A NEW ITERATIVE ALGORITHM FOR PROBABILISTIC PERFORMANCE MEASURE

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ABSTRACT

Compared to traditional Reliability Index Approach (RIA), Performance Measure Approach (PMA) is considered to be more efficient and stable for evaluation of probabilistic constraints in reliability-based design optimization of structures. In PMA, the probabilistic performance measure is obtained through locating the minimum performance target point (MPTP) with the specified target reliability index in standard normal space. The advanced mean-value (AMV) method is well suitable for locating MPTP due to its simplicity and efficiency. However, the iterative sequence may converge very slowly, or oscillate and fail to converge if the performance function is highly nonlinear. Several modified algorithms were suggested to enhance the convergence of AMV, but their implementation is complicated and the prior knowledge of convexity or concavity of the performance function is needed. In this paper an easy iterative algorithm, which introduces a "new" step size to control the convergence of the sequence, is proposed. This step size is new because it may be constant during the iteration or decreases successively using a self-adjust strategy. It is proved that the AMV method is a special case of this proposed algorithm when the step size tends to infinity. Numerical results of several nonlinear performance functions indicate that the proposed algorithm is effective and as simple as the AMV but more robust.

KEYWORDS

Performance Measure Approach, Probabilistic performance measure, Advanced mean-value method, Iterative algorithm.

INTRODUCTION

Uncertainties are observed in material and geometric properties and external loads during structures' lifetime. Consequently, the reliability-based design optimization (RBDO) is indispensable and rational (Tu et al 1999; Yang and Gu 2004; Chiralaksanakul and Mahadevan 2005; Cheng et al, 2006; Zou and Mahadevan 2006). The RBDO formulation is usually expressed as the minimization of objective function under probabilistic constraints and the effectiveness, robustness and efficiency of the assessment of probabilistic constraints are the key of smooth implementation for RBDO. There are two approaches to evaluate probabilistic constraints: reliability index approach (RIA) and performance measure approach (PMA) (Tu et al 1999). The latter is also referred to as inverse reliability analysis (Kiureghian et al 1994; Li and Foschi 1998).

In RIA, the sub-problem of constraint evaluation is transformed to check if the reliability index β is greater than the specified target reliability index and β is the distance between the origin and the most probable failure point (MPFP) on the limit state surface in the standard normal random variable space. Whereas, in PMA the constraint evaluation is to check if the probabilistic performance measure (PPM) G_p is greater than zero and G_p is the performance function value at the minimum performance target point (MPTP) with the specified target reliability index in the standard normal space. Compared with RIA, it is thought that PMA has higher efficiency and superior numerical stability (Tu et al 1999; Lee et al 2002; Youn et al 2003; Du et al 2004; Lee and Lee 2005; Yi et al 2008). The iterative scheme of advanced mean value (AMV) is well suitable for PMA due to the simplicity and efficiency. However, for some nonlinear performance functions the iterative sequences of AMV formulation could yield non-convergence solutions such as the periodic oscillation observed in the references (Lee et al 2002; Youn et al 2003; Youn and Choi 2004; Du et al 2004). Hence, several improved algorithms aiming to enhance the convergence of AMV were suggested. Youn and his research team proposed conjugate mean value method (CMV) (Youn et al 2003), hybrid mean value method (HMV) (Youn and Choi 2004), and enhanced hybrid mean value method (EHMV) (Youn et al 2005a). However, these algorithms need the prior knowledge including normal vectors of several iterative points and convexity or concavity of the performance function, and require complicated implementation (Youn et al 2005a; Youn et al 2005b). Furthermore, they could still fail to converge for some problems. Yang & Yi (2009) employed the stability transformation method of chaos control (CC) to achieve the oscillation, bifurcation and chaos control for the solution of the AMV iterative procedure. Although the CC method performs well for nonlinear performance functions, it is computationally inefficient. Accordingly Meng et al (2015) proposed a modified chaos control (MCC) method, which improved the convergence by extending the iterative point of the CC method to the constraint boundary. But for both CC and MCC method, it is not easy to select the appropriate factor and involutory matrix, especially for high-dimensional random space.

