



## On the Impact of Various Formulations of the Boundary Condition within Numerical Option Valuation by DG Method

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**Abstract.** Options, a crucial type of financial instrument, are very challenging as concerns both, the application and valuation. A key property of (exotic) options is to provide a tool to manage the market risk coming from everyday innovations at the market. Due to the complexity of underlying processes and/or payoff functions valuation via numerical methods is often inevitable. The flexibility in terms of model assumptions often brings high time costs so that it can be useful to reduce the space on which the computation is executed in order to keep both the computation time and calculation error at acceptable levels. Efficient formulation of the boundary conditions of option valuation formula is one of such approaches. In this paper we focus on the impact of Dirichlet, Neumann and transparent boundary conditions when the valuation formula is discretized by the discontinuous Galerkin method combined with the implicit Euler scheme for the temporal discretization. The numerical results are presented using real data of DAX index options.

### 1. Introduction

Financial markets allow a smooth functioning of economic relations by transferring of funds, liquidity, maturity and risk, which is realized by various types of financial instruments with the help of financial intermediaries such as banks, insurance companies or security firms. Obviously, these activities proceed smoothly only if the valuation of particular financial instruments is transparent and efficient in terms of time costs.

An important group of financial instruments with complex payoff functions and no less challenging valuation is referred to as exotic options. The term option indicates that at maturity time the holder (party in the long position) can decide whether he or she exercises the right (to trade the underlying asset of the option under given conditions) or not. Moreover, the term exotic comes from the fact that the contract conditions are rather complex. The complexity can concern several non-linearities and/or gaps in the payoff function, conditionality on the previous evolution of the underlying asset price (a so called path), etc. These features, at one hand, complicate the valuation procedure but on the other hand provide more efficient tools

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for risk management or, alternatively, speculation, since in this way special needs of market participants can be fit perfectly. Obviously, the more complex the payoff function is, the more advanced valuation methods we need.

Actually, option valuation used to be rather heuristic for a long time and nobody was sure how to relate the expectations about future evolution of the underlying price – or, more generally, underlying factors – to the fair price of option. Moreover, lack of knowledge about efficient option valuation was probably the main reason why there was (almost) no supply of exotic options. However, everything changed in early 70's after introducing a hedging strategy of options by replicating portfolio consisting of a risk-free bond and the option underlying asset in two papers by Black and Scholes (1973) and Merton (1973).

Obviously, such strategy has many drawbacks, including requirements on continues time rebalancing of the portfolio with infinitesimal changes in the positions, the assumption of normal distribution with fixed risk-free rate and volatilities, etc. Despite that, an elegant approach to option valuation based on the solution of partial differential equations (PDE), which describe increments in the value of the option and its underlying asset, under boundary conditions (b.c.) and terminal conditions specified by the option payoff function, see Black and Scholes (1973) and Merton (1973) for more details, won recognition and the approach started to be used in both, academy and practice, and two of the authors received the Nobel price (the third one passed away meantime).

The subsequent research in option pricing by huge number of authors focused on extending the valuation approach to be valid when some of the original assumptions are relaxed and even for more complex payoff functions and/or underlying distributions that fit real observations of market price returns better, see e.g. Cont and Tankov (2003) for a some of the most interesting extensions via Lévy models.

Soon, it became clear that analytical solution of the PDE under the assumption of more complex processes and payoff functions will not be easy and closed form option valuation formula will be hardly available. Thus, the researchers focused on numerical approximations to the option valuation problem, including stochastic simulation approach (i.e. the realization of a large number of pseudo/quasi-random scenarios of the underlying distribution, see e.g. Glasserman, 2003) and numerical solution to the approximation of the PDE via a group of finite elements methods (see e.g. Topper (2005) for a review of some approaches). Obviously, the approximation can lead to correct results only if carried out in the right way and in line with the model assumptions. Moreover, it is important to optimize particular steps of the approximation so that the results are not only correct but also attainable in a reasonable time, since at financial markets even milliseconds can cost a fortune.

