The Logic of Discrete Qualitative Relations

Giulia Sindoni[1] and John G. Stell[2]

1 **School of Computing, University of Leeds, Leeds, UK**
`scgsi@leeds.ac.uk`

2 **School of Computing, University of Leeds, Leeds, UK**
`j.g.stell@leeds.ac.uk`

Abstract

We consider a modal logic based on mathematical morphology which allows the expression of mereotopological relations between subgraphs. A specific form of topological closure between graphs is expressible in this logic, both as a combination of the negation $\neg$ and its dual $\lrcorner$, and as modality, using the stable relation Q, which describes the incidence structure of the graph. This allows to define qualitative spatial relations between discrete regions, and to compare them with earlier works in mereotopology, both in the discrete and in the continuous space.

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1 Introduction

Qualitative spatial relations have a long history with two major strands: the Region-Connection Calculus (RCC) of Randell et al. [10] and the 9-intersection approach of Egenhofer et al. [5]. These were initially intended to model 'continuous', or more precisely 'dense', space that can be subdivided indefinitely often. The RCC is defined in terms of first order axioms based on a primitive predicate of connection. The intersection-based theories, on the other hand, evaluate spatial scenarios through a matrix of statements that pairs of features of two regions have non-empty intersection. In the 9-intersection case a pair of regions A, B is given a spatial relationship by considering the interior, boundary, and exterior of each region and obtaining a matrix of truth values from the emptiness or non-emptiness of the 9 possible intersections between the three features of A and of B.

The mathematical discipline of *topology* provides one theory of space, related to the qualitative approach in various ways [17]. Topology is different from geometry as the former studies the properties of the space that are preserved under continuous deformation, while the latter includes shapes, relative positions and sizes of figures.

The qualitative spatial approach of the RCC is based on mereotopology, which includes mereology [13], the theory of parts and wholes.

Mereology alone is not expressive enough to be useful in Qualitative Spatial Reasoning. Besides the generic notion of parthood one needs also to be able to distinguish between central and peripheral parthood. Other desirable notions are those of connection and apartness. To express them a primitive relation of connection is usually introduced, stipulated to be symmetric and reflexive. Using parthood and connection as primitive, other important additional relations can be expressed. They are: 'X is disconnected from Y', 'X externally connected to Y', 'X is tangential part of Y', and 'X is a non-tangential part of Y'. This gives rise to what is known as RCC8. The only two relations taken as primitive are parthood and connection, being all the other relations reducible to logical formulae containing these

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two. Systems of this sort are known as *mereotopologies.* A comprehensive analysis of models of mereotopological theories in terms of a relation of connection interpreted in a topological space has been presented by Cohn and Varzi [3].

Another direction in the modelling of qualitative relations in continuous space was initiated by Bloch [2], who combined modal logic with the image processing techniques of mathematical morphology [9]. Bloch demonstrated that a modal logic associated with mathematical morphology could be used to express qualitative spatial relations. More recent developments in mathematical morphology have seen much interest in applying the techniques in discrete spaces. These spaces generally consist of graphs in the sense of a set of nodes together with a binary relation of adjacency between the nodes. This kind of discrete space is exactly that investigated by Galton [6], [7], who considered how mereotopological notions could be developed for discrete space. Galton's notion of discrete space is the one of adjacency space: sets of nodes linked by a reflexive and symmetric relation of adjacency $\sim$. This is, in turn, based on the work on *digital topology* of Rosenfeld [12]. The main concern of digital topology is the study of topological properties of (subsets of) digital pictures, arrays of lattice points having positive integers coordinates (x, y). Here, given a point of coordinates (x, y), one can consider its *orthogonal adjacencies*, so those points sharing one of the coordinates with the point considered. Or one can also consider its *orthodiagonal adjacencies*, consisting of its orthogonal adjacencies with its four diagonal adjacent point. These constructions are the adjacency spaces $(\mathbb{Z}, \sim_4)$ and $(\mathbb{Z}, \sim_8)$.

Discrete space presents some notable challenges for mereotopology. For example, the usual definition of part in terms of connection leads inescapably, in the presence of atomic regions, to the conclusion that some regions will be parts of their complements. This may call for a different understanding of notions such as complement and part, or alternatively for novel techniques for developing mereotopological theories. In this paper we present new results on the mereotopology of discrete space using a recently developed modal logic [15] with a semantics based on morphological operations on graphs. This allows us use the approach suggested by Bloch for expressing qualitative relationships but in the very different setting of discrete space. Our results are related to the algebraic approach advocated by Stell and Worboys at COSIT twenty years ago [16]. This work took the bi-intuitionistic algebra of subgraphs that Lawvere [8] noted and showed its relevance to qualitative relations in discrete space. The current paper extends this significantly through its use of the modal logic [15] thus allowing us to adapt Bloch's insights about the use of morphological operations to the discrete case.