In this paper, an easy iterative algorithm, which introduces a “new” step size to control the convergence of the sequence, is proposed. This step size is new because it may be constant during the iteration or decreases successively using a self-adjust strategy. It is proved that the AMV method is a special case of this proposed algorithm when the step size tends to infinity. Numerical results of several nonlinear performance functions, including an engineering application, indicate that the proposed algorithm is effective and as simple as the AMV but more robust.

PROPOSED ITERATION ALGORITHM

Basic iteration formulation

To measure probabilistic performance using inverse reliability analysis, one first needs to transform the original random vector $\mathbf{x}$ to a standard Gaussian vector $\mathbf{u}$ (zero means, unit variance and independent components), expressed as $\mathbf{u} = \mathrm{T}(\mathbf{x})$ or $\mathbf{x} = \mathrm{T}^{-1}(\mathbf{u})$. Then the performance function $G(\mathbf{d},\mathbf{x}) = G\left(\mathbf{d},\mathrm{T}^{-1}(\mathbf{u})\right) = g(\mathbf{d},\mathbf{u})$, in which $\mathbf{d}$ is the design variable vector, representing either deterministic physical quantities or parameters of the probability distribution of the random variables (e.g. the mean values or standard deviations of the random variables). The performance function $G(\mathbf{d},\mathbf{x}) < 0$ denotes the failure domain. The probabilistic performance measure $G_p(\mathbf{d})$ can be obtained from an optimization problem in standard normal **u**-space (Tu et al 1999; Lee et al 2002),

$$\begin{aligned}&\text{for any given } \mathbf{d} \quad \text{find} \quad \mathbf{u}^*\text{, such that}\\ &\text{minimize} \quad g(\mathbf{d},\mathbf{u})\\ &\text{s.t.} \quad \|\mathbf{u}\| = \beta_t\end{aligned} \tag{1}$$

where $\mathbf{u}^*$ is the minimum performance target point (MPTP) on the sphere surface with the target reliability index β_t and PPM $G_p(\mathbf{d}) = g(\mathbf{d},\mathbf{u}^*) = G(\mathbf{d},\mathbf{x}^*)$. Hereafter **d** will be omitted for concision.

In PMA, AMV algorithm, which can be derived from the KKT conditions of the optimization formulation (1), is commonly used because of its simplicity and efficiency. The iterative formula of AMV is expressed as Eq. (2) and Figure 1 shows the iterative procedure of AMV method.

$$\mathbf{u}^{k+1} = \beta_t \mathbf{n}(\mathbf{u}^k) \qquad \mathbf{n}(\mathbf{u}_k) = -\frac{\nabla_{\mathbf{U}} g(\mathbf{u}^k)}{\|\nabla_{\mathbf{U}} g(\mathbf{u}^k)\|} \tag{2}$$

where $\mathbf{u}^k = (u_1^k, u_2^k, \cdots, u_n^k)$ is the k-th iteration point on the target reliability surface. $\nabla_{\mathbf{U}} g(\mathbf{u}^k) = (\frac{\partial g}{\partial u_1^k}, \frac{\partial g}{\partial u_2^k}, \cdots, \frac{\partial g}{\partial u_n^k})$ is the gradient vector of performance function at point $\mathbf{u}^k$. $\mathbf{n}(\mathbf{u}^k)$ is the negative gradient direction at point $\mathbf{u}^k$, as shown in Figure 1.