In this respect we should note one particular feature related to numerical solution of option's PDE and approximation of its solution – since the domain on which the numerical computation is supposed to be done can be unbounded, i.e. underlying asset prices such as stocks can, in theory, attain infinite values, it is inevitable to specify some strict boundary conditions. Our aim in this paper is motivated by such need and thus we analyze the impact of three basic formulations of option boundary conditions for both European call and put options, assuming the discontinuous Galerkin (DG) approach (see Cockburn et al. (2000) and Rivière (2008)), in particular the Dirichlet, Neumann and transparent b.c., and examine the computational error and demandingness. Since the left boundary, i.e. the underlying asset price has no value, is generally quite simple and has very low impact on the computational demandingness, we focus especially on the right boundary, where the underlying asset price should approach to infinity. We assume plain vanilla European call and put options (i.e. not exotic feature) under normal distribution since analytical solutions to the valuation problem are available, i.e. it is easy to compare the numerical result with assumed value of a given option.

We proceed as follows. In Section 2 we review theoretical background of options and their valuation and then in Section 3 we provide the basics behind the Black and Scholes model, the PDE and boundary conditions. Next, in Section 4 we recall the discretization scheme based on the discontinuous Galerkin approach. Finally, a numerical analysis using DAX options market data is provided.

## 2. Options and their valuation

As it has already been stated above, an option is a unique type of financial derivative, which has a nonlinear and sometimes even discontinuous payoff function. Such property comes from the fact that the option holder – the party in the long position – can decide to realize the option, i.e. to exercise it, which mostly means to buy (call option) or sell (put option) the underlying asset (stock, bond, foreign currency, etc.) at, for European options, or, at or prior, for American options, maturity by paying the strike price.

Obviously, the option rights are realized only if it is efficient (i.e., profitable). It indicates that the option payoff is either positive or zero – it can never be negative. By contrast, the payoff of the party in the short position (option seller) is either negative or zero – it can never be positive. This is the source of the nonlinearity, which can make the option valuation more difficult when compared to the linear instruments, such as forwards or swaps.

The efficient approach to option valuation depends on its type, i.e. how complex the payoff function is, and what are the assumptions about the distribution of the asset prices, which underlines the option. The most standard assumption is to suppose that the prices follow lognormal distribution and thus to use the normal (Gaussian) distribution for the log-returns (see e.g. Black and Scholes, 1973). Although a long time ago it has already been shown that such assumption is not confirmed by real observations at financial markets, see e.g. Mandelbrot and Taylor (1967), the market practice is still closely related to the normal distribution and Black and Scholes model. However, in order to make the model consistent with market prices of traded options, the practice is to use a so called implied volatility instead of the historical one (i.e. calculated from past returns). It works in such a way that the valuation formula due to Black and Scholes is reverted and the price of an option obtained from the market is used as an input variable. Then, the output is the implied volatility, which is clearly not used for pricing of the same option (why do the same thing again), but it is utilized to construct (see Benko et al., 2007) the implied volatility curve (in dependency on moneyness) or surface (combinations of moneyness and maturity), which is further used to price options written on the same underlying asset, but with different parameters than liquid traded options (moneyness or maturity or both), or even to price exotic options.

Unfortunately, the Black and Scholes model is mostly useless for valuation of American options as well as for many exotic options. In such cases a numerical approximation must be run. Here, in order to get the solution and to keep time costs at sufficient level as well we do not run the calculation procedure on the complete space, but we utilize the knowledge which we have about the option value in dependency on assumed variables. In the most basic case, the variables are the maturity time and the underlying asset price. In the latter case, we utilize the principles of financial quantities, such as stocks – if its price drops to zero once, it cannot be positive anymore. Similarly, if it reach sufficient level, we do not suppose that it can drop below some threshold again in a reasonable time. Obviously, the impact on the option value is opposite for call and put options due to their inverse rights (buy versus sell). The impact of the time also differs for call and put options as we illustrate at Figure 1. It is apparent that the call option value is always above its payoff function, while for the put option there exist a point from which to the left the option value is lower than the payoff. This is called a negative time value and it is a crucial property since it makes exercising of (especially) put options prior the maturity efficient. In the next section, these findings will be utilized to formulate boundary conditions under which option's PDE is going to be solved.