Both the mereotopological work of Galton [6] and the morphological investigations of Cousty et al. [4] take place in a setting where space consists of nodes which may or may not be linked by edges. Galton's adjacency spaces can be regarded as graphs. Anyway there is a notable difference between theory of adjacency space and graph theory, as Galton underlines [7]. A substructure of an adjacency space can be specified just in terms of nodes, two nodes being connected by only one edge, or relation of adjacency. This is not true in the general setting of a graph, where multiple edges may occur between two nodes, and, therefore, different subgraphs sharing the same set of nodes may be considered. Cousty et al. find indeed that edges need to play a more central role, and make the key observation that sets of nodes which differ only in their edges need to be regarded as distinct. The logic used in the present paper takes its semantics in a setting where regions in a graph are more general still. We allow graphs to have multiple edges between the same pair of nodes, thus using a structure sometimes called a multi-graph. This generality appears important in practical examples, such as needing to model two distinct roads between the same endpoints, or distinct rail connections between the same two stations.

The contribution of the present paper is thus to develop the interaction between modality and morphology identified by Bloch but in the discrete setting. In doing so we are able to show how this relates to earlier work in mereotopology both in the discrete and in the continuous case. We start in Section 2 by reviewing the framework of Cohn and Varzi and showing that the discrete connection of Galton's work lies outside this framework. In Section 3 we review the semantics of a multi-modal logic where formulas are interpreted as subgraphs. This is used in Section 4 to express qualitative spatial relations within the logic. We provide conclusions in Section 5.

2 Connection in Continuous and Discrete Space

In this section we review the approach of Cohn and Varzi [3] and show that it needs to be generalized if it is to capture the notion of discrete connection defined by Galton [6].

2.1 Mereotopological Connection from Topological Closure

Giving a set A, a topology τ on A is usually given as a collection of subsets of A which is closed under finite intersections and arbitrary unions. The set A together with the topology τ on A is a topological space, and the elements of τ are the open sets of the space. A set is closed if and only if it is the complement of an open set. An alternative formulation of topology, dual to the one in terms of open sets, can be given. Closed sets are the fundamental elements, and a topology on a set A is a collection of subsets that is closed under finite union and arbitrary intersection [14].

Cohn and Varzi [3] give three definitions of connection which depend on the notion of topological closure.

▶ **Definition 1.** A closure operator on a set A is a function c associating with each $x \subseteq A$ a set $c(x) \subseteq A$, which satisfies the following axioms (Kuratowski axioms) for all $x, y \subseteq A$.
K1. $c(\varnothing) = \varnothing$.
K2. $x \subseteq c(x)$.
K3. $c(c(x)) \subseteq c(x)$.
K4. $c(x \cup y) = c(x) \cup c(y)$.

Given a set A together with an operator c satisfying $K1$-$K4$ axioms is equivalent to specifying a topological space in terms of open sets or in terms of closed sets. The closed sets of a topological space correspond to the sets $x \subseteq A$ for which $c(x) = x$.

▶ **Definition 2.** Let c be a topological closure on A. Three binary relations of connection between subsets $x, y \subseteq A$ are defined as follows.
1. $C_1(x, y) \Leftrightarrow x \cap y \neq \varnothing$.
2. $C_2(x, y) \Leftrightarrow c(x) \cap y \neq \varnothing$ or $x \cap c(y) \neq \varnothing$.
3. $C_3(x, y) \Leftrightarrow c(x) \cap c(y) \neq \varnothing$.

Cohn and Varzi use a setting where a mereotopological theory might allow only certain subsets of the topological space as its entities. For example, they draw a line between theories which allow as elements in the domain of quantification *boundary* elements, which, intuitively are elements with an empty interior, such as points, lines and surfaces, and theories which exclude such boundary elements. However, the spatial relationship of connection is defined in a way that is applicable to arbitrary subsets of the topological space.

The Region-Connection Calculus (RCC) is one of the most popular theories in qualitative spatial reasoning. An important models of the RCC is to take regions to be non-empty regular closed subsets of $\mathbb{R}^2$, with the usual topology. A subset is called regular closed when it is equal to the closure of its interior. In particular, this means that although a single point, or a line including its endpoints, is a closed set in $\mathbb{R}^2$, it is not regular closed, as its interior is empty. Therefore, such elements are not considered as regions in this context, and the RCC belongs to those theories which do not allow boundary elements in their domain. In the regular-closed model of RCC, all three connections above yield the same relation between regions, and connection means sharing at least one point. External connection, or abutting in the language of Cohn and Varzi, is distinguished from connection as regions that abut share points in this model but do not share regions. However, in other models of RCC and in other mereotopological systems the three notions of connection can have substantially different properties.