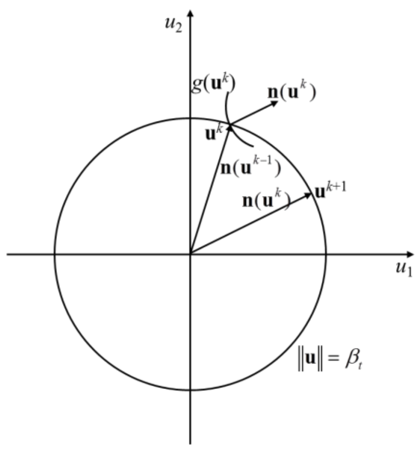

Figure 1 Iterative procedure of AMV method

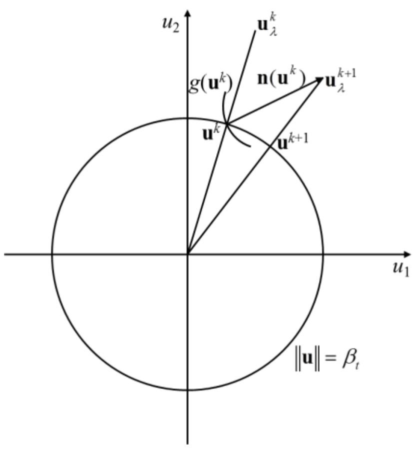

Figure 2 Iterative procedure of the proposed method

It is found that if the limit surface is flat, AMV method converges fast. However, for some nonlinear performance functions the iterative sequence of AMV could yield non-convergence solution such as the periodic oscillation observed in the references (Lee et al 2002; Youn et al 2003; Youn and Choi 2004; Du et al 2004). Here a "new" step size parameter to control the convergence of the sequence is proposed.

As shown in Figure 2, $\mathbf{u}^k$ and $\mathbf{n}(\mathbf{u}_k)$ have the same meanings as that in Figure 1. We can get a point

$\mathbf{u}_{\lambda}^{k+1}$ by moving a step with size λ (>0) along the negative gradient direction from point $\mathbf{u}^{k}$. The $k+1$-th iteration point, $\mathbf{u}^{k+1}$, is the intersection point of the target reliability sphere surface and the line, which connects the point $\mathbf{u}_{\lambda}^{k+1}$ to the origin. The expressions are as follows:

$$\mathbf{u}_{\lambda}^{k+1} = \mathbf{u}^{k} - \lambda \nabla_{\mathrm{U}} g(\mathbf{u}^{k}) \qquad \mathbf{u}^{k+1} = \beta_t \frac{\mathbf{u}_{\lambda}^{k+1}}{\left\|\mathbf{u}_{\lambda}^{k+1}\right\|} \tag{3}$$

It can be seen in Figure 2 that $\mathbf{u}_{\lambda}^{k+1} = \mathbf{u}^{k}$ if $\lambda = 0$, which means the iteration points are in fixed position and the iteration procedure will never converge to the MPTP. So we must set $\lambda > 0$. If $\lambda \to \infty$,

$$\begin{aligned}\lim_{\lambda\to\infty} u_i^{k+1} &= \lim_{\lambda\to\infty} \beta_t \frac{u_i^k - \lambda \dfrac{\partial g(\mathbf{u}^k)}{\partial u_i}}{\sqrt{\sum_{i=1}^{n} (u_i^k - \lambda \dfrac{\partial g(\mathbf{u}^k)}{\partial u_i})^2}} = \lim_{\lambda\to\infty} \beta_t \frac{u_i^k / \lambda - \dfrac{\partial g(\mathbf{u}^k)}{\partial u_i}}{\sqrt{\sum_{i=1}^{n} (u_i^k / \lambda - \dfrac{\partial g(\mathbf{u}^k)}{\partial u_i})^2}} \\ &= -\beta_t \frac{\dfrac{\partial g(\mathbf{u}^k)}{\partial u_i}}{\sqrt{\sum_{i=1}^{n} [\dfrac{\partial g(\mathbf{u}^k)}{\partial u_i}]^2}} \qquad (i = 1, 2, \cdots, n)\end{aligned} \tag{4}$$

This is just the iteration formulation of the AMV method shown in Eq. (2). It means the AMV method is a special case of this proposed algorithm when the step size tends to infinity.