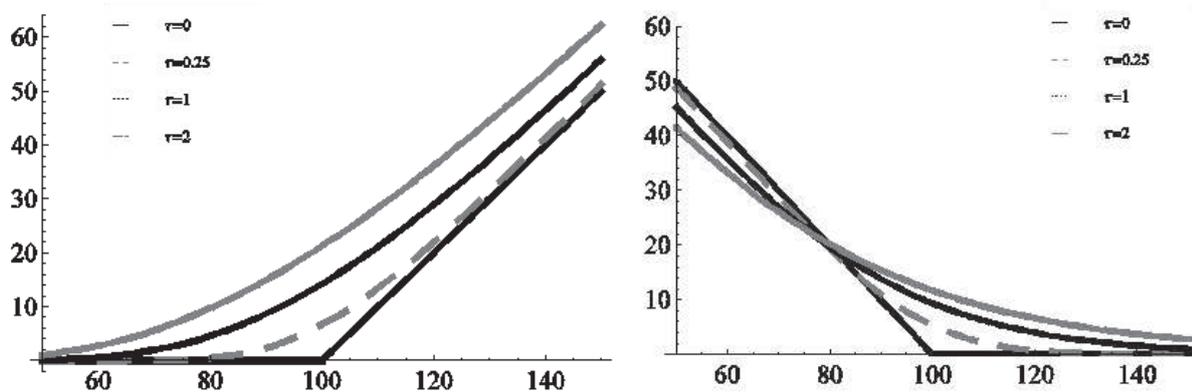


Figure 1: Relation of call (on the left) and put (on the right) option values to the payoff function ( $\tau = 0$ ) in dependency of increasing time to maturity  $\tau$  with strike  $K = 100$

### 3. Black-Scholes model

We now proceed to the formal definition of the option valuation model due to Black and Scholes (1973) so that we can show how to discretize it later. Let  $\Omega = (0; S_{max})$ ,  $0 < S_{max} < \infty$  be bounded open interval and  $T > 0$  stands for the maturity. We denote by  $x$  the price of the underlying asset (e.g. stock) and by  $t$  the time to maturity of the option. The price  $u : Q_T := \Omega \times (0; T) \rightarrow [0; \infty)$  of the plain vanilla option satisfies the Black-Scholes (BS) partial differential equation (for more details on options and their valuation see Tichý (2011) and reference therein):

$$\frac{\partial}{\partial t} u(x, t) - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} u(x, t) - rx \frac{\partial}{\partial x} u(x, t) + ru(x, t) = 0 \quad \text{in } Q_T. \tag{1}$$

Market parameters affecting the price are the risk-free interest rate  $r$  and volatility  $\sigma$  of returns of the underlying asset price  $x$ . In real markets, these values vary with time, but to keep the model and analysis simple, we assume  $r$  and  $\sigma$  to be constant. Besides that, we should recall a so called Greeks, delta and gamma. While delta measures the sensitivity to changes in the price of the underlying asset and it can be calculated as the derivative of  $u$  regarding to the underlying price  $x$ , i.e.  $\Delta = \frac{\partial u}{\partial x}$ , the rate of change of  $\Delta$  is called gamma, defined as  $\Gamma = \frac{\partial^2 u}{\partial x^2}$ , which is, in fact, the second order sensitivity of the option value on the underlying asset price.

The BS model is closed with the initial and boundary conditions. The initial condition is given by a payoff function corresponding to the terminal state if we consider the reversal time, i.e. for plain vanilla options we get

$$u(x, 0) = \begin{cases} \max(x - K, 0) & (\text{call}) \\ \max(K - x, 0) & (\text{put}) \end{cases} \quad x \in \Omega, \tag{2}$$

where the symbol  $K$  stands for the strike price.

Without loss of generality, we fix the Dirichlet boundary condition at the left boundary node  $x = 0$ , i.e.,

$$u(0, t) = \begin{cases} 0 & (\text{call}) \\ Ke^{-rt} & (\text{put}), \end{cases} \quad t \in (0; T). \tag{3}$$

and investigate the behavior of Dirichlet, Neumann and transparent b.c. at the right boundary node  $x = S_{max}$ . The generalization to the opposite situation is straightforward due to the put-call parity.

We consider the following three kinds of b.c. In the first instance, we employ the standard approach via Dirichlet and Neumann boundary conditions based on the asymptotic behavior of the exact solution as  $x \rightarrow +\infty$ , i.e.,

$$u(S_{max}, t) = \begin{cases} S_{max} - Ke^{-rt} & (call) \\ 0 & (put), \end{cases} \quad t \in (0; T), \quad (Dirichlet) \tag{4}$$

$$\frac{\partial}{\partial x} u(S_{max}, t) = \begin{cases} 1 & (call) \\ 0 & (put), \end{cases} \quad t \in (0; T), \quad (Neumann) \tag{5}$$

The main disadvantage of these two aforementioned approaches is the necessity of a sufficiently large computational domain in the vicinity of the strike price in order to suppress the approximation errors arising from the replacement of option values at the finite price  $S_{max}$  by asymptotic values at infinity. On the other hand, the large computational domain implies a high data requirement for the resulting numerical schemes, which is not desirable.