The contribution of Cohn and Varzi is to have provided a framework within which numerous mereotopological notions are expressible by varying the notion of connection used as well as the two key derived notions of part and fusion. In the case of part, the three connections yield three parthoods as follows.

$$P_i(x, y) \Leftrightarrow \forall z(C_i(z, x) \rightarrow C_i(z, y))\,.$$

Although Cohn and Varzi [3, p359] aim for neutrality with respect to density of space, that is whether space can be repeatedly sub-divided *ad infinitum*, we shall see next that the use of topological closure prevents the framework including one of the most straightforward examples of connection in a discrete space.

2.2 Galton's Discrete Connection

Galton [6] studied a notion of connection between subsets of a particular kind of discrete space. The spatial setting is a set N together with a relation of adjacency $\alpha \subseteq N \times N$. The relation α is symmetric and reflexive, but not transitive. Connection, C_α, is defined for subsets $x, y \subseteq N$ by $C_\alpha(x, y)$ if there are $a \in x$ and $b \in y$ such that $(a, b) \in \alpha$. We shall show next that there are spaces N, α where this connection is not expressible as any C_i, in the sense of Cohn and Varzi, for any topological closure on N. A specific example appears in Figure 1 where the links indicate adjacencies between distinct elements of the five element set $N = \{m, n, p, q, r\}$.

First, C_α cannot be C_1 as two adjacent nodes give disjoint singleton subsets which are C_α connected. So suppose that $C_\alpha = C_2$ for some topological closure c. If k is any node in N then $\{k\}$ is C_α connected to no singletons except those $\{k'\}$ such that $\alpha(k, k')$. Thus $c(\{k\})$ contains only nodes which are adjacent to k. Hence for the specific nodes m and n we have $c(\{m\}) \subseteq \{r, m, n\}$ and $c(\{n\}) \subseteq \{m, n, p\}$. Now $\{m\}$ and $\{n\}$ are connected in the connection C_α so if they are C_2 connected we must have $n \in c(\{m\})$ or $m \in c(\{n\})$. Consider first the case that $n \in c(\{m\})$. This implies $\{n\} \subseteq c(\{m\})$ so that $c(\{n\}) \subseteq c(c(\{m\})) \subseteq c(\{m\})$. Thus $p \notin c(\{n\})$ and $c(\{n\}) \subseteq \{m, n\}$. But $\{n\}$ and $\{p\}$ are connected in C_α, so $n \in c(\{p\})$, and hence $c(\{n\}) \subseteq c(\{p\})$. As $m \notin c(\{p\})$ we conclude $c(\{n\}) = \{n\}$ in the case that $n \in c(\{m\})$. In the case that $m \in c(\{n\})$ we conclude that $c(\{m\}) = \{m\}$. Thus in either case one of the sets $\{m\}$ and $\{n\}$ is a closed set, and they cannot both be closed since they need to be C_2 connected.

This applies to each pair of adjacent nodes in N; one of them is a closed set and the other is not. With an odd number of nodes in total this is a contradiction. Hence no such topological closure, c, can generate a C_2 connection equal to C_α. There remains the possibility that C_α is of the form C_3. Suppose then that some topological closure on N generates C_α as C_3. We must have $c(\{m\}) \cap c(\{n\}) \neq \varnothing$. For similar reasons to the C_2

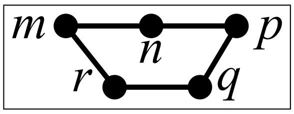

Figure 1 A discrete space where connection cannot be defined in terms of topological closure.

case we must have $c(\{m\}) \subseteq \{r, m, n\}$ and $c(\{n\}) \subseteq \{m, n, p\}$, so to obtain the non-empty intersection either $m \in c(\{n\})$ or $n \in c(\{m\})$. In the case that $n \in c(\{m\})$ we get $p \notin c(\{n\})$ as $c(\{n\}) \subseteq c(\{m\})$. But n and p are adjacent so are connected singletons and $c(\{n\})$ and $c(\{p\})$ must intersect and the only possibility for this intersection is n. Now if $m \in c(\{n\})$ we get $m \in c(c(\{p\}))$ which would make m and p adjacent. Hence $c(\{n\})$ can only be $\{n\}$. It is straightforward to continue to a contradiction as in the C_2 case.

