquely determined trigonometric polynomial of degree* $\leq [\frac{M}{2}] = m = n-1$ *which interpolates the function* $f \in C_{2\pi}$ *at points* X_M*.*

b) *The (trigonometric) Lagrange interpolation, even case: If*

$$M = 2m, \quad n = m+1 = \frac{M}{2} + 1 \quad \text{and} \quad \alpha = \frac{M-n}{n} = 1 - \frac{2}{n}$$

then $S_{n,M}^{\varphi_\alpha} = L_M$*, i.e.* $S_{n,M}^{\varphi_\alpha} f$ *is a (not uniquely determined) trigonometric polynomial of degree* $\leq \frac{M}{2} = m = n-1$ *which interpolates the function* $f \in C_{2\pi}$ *at points* X_M*.*

c) *The (trigonometric) Hermite–Fejér interpolation: Fix an arbitrary natural number M. Let $n = M$, $\alpha = 0$ and $f \in C_{2\pi}$. Then $S_{n,M}^{\varphi_\alpha} = H_M$, i.e. $S_{n,M}^{\varphi_\alpha} f$ is the uniquely determined trigonometric polynomial of degree $\leq (n-1)$ which satisfies the Hermite–Fejér interpolation properties (see Subsection 3.1.2).*

Proof. **a)** The degree of the polynomial $S_{n,M}^{\varphi_\alpha} f$ in this case is $n - 1 = \left[\frac{M}{2}\right]$, and it interpolates f at points X_M because condition *b)* of Theorem 3.4 holds.

b) The degree of polynomial $S_{n,M}^{\varphi_\alpha} f$ in this case is $n - 1 = \frac{M}{2} = \left[\frac{M}{2}\right]$, because M is an even number.

Also we have

$$\alpha = \frac{n-2}{n} = \frac{2(n-1)-n}{n} = \frac{M-n}{n},$$

so condition *a)* of Theorem 3.4 holds, meaning that $S_{n,M}^{\varphi_\alpha} f$ interpolates f.

c) It was shown (cf. [39, Theorem 1, p. 142]) that $S_{M,M}^{\varphi} f$ is the Hermite–Fejér interpolation polynomial if φ is the Fejér summation function φ_0. From Theorem 3.4 condition *b)* cannot hold, so necessarily $\alpha = \frac{M-n}{n} = 0$, meaning $M = n$. □

Note that here we obtained the Lagrange interpolation polynomials by using summation functions different from the ones in Subsection 3.1.1.

The approximation properties of de la Vallée Poussin type interpolations will be discussed in the next section.

3.2 Approximation properties

Here we present our results regarding the approximation properties of de la Vallée Poussin type interpolation operators $S_{n,M}^{\varphi_\alpha}$, $(\alpha \in [0,1), n \in \mathbb{N}, M \in \mathbb{N}^+)$, introduced in Subsection 3.1.3. We remark that similar types of operators were investigated (among others) in [18]. Most of our results are generalizations of the de la Vallée Poussin type interpolations introduced and discussed by Bernstein in his paper [4].

We investigated the projection properties thoroughly for the de la Vallée Poussin sums of single- and multivariate trigonometric Fourier series. Let us present the following result regarding the projection property of $S_{n,M}^{\varphi_\alpha}$.

Theorem 3.6. *Let $\alpha \in [0,1)$ be an arbitrary number and suppose that $n \leq M$. If*

$$0 \leq s \leq \min\{n\alpha, M-n\},$$

then for every trigonometric polynomial $T \in \mathcal{T}_s$ the identity

$$\big(S_{n,M}^{\varphi_\alpha} T\big)(x) = T(x) \qquad (x \in \mathbb{R})$$

holds.

Proof. Let us consider an arbitrary polynomial

$$T(x) = \sum_{l=-s}^{s} c_l e^{ilx} \qquad (x \in \mathbb{R}).$$

Then

$$\big(S_{n,M}^{\varphi_\alpha} T\big)(x) = \sum_{j=-n}^{n} \varphi_\alpha\big(\frac{j}{n}\big)\hat{T}_M(j)e^{ijx},$$

where

$$\hat{T}_M(j) = \frac{1}{M}\sum_{l=-s}^{s} c_l \sum_{k=0}^{M-1} e^{ilx_{k,M}} e^{-ijx_{k,M}}.$$

We have $s \leq M-n$ and $s \leq n\alpha < n$, which means that

$$s-(-s) < M-n+n = M,$$

so in $\hat{T}_M(j)$ the inner sum is 1 if $l=j$ and 0 otherwise.

From this we get

$$\big(S_{n,M}^{\varphi_\alpha} T\big)(x) = \sum_{j=-s}^{s} \varphi_\alpha\big(\frac{j}{n}\big) \cdot c_j e^{ijx}.$$

Since

$$s \leq n\alpha \Leftrightarrow \frac{s}{n} \leq \alpha$$

then $\varphi_\alpha\left(\frac{j}{n}\right) = 1$ for every $-s \le j \le s$. □

Next, we give a two-sided estimation of the operator norm

$$\begin{aligned} \|S_{n,M}^{\varphi_\alpha}\| &= \sup\{\|S_{n,M}^{\varphi_\alpha} f\|_\infty : \|f\|_\infty = 1\} = \\ &= \max_{x\in[0,2\pi)} \left\{ \frac{2}{M} \sum_{k=0}^{M-1} \left| D_n^{\varphi_\alpha}(x - x_{k,M}) \right| \right\} \end{aligned} \tag{3.1}$$

in the case when the operator $S_{n,M}^{\varphi_\alpha}$ is interpolatory. We will present the proof of the following theorem in the final section of this chapter, as it is quite lengthy.

Theorem 3.7.

$$\|S_{n,M}^{\varphi_\alpha}\| = \frac{2}{\pi} \log N + O(1),$$

i.e. to any interpolatory operator $S_{n,M}^{\varphi_\alpha}$ *there exist positive constants* c_1, c_2 *independent of* n *and* M *such that the following inequalities hold*

$$\frac{2}{\pi} \log N + c_1 \le \|S_{n,M}^{\varphi_\alpha}\| \le \frac{2}{\pi} \log N + c_2,$$

for every above numbers n, M*, where* $N := \frac{M}{2n-M}$.

So now we have the precise norm of these operators.

As before, let us denote the error of the n-th degree best approximation of $f \in C_{2\pi}$ by

$$E_n(f) := \inf_{T\in\mathcal{T}_n} \|f - T\|_\infty.$$

Choose the index sequences $(n_k, k \in \mathbb{N})$ and $(M_k, k \in \mathbb{N})$ arbitrarily. We shall investigate the convergence of the operator sequence

$$S_{n_k,M_k}^{\varphi_\alpha} : \left(C_{2\pi}, \|\cdot\|_\infty\right) \to \left(C_{2\pi}, \|\cdot\|_\infty\right) \qquad (k \in \mathbb{N}). \tag{3.2}$$

Using the estimations above, we have (cf. [4, p. 150]) the following result.

Theorem 3.8. *Let* $f \in C_{2\pi}$*. Consider the trigonometric polynomial sequence* $S_{n_k,M_k}^{\varphi_{\alpha_k}} f$ $(k \in \mathbb{N})$*. Suppose that* *i)* $M_k \to +\infty$ *(*$k \to +\infty$*) and let*

ii) $\alpha_k = \dfrac{M_k - n_k}{n_k}, \quad N_k = \dfrac{M_k}{2n_k - M_k} \quad (k \in \mathbb{N}).$

Then $S^{\varphi_{\alpha_k}}_{n_k,M_k} f$ *interpolates* f *at the points* X_{M_k} *for every* $k \in \mathbb{N}$*, and*

$$\|S^{\varphi_{\alpha_k}}_{n_k,M_k} f - f\|_\infty \le \left\{\frac{2}{\pi}\log N_k + O(1)\right\} \cdot E_{M_k-n_k}(f) \qquad (f \in C_{2\pi}).$$

Proof. Let

$$\rho_k := \|S^{\varphi_{\alpha_k}}_{n_k,M_k} f - f\|_\infty.$$

We denote by $Q_{M_k-n_k}$ the trigonometric polynomial of degree $M_k - n_k$ for which

$$\|f - Q_{M_k-n_k}\|_\infty = E_{M_k-n_k}(f).$$

Consider the inequality

$$\begin{aligned} &\left|\left(S^{\varphi_{\alpha_k}}_{n_k,M_k} f\right)(x) - f(x)\right| \le \\ &\le \left|\left(S^{\varphi_{\alpha_k}}_{n_k,M_k} f\right)(x) - Q_{M_k-n_k}(x)\right| + \left|Q_{M_k-n_k}(x) - f(x)\right|. \end{aligned} \tag{3.3}$$

Using Theorem 3.6 the interpolatory polynomial can be written in the following form:

$$\left(S^{\varphi_{\alpha_k}}_{n_k,M_k} f\right)(x) = Q_{M_k-n_k}(x) + \left(S^{\varphi_{\alpha_k}}_{n_k,M_k}(f - Q_{M_k-n_k})\right)(x).$$

Using Theorem 3.7 leads to the following inequality:

$$\left|\left(S^{\varphi_{\alpha_k}}_{n_k,M_k} f\right)(x) - Q_{M_k-n_k}(x)\right| \le \left\{\frac{2}{\pi}\log N + O(1)\right\} \cdot E_{M_k-n_k}(f),$$

which together with (3.3) gives

$$\begin{aligned} \rho_k &\le \left\{1 + \frac{2}{\pi}\log N + O(1)\right\} \cdot E_{M_k-n_k}(f) = \\ &= \left\{\frac{2}{\pi}\log N + O(1)\right\} \cdot E_{M_k-n_k}(f). \end{aligned}$$

□

Corollary 3.9. *Let $f \in C_{2\pi}$. Consider the polynomial sequence $S^{\varphi_{\alpha_k}}_{n_k,M_k} f$ $(k \in \mathbb{N})$. Suppose that conditions i) and ii) of Theorem 3.8 hold, and*

iii) the sequence

$$N_k = \frac{M_k}{2n_k - M_k} \qquad (k \in \mathbb{N})$$

is bounded.

Then the sequence $S^{\varphi_{\alpha_k}}_{n_k,M_k} f$ $(k \in \mathbb{N})$ uniformly converges to f.

By Theorem 3.8, the order of the convergence is near the best approximation.

For a sequence of the (trigonometric) Lagrange interpolation polynomials, if $M_k \to +\infty$ $(k \to +\infty)$ then the sequence

$$N_k = \frac{M_k}{2n_k - M_k} = \left[\frac{M_k}{2}\right]$$

is not bounded, so we do not have uniform convergence for a sequence of operators $(L_M, M \in \mathbb{N}^+)$, but Theorem 3.7 with the above values yield

$$\|L_M\| = \log M + O(1),$$

so we deduced Proposition 3.2 from our results.

A sequence of the (trigonometric) Hermite–Fejér polynomials satisfies conditions *i)-iii)* and therefore $H_M f$ uniformly converges to f for any $f \in C_{2\pi}$, so we obtain the results presented in Subsection 3.1.2.

Moreover for any fixed $0 \le \alpha < 1$, if conditions *i)* and *ii)* hold for a sequence $S^{\varphi_\alpha}_{n_k,M_k} f$ then

$$N_k = \frac{M_k}{2n_k - M_k} = \frac{1+\alpha}{1-\alpha}$$

is constant (i.e. bounded), so $S^{\varphi_\alpha}_{n_k,M_k} f$ uniformly converges to f.