Therefore, we additionally mention the alternative sophisticated treatment of boundary condition inspired by the original works Keller and Givoli (1989) in the field of computational physics where they combined the both conflicting requirements (i.e. a domain length and data demands). These resulting transparent boundary conditions permit us to compute the solution for an unbounded domain in a bounded domain with no errors. Here we introduce only the transparent boundary condition for the put option, because the opposite situation with a call option can be again easy derived according to the put-call parity relation. This transparent boundary condition reads as

$$u(S_{max}, t) = -\sqrt{\frac{1}{2\pi}} \int_0^t \left( \frac{r - \sigma^2/2}{2\sigma} u(S_{max}, s) + \sigma S_{max} \frac{\partial}{\partial x} u(S_{max}, s) \right) \frac{e^{-\frac{(r+\sigma^2/2)^2(t-s)}{2\sigma^2}}}{\sqrt{t-s}} ds, \quad t \in (0; T), \quad (transparent) \tag{6}$$

From the mathematical point of view the problem (1)-(6) represents a convection-diffusion-reaction equation equipped with a set of appropriate boundary conditions (3)-(6) prescribed at endpoints of  $\Omega$  and with the initial condition given by the payoff function (2). The conditions (2)-(6) depend on the type of the option.

#### 4. Discretization

We recall discretization schemes, based on the discontinuous Galerkin method, for the numerical solution of the European option valuation problem introduced in Hozman (2012, 2013). We use a method of lines at first, i.e. the discretization is carried out with respect to the space variables and time is treated continuously. The standard DG method uses piecewise, generally discontinuous approximations, and the semi-discrete solution  $u_h(t)$  is sought in the space

$$S_h \equiv \{v_h \in L^2(\Omega); v_h|_{I_k} \in P_p(I_k) \quad \forall I_k \in T_h\}, \tag{7}$$

where  $T_h$  is a family of partitions of the domain  $[0; S_{max}]$  and  $P_p(I_k)$  denotes the space of all polynomial functions of order less or equal to  $p$  defined on subintervals  $I_k$  of length  $h$ .

Then we end up with the following DG formulation for the semi-discrete solution  $u_h(t)$  represented by the system of ordinary differential equations:

$$\frac{d}{dt} (u_h(t), v_h) + B_h(u_h(t), v_h) = l_h(v_h)(t) \quad \forall v_h \in S_h, \quad \forall t \in (0; T) \tag{8}$$

with initial condition  $u_h(0) = u_h^0$ , where  $u_h^0$  denotes the  $S_h$ -approximation of the initial condition  $u(x, 0)$ . The notation  $(\cdot, \cdot)$  is the inner product in  $L^2(\Omega)$ , the form  $B_h$  involves the semi-discrete variants of diffusion, convection, penalization and reaction terms, and the form  $l_h$  arises from the prescribed boundary conditions. For more details see Hozman (2012, 2013).

In order to obtain the discrete solution, it is necessary to equip scheme (8) with suitable solvers for the time integration. Therefore, we employ the implicit Euler scheme with respect to the time coordinate, giving the first order convergence in time with the unconditional stability and no limitations on the length of the time step. For simplicity, we consider a partition of the interval  $[0; T]$  with a constant time step  $\tau \equiv T/M$ . Let  $u_h^l$  stand for the approximate solution of  $u_h(l\tau), l = 0, \dots, M$ , of the problem (8). The values  $u_h^l$  at given time levels are computed according to the following formula

$$\frac{1}{\tau}(u_h^l - u_h^{l-1}, v_h) + B_h(u_h^l, v_h) = I_h(v_h)(l\tau) \quad \forall v_h \in S_h, l = 1, \dots, M \tag{9}$$

with starting data  $u_h^0$ . Since the problem (1) is linear, the implicit treatment in (9) also leads to a linear algebraic problem at each time level.