Determination of the Step Size

When $\left\|\mathbf{u}^{k+1} - \mathbf{u}^{k}\right\| > 0$, one can always find a proper value of λ, which makes $\left\|\mathbf{u}^{k+1} - \mathbf{u}^{k}\right\| > \left\|\mathbf{u}^{k+2} - \mathbf{u}^{k+1}\right\|$. Suppose $\left\|\mathbf{u}^{k+2} - \mathbf{u}^{k+1}\right\| = t^{k+1}\left\|\mathbf{u}^{k+1} - \mathbf{u}^{k}\right\|$, where $0 < t^{k+1} < 1$. There must be a sequence of λ which makes

$$\begin{aligned}&\lim_{k\to\infty}\left\|\mathbf{u}^{k+2} - \mathbf{u}^{k+1}\right\| = \lim_{k\to\infty} t^{k+1}\left\|\mathbf{u}^{k+1} - \mathbf{u}^{k}\right\| = \lim_{k\to\infty} t^{k+1}t^{k}\left\|\mathbf{u}^{k} - \mathbf{u}^{k-1}\right\| \\ &= \lim_{k\to\infty} t^{k+1}t^{k}t^{k-1}\left\|\mathbf{u}^{k-1} - \mathbf{u}^{k-2}\right\| = \cdots = \lim_{k\to\infty} t^{k+1}t^{k}t^{k-1}\cdots t^{2}\left\|\mathbf{u}^{2} - \mathbf{u}^{1}\right\| = 0\end{aligned} \tag{5}$$

Eq. (5) means the proposed method can always converge when setting a appropriate value of λ.

Then how to set a appropriate value for step size λ to guarantee convergence of the iteration process? The appropriate value for λ depends on the order of nonlinearity of the performance function in standard normal space. λ should be small if the order of nonlinearity of the performance function is high. Computational results of the following examples show that performance functions with different nonlinear degree have relevant maximum values $\lambda_{\max}$ and the iterative sequence can converge only when $0 < \lambda < \lambda_{\max}$. When the nonlinear degree is relatively low, $\lambda_{\max} = \infty$ and the AMV method, the special case of the proposed algorithm, can be used. However the performance function's nonlinear degree is usually unpredictable and even if it can be predictable, the quantity relation between $\lambda_{\max}$ and the nonlinear degree is impossible to constitute. Consequently we must "try" in practical application and this troublesome trying work can be done by computers. Firstly, a relatively large value is set for λ. If $\left\|\mathbf{u}^{k+2}-\mathbf{u}^{k+1}\right\|>\left\|\mathbf{u}^{k+1}-\mathbf{u}^{k}\right\|$ for a certain iteration in the iterative process, set $\lambda=\lambda/c$. Here *c*, named adjusting coefficient for step size, is a constant greater than 1. Computational examples indicate good results can be obtained when *c* is set between 2.2~2.6. Of course there is no need to adjust λ if the initial value of λ is less than $\lambda_{\max}$. In addition, the smaller the step size is, the more iterations are needed. This proposed method can be named as step-size-adjustment iterative algorithm.

Iterative Process

The proposed algorithm consists of the following steps.
Step 1: Transform the original random variable vector **x** into a standard normal vector **u**.
Step 2: Set k=0 and determine initial iterative point $\mathbf{u}^0$ (we usually set $\mathbf{u}^0$=0), initial step size λ and adjustment factor for step size c. Generally, we set $0<\lambda<10$ and 2.2<c<2.6. ε is convergence precision.
Step 3: Calculate the gradient vector of performance function $\nabla_{\mathbf{U}} g(\mathbf{u}^k)$ at point $\mathbf{u}^k$ according to finite difference method.
Step 4: Calculate $\mathbf{u}^{k+1}$ from Eq. (3)
Step 5: If $\left\|\mathbf{u}^{k+1}-\mathbf{u}^{k}\right\|<\varepsilon$, stop the iteration and $\mathbf{u}^{k+1}$ is MPTP. Otherwise, go to step 6.
Step 6: If $\left\|\mathbf{u}^{k+1}-\mathbf{u}^{k}\right\|>\left\|\mathbf{u}^{k+2}-\mathbf{u}^{k+1}\right\|$, return to step 3. Otherwise, decrease the step size using $\lambda=\lambda/c$, and then return to step 3.

ILLUSTRATIVE EXAMPLES

In this section, several examples including an engineering application with different nonlinear degree of the performance functions are given to demonstrate the effectiveness, robustness and efficiency of the proposed method.