In the final section of this chapter, we present the proof of Theorem 3.7.

3.3 Proof of Theorem 3.7

$S^{\varphi_\alpha}_{n,M}$ has the interpolation property, so considering our remark for Theorem 3.4 we can assume that

$$\alpha = \frac{M}{n} - 1 = \frac{M-n}{n} \tag{3.4}$$

and

$$\left[\frac{M}{2}\right] < n \le M.$$

Let $H := 2n - M$. Observe that M and H have the same parity and $1 \le H \le M$.

Let $f \in C_{2\pi}$. First we give two different forms of the polynomial

$$\begin{aligned}\left(S^{\varphi_\alpha}_{n,M} f\right)(x) &= \frac{2}{M}\sum_{k=0}^{M-1} f(x_{k,M}) D^{\varphi_\alpha}_n(x - x_{k,M}) = \\ &= \frac{2}{M}\sum_{k=0}^{M-1} f(x_{k,M})\left\{\frac{1}{2} + \sum_{j=1}^{n}\varphi_\alpha\left(\frac{j}{n}\right)\cos j(x - x_{k,M})\right\}.\end{aligned}$$

Lemma 3.10. *The polynomial $S^{\varphi_\alpha}_{n,M} f$ has the form*

$$\left(S^{\varphi_\alpha}_{n,M} f\right)(x) = \frac{1}{MH}\sum_{k=0}^{M-1} \frac{\sin\frac{M}{2}(x - x_{k,M})\sin\frac{H}{2}(x - x_{k,M})}{\sin^2\frac{x - x_{k,M}}{2}} f(x_{k,M}).$$

Proof. Via induction one can easily prove that

$$\frac{\sin hx}{\sin x} = \begin{cases} 2\cos x + 2\cos 3x + \ldots + 2\cos(h-1)x, & h \text{ even}, \\ 1 + 2\cos 2x + \ldots + 2\cos(h-1)x, & h \text{ odd}. \end{cases} \tag{3.5}$$

Considering (1.18) we have to show that

$$\frac{\sin\frac{M}{2}(x - x_{k,M})\sin\frac{H}{2}(x - x_{k,M})}{2H\sin^2\frac{x - x_{k,M}}{2}} = D^{\varphi_\alpha}_n(x - x_{k,M}). \tag{3.6}$$

Since

$$\alpha = \frac{M-n}{n} = \frac{\frac{M-(2n-M)}{2}}{n} = \frac{\frac{M-H}{2}}{n}$$

and

$$\frac{1-\frac{j}{n}}{1-\alpha} = \frac{1-\frac{j}{n}}{1-\left(\frac{M}{n}-1\right)} = 1 - \frac{j-\frac{M-H}{2}}{H},$$

thus the used values of φ_α are

$$\varphi_\alpha\left(\frac{j}{n}\right) = \begin{cases} 1, & j = 1, 2, \ldots, \frac{M-H}{2}; \\ 1 - \frac{j-\frac{M-H}{2}}{H}, & j = \frac{M-H}{2}+1, \ldots, n. \end{cases} \tag{3.7}$$

First, let us assume that $M = 2m+1$ and $H = 2h+1$, so $n = m+h+1$. The following identity was showed by S.N. Bernstein in [4, p. 147]:

$$\begin{aligned} \frac{1}{2(2h+1)} \frac{\sin\frac{2m+1}{2}x \sin\frac{2h+1}{2}x}{\sin^2\frac{x}{2}} &= \frac{1}{2(2h+1)}\Big[(2h+1)+ \\ +2(2h+1)\sum_{j=1}^{m-h}\cos jx + 2\sum_{j=m-h+1}^{m+h}&(m+h+1-j)\cos jx\Big]. \end{aligned} \tag{3.8}$$

Observe that (3.8) completes the proof of this case as the coefficients of cosines are the needed $\varphi_\alpha\left(\frac{j}{m+h+1}\right)$ values.

Now consider the case $M = 2m$ and $H = 2h$ (cf. [4, p. 151]). This means $n = m+h$. By (1.18) now we have to show that

$$\begin{aligned} D_n^{\varphi_\alpha}(x) = \frac{1}{2} + \sum_{j=1}^{n}\varphi_\alpha\left(\frac{j}{n}\right)\cos jx &= \frac{\sin mx \sin hx}{4h\sin^2\frac{x}{2}} = \\ &= \frac{1}{h}\cdot\frac{\sin mx}{2\tan\frac{x}{2}}\cdot\frac{\sin hx}{\sin x}, \end{aligned} \tag{3.9}$$

where the middle fraction is (cf. [62, p. 50])

$$\frac{\sin mx}{2\tan\frac{x}{2}} = \frac{1}{2} + \sum_{j=1}^{m}\cos jx - \frac{1}{2}\cos mx.$$

For the last fraction we shall use (3.5). We consider two cases:

i) If $h = 2l + 1$ then the right side of (3.9) becomes

$$\frac{1}{2l+1} \cdot \left(\frac{1}{2} + \sum_{j=1}^{m} \cos jx - \frac{1}{2} \cos mx \right) \cdot \left(1 + 2 \sum_{j=1}^{l} \cos 2jx \right).$$

We multiply the second and third terms and use the trigonometric identity for $\cos px \cdot \cos qx$ $(p, q \in \mathbb{N})$. The result becomes

$$\frac{1}{2l+1} \left[\frac{2l+1}{2} + (2l+1) \sum_{j=1}^{m-2l-1} \cos jx + \sum_{j=m-2l}^{m+2l} \frac{m+2l+1-j}{2} \cos jx \right].$$

Here if we write $l = \frac{h-1}{2}$ back, we get the wanted $\varphi_\alpha\left(\frac{j}{m+h}\right)$ coefficients for the cosines.

ii) If $h = 2l$ then the right side of (3.9) becomes

$$\frac{1}{2l} \cdot \left(\frac{1}{2} + \sum_{j=1}^{m} \cos jx - \frac{1}{2} \cos mx \right) \cdot \left(2 \sum_{j=1}^{l} \cos(2j-1)x \right).$$

Again, we multiply the second and third terms and use the trigonometric identity. The result becomes

$$\frac{1}{2l} \left[\frac{2l}{2} + 2l \sum_{j=1}^{m-2l} \cos jx + \sum_{j=m-2l+1}^{m+2l-1} \frac{m+2l-j}{2} \cos jx \right].$$

Here if we write $l = \frac{h}{2}$ back, we get the wanted $\varphi_\alpha\left(\frac{j}{m+h}\right)$ coefficients for the cosines.

Cases *i)* and *ii)* together yield (3.9), so the proof of the lemma is complete. □

Also from the form (1.18), by using the trigonometric identity

$$\cos j(x - x_{k,M}) = \cos jx \cos jx_{k,M} + \sin jx \sin jx_{k,M},$$

one can get the following simple form of $S_{n,M}^{\varphi_\alpha} f$ (cf. [4, p. 147]).

Lemma 3.11.

$$\big(S^{\varphi_\alpha}_{n,M} f\big)(x) = \frac{A_0}{2} + \sum_{j=1}^{n} A_j \cos jx + B_j \sin jx, \tag{3.10}$$

where

$$A_j = \varphi_\alpha\Big(\frac{j}{n}\Big) \cdot \frac{2}{M} \sum_{k=0}^{M-1} f(x_{k,M}) \cos jx_{k,M}, \qquad (j \le n);$$

$$B_j = \varphi_\alpha\Big(\frac{j}{n}\Big) \cdot \frac{2}{M} \sum_{k=0}^{M-1} f(x_{k,M}) \sin jx_{k,M}, \qquad (j \le n).$$

With these lemmas at hand, we can show the following identity (cf. [4, p. 148]).

Lemma 3.12. *If $M, H \in \mathbb{N}$, $1 \le H \le M$ and X_M is the point system defined by* (1.10) *then*

$$\frac{1}{MH} \sum_{k=0}^{M-1} \frac{\sin^2 \frac{H}{2}(x - x_{k,M})}{\sin^2 \frac{x - x_{k,M}}{2}} = 1 \qquad (x \in \mathbb{R}).$$

Proof. From (3.6) and (3.7) with $M = H$ we obtain that

$$\frac{1}{MH} \sum_{k=0}^{M-1} \frac{\sin^2 \frac{H}{2}(x - x_{k,M})}{\sin^2 \frac{x - x_{k,M}}{2}} = \frac{2}{M} \sum_{k=0}^{M-1} \Big[\frac{1}{2} + \sum_{j=1}^{H-1} \frac{H-j}{H} \cos j(x - x_{k,M})\Big].$$

By changing the order of sums this equals to

$$\frac{2}{M}\Big[\Big(\sum_{k=0}^{M-1} \frac{1}{2}\Big) + \sum_{j=1}^{H-1}\Big(\frac{H-j}{H} \cdot \sum_{k=0}^{M-1} \cos j(x - x_{k,M})\Big)\Big]. \tag{3.11}$$

Since X_M is an equidistant point system, one can see that the following equation holds:

$$\sum_{k=0}^{M-1} \cos j(x - x_{k,M}) = 0 \qquad (x \in \mathbb{R}, j = 1, 2, \ldots, M-1).$$

Using this in (3.11) we get our statement. □

Now we prove the following statement regarding the operator norm (cf. [4, pp. 148-150]).

Lemma 3.13. *Consider the interpolatory operator* $S_{n,M}^{\varphi_\alpha}$ *and* $N = \frac{M}{2n-M}$.

a) If $N \in \mathbb{N}$ *then the norm of the operator is*

$$\|S_{n,M}^{\varphi_\alpha}\| = \frac{1}{N}\sum_{\varrho=0}^{N-1}\frac{1}{\sin\frac{1+2\varrho}{2N}\pi} = \frac{1}{N}\Big[\frac{1}{\sin\frac{1}{2N}\pi} + \frac{1}{\sin\frac{3}{2N}\pi} + .. + \frac{1}{\sin\frac{2N-1}{2N}\pi}\Big].$$

b) If N *is not a natural number then the norm can be estimated as*

$$\frac{1}{[N]}\sum_{\varrho=0}^{[N]-1}\frac{1}{\sin\frac{1+2\varrho}{2[N]}\pi} \le \|S_{n,M}^{\varphi_\alpha}\| \le \frac{1}{[N+1]}\sum_{\varrho=0}^{[N]}\frac{1}{\sin\frac{1+2\varrho}{2[N+1]}\pi}.$$

Proof. First let us assume that $N := \frac{M}{H} \in \mathbb{N}$. Using this, the index k in (3.1) can be expressed as $k = \lambda N + \varrho$, where $\lambda = 0, 1, .., H-1$ and $\varrho = 0, 1, .., N-1$.