Further, we introduce the matrix representation of the discrete problem (9). Let  $\{v_j\}_{j=1}^{DOF}$  be the standard basis of  $S_h$ ,  $DOF = \dim(S_h)$ , then  $u_h^l$  can be written as  $u_h^l(x) = \sum_{j=1}^{DOF} \xi_j^l v_j(x), x \in \Omega$ , for vector of real coefficients  $\mathcal{W}^l = (\xi_1^l, \dots, \xi_{DOF}^l)^T$ . If the expressions above for  $u_h^l$  are inserted into equation (9) we receive the sparse matrix equation

$$(\mathbb{M} + \tau \mathbb{B})\mathcal{W}^l = \mathbb{M}\mathcal{W}^{l-1} + \tau \mathcal{F}^l, l = 1, \dots, M, \tag{10}$$

where the matrix  $\mathbb{M}$  is related to the mass matrix, the matrix  $\mathbb{B}$  to the form  $B_h$  and the vector  $\mathcal{F}^l$  represents the right-hand side at time  $t = l\tau$ , respectively.

#### 4.1. Discretization of boundary conditions

Since Dirichlet and Neumann b.c. are given by simple explicit formulae (4) and (5), their discretization is straightforward. Therefore, the attention is paid to the transparent boundary condition treatment only. As mentioned before we consider the transparent boundary condition for a put option given by (6), which can be approximated according to Achdou and Pironneau (2005) by the following formula:

$$\begin{aligned} \sqrt{2\pi}u(S_{max}, t) + 2\sqrt{\tau} \left( \frac{r - \sigma^2/2}{2\sigma} u(S_{max}, t) + \sigma S_{max} \frac{\partial}{\partial x} u(S_{max}, t) \right) \\ = - \int_0^{t-\tau} \left( \frac{r - \sigma^2/2}{2\sigma} u(S_{max}, s) + \sigma S_{max} \frac{\partial}{\partial x} u(S_{max}, s) \right) \frac{e^{-\frac{(r+\sigma^2/2)^2(t-s)}{2\sigma^2}}}{\sqrt{t-s}} ds \end{aligned} \tag{11}$$

Condition (11) can be subsequently rewritten as a special case of Robin boundary condition, which reads as

$$\frac{1}{2}\sigma^2 S_{max}^2 \frac{\partial}{\partial x} u(S_{max}, t) + \left( \frac{\sqrt{2\pi}\sigma S_{max}}{4\sqrt{\tau}} + \frac{S_{max}(r - \sigma^2/2)}{4} \right) u(S_{max}, t) = \frac{\sigma S_{max}}{4\sqrt{\tau}} g(S_{max}, t) \tag{12}$$

where function  $g(S_{max}, t)$  denotes the right-hand side of (11) and its discretization by the rectangular rule on a uniform space-time grid of steps  $h$  and  $\tau$  gives

$$g(S_{max}, l\tau) \approx \sum_{m=0}^{l-1} \frac{e^{-\frac{(r+\sigma^2/2)^2(l-m)\tau}{2\sigma^2}}}{\sqrt{l-m}} \left( -\frac{\sqrt{\tau}(r - \sigma^2/2)}{2\sigma} \xi_{DOF}^m + \frac{\sqrt{\tau}\sigma S_{max}}{h} \xi_{DOF-1}^m - \frac{\sqrt{\tau}\sigma S_{max}}{h} \xi_{DOF}^m \right). \tag{13}$$

Let us note that it is required to store all values on the last subinterval  $[S_{max} - h; S_{max}]$  from all previous time levels in order to evaluate the function  $g$  at new time level according to (13).

### 5. Numerical experiment

Now, after defining the discretization scheme and deriving the boundary conditions, we can proceed to the comparison of their impact on the error in the option price estimation due to problem (1). For this purpose we assume real market data – DAX option prices on a given day with various strike prices and the volatility obtained from the Black and Scholes model. In fact, the collection of market data includes both, the option prices and implied volatilities, but not the risk-free rate for a given maturity. Therefore, we use the rates that are implied by the two aforementioned quantities, which should assure the convergence of the results to the market prices.