Example 1 A performance function is given as (Youn et al (2003)) :

$$G=-\exp(x_1-7)-x_2+10$$

This problem contains two independent random variables **x**=[x_1 x_2] with normal distribution x_i~N(6.0,0.8), i=1,2. The target reliability index is $\beta_t=3.0$.

Table 1 shows the results of PPMs and corresponding iterations using the proposed method with different λ and using the AMV algorithm. The AMV algorithm can converge because the nonlinear degree of the performance function is relatively low. In other words, $\lambda_{\max}=\infty$ for this problem. From Table 1, we can see that all the resulting PPMs are the same, but the iterations are different when using the proposed method with different λ. The larger the value of λ is, the less iterations are needed to converge. The number of iterations is the same with that of the AMV algorithm when λ is greater than 10 and the proposed method retrogresses to the AMV method.

Table 1 Results of PPM and iterations for different λ

λ	0.05	0.1	0.5	1	5	10	>10	AMV	CMV(Youn et al (2003))
Iterations	145	80	23	15	8	8	7	7	12
G_p	-0.354	-0.356	-0.358	-0.358	-0.358	-0.358	-0.358	-0.358	-0.358

Example 2 Consider the following performance function(Meng et al (2015)):

$$G(\mathbf{x}) = 0.3x_1^2 x_2 - x_2 + 0.8x_1 + 1$$

$$x_1 \sim N(1.2, 0.42) \quad x_2 \sim N(1.0, 0.42) \quad \beta_t = 6.0$$

The iterative results of different approaches are listed in Table 2. It is seen that the AMV algorithm fails to converge while all the other methods can converge. However, the proposed method converges to MPTP accurately after only five iterations and it is much more efficient than other algorithms. **Figure 3** shows the iterative history of the proposed method in standard normal space.

Table 2 Results of different approaches in Example 2

	G_p	**u***	Iterations	Function evaluations
AMV	-----			
HMV	-2.2212	(-2.911, 5.246)	114	456
CC $\lambda = 0.5$	-2.2290	(-3.116, 5.127)	14	56
MCC Meng et al (2015)	-2.2162	(-2.860, 5.275)	16	64
Proposed method $\lambda = 5$	-2.2293	(-3.108, 5.132)	5	20

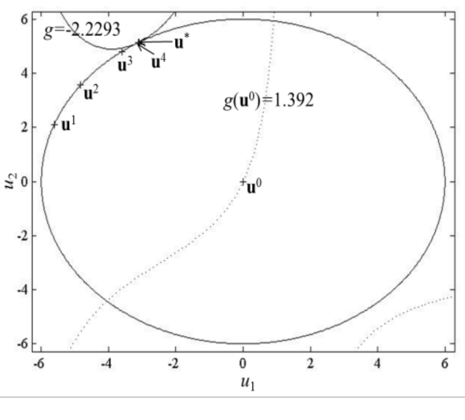

Figure 3 Iterative history in Example 2

Example 3 A performance function with high nonlinear degree is given as(Yang & Yi (2009)):

$$G(\mathbf{x}) = x_1^3 + x_1^2 x_2 + x_2^3 - 18$$

$$x_1 \sim N(10,5) \quad x_2 \sim N(9.9,5) \quad \beta_t = 3.0$$

The AMV method cannot converge while good results can be obtained using the proposed method with the step size adjusted automatically. Using initial step size $\lambda = 10.0$ and adjustment factor c=2.5, the iteration history is listed in Table 3. It can be seen that the step size λ is adjusted 7 times during 19 iterations and is decreased from 10 to 0.01638 finally. The iterative process converges to the MPTP accurately after 20 iterations. The probabilistic performance measure is $G_p = -76.035$ and the MPTP is $\mathbf{u}^* = (-1.0595, -2.8067)^T$. Compared with chaos control method(CC) in Yang & Yi (2009), which needs 95 iterations to converge, the proposed method is more effective.