For the norm of operator $S_{n,M}^{\varphi_\alpha}$ we get

$$\|S_{n,M}^{\varphi_\alpha}\| = \max_{x\in[0,2\pi)}\Big\{\frac{1}{MH}\sum_{\varrho=0}^{N-1}\sum_{\lambda=0}^{H-1}\Big|\frac{\sin\frac{M}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})\cdot\sin\frac{H}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})}{\sin^2\frac{1}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})}\Big|\Big\}.$$

Let

$$A(x) := \frac{1}{MH}\sum_{\varrho=0}^{N-1}\sum_{\lambda=0}^{H-1}\Big|\frac{\sin\frac{M}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})\cdot\sin\frac{H}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})}{\sin^2\frac{1}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})}\Big|.$$

Then, we have

$$\left|\sin\frac{M}{2}\left(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H}\right)\right| = \left|\sin\left(\frac{M}{2}x+\varrho\pi+N\lambda\pi\right)\right| =$$
$$= \left|\sin\frac{M}{2}x\cdot\cos\left(\varrho+N\lambda\right)\pi+\cos\frac{M}{2}x\cdot\sin\left(\varrho+N\lambda\right)\pi\right| = \left|\sin\frac{M}{2}x\right|,$$

so

$$A(x) = \frac{\left|\sin\frac{M}{2}x\right|}{MH}\sum_{\varrho=0}^{N-1}\sum_{\lambda=0}^{H-1}\frac{\left|\sin\frac{H}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})\right|}{\sin^2\frac{1}{2}(x+\frac{2\varrho\pi}{M}+\frac{2\lambda\pi}{H})}.$$

Additionally,

$$A(x) = \frac{\left|\sin \frac{M}{2}x\right|}{MH} \sum_{\varrho=0}^{N-1} \sum_{\lambda=0}^{H-1} \frac{1}{\left|\sin \frac{H}{2}(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H})\right|} \cdot \frac{\sin^2 \frac{H}{2}(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H})}{\sin^2 \frac{1}{2}(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H})},$$

where

$$\left|\sin \frac{H}{2}(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H})\right| = \left|\sin \left(\frac{H}{2}(x + \frac{2\varrho\pi}{M}) + \lambda\pi\right)\right| =$$

$$\left|\sin \frac{H}{2}(x + \frac{2\varrho\pi}{M}) \cdot \cos \lambda\pi + \cos \frac{H}{2}(x + \frac{2\varrho\pi}{M}) \cdot \sin \lambda\pi\right| = \left|\sin \frac{H}{2}(x + \frac{2\varrho\pi}{M})\right|,$$

implying that

$$A(x) = \frac{\left|\sin \frac{M}{2}x\right|}{MH} \sum_{\varrho=0}^{N-1} \frac{1}{\left|\sin \frac{H}{2}(x + \frac{2\varrho\pi}{M})\right|} \sum_{\lambda=0}^{H-1} \frac{\sin^2 \frac{H}{2}(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H})}{\sin^2 \frac{1}{2}(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H})}.$$

Here, considering Lemma 3.12 in the case $M = H$, the inner sum becomes H^2, thus

$$A(x) = \frac{H}{M}\left|\sin \frac{M}{2}x\right| \sum_{\varrho=0}^{N-1} \frac{1}{\left|\sin \frac{H}{2}(x + \frac{2\varrho\pi}{M})\right|}.$$

Finally, with the $x = \frac{y}{H}$ substitution, by $\frac{M}{H} = N$ we have

$$\max_{x\in[0,2\pi)} A(x) = \max_{y\in[0,2\pi H)} \left\{\left|\frac{\sin \frac{N}{2}y}{N}\right| \sum_{\varrho=0}^{N-1} \frac{1}{\left|\sin(\frac{y}{2} + \frac{\varrho\pi}{N})\right|}\right\} =$$

$$= \max_{x\in[0,2\pi)} \left\{\left|\frac{\sin \frac{N}{2}x}{N}\right| \sum_{\varrho=0}^{N-1} \frac{1}{\left|\sin(\frac{x}{2} + \frac{\varrho\pi}{N})\right|}\right\}.$$

Now let us have a closer look at the function

$$B_N(x) := \left|\frac{\sin \frac{N}{2}x}{N}\right| \sum_{\varrho=0}^{N-1} \frac{1}{\left|\sin(\frac{x}{2} + \frac{\varrho\pi}{N})\right|}.$$

Using simple evaluation one can easily see that B_N is $\frac{2\pi}{N}$-periodical and

$$B_N\left(\frac{2\pi}{N} - x\right) = B_N(x), \qquad x \in \left[0, \frac{2\pi}{N}\right],$$

so B_N is even within each period.

Also observe that $B_N(x) \geq 1$ and $B_N(0) = 1$. Now if the function would have a local maximal value on the interval $\left(0, \frac{\pi}{N}\right)$, then due to its parity there would be at least 4 local extremas on the interval $\left[0, \frac{2\pi}{N}\right)$, and by its periodicity there would be at least $4N$ extremas on the interval $[0, 2\pi)$, which is not possible. So the function takes his only maximal value on the interval $\left[0, \frac{2\pi}{N}\right)$ at the point $x = \frac{\pi}{N}$, and due to periodicity

$$\|S_{n,M}^{\varphi_\alpha}\| = \max_{x\in[0,2\pi)} B_N(x) = \frac{1}{N}\sum_{\varrho=0}^{N-1}\frac{1}{\sin\frac{1+2\varrho}{2N}\pi}.$$

Now suppose that N is not an integer. As above, for the norm of the operator $S_{n,M}^{\varphi_\alpha}$ we have

$$\|S_{n,M}^{\varphi_\alpha}\| = \max_{x\in[0,2\pi)}\left\{\frac{1}{MH}\sum_{k=0}^{M-1}\left|\frac{\sin\frac{M}{2}(x-x_{k,M})\sin\frac{H}{2}(x-x_{k,M})}{\sin^2\frac{x-x_{k,M}}{2}}\right|\right\}. \tag{3.12}$$

Here we can get the lower bound

$$\|S_{n,M}^{\varphi_\alpha}\| \geq \frac{1}{[N]}\sum_{\varrho=0}^{[N]-1}\frac{1}{\sin\frac{1+2\varrho}{2[N]}\pi}$$

by discarding some members of the sum and taking similar steps as in the previous case.

We can also get the upper bound

$$\|S_{n,M}^{\varphi_\alpha}\| \leq \frac{1}{[N+1]}\sum_{\varrho=0}^{[N]}\frac{1}{\sin\frac{1+2\varrho}{2[N+1]}\pi}$$

in a similar way. □

To complete the proof of Theorem 3.7, let $N \in \mathbb{N}$ and

$$L := \frac{1}{N}\sum_{\varrho=0}^{N-1}\frac{1}{\sin\frac{1+2\varrho}{2N}\pi}.$$

Using that $L_N = \frac{2}{\pi} \log N + O(1)$ (cf. [4, (13)] or [37, p. 108]) we get the desired estimation. If N is a noninteger value, we obtain the statement in a similar way using part *b)* of Lemma 3.13. $\square$

Chapter 4

Weighted interpolation on the roots of Chebyshev polynomials

In this chapter, we are establishing a connection between the trigonometric interpolations, obtained as sums of discrete (trigonometric) Fourier series, and algebraic interpolations on the closed interval $[-1, 1]$. It is known that the results concerning the former transfer naturally to the case of algebraic interpolation on the roots of first kind of Chebyshev polynomials. Now we are considering a more general approach, using the roots of all four kinds of Chebyshev polynomials for the point systems.

Achieving uniform convergence on the whole interval [-1,1] is problematic in these cases because of unpleasant behaviour near the endpoints (regarding the details we recommend the work [22]), but two slightly different approaches are known to deal with this. The first is to supplement the problematic point systems with suitable endpoints (see e.g. [40] and our own work [28]). The other technique is multiplying the functions by suitable weight functions before dealing with the problem, thus considering the convergence in weighted spaces of continuous functions (see e.g. [22], [44] and [45]). In this chapter we follow this latter method.

With a similar train of thought presented in [40], starting from discrete (algebraic) Fourier series we construct discrete interpolation processes on the roots of four kinds of Chebyshev polynomials generated by suitable summation functions $\varphi \in \Phi_+$. We prove a general result similar to the Natanson–Zuk theorem, stating that if the cosine transform of φ is integrable then these processes are uniformly convergent on the whole interval $[-1,1]$ in some weighted spaces of continuous functions. We also examine necessary and sufficient conditions for the interpolation. As applications, we obtain various new results for the Lagrange interpolation and its arithmetic means; the Grünwald, the de la Vallée Poussin and the Hermite–Fejér interpolation. All of the presented results are from our work [29].

4.1 Preliminaries

Let $C(I)$ represent the linear space of continuous functions defined on an interval $I \subset \mathbb{R}$,

$$w_{\gamma,\delta}(x) := (1-x)^{\gamma}(1+x)^{\delta} \qquad \big(x \in [-1,1],\ \gamma,\delta \geq 0\big)$$

be a weight function and define the weighted function space

$$C_{w_{\gamma,\delta}} := \Big\{ f \in C(-1,1) \mid \lim_{\pm 1}(f w_{\gamma,\delta}) = 0 \Big\},$$

if $\gamma, \delta > 0$. Otherwise, if $\gamma = 0$ (respectively $\delta = 0$) let $C_{w_{\gamma,\delta}}$ consists of all continuous functions on $(-1,1]$ (respectively on $[-1,1)$) and

$$\lim_{-1}(f w_{\gamma,\delta}) = 0 \qquad \big(\text{resp.} \quad \lim_{1}(f w_{\gamma,\delta}) = 0\big).$$

Finally , if $\gamma = \delta = 0$ (i.e. $w_{\gamma,\delta} \equiv 1$) then let $C_{w_{\gamma,\delta}} = C[-1,1]$.

Then

$$\|f\|_{w_{\gamma,\delta}} := \|f w_{\gamma,\delta}\|_\infty := \max_{x \in [-1,1]} |(f w_{\gamma,\delta})(x)| \qquad (f \in C_{w_{\gamma,\delta}})$$

is a norm on $C_{w_{\gamma,\delta}}$ and $(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}})$ is a Banach space.

If $X_M := \{x_{k,M}\} \subset (-1,1)$ $(M \in \mathbb{N}^+)$ is an interpolatory matrix, that is

$$-1 < x_{M,M} < x_{M-1,M} < \cdots < x_{2,M} < x_{1,M} < 1$$

and $f : [-1, 1] \to \mathbb{R}$ is a given function then we denote the Lagrange interpolation polynomial of f on X_M by $L_M(f, X_M, \cdot)$.

Using [52, Theorem 2.2] we have a *Faber type result* (cf. Corollary 3.1 for the trigonometric Lagrange interpolation) for the weighted approximation of the Lagrange interpolation, namely if $\gamma, \delta \geq 0$ then for the matrix of nodes X_M there exists a function $f \in C_{w_{\gamma,\delta}}$ for which the relation

$$\|f - L_M(f, X_M, \cdot)\|_{w_{\gamma,\delta}} \to 0 \quad \text{as} \quad M \to +\infty \tag{4.1}$$

does not hold.

Therefore, as before, we can ask how to construct such discrete processes which are uniformly convergent in suitable spaces of continuous functions.

One possibility to achieve this aim is to loosen the strict condition on the degree of interpolating polynomials (see [37, Chapter II], [43], [51], [17]). The success of a construction like this strongly depends on the matrix of nodes.

Another way to obtain uniformly convergent processes is to consider suitable sums of the Lagrange interpolation polynomials (see [44], [45]).

Here we use a mixture of the above techniques, analogue to the summation of the discrete trigonometric Fourier series (see Chapter 1), to obtain wide classes of uniformly convergent weighted processes on the roots of the four kinds of Chebyshev polynomials using a summation function φ.