### 5.1. Parameter setting

In particular, we use market data of the DAX call and put options as available for September 15, 2011, including selected Greeks (delta and gamma) and implied volatilities, which are supposed to be computed by inverting the Black-Scholes formula (1973):

$$u(x, 0) = \begin{cases} x\phi(d) - Ke^{-rt}\phi(d - \sigma\sqrt{t}) & (\text{call}) \\ -x\phi(-d) + Ke^{-rt}\phi(-d + \sigma\sqrt{t}) & (\text{put}), \end{cases} \quad \text{with } d = \frac{\ln(x/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}, \quad (14)$$

where  $\phi$  is the distribution function of the standard normal distribution. On a given day, the current price is 5508.238 and the strike prices run from 1000 to 6000 with span of 1000. The remaining maturity of all options is approximately half of the year. In order to easily illustrate the influence of the different treatment of boundary conditions we set the maximal underlying price  $S_{max} = 7000$ , i.e. the computational domain  $\Omega = (0; 7000)$ . All options have expiration date  $T = 0.50137$  and the calculated risk-free interest rate  $r = 0.0176$ . The computations are carried out on a sequence of different strike prices for all three types of boundary conditions. The whole algorithm is implemented in MATLAB and uses piecewise linear approximations on a uniform partition of  $\Omega$ . In order to ensure a high resolution of the approximate solution and its observed sensitivity measures, a sufficiently fine space-time grid is set up with equidistant mesh step  $h = 20.0$  and time step  $\tau = 1/1200$ . Further, the resulting linear algebraic problem (10) is numerically solved by the restarted GMRES solver. The restart is carried out after 50 iterations and the iterative process is stopped when the discrete  $l^2$ -norm of the residuum is smaller than  $10^{-8}$ .

### 5.2. Analysis of the influence of boundary conditions

Now we can proceed to the investigation of the influence of Dirichlet, Neumann and transparent b.c. on the behavior of the call (Table 1) and put option (Table 2) price  $u_n$ . The first panel in Table 1 shows results obtained for call option price if Dirichlet b.c. is supposed. It is apparent that the implied volatilities are identical for deep ITM options and that the relative error in the option price is very high especially for  $K = 6000$  and that there is a break-even point for  $K = 3000$ , since both the absolute and relative errors are the lowest here.

Table 1: The values of the vanilla call option with different strike prices for particular boundary conditions in comparison with reference option values

Results for Dirichlet condition (call)					
strike price	implied volatility	option price	$u_h(T)$ (Dirichlet)	absolute error	relative error (%)
1000	0.4623	4494.1	4517.022833	22.924833	0.5101
2000	0.4623	3509.8	3526.125805	16.325805	0.4651
3000	0.4623	2549.9	2549.956502	0.056502	0.0022
4000	0.4222	1638.8	1636.914953	1.885047	0.1150
5000	0.3557	845.7	839.581402	6.118598	0.7235
6000	0.2886	279.0	265.973621	13.026379	4.6690
Results for Neumann condition (call)					
strike price	implied volatility	option price	$u_h(T)$ (Neumann)	absolute error	relative error (%)
1000	0.4623	4494.1	4517.022688	22.924688	0.5101
2000	0.4623	3509.8	3526.127648	16.327648	0.4652
3000	0.4623	2549.9	2550.314700	0.414699	0.0163
4000	0.4222	1638.8	1640.203269	1.403269	0.0856
5000	0.3557	845.7	849.478145	3.778145	0.4467
6000	0.2886	279.0	289.910294	10.910294	3.9105
Results for transparent condition (call)					
strike price	implied volatility	option price	$u_h(T)$ (transparent)	absolute error	relative error (%)
1000	0.4623	4494.1	4516.946714	22.848714	0.5084
2000	0.4623	3509.8	3526.045350	16.245350	0.4629
3000	0.4623	2549.9	2550.063086	0.163086	0.0064
4000	0.4222	1638.8	1638.613858	0.186142	0.0114
5000	0.3557	845.7	844.840792	0.859208	0.1016
6000	0.2886	279.0	278.704882	0.295118	0.1058

Almost identical observations are apparent in the panel 2, which shows the results for Neumann b.c., though it seems that the error is mostly slightly lower when compared to the results in panel 1. Finally, in panel 3 we can observe approximately the same results for deep ATM options, while error recorded for close to ATM options, which are often more important for the risk management purposes, is significantly lower than we observed for both previously applied b.c. Moreover, as concerns the relative error, it is even lower than for deep ITM options. Similar results can be observed in Table 2. Since this time the put options are considered, most of the options are deep OTM and only the last line shows a (mild) ITM option. It follows that the implied volatilities differ for particular strike levels and that even the highest put option price is lower than all call prices, except the only one which is OTM. Obviously, it also indicates that the absolute error is quite low this time.