Table 3 Iterative history of the proposed method in Example 3

k	u_1^k	u_2^k	G^k	λ^k	k	u_1^k	u_2^k	G^k	λ^k
0	0	0	2942.299	10.0000	10	-1.0406	-2.8137	-75.987	0.04096
1	-2.3526	-1.8615	-21.431	10.0000	11	-1.0751	-2.8007	-75.971	0.01638
2	-2.5989	-1.4985	-9.3151	10.0000	12	-1.0349	-2.8159	-75.945	0.01638
3	-1.2876	-2.7096	-67.648	4.0000	13	-1.0731	-2.8015	-75.984	0.01638
4	-0.6728	-2.9236	-38.553	1.6000	14	-1.0441	-2.8124	-76.007	0.01638
5	-1.5929	-2.5422	-43.421	0.6400	15	-1.0659	-2.8042	-76.020	0.01638
6	-0.1262	-2.9973	226.29	0.2560	16	-1.0494	-2.8105	-76.027	0.01638
7	-2.1322	-2.1104	-18.851	0.2560	17	-1.0619	-2.8058	-76.031	0.01638
8	-2.2573	-1.9760	-20.096	0.1024	18	-1.0525	-2.8093	-76.033	0.01638
9	-2.7054	-1.2965	20.551	0.04096	19	-1.0595	-2.8067	-76.035	0.01638

Example 4 A slab-column structure is shown in Figure. 4. The structure is made of the same material with density 7.8e3 kg/m , elasticity modulus 2e11Pa and Poisson's ratio 0.3.The thickness of the slabs is 0.2 m. ANSYS is used to model and analyze the structure . The slabs are modelled by Shell63 elements and the columns are modelled by beam4 elements. The columns are fix supported at the bottom. The foursquare columns are divided into three segments by the slabs and the side lengths of the three segments are k_i*0.5 m (i=1,2,3). k_1, k_2 and k_3 are independent random variables and their stochastic information is given in Table 4. The performance function is $G = \omega_0 - \omega$, in which $\omega_0 = 4.1072$ Hz. The target reliability index is $\beta_t = 3.0$.

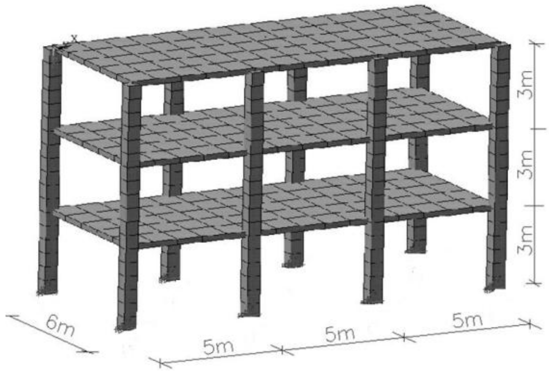

Figure 4 The slab-column structure

Table 4 The stochastic information for Example 4

Variable	Distribution	Mean	std. dev.
k_1	Normal	1.2962	0.2
k_2	Normal	0.9917	0.2
k_3	Normal	0.6627	0.2

The AMV method converged after 11 iterations and 46 function evaluations. The proposed method, with initial step size $\lambda = 50.0$ and adjustment factor c=2.5, converged after 6 iterations and 32 function evaluations. The obtained PPM is G_p=0.0159 and the MPTP is **u***=(1.1931,1.1173,1.2403). The CC method converged to almost the same result after 15 iterations and 62 function evaluations. The results are listed in Table 5.

Table 5 Results using different methods in Example 4

	Iterations	Function Evaluations	G_p	$\mathbf{u}^*$
AMV	11	46	0.0163	(1.1955,1.0988,1.2444)
Proposed method	6	32	0.0159	(1.1931,1.1173,1.2403)
CC	15	62	0.0160	(1.1927,1.1042,1.2429)

CONCLUSIONS

When locating MPTP in PMA, the AMV method may converge very slowly, or oscillate and fail to converge if the performance function is highly nonlinear. In this paper an easy iterative algorithm, which introduces a "new" step size to control the convergence of the sequence, is proposed. This step size is new because it may be constant during the iteration or decreases successively several times during the whole iteration process using an easy self-adjust strategy. It is proved that the AMV method is a special case of this proposed algorithm when the step size tends to infinity. Numerical results of several nonlinear performance functions, including an engineering application, indicate that the proposed algorithm is effective and as simple as the AMV but more robust.

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