Let $w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$ be a Jacobi weight ($\alpha, \beta > -1$) and consider the sequence of *orthonormal* polynomials $p_n(x) := p_n^{(\alpha,\beta)}(x)$ having positive main coefficients ($n \in \mathbb{N}$) with respect to the weight $w_{\alpha,\beta}$:

$$\int_{-1}^{1} p_n^{(\alpha,\beta)}(x) p_m^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)\, dx = \delta_{m,n} \qquad (m, n \in \mathbb{N}). \tag{4.2}$$

Let us denote by

$$X_M(w_{\alpha,\beta}) := \{x_{k,M} := x_{k,M}(w_{\alpha,\beta}) : k = 1, 2, \ldots, M\}, \qquad (M \in \mathbb{N}^+)$$

the M different roots of $p_M(w_{\alpha,\beta},\cdot)$, indexed in decreasing order.

The Lagrange interpolation polynomial of a function $f \in C_{w_{\alpha,\beta}}$ on $X_M(w_{\alpha,\beta})$ ($M \in \mathbb{N}^+$) will be denoted by $L_M(f, X_M(w_{\alpha,\beta}), \cdot)$ and can be expressed (see [38, Theorem 3.2.2 and 3.4.6]) as

$$L_M(f, X_M(w_{\alpha,\beta}), x) = \sum_{j=0}^{M-1} c_{j,M}(f) p_j^{(\alpha,\beta)}(x) \qquad (x \in [-1,1]), \tag{4.3}$$

where

$$c_{j,M}(f) := c_{j,M}(f, w_{\alpha,\beta}) = \sum_{k=0}^{M} f(x_{k,M}) p_j(x_{k,M}) \lambda_{k,M}(w_{\alpha,\beta}) \tag{4.4}$$

for $j = 0, 1, \ldots, M-1$, and $\lambda_{k,M} := \lambda_{k,M}(w_{\alpha,\beta})$ denote the Christoffel numbers with respect to the weight $w_{\alpha,\beta}$.

The definition of the coefficients $c_{j,M}(f, w_{\alpha,\beta})$ may be extended for all $j \in \mathbb{N}$ by the formula above, and the series

$$\sum_{j\in\mathbb{N}} c_{j,M}(f) p_j^{(\alpha,\beta)}$$

can be considered as a *discrete (algebraic) Fourier series* of f.

For the algebraic case, we must consider a slightly different set of summation functions, as defined below.

Let us denote by Φ_+ the set of *summation functions* $\varphi : [0, +\infty) \to \mathbb{R}$ satisfying the requirements

(i) $\operatorname{supp} \varphi \subset [0, 1)$,

(ii) $\lim_{t\to 0+0} \varphi(t) = \varphi(0) = 1$,

(iii) the limits

$$\varphi(t_0 \pm 0) := \lim_{t\to t_0 \pm 0} \varphi(t)$$

exist and finite in every $t_0 \in (0, +\infty)$,

(iv) for all $t > 0$ we have $\varphi(t) \in [\varphi(t-0), \varphi(t+0)]$.

Notice that any $\varphi \in \Phi_+$ is Riemann integrable on $[0,1]$.

The next section contains the construction of φ-summations for the parameters $|\alpha| = |\beta| = \frac{1}{2}$. We discuss the convergence and the interpolation property of these processes in general.

4.2 General results

From now on, we shall consider only the special cases

$$|\alpha| = |\beta| = \tfrac{1}{2}, \tag{4.5}$$

i.e. the node systems $X_M(w_{\alpha,\beta})$ contains the roots of one of the four kinds of Chebyshev polynomials. With the notations $x = \cos\vartheta$, $x \in [-1,1]$, $\vartheta \in$ $\in [0,\pi]$ we recall the orthonormal *first, second, third and fourth kind Chebyshev polynomials*, respectively:

$$p_n^{(-\frac{1}{2},-\frac{1}{2})}(x) = \sqrt{\tfrac{2}{\pi}}T_n(x) = \sqrt{\tfrac{2}{\pi}}\cos n\vartheta \tag{4.6}$$

if $n \in \mathbb{N}^+$ and

$$p_0^{(-\frac{1}{2},-\frac{1}{2})}(x) = \sqrt{\tfrac{1}{\pi}}T_0(x) = \sqrt{\tfrac{1}{\pi}}.$$

$$p_n^{(\frac{1}{2},\frac{1}{2})}(x) = \sqrt{\tfrac{2}{\pi}}U_n(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(n+1)\vartheta}{\sin\vartheta}, \tag{4.7}$$

$$p_n^{(-\frac{1}{2},\frac{1}{2})}(x) = \sqrt{\tfrac{1}{\pi}}V_n(x) = \sqrt{\frac{1}{\pi}}\,\frac{\cos(2n+1)\frac{\vartheta}{2}}{\cos\frac{\vartheta}{2}}, \tag{4.8}$$

$$p_n^{(\frac{1}{2},-\frac{1}{2})}(x) = \sqrt{\tfrac{1}{\pi}}W_n(x) = \sqrt{\frac{1}{\pi}}\,\frac{\sin(2n+1)\frac{\vartheta}{2}}{\sin\frac{\vartheta}{2}}. \tag{4.9}$$

For these α and β, let us define the values

$$\gamma := \frac{\alpha}{2} + \frac{1}{4} \quad \text{and} \quad \delta := \frac{\beta}{2} + \frac{1}{4}. \tag{4.10}$$

For a function $f \in C_{w_{\gamma,\delta}}$ and a fixed summation function $\varphi \in \Phi_+$ we define the the following polynomials

$$S^{\varphi}_{n,M}\left(f, X_M(w_{\alpha,\beta}), x\right) := \sum_{j=0}^{n} \varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) c_{j,M}(f) p_j^{(\alpha,\beta)}(x),$$
$$\left(x \in [-1,1];\ M \in \mathbb{N}^+;\ n \in \mathbb{N}\right), \tag{4.11}$$

where the coefficients $c_{j,M}(f)$ are given by (4.4). The degree of this polynomial is $\leq n$. Note that the above polynomials have simple explicit, easily computable forms (the exact roots are known).

We remark that the "usual" way to define φ summation polynomials would be by the formula (cf. e.g. [42], [40])

$$\sum_{j=0}^{n} \varphi\left(\frac{j}{n}\right) c_{j,M}(f) p_j^{(\alpha,\beta)}(x) \qquad \left(x \in [-1,1],\ f \in C_{w_{\gamma,\delta}},\ n \in \mathbb{N}\right).$$

Now we examine some properties of polynomials (4.11).

4.2.1 On the coefficients $c_{j,M}(f)$

First we take a closer look at the coefficients $c_{j,M}(f, w_{\alpha,\beta})$ (see (4.4)), if $|\alpha| = |\beta| = \frac{1}{2}$. In the case of discrete trigonometric Fourier series, many general results are based on the nice symmetry properties of the discrete Fourier coefficients $\hat{f}_M(j)$ defined by (1.14). Here we prove similar properties for the coefficients $c_{j,M}(f, w_{\alpha,\beta})$.

Lemma 4.1. *Let us fix the positive integer M. For any $x_{k,M} \in X_M(w_{\alpha,\beta})$ ($k = 1, 2, \ldots, M$) and $j = 0, 1, \ldots, M-1$ we have*

$$p_j^{(\alpha,\beta)}(x_{k,M}) = -p_{2M-j}^{(\alpha,\beta)}(x_{k,M}), \tag{4.12}$$

and

$$p_M^{(\alpha,\beta)}(x_{k,M}) = 0. \tag{4.13}$$

For a function $f \in C_{w_{\gamma,\delta}}$ *the coefficients* $c_{j,M}(f, w_{\alpha,\beta})$ *have the properties*

$$c_{j,M}(f) = -c_{2M-j,M}(f) \qquad (j = 0, 1, \ldots, M-1), \tag{4.14}$$

and

$$c_{M,M}(f) = 0. \tag{4.15}$$

Proof. (4.13) obviously holds since the elements of $X_M(w_{\alpha,\beta})$ are the roots of $p_M^{(\alpha,\beta)}$. The equality (4.12) follows from certain trigonometric identities. The proofs are similar in each four cases for α, β. We shall discuss only the case $\alpha = \beta = \frac{1}{2}$, when for $k = 1, 2, \ldots, M$ we have (see (4.7))

$$X_M(w_{\frac{1}{2},\frac{1}{2}}) \ni x_{k,M} = \cos\vartheta_{k,M} = \cos\frac{k}{M+1}\pi.$$

Since for any $j = 0, 1, \ldots, M-1$ we have

$$\begin{aligned}\sin(j+1)\vartheta_{k,M} &= \sin\Big[\big(2M+2-(2M-j+1)\big)\frac{k\pi}{M+1}\Big] = \\ &= -\sin(2M-j+1)\vartheta_{k,M},\end{aligned}$$

consequently by (4.7)

$$\begin{aligned}p_j^{(\frac{1}{2},\frac{1}{2})}(x_{k,M}) = p_j^{(\frac{1}{2},\frac{1}{2})}(\cos\vartheta_{k,M}) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}} = \\ = -\sqrt{\frac{2}{\pi}}\,\frac{\sin(2M-j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}} = -p_{2M-j}^{(\frac{1}{2},\frac{1}{2})}(\cos\vartheta_{k,M}) = -p_{2M-j}^{(\frac{1}{2},\frac{1}{2})}(x_{k,M}),\end{aligned}$$

which proofs (4.12).

Now from the definition of the coefficients (4.4) immediately follow (4.14) and (4.15). □

4.2.2 Discrete orthogonality

It is possible to convert the (continuous) orthogonality relationship (4.2) with respect to the system $\big(p_n^{(\alpha,\beta)},\ n \in \mathbb{N}\big)$, into a discrete orthogonality relationship simply by replacing the integral with a certain sum. This result is similar to (1.12), the (discrete) orthogonality of the trigonometric system.

For the four kinds of orthonormal Chebyshev polynomials the following *discrete orthogonality* properties hold:

Lemma 4.2. *For a fixed $M \in \mathbb{N}^+$ and $i, j = 0, 1, \ldots, M-1$ we have*

$$\sum_{k=1}^{M} p_i^{(\alpha,\beta)}(x_{k,M})\, p_j^{(\alpha,\beta)}(x_{k,M})\, w_{\gamma,\delta}^2(x_{k,M})\, C_M(w_{\alpha,\beta}) = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j, \end{cases}$$

where $x_{k,M} \in X_M(w_{\alpha,\beta})$ and

$$C_M(w_{\alpha,\beta}) = \begin{cases} \dfrac{\pi}{M}, & \text{if} \quad \alpha = \beta = -\frac{1}{2} \\ \dfrac{\pi}{M+1}, & \text{if} \quad \alpha = \beta = \frac{1}{2} \\ \dfrac{2\pi}{2M+1}, & \text{otherwise.} \end{cases}$$

We remark that with suitable notations, it is possible to give a more compact form of this result. We chose this form because it holds some details proving to be useful later on.

Proof. From the Gauss–Jacobi quadrature formula (see [38, Theorem 3.4.1]) we have the following discrete orthogonality relation for $i + j \leq 2M - 1$

$$\int_{-1}^{1} p_i^{(\alpha,\beta)}(x) p_j^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)\, dx = \sum_{k=1}^{M} p_i^{(\alpha,\beta)}(x_{k,M})\, p_j^{(\alpha,\beta)}(x_{k,M})\, \lambda_{k,M} = \delta_{i,j}$$

where $\lambda_{k,M}$'s are the Christoffel numbers, for which in the cases $|\alpha| = |\beta| = \frac{1}{2}$ by [38, pp. 352–353] we have

$$\lambda_{k,M}(w_{\alpha,\beta}) = C_M(w_{\alpha,\beta}) \cdot w_{\gamma,\delta}^2(x_{k,M}) \qquad (k = 1, \ldots, M), \tag{4.16}$$

which proves the statement. □

4.2.3 Kernel function of the weighted operator

Our aim is to examine the approximation properties of $S^{\varphi}_{n,M}$ in the weighted space $(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}})$. Essentially, this means that for $f \in C_{w_{\gamma,\delta}}$ we have to estimate the expression

$$\|f - S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), \cdot)\|_{w_{\gamma,\delta}} =$$
$$= \max_{x\in[-1,1]} \big| f(x)\, w_{\gamma,\delta}(x) - S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), x) w_{\gamma,\delta}(x) \big|.$$

In other words, we approximate the function $f w_{\gamma,\delta}$ with the weighted polynomial $S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), \cdot)\, w_{\gamma,\delta}$.