Table 2: The values of the vanilla put option with different strike prices for particular boundary conditions in comparison with reference option values

Results for Dirichlet condition (put)					
strike price	implied volatility	option price	$u_h(T)$ (Dirichlet)	absolute error	relative error (%)
1000	0.7746	0.3	0.350480	0.050480	16.8267
2000	0.6708	8.6	8.603413	0.003413	0.0397
3000	0.5613	41.3	40.174430	1.125570	2.7254
4000	0.4594	123.0	119.386033	3.613967	2.9382
5000	0.3774	322.5	313.877328	8.622672	2.6737
6000	0.3090	748.4	729.901892	18.498108	2.4717
Results for Neumann condition (put)					
strike price	implied volatility	option price	$u_h(T)$ (Neumann)	absolute error	relative error (%)
1000	0.7746	0.3	0.377727	0.077727	25.9090
2000	0.6708	8.6	9.376756	0.776756	9.0320
3000	0.5613	41.3	43.238700	1.938700	4.6942
4000	0.4594	123.0	126.239046	3.239046	2.6334
5000	0.3774	322.5	328.945882	6.445882	1.9987
6000	0.3090	748.4	764.950868	16.550868	2.2115
Results for transparent condition (put)					
strike price	implied volatility	option price	$u_h(T)$ (transparent)	absolute error	relative error (%)
1000	0.7746	0.3	0.365424	0.065424	21.8080
2000	0.6708	8.6	9.025768	0.425768	4.9508
3000	0.5613	41.3	41.837685	0.537685	1.3019
4000	0.4594	123.0	123.081728	0.081728	0.0664
5000	0.3774	322.5	321.955844	0.544156	0.1687
6000	0.3090	748.4	748.578996	0.178996	0.0239

By contrast, if we focus on the relative error, it is mostly higher than for call options, which might lead to significant losses when compared to the expected return. More importantly, the transparent b.c. beats the other two approaches significantly for all strike levels, except the deepest OTM, which might be given by a very low price and potentially a high sensitivity to the formulation of the discretization, which might be further analyzed by changing the discretization scheme.

Next, we examine also the sensitivity measures, commonly known as the Greeks, which are very important tools in risk management. The presented method (9) is well suited for calculating the Greeks since it gives a polynomial approximation in the spatial variables. In the simplest case of a linear approximation, the second order derivatives (i.e.,  $\Gamma$ ) are computed by the forward difference formula for values of  $\Delta$ . The approximate values of option price and their sensitivity measures are compared with exact ones given at underlying's price  $x = 5508.238$ , according to the DAX data file on a given day.