Now we derive an alternative form of the weighted operator $S^{\varphi}_{n,M}\, w_{\gamma,\delta}$. From (4.11) and (4.4) we have

$$S^{\varphi}_{n,M}(f, X_M, x) w_{\gamma,\delta}(x) =$$
$$= \sum_{k=1}^{M} \big(w_{\gamma,\delta} f\big)(x_{k,M}) \cdot K^{\varphi}_{n,M}(w_{\alpha,\beta}, w_{\gamma,\delta}, x_{k,M}, x) \cdot \frac{\lambda_{k,M}}{w^2_{\gamma,\delta}(x_{k,M})}\,, \tag{4.17}$$

where the kernel function $K^{\varphi}_{n,M}$ is defined as

$$K^{\varphi}_{n,M}(w_{\alpha,\beta}, w_{\gamma,\delta}, x_{k,M}, x) := K^{\varphi}_{n,M}(x_{k,M}, x) :=$$
$$:= \sum_{j=0}^{n} \varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) \cdot \big(w_{\gamma,\delta} p_j^{(\alpha,\beta)}\big)(x_{k,M}) \cdot \big(w_{\gamma,\delta} p_j^{(\alpha,\beta)}\big)(x) \tag{4.18}$$

for an $x_{k,M} \in X_M$.

Now we establish another important connection to the trigonometric summations, as we are going to show that this kernel can be expressed by the (Dirichlet) kernel of the φ-sum of the partial sum of trigonometric Fourier series, defined by (1.4).

Lemma 4.3. *Let us fix $n \in \mathbb{N}$, the positive integer M and the node $x_{k,M} = = \cos\vartheta_{k,M} \in X_M(w_{\alpha,\beta})$. For $\vartheta \in [0,\pi]$ we have*

$$K_{n,M}^{\varphi}(\cos\vartheta_{k,M}, \cos\vartheta) =$$

$$= \frac{1}{2\pi} \cdot \begin{cases} D_n^{\varphi}(\vartheta + \vartheta_{k,M}) + D_n^{\varphi}(\vartheta - \vartheta_{k,M}), & \text{if} \quad \alpha = \beta = -\frac{1}{2} \\ D_{n+2}^{\varphi}(\vartheta - \vartheta_{k,M}) - D_{n+2}^{\varphi}(\vartheta + \vartheta_{k,M}), & \text{if} \quad \alpha = \beta = \frac{1}{2} \\ D_{n+1}^{\varphi}(\vartheta + \vartheta_{k,M}) + D_{n+1}^{\varphi}(\vartheta - \vartheta_{k,M}), & \text{if} \quad \alpha = -\frac{1}{2},\ \beta = \frac{1}{2} \\ D_{n+1}^{\varphi}(\vartheta - \vartheta_{k,M}) - D_{n+1}^{\varphi}(\vartheta + \vartheta_{k,M}), & \text{if} \quad \alpha = \frac{1}{2},\ \beta = -\frac{1}{2}, \end{cases}$$

where D_n^{φ} is the kernel defined by (1.4), i.e.

$$D_n^{\varphi}(\vartheta) = 1 + 2\sum_{j=1}^{n} \varphi\left(\frac{j}{n}\right) \cos j\vartheta.$$

Proof. The proof is based on the trigonometric form of the polynomials $p_j^{(\alpha,\beta)}$. It is similar in each four cases for α, β, so we give the proof only for $\alpha = \beta = \frac{1}{2}$. In this case $\gamma = \delta = \frac{1}{2}$, so for $j = 0, 1, \ldots, n$ we have

$$\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) = \varphi\left(\frac{j+1}{n+2}\right).$$

From (4.7) and (4.18) with $x =: \cos\vartheta$, ($\vartheta \in [0,\pi]$) we obtain

$$K_{n,M}^{\varphi}(\cos\vartheta_{k,M}, \cos\vartheta) = \frac{2}{\pi}\sum_{j=0}^{n} \varphi\left(\frac{j+1}{n+2}\right) \sin(j+1)\vartheta_{k,M} \sin(j+1)\vartheta,$$

since

$$w_{\frac{1}{2},\frac{1}{2}}(\cos\vartheta) \cdot p_j^{(\frac{1}{2},\frac{1}{2})}(\cos\vartheta) = \sin\vartheta \cdot \sqrt{\frac{2}{\pi}}\,\frac{\sin(j+1)\vartheta}{\sin\vartheta} = \sqrt{\frac{2}{\pi}}\,\sin(j+1)\vartheta,$$

where $j = 0, 1, \ldots, n$. Thus we get

$$K^{\varphi}_{n,M}(\cos\vartheta_{k,M}, \cos\vartheta) =$$
$$= \frac{1}{\pi}\Big[\sum_{j=1}^{n+1}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta-\vartheta_{k,M}) - \sum_{j=1}^{n+1}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta+\vartheta_{k,M})\Big],$$

and using the fact that $\varphi(1) = 0$, this expression for $K^{\varphi}_{n,M}(\cos\vartheta_{k,M}, \cos\vartheta)$ also equals to

$$\frac{1}{2\pi}\Big[1 + 2\sum_{j=1}^{n+2}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta-\vartheta_{k,M}) - 1 - 2\sum_{j=1}^{n+2}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta+\vartheta_{k,M})\Big],$$

consequently

$$K^{\varphi}_{n,M}(\cos\vartheta_{k,M}, \cos\vartheta) = \frac{1}{2\pi}\left[D^{\varphi}_{n+2}(\vartheta-\vartheta_{k,M}) - D^{\varphi}_{n+2}(\vartheta+\vartheta_{k,M})\right]. \qquad \square$$

4.2.4 Uniform convergence

As before, from the two-parameter operator family $\left(S^{\varphi}_{n,M}, n, M \in \mathbb{N}\right)$ we can choose a one-parameter family using two arbitrary index sequences $(n_m,\ m \in \mathbb{N})$ for the degree, and $(M_m,\ m \in \mathbb{N})$ for the number of nodes. Thus we obtain a sequence of bounded linear operators

$$S^{\varphi}_{n_m,M_m} : C_{w_{\gamma,\delta}} \to \mathcal{P}_{n_m} \qquad (m \in \mathbb{N}), \tag{4.19}$$

where $\mathcal{P}_m$ denotes the linear space of algebraic polynomials of degree $\leq m$.

Denote by $L^1(\mathbb{R}^+)$ the linear space of measurable functions $g : \mathbb{R}^+ \to \mathbb{R}$ for which the Lebesgue integral $\int_{\mathbb{R}^+} |g|$ is finite. The function

$$\|g\|_{L^1(\mathbb{R}^+)} := \int_{\mathbb{R}^+} |g| \qquad \left(g \in L^1(\mathbb{R}^+)\right)$$

is a norm on $L^1(\mathbb{R}^+)$ and $\left(L^1(\mathbb{R}^+), \|\cdot\|_{L^1(\mathbb{R}^+)}\right)$ is a Banach space.

The cosine transform of $g \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{g}_c(x) := \frac{1}{\pi} \int_0^{+\infty} g(t) \cos(tx)\, dt \qquad \left(x \in \mathbb{R}^+\right).$$

The following theorem shows that if the cosine transform of the summation function φ is Lebesgue integrable on $\mathbb{R}^+ := [0, +\infty)$ then a sequence of polynomials (4.11) tends to f uniformly for any f from the weighted space $C_{w_{\gamma,\delta}}$.

Theorem 4.4. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ, δ) *is given by* (4.10). *Suppose that* $\varphi \in \Phi_+$ *and*

$$n_m \to +\infty \ (m \to +\infty) \quad \text{and} \quad n_m \le 2M_m.$$

If $\hat{\varphi}_c \in L^1(\mathbb{R}^+)$ *then for any* $f \in C_{w_{\gamma,\delta}}$ *we have*

$$\|f - S^{\varphi}_{n_m,M_m}(f, X_{M_m}(w_{\alpha,\beta}), \cdot)\|_{w_{\gamma,\delta}} \to 0 \qquad (m \to +\infty), \tag{4.20}$$

where the polynomials $S^{\varphi}_{n_m,M_m}$ *are defined by* (4.11).

Compare this result with the Natanson–Zuk theorem (see Theorem 1.5) and its discrete version Theorem 1.13. We present the proof in Section 4.4.

The direct verification of $\hat{\varphi} \in L^1(\mathbb{R}^+)$ is generally not easy, but the following sufficient condition is a simple consequence of [25, p. 176].

Theorem A. *If* $g : \mathbb{R}^+ \to \mathbb{R}$ *is a continuous function supported in* $[0, 1]$, *and* $g \in \operatorname{Lip} \eta$ $(\eta > 1/2)$ *on* $[0, 1]$ *then* $\hat{g}_c \in L^1(\mathbb{R}^+)$.

Using these results one can easily choose the summation function φ such that the conditions of Theorem A hold, and construct many uniformly convergent discrete processes with simple computable explicit forms.

4.2.5 Interpolatory properties

We also investigate the interpolatory properties of the polynomials (4.11). The following theorem states that these polynomials interpolate the function $f \in C_{w_{\gamma,\delta}}$

at the points $X_M(w_{\alpha,\beta})$, i.e.

$$f(x_{k,M}) = S^{\varphi}_{n_k,M_k}(f, X_M(w_{\alpha,\beta}), x_{k,M}) \qquad (x_{k,M} \in X_M(w_{\alpha,\beta}))$$

if and only if some values of the summation function φ are symmetrical to the center $(x_0, 1/2)$, where

$$x_0 = \frac{M+\gamma+\delta}{n+2\gamma+2\delta},$$

so we obtain a result analogue to Theorem 1.14.

Theorem 4.5. *Let $|\alpha| = |\beta| = \frac{1}{2}$ and $\gamma, \delta \geq 0$ arbitrary real numbers, moreover suppose that $M \geq 2$, $M \leq n \leq 2M$ $(n, M \in \mathbb{N}^+)$ and $\varphi \in \Phi_+$. The polynomial $S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), x)$ interpolates the function $f \in C_{w_{\gamma,\delta}}$ at the points $X_M(w_{\alpha,\beta})$ if and only if*

$$\varphi\left(\frac{j+\delta+\gamma}{n+2\delta+2\gamma}\right) + \varphi\left(\frac{2M-j+\delta+\gamma}{n+2\delta+2\gamma}\right) = 1$$

for every $j = 0, 1, \ldots, n$, $j \neq M$.

Proof. Let $f \in C_{w_{\gamma,\delta}}$ and $M \geq 2$, $M \in \mathbb{N}$. Using (4.14) and the fact that $M \leq n \leq$ $\leq 2M$ we can write the polynomial $S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), x)$ (see (4.11)) in the form

$$S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), x) =$$
$$= \sum_{j=0}^{M-1} c_{j,M}(f) \left[\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) p_j(x) - \varphi\left(\frac{2M-j+\gamma+\delta}{n+2\gamma+2\delta}\right) p_{2M-j}(x)\right],$$

since for $0 \leq j < 2M-n$ we have $n < 2M-j \leq 2M$, and

$$\varphi\left(\frac{2M-j+\gamma+\delta}{n+2\gamma+2\delta}\right) = 0.$$

Now for an arbitrary $x_{i,M} \in X_M(w_{\alpha,\beta})$, $(i = 1, 2, \ldots, M)$ by (4.12) we get

$$S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), x_{i,M}) =$$
$$= \sum_{j=0}^{M-1} c_{j,M}(f) \cdot \left[\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) + \varphi\left(\frac{2M-j+\gamma+\delta}{n+2\gamma+2\delta}\right)\right] \cdot p_j(x_{i,M}),$$

and considering (4.3), $S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), x_{i,M})$ also equals to

$$L_M(f, X_M(w_{\alpha,\beta}), x_{i,M})+$$
$$+\sum_{j=0}^{M-1} c_{j,M}(f) \cdot \left[\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) + \varphi\left(\frac{2M-j+\gamma+\delta}{n+2\gamma+2\delta}\right) - 1\right] \cdot p_j(x_{i,M}).$$

Since

$$L_M(f, X_M(w_{\alpha,\beta}), x_{i,M}) = f(x_{i,M}) \qquad (\forall\ x_{i,M} \in X_M(w_{\alpha,\beta})),$$

the equation

$$S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), x_{i,M}) = f(x_{i,M})$$

holds for every $x_{i,M} \in X_M(w_{\alpha,\beta})$ if and only if the polynomial

$$\sum_{j=0}^{M-1} c_{j,M}(f) \cdot \left[\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) + \varphi\left(\frac{2M-j+\gamma+\delta}{n+2\gamma+2\delta}\right) - 1\right] \cdot p_j(x)$$

equals to zero at every point $x_{i,M} \in X_M(w_{\alpha,\beta})$, so it has M distinct roots and its degree $\leq M-1$, consequently it is the zero polynomial.

So $S^{\varphi}_{n,M}(f, X_M(w_{\alpha,\beta}), \cdot)$ interpolates f if and only if

$$\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) + \varphi\left(\frac{2M-j+\gamma+\delta}{n+2\gamma+2\delta}\right) - 1 = 0$$

for $j = 0, 1, \ldots, M-1$. This completes the proof. □

4.3 Results in special cases

Now we present specific interpolations which can be considered as the weighted variants of some classical interpolation methods. We encourage the reader to compare the summation functions φ used in this section with the ones presented in Chapters 1 and 3.

First, for a function $f \in C_{w_{\gamma,\delta}}$ and $M \in \mathbb{N}^+$ the *Lagrange interpolation* polynomials $L_M(f, X_M(w_{\alpha,\beta}), \cdot)$ can be obtained as special cases of (4.11). Indeed, let $n := M$ and

$$\varphi_L(t) := \begin{cases} 1, & \text{if } t \in [0,1) \\ 0, & \text{if } t \in [1,+\infty). \end{cases}$$

Using Theorem 4.5 it is clear that $S^{\varphi_L}_{M,M}(f, X_M(w_{\alpha,\beta}), \cdot)$ interpolates f at the points $X_M(w_{\alpha,\beta})$, and the degree of the summation polynomial cannot exceed $M-1$ (since $c_{M,M}(f) = 0$, see Lemma 4.12), so it must be the Lagrange interpolation polynomial of f.

As we have already mentioned (see (4.1)), the sequence of these polynomials generally does not tend uniformly to f in $(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}})$.

4.3.1 Arithmetic means of Lagrange interpolation

Let $M \in \mathbb{N}^+$ and for $m = 0, 1, \ldots, M-1$ define the polynomials

$$L_{m,M}(f, X_M(w_{\alpha,\beta}), x) := \sum_{j=0}^{m} c_{j,M}(f) p_j^{(\alpha,\beta)}(x), \qquad (f \in C_{w_{\gamma,\delta}}, x \in [-1,1]).$$

Note that $L_{M-1,M}(f, X_M, \cdot)$ is the Lagrange interpolation polynomial.

The arithmetic means of Lagrange interpolation are defined by the formula

$$\sigma_M(f, X_M(w_{\alpha,\beta}), \cdot) := \frac{1}{M+\gamma+\delta} \sum_{m=0}^{M-1} L_{m,M}(f, X_M(w_{\alpha,\beta}), \cdot).$$

Theorem 4.6. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ, δ) *is given by* (4.10). *Then for any* $f \in C_{w_{\gamma,\delta}}$ *we have*

$$\lim_{M\to+\infty} \|f - \sigma_M(f, X_M(w_{\alpha,\beta}), \cdot)\|_{w_{\gamma,\delta}} = 0.$$

Theorem 4.6 is a discrete version of Fejér's theorem about the arithmetic means of Fourier series (see Corollary 1.6). Analogue results in interpolation theory are due to S. N. Bernstein [2] and J. Marcinkiewicz [20] in the unweighted case.

We remark that the same result was already obtained in [45] (for more general parameters $\alpha, \beta, \gamma, \delta$), but our proof differs from the one presented there. A similar result was introduced for the four kinds of Chebyshev nodes in [40], where the author supplemented the node systems with additional points instead of using weights.

We also note that by Theorem 4.5, $\sigma_M(f, X_M(w_{\alpha,\beta}), \cdot)$ does not interpolate f at the points of $X_M(w_{\alpha,\beta})$.

Proof of Theorem 4.6. A simple calculation shows that

$$\sigma_M(f, X_M(w_{\alpha,\beta}), \cdot) = S^{\varphi_F}_{2M,M}(f, X_M(w_{\alpha,\beta}), \cdot),$$

where

$$\varphi_F(t) := \begin{cases} 1 - 2t, & \text{if } t \in [0, \frac{1}{2}] \\ 0, & \text{if } t \in (\frac{1}{2}, +\infty). \end{cases}$$

For the cosine transform of φ_F we have

$$\hat{\varphi}_F(x) = \frac{1}{4\pi} \left(\frac{\sin(x/4)}{x/4} \right)^2,$$

consequently $\hat{\varphi}_F(x) \in L^1(\mathbb{R}^+)$, and by Theorem 4.4, our proof is complete. □

4.3.2 Grünwald–Rogosinski type processes

Let us consider the summation function

$$\varphi_G(t) := \begin{cases} \cos t\pi, & \text{if } t \in [0, \frac{1}{2}] \\ 0, & \text{if } t \in (\frac{1}{2}, +\infty). \end{cases}$$

Theorem 4.7. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ, δ) *is given by* (4.10) *and suppose that* $f \in C_{w_{\gamma,\delta}}$.

(i) *For* φ_G *we have the Rogosinski type average of Lagrange interpolation, i.e. for* $f \in C_{w_{\gamma,\delta}}$ *the relation*

$$w_{\gamma,\delta} S_{2M,M}^{\varphi_G}(f, X_M(w_{\alpha,\beta}), x) =$$
$$= \frac{1}{2}\Big\{\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M(w_{\alpha,\beta}), x_+) + \mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M(w_{\alpha,\beta}), x_-)\Big\}$$

holds, where

$$\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M(w_{\alpha,\beta}), \cdot) := w_{\gamma,\delta} \cdot L_M(f, X_M(w_{\alpha,\beta}), \cdot)$$

and

$$x_\pm := x \cos t_M \pm \sqrt{1-x^2} \sin t_M, \qquad t_M := \frac{\pi}{2(M+\gamma+\delta)}.$$

(ii) *For these polynomials we have*

$$\lim_{M \to +\infty} \|f - S_{2M,M}^{\varphi_G}(f, X_M(w_{\alpha,\beta}), \cdot)\|_{w_{\gamma,\delta}} = 0.$$

If $\alpha = \beta = -\frac{1}{2}$ and $\gamma = \delta = 0$, then we obtain Grünwald's classical result [15] for first kind Chebyshev roots in the unweighted case.

M.S. Webster [55] proved that for $\alpha = \beta = \frac{1}{2}$ the uniform convergence (without weight) is true only for closed subintervals of $(-1, 1)$. In [54] P. Vértesi generalized Webster's result for arbitrary $\alpha, \beta > -1$. Theorem 4.7 shows that for $|\alpha| = |\beta| = \frac{1}{2}$ the uniform convergence holds on the whole interval $[-1, 1]$, if we use suitable weight function.

We also note that by Theorem 4.5, $S_{2M,M}^{\varphi_G}(f, X_M(w_{\alpha,\beta}), \cdot)$ does not interpolate f at the points of $X_M(w_{\alpha,\beta})$.

Proof of Theorem 4.7. The verification of **(i)** is based on the trigonometric form of the polynomials $p_j^{(\alpha,\beta)}$. It is similar in each four cases for α, β, so we give the proof only for $\alpha = \beta = \frac{1}{2}$. In this case $\gamma = \delta = \frac{1}{2}$.

Using the notation $x =: \cos\vartheta$, $(\vartheta \in [0, \pi])$ a simple calculation shows that

$$x_\pm = \cos(\vartheta \mp t_M).$$

Now by (4.7) for $j = 0, 1, \dots, M-1$ we have

$$\begin{gathered}\big(w_{\frac{1}{2},\frac{1}{2}} p_j^{(\frac{1}{2},\frac{1}{2})}\big)(x_+) + \big(w_{\frac{1}{2},\frac{1}{2}} p_j^{(\frac{1}{2},\frac{1}{2})}\big)(x_-) = \\ = \sin(\vartheta - t_M) \cdot \frac{\sin\big[(j+1)(\vartheta - t_M)\big]}{\sin(\vartheta - t_M)} + \sin(\vartheta + t_M) \cdot \frac{\sin\left[(j+1)(\vartheta + t_M)\right]}{\sin(\vartheta + t_M)} = \\ = 2\cos(j+1)t_M \cdot \sin(j+1)\vartheta,\end{gathered}$$

and thus (by (4.3))

$$\begin{aligned}\frac{1}{2}\Big\{&\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M(w_{\alpha,\beta}), x_+) + \mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M(w_{\alpha,\beta}), x_-)\Big\} = \\ &= \frac{1}{2}\sum_{j=0}^{M-1} 2\cos(j+1)t_M \cdot c_{j,M}(f)\sin(j+1)\vartheta = \\ &= \sin\vartheta \sum_{j=0}^{2M} \varphi\left(\frac{j+1}{2M+2}\right) \cdot c_{j,M}(f)\frac{\sin(j+1)\vartheta}{\sin\vartheta},\end{aligned}$$

where

$$\varphi\left(\frac{j+1}{M+2}\right) = \cos(j+1)t_M = \cos\frac{(j+1)\pi}{2M+2}$$

for $j = 0, 1, \dots, M-1$, and $\varphi\left(\dfrac{j+1}{M+2}\right) = 0$, otherwise.