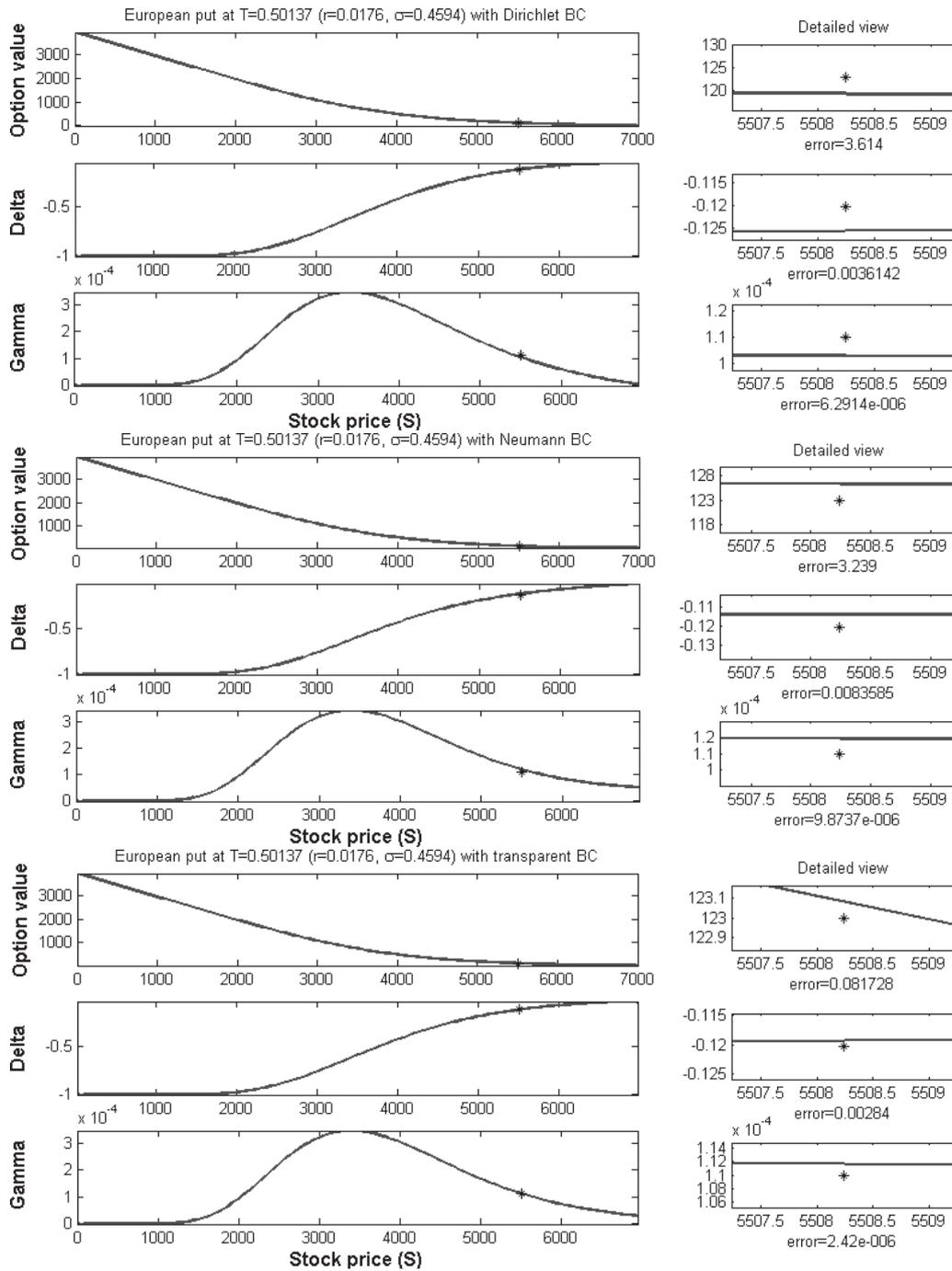


Figure 2: The prices of the vanilla put option and the Greeks with  $K = 4000$  for Dirichlet (top), Neumann (middle) and transparent (bottom) b.c. together with detail (right) in the neighborhood of the exact values (star)

Figure 2 depicts the case  $K = 4000$  and captures the differences in the values of option, delta and gamma with respect to the different treatment of b.c. in comparison with the exact values. One can clearly observe that the transparent boundary conditions produce the smallest absolute errors, which further stress the advantageous of this approach. From more detailed analysis it is clear that if the proportion  $\frac{S_{max}}{K} \rightarrow 1^+$  then

it is better and almost essential to use transparent boundary conditions. On the other hand, as  $K \rightarrow 0^+$  the treatment of Dirichlet b.c. produces the smallest errors, though the differences are not so high.

## 6. Conclusion

Functioning financial markets are based on transparent and efficient pricing of financial instruments, including those with complex payoff functions. An inevitable step for correct valuation of exotic options via a particular technique is the examination of a given approach for plain vanilla options, which are very liquid and for which closed form valuation formula is available.

In this paper we focused on discretization scheme based on the discontinuous Galerkin approach for numerical valuation of plain vanilla European options and in particular on the impact of the choice of the way in which one of the boundary conditions is formulated – in particular, the one which bounds a potentially infinite underlying asset price.

Based on the real data of European call and put option prices on German DAX index collected on September 15, 2011 it was shown that in most cases, the best result in terms of the lowest error in the differences in the price as well as the Greeks is provided by the so called transparent boundary condition, except some special cases.

Such conclusion is very promising since the suggested approach leads to low error (relative/absolute) at low time costs, which is an inevitable condition for efficient usage of financial derivatives with more complex payoff functions. The further analysis might be focused on more precise identification of conditions under which the Dirichlet b.c. outperforms the transparent condition and what might be the impact if discontinuity in the payoff is considered.

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