Consequently

$$\begin{aligned}\frac{1}{2}\Big\{&\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M(w_{\alpha,\beta}), x_+) + \mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M(w_{\alpha,\beta}), x_-)\Big\} = \\ &= w_{\frac{1}{2},\frac{1}{2}} \cdot S^{\varphi G}_{2M,M}(f, X_M(w_{\frac{1}{2},\frac{1}{2}}), x).\end{aligned}$$

(ii) For the cosine transform of φ_G we have

$$\hat{\varphi}_G(x) = \frac{\sin(x-\pi)/2}{x^2-\pi^2} \qquad (x \in \mathbb{R}^+),$$

so $\hat{\varphi}_G \in L^1(\mathbb{R}^+)$. By Theorem 4.4, we obtain our statement. □

4.3.3 De la Vallée Poussin type interpolation

Fix a number $\kappa \in (0,1)$ and let

$$\varphi_\kappa := \begin{cases} 1, & \text{if } t \in [0, \frac{1-\kappa}{2}) \\ -\frac{1}{\kappa}\left(t - \frac{1+\kappa}{2}\right), & \text{if } t \in [\frac{1-\kappa}{2}, \frac{1+\kappa}{2}] \\ 0, & \text{if } t \in (\frac{1+\kappa}{2}, +\infty). \end{cases}$$

Theorem 4.8. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ, δ) *is given by* (4.10) *and suppose that* $f \in C_{w_{\gamma,\delta}}$.

(i) *For any fixed* $\kappa \in (0,1)$ *and* $M \in \mathbb{N}^+$, *the degree of the polynomial*

$$S^{\varphi_\kappa}_{2M,M}(f, X_M(w_{\alpha,\beta}), \cdot)$$

is $\leq M(1+\kappa)$ *and it interpolates* f *at the points of* $X_M(w_{\alpha,\beta})$.

(ii) *For any* $f \in C_{w_{\gamma,\delta}}$ *we have*

$$\lim_{M\to+\infty} \|f - S^{\varphi_\kappa}_{2M,M}(f, X_M(w_{\alpha,\beta}), \cdot)\|_{w_{\gamma,\delta}} = 0.$$

For the values $\kappa = 1$ and $\kappa = 0$, we would obtain the Lagrange interpolation and the weighted Hermite–Fejér type interpolation (see in the next subsection), respectively. In trigonometric interpolation, S. N. Bernstein has analogue results [4] for a class of interpolatory polynomials.

We remark that Theorem 4.8 can also be considered as a discrete algebraic version of the de la Vallée Poussin summation of (trigonometric) Fourier series and discrete

Fourier series. While we exhaustively investigated those processes in the previous chapters, the discrete algebraic versions of our results are not yet available.

This result also shows similarity to a result of P. Erdős [10, Theorem 1] in the classical (unweighted) case, where he proved that if the interpolatory point system $(X_M,\ M \in \mathbb{N}^+)$ is such that the fundamental polynomials of Lagrange interpolation are uniformly bounded, then for any $f \in C[-1,1]$ there exists a sequence of polynomials Q_M $(M \in \mathbb{N}^+)$ of degree $\le M(1+\kappa)$ tending uniformly to f, and Q_M interpolates f at the points of X_M for every $M \in \mathbb{N}^+$. For our four point systems, we now have a weighted analogue of this result.

Proof of Theorem 4.8. An easy calculation shows that

$$\hat{\varphi}_\kappa(x) = \frac{1}{2(1-\kappa)\pi} \frac{\sin^2(x/2) - \sin^2(1+\kappa)x}{(x/2)^2} \qquad (x \in \mathbb{R}^+),$$

so $\hat{\varphi}_\kappa \in L^1(\mathbb{R}^+)$, and also

$$\varphi_\kappa(t) + \varphi_\kappa(1-t) = 1 \qquad (t \in [0,1]),$$

thus from Theorem 4.4 and 4.5 we obtain the statement. □

4.3.4 Weighted Hermite–Fejér type interpolation

Let us define the summation function

$$\varphi_H(t) := \begin{cases} 1-t, & \text{if } t \in [0,1] \\ 0, & \text{if } t \in (1,+\infty). \end{cases}$$

The next theorem states that the weighted Hermite–Fejér type interpolatory polynomials can be obtained by using suitable summation function.

Theorem 4.9. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ,δ) *is given by* (4.10) *and suppose that* $f \in C_{w_{\gamma,\delta}}$.

(i) *For any* $M = 1, 2, \dots$ *the polynomials*

$$S^{\varphi_H}_{2M,M}\big(f, X_M(w_{\alpha,\beta}), x\big) = \sum_{j=0}^{2M} \left(1 - \frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) c_{j,M}(f)\, p_j^{(\alpha,\beta)}(x)$$

(*see* (4.4) *and* (4.11)) *satisfy the following Hermite–Fejér type interpolatory properties*

$$S^{\varphi_H}_{2M,M}(f, X_M(w_{\alpha,\beta}), x_{k,M}) = f(x_{k,M}), \tag{4.21}$$

$$\big(w_{\gamma,\delta} S^{\varphi_H}_{2M,M}(f, X_M(w_{\alpha,\beta}), \cdot)\big)'(x_{k,M}) = 0, \tag{4.22}$$

for all $x_{k,M} \in X_M(w_{\alpha,\beta})$.

(ii) *For any* $f \in C_{w_{\gamma,\delta}}$ *we have*

$$\lim_{M\to+\infty} \|f - S^{\varphi_H}_{2M,M}(f, X_M(w_{\alpha,\beta}), \cdot)\|_{w_{\gamma,\delta}} = 0.$$

If $\alpha = \beta = -\frac{1}{2}$ and $\gamma = \delta = 0$, then we obtain Fejér's classical result for first kind Chebyshev roots in the unweighted case. (See e.g. [37, p. 165], [53].)

In [17] Ágota P. Horváth proved a general convergence theorem for the above type weighted Hermite–Fejér interpolation process on $\varrho(w)$-normal point systems (especially on Jacobi roots, see [17, Example (2)]); but Theorem 2 of her paper does not contain our Theorem 4.9.

G. Mastroianni and J. Szabados [23] investigated an other type weighted Hermite–Fejér interpolation process based on Jacobi nodes.

Proof of Theorem 4.9. **(i)** The summation function φ_H obviously satisfies the symmetry property of Theorem 4.5, which proves the interpolatory properties (4.21).

For the proof of (4.22) we shall use the following result regarding some values of the derivatives of the functions $w_{\gamma,\delta} p_j^{(\alpha,\beta)}$.

Lemma 4.10. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ, δ) *is given by* (4.10). *Fix the positive integer* M. *Then for any node* $x_{k,M} \in X_M(w_{\alpha,\beta})$ *we have*

$$(2M - j + \gamma + \delta)\Big(w_{\gamma,\delta} p_j^{(\alpha,\beta)}\Big)'(x_{k,M}) = (j + \gamma + \delta)\Big(w_{\gamma,\delta} p_{2M-j}^{(\alpha,\beta)}\Big)'(x_{k,M}),$$

where $j = 0, 1, \ldots, M - 1$.

Proof. The proof is based on the trigonometric form of the polynomials $p_j^{(\alpha,\beta)} =: p_j$, and is similar in each four cases for α, β, so we give the proof only for $\alpha = \beta = \frac{1}{2}$. In this case $\gamma = \delta = \frac{1}{2}$,

$$X_M(w_{\frac{1}{2},\frac{1}{2}}) \ni x_{k,M} = \cos\frac{k\pi}{M+1} =: \cos\vartheta_{k,M},$$

and by (4.7)

$$\big(w_{\frac{1}{2},\frac{1}{2}}\, p_j\big)(x) = \sin\big((j+1)\arccos x\big), \qquad (j = 0, 1, \ldots, M-1).$$

For an arbitrary $j = 0, 1, \ldots, M - 1$ and $\vartheta \in [0, \pi]$ we have

$$\Big(w_{\frac{1}{2},\frac{1}{2}}\, p_j\Big)'(\cos\vartheta) = \sqrt{\frac{2}{\pi}}\frac{(j+1)\cdot\cos(j+1)\vartheta}{\sin\vartheta},$$

and since

$$\cos\frac{(j+1)k\pi}{M+1} = \cos\frac{(2M+2-(2M-j+1))k\pi}{M+1} = \cos\frac{(2M-j+1)k\pi}{M+1},$$

thus

$$\Big(w_{\frac{1}{2},\frac{1}{2}}\, p_j\Big)'(x_{k,M}) = \sqrt{\frac{2}{\pi}}\frac{(j+1)\cdot\cos(2M-j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}}, \qquad (x_{k,M} \in X_M).$$

Observe that the expression on the right side equals to

$$\frac{j+1}{2M-j+1}\cdot\sqrt{\frac{2}{\pi}}\frac{(2M-j+1)\cdot\cos(2M-j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}} =$$
$$= \frac{j+1}{2M-j+1}\Big(w_{\frac{1}{2},\frac{1}{2}}\, p_{2M-j}\Big)'(x_{k,M}),$$

proving our statement. □

Let $\varphi := \varphi_H$. Then we have

$$\big(w_{\gamma,\delta}S^{\varphi}_{2M,M}(f, X_M(w_{\alpha,\beta}), \cdot)\big)' = \sum_{j=0}^{2M} \varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) c_{j,M}(f)\cdot(w_{\gamma,\delta}p_j)'.$$

By (4.14), this equals to

$$\sum_{j=0}^{M-1}\Big[\varphi\big(\tfrac{j+\gamma+\delta}{2M+2\gamma+2\delta}\big)(w_{\gamma,\delta}p_j)' - \varphi\big(\tfrac{2M-j+\gamma+\delta}{2M+2\gamma+2\delta}\big)(w_{\gamma,\delta}p_{2M-j})'\Big]\cdot c_{j,M}(f).$$

Using

$$\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) + \varphi\left(1-\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) = 1.$$

and Lemma 4.10 together leads us to

$$\big(w_{\gamma,\delta}S^{\varphi}_{2M,M}(f, X_M, \cdot)\big)'(x_{k,M}) =$$
$$= \sum_{j=0}^{M-1}\left[\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) - \frac{2M-j+\gamma+\delta}{j+\gamma+\delta}\left(1-\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)\right)\right]\cdot$$
$$\cdot c_{j,M}(f)\big(w_{\gamma,\delta}p_j\big)'(x_{k,M}),$$

which equals to 0 for every $x_{k,M} \in X_M$ if

$$\frac{2M+2\gamma+2\delta}{j+\gamma+\delta}\cdot\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) - \frac{2M-j+\gamma+\delta}{j+\gamma+\delta} = 0$$

for $j = 0, 1, \ldots, M-1$, or in another form,

$$\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) = 1 - \frac{j+\gamma+\delta}{2M+2\gamma+2\delta}, \qquad (j = 0, 1, \ldots, M-1).$$

Since φ_H satisfies this condition and the interpolatory condition as well, so the proof of (4.22) is complete.

(ii) Since

$$\hat{\varphi}_H(x) = \frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2 \qquad (x \in \mathbb{R}^+)$$

belongs to $L^1(\mathbb{R}^+)$, therefore by Theorem 4.4 we obtain the statement. ☐

4.4 Proof of Theorem 4.4

Let $|\alpha| = |\beta| = \frac{1}{2}$. We shall use the Banach–Steinhaus theorem. The polynomials $\left\{p_i^{(\alpha,\beta)} \ : \ i \in \mathbb{N}\right\}$ form a closed system in the space $(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}})$ (see e.g. [44, Section 3]), therefore we have to show that

$$\left\|S^{\varphi}_{n_m,M_m}(p_i, X_{M_m}, \cdot) - p_i\right\|_{w_{\gamma,\delta}} \to 0 \qquad (m \to +\infty) \tag{4.23}$$

for every fixed $i \in \mathbb{N}$, moreover the norms of the operators $S^{\varphi}_{n_m,M_m}$ is uniformly bounded, i.e. there exists $c > 0$ independent of m such that

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} \le c \qquad (m \in \mathbb{N}), \tag{4.24}$$

where

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} := \sup_{\|f\|_{w_{\gamma,\delta}}=1} \left\{\|S^{\varphi}_{n_m,M_m}(f, X_{M_m}, \cdot)\|_{w_{\gamma,\delta}} \ : \ f \in C_{w_{\gamma,\delta}}\right\}.$$

In order to prove (4.23), let us fix $i \in \mathbb{N}$ and assume that m is large enough, i.e. $\min\{M_m, n_m\} > i$. Now by Lemma 4.2, for $j = 0, 1, \ldots, M_m - 1$ we have

$$c_{j,M_m}(p_i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \ne j, \end{cases}$$

so considering $n_m \le 2M_m$ and (4.14), the equality

$$S^{\varphi}_{n_m,M_m}(p_i, X_{M_m}, \cdot) =$$
$$= \varphi\left(\frac{i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) p_i - \varphi\left(\frac{2M_m - i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) p_{2M_m-i}$$

holds. It is clear that

$$\lim_{m\to+\infty} \varphi\left(\frac{i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) = \varphi(0) = 1.$$

Since $n_m \le 2M_m$ $(m \in \mathbb{N})$ and

$$\liminf_{m\to+\infty} \frac{2M_m - i+\gamma+\delta}{n_m+2\gamma+2\delta} = \liminf_{m\to+\infty}\left(\frac{2M_m+2\gamma+2\delta}{n_m+2\gamma+2\delta} - \frac{i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) \ge 1,$$

moreover $\varphi(x) = 0$ if $x \ge 1$, thus we have

$$\lim_{m\to+\infty} \varphi\left(\frac{2M_m - i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) = 0,$$

therefore we proved (4.23).

Next we show (4.24). Using (4.16), (4.17) and $C_{M_m} = C_{M_m}(w_{\alpha,\beta})$, the norm can be expressed as

$$\|S^{\varphi}_{n_m,M_m}(f, X_{M_m}, \cdot)\|_{w_{\gamma,\delta}} =$$
$$= \left\|\sum_{k=1}^{M_m} w_{\gamma,\delta}(x_{k,M_m}) f(x_{k,M_m}) \cdot C_{M_m} \cdot K^{\varphi}_{n_m,M_m}(x_{k,M_m}, \cdot)\right\|_{\infty},$$

so if $\|f\|_{w_{\gamma,\delta}} = \sup_{x\in[-1,1]} |\left(w_{\gamma,\delta} f\right)(x)| = 1$, then we obtain

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} = \sup_{x\in[-1,1]} C_{M_m} \sum_{k=0}^{M_m} \left|K^{\varphi}_{n_m,M_m}(x_{k,M_m}, x)\right|.$$

By Lemma 4.3 the kernel can be uniformly expressed as

$$K^{\varphi}_{n_m,M_m}(\vartheta_{k,M_k}, \vartheta) = \frac{1}{2\pi}\left[D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta - \vartheta_{k,M_k}) \pm D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta + \vartheta_{k,M_k})\right],$$

so

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} \le$$
$$\le \frac{C_{M_m}}{2\pi} \max_{\vartheta\in[0,\pi]} \sum_{k=1}^{M_m} \left\{\left|D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta + \vartheta_{k,M_k})\right| + \left|D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta - \vartheta_{k,M_k})\right|\right\}.$$

Let

$$\|D_n^{\varphi}\|_{M,1} := \frac{1}{2M} \sup_{\vartheta \in [0,\pi]} \sum_{k=1}^{M} \big\{ |D_n^{\varphi}(\vartheta + \vartheta_{k,M})| + |D_n^{\varphi}(\vartheta - \vartheta_{k,M})| \big\}.$$

Then

$$\|D_n^{\varphi}\|_{M,1} \le \left(1 + \frac{2n\pi}{M}\right) \|D_n^{\varphi}\|_1 := \left(1 + \frac{2n\pi}{M}\right) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n^{\varphi}(t)|\, dt$$

(see [48, pp. 242]) and

$$\sup_{n\in\mathbb{N}} \|D_n^{\varphi}\|_1 = \|\hat{\varphi}_c\|_{L^1(\mathbb{R}^+)}$$

(see Theorem 2 in §24 of [25]). Consequently if $\hat{\varphi}_c \in L^1(\mathbb{R}^+)$ and $n_m \le 2M_m$, then there exists $c > 0$ such that

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} \le C_{M_m} \frac{M_m}{\pi} \left(1 + \frac{n_m + 2\gamma + 2\delta}{M_m} \pi\right) \|\hat{\varphi}_c\|_{L^1(\mathbb{R}^+)} < c,$$

since $C_{M_m}(w_{\alpha,\beta}) \le \frac{\pi}{M_m}$ for any $|\alpha| = |\beta| = \frac{1}{2}$. This completes the proof of (4.24) and consequently of Theorem 4.4. $\square$

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Summary

The focus of our work is the uniform convergence of different de la Vallée Poussin type summations. We encounter this topic in theories of classical and multivariate trigonometric Fourier series, discrete Fourier series and trigonometric interpolation, and finally algebraic interpolation.

In the first chapter we discuss the historical background of our study, establish the most important notations and definitions and recall some fundamental results on which the later chapters are based upon.

In the second chapter we deal with the de la Vallée Poussin means of the triangular partial sums of multivariate Fourier series. We determine the exact order of the corresponding operator norms. The lower estimation of these norms will be extended to a class of projection operators having similar projection properties.

In the third chapter we discuss some classic methods of trigonometric interpolation, mainly the Lagrange and Hermite–Fejér interpolations. We also define the de la Vallée Poussin sums, which will be used as a tool to describe the transition between these two classic methods. We give general properties, the precise operator norm and (uniform) convergence order for these cases.

In the final chapter, we are establishing a connection between the trigonometric interpolations and the algebraic interpolations on the closed interval $[-1, 1]$. We construct discrete interpolation processes on the roots of four kinds of Chebyshev polynomials generated by suitable summation functions. We investigate convergence in some weighted spaces of continuous functions. We also examine necessary and sufficient conditions for the interpolation, and discuss specific applications.

Összefoglalás

Értekezésünk témája különböző de la Vallée Poussin típusú szummációs módszerek tanulmányozása. Ez a témakör a klasszikus és többváltozós trigonometrikus Fourier sorok, a diszkrét Fourier sorok és a trigonometrikus interpoláció, valamint az algebrai interpoláció elméletében egyaránt vizsgálható.

Az első fejezetben kutatásunk történeti hátterét ismertetjük, bevezetjük a fontosabb jelöléseket és fogalmakat, és felidézünk olyan alapvető eredményeket, melyeken a későbbi fejezetek alapszanak.

A mśodik fejezetben a háromszögösszegű többváltozós Fourier sorok de la Vallée Poussin közepeit vizsgáljuk. Meghatározzuk a kapcsolódó operátorok normáinak pontos nagyságrendjét. Alsó becslésünket hasonló projekciós tulajdonságokkal bíró operátorokra is kiterjesztjük.

A harmadik fejezetben ismertetjük a trigonometrikus interpoláció néhány klasszikus módszerét, különös tekintettel a Lagrange és az Hermite–Fejér interpolációkra. Bevezetjük a de la Vallée Poussin összegeket, melyek eszközként szolgálnak az említett módszerek közti átmenet vizsgálatában. Az általános tulajdonságok mellett pontos operátornormát és konvergenciarendet adunk.

Az utolsó fejezetben kapcsolatot teremtünk a trigonometrikus interpolációk és a $[-1, 1]$ intervallumon értelmezett algebrai interpolációk között. Diszkrét interpolációs eljárásokat konstruálunk a négy fajta Csebisev polinom gyökein, szummációs függvények segítségével. A konvergenciát bizonyos súlyozott függvényterekben vizsgáljuk. Az interpoláció szükséges és elégséges feltételeit, valamint speciális alkalmazásokat is tárgyalunk.

[1]ADATLAP
a doktori értekezés nyilvánosságra hozatalához

I. A doktori értekezés adatai
A szerző neve: Németh Zsolt
MTMT-azonosító: 10035377
A doktori értekezés címe és alcíme: De la Vallée Poussin type approximation methods
DOI-azonosító[2]: 10.15476/ELTE.2016.022
A doktori iskola neve: ELTE IK Informatika Doktori Iskola
A doktori iskolán belüli doktori program neve: Numerikus és szimbolikus számítások
A témavezető neve és tudományos fokozata: Szili László, a matematikai tudomány kandidátusa
A témavezető munkahelye: ELTE IK Numerikus Analízis Tanszék

II. Nyilatkozatok
1. A doktori értekezés szerzőjeként[3]

a) hozzájárulok, hogy a doktori fokozat megszerzését követően a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az ELTE Digitális Intézményi Tudástárban. Felhatalmazom az ELTE IK Informatika Doktori Iskola hivatalának ügyintézőjét Ríz-Herczeg Krisztinát, hogy az értekezést és a téziseket feltöltse az ELTE Digitális Intézményi Tudástárba, és ennek során kitöltse a feltöltéshez szükséges nyilatkozatokat.

b) kérem, hogy a mellékelt kérelemben részletezett szabadalmi, illetőleg oltalmi bejelentés közzétételéig a doktori értekezést ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;[4]

c) kérem, hogy a nemzetbiztonsági okból minősített adatot tartalmazó doktori értekezést a minősítés (*dátum*)-ig tartó időtartama alatt ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;[5]

d) kérem, hogy a mű kiadására vonatkozó mellékelt kiadó szerződésre tekintettel a doktori értekezést a könyv megjelenéséig ne bocsássák nyilvánosságra az Egyetemi Könyvtárban, és az ELTE Digitális Intézményi Tudástárban csak a könyv bibliográfiai adatait tegyék közzé. Ha a könyv a fokozatszerzést követőn egy évig nem jelenik meg, hozzájárulok, hogy a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban.[6]

2. A doktori értekezés szerzőjeként kijelentem, hogy

a) az ELTE Digitális Intézményi Tudástárba feltöltendő doktori értekezés és a tézisek saját eredeti, önálló szellemi munkám és legjobb tudomásom szerint nem sértem vele senki szerzői jogait;

b) a doktori értekezés és a tézisek nyomtatott változatai és az elektronikus adathordozón benyújtott tartalmak (szöveg és ábrák) mindenben megegyeznek.

3. A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének plágiumkereső adatbázisba helyezéséhez és plágiumellenőrző vizsgálatok lefuttatásához.

Kelt: 2016. február 22.

a doktori értekezés szerzőjének aláírása

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[1] Beiktatta az Egyetemi Doktori Szabályzat módosításáról szóló CXXXIX/2014. (VI. 30.) Szen. sz. határozat. Hatályos: 2014. VII.1. napjától.
[2] A kari hivatal ügyintézője tölti ki.
[3] A megfelelő szöveg aláhúzandó.
[4] A doktori értekezés benyújtásával egyidejűleg be kell adni a tudományági doktori tanácshoz a szabadalmi, illetőleg oltalmi bejelentést tanúsító okiratot és a nyilvánosságra hozatal elhalasztása iránti kérelmet.
[5] A doktori értekezés benyújtásával egyidejűleg be kell nyújtani a minősített adatra vonatkozó közokiratot.
[6] A doktori értekezés benyújtásával egyidejűleg be kell nyújtani a mű kiadásáról szóló kiadói szerződést.