3 Modal Logic with Graph Morphology Semantics

3.1 Classical Modal Logic

The syntax of classical propositional modal logic provides propositional variables $p, q, r, \ldots$, the usual logical connectives $\vee$, $\wedge$, $\to$, $\neg$, and the modalities $\Diamond$, and $\Box$. Formulae are defined by stipulating that propositional variable are formulae, and if φ, ψ are formulae then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\neg\varphi$, $\Diamond\varphi$, and $\Box\varphi$. The semantics for this logic allows an interpretation of atomic propositions as subsets of a set of 'worlds' and formulae correspond to subsets constructed out of these. While an abstract set has no spatial structure by itself, we shall see that a more elaborate logic has a natural semantics in which formulae correspond to subgraphs of a graph. This means that spatial relations between subgraphs can be expressed in the logic. Before introducing this logic we need to review the connection between classical propositional modal logic and the morphological operations of dilation and erosion.

Kripke semantics for propositional modal logic is based on a binary relation on a set of worlds, W (see [1] for an introduction to Kripke semantics). Propositional variables are then interpreted as subsets of W, and truth and falsity in the language, often denoted $\top$ and $\bot$ are interpreted respectively as W and $\varnothing$. In this setting the logical connectives $\vee$, $\wedge$, $\neg$ are interpreted as the set-theoretic operations of union, intersection and complement. Once we are given a subset $[\![p]\!] \subseteq W$ for each propositional variable p, we can assign to each non-modal formula φ its interpretation as a subset $[\![\varphi]\!] \subseteq W$. Implication $\to$ is handled by defining $[\![\varphi \to \psi]\!] = -[\![\varphi]\!] \cup [\![\psi]\!]$ where $-$ is set-theoretic complement. This means $[\![\varphi \to \psi]\!]$ holds in a given interpretation if and only if $[\![\varphi]\!] \subseteq [\![\psi]\!]$ making a connection between the logic and the mereological assertion that one set is a part of another.

The semantics of the modalities $\Diamond$ and $\Box$ can easily be expressed in terms of the morphological operations of dilation and erosion which are defined as follows.

▶ **Definition 3.** For any subset $X \subseteq W$, and any relation $R \subseteq W \times W$, we define:

- **Dilation:** $X \oplus R = \{w \in W \mid \exists x(x\ R\ w \text{ and } x \in X)\}$,
- **Erosion:** $R \ominus X = \{w \in W \mid \forall x(w\ R\ x \text{ implies } x \in X)\}$.

In order to understand how dilation and erosion work, we do an example

▶ **Example 4.** Let $W = \{a, b, c, d, e\}$ and $X = \{a, b, c\}$ and $R = \{(a, a), (a, b), (c, d), (e, c)\}$. Then $X \oplus R = \{a, b, d\}$ and $R \ominus X = \{a, e\}$.

Using these operations we then define $[\![\Diamond\varphi]\!] = [\![\varphi]\!] \oplus \breve{R}$ and $[\![\Box\varphi]\!] = R \ominus [\![\varphi]\!]$. Note the use of the converse $\breve{R}$; that is $\Diamond\varphi$ holds at worlds accessible from worlds in $[\![\varphi]\!]$ via the converse of the accessibility relation.

3.2 Graphs and Relations on Graphs

We move now to consider not merely subsets of a set but subgraphs of a graph. This builds on the use of algebraic operations on subgraphs as introduced to the COSIT community by Stell and Worboys [16], but now in the context of a modal logic which allows the expression of mereotopological relations between subgraphs. The logic itself appears in [15] in a more general context, but the applications to discrete spatial representation have not been investigated before. We need first to explain what we mean by a graph.

A graph in which there are potentially multiple edges between nodes, and potentially (multiple) loops on the nodes can be defined a set W, thought of as consisting of all the nodes and edges together, with a relation $Q \subseteq W \times W$. This relation relates every edge to its incident nodes and no other elements of W are related. Thus $w\ Q\ v$ holds iff w is an edge incident with node v. From Q we derive its reflexive closure, which we denote by H. Given just W and H we can distinguish nodes from edges as a node is an element of W related only to itself by H, whereas an edge must be related both to itself and at least one other element of W. The subgraphs of a graph (W, H) are the subsets which for each edge include all the incident nodes. A set $X \subseteq W$ will be a subgraph iff $X \oplus H \subseteq X$ or equivalently $X \oplus Q \subseteq X$.

The algebra of subgraphs, already noted by Lawvere as cited in [16], provides unions and intersections of subgraphs but most significantly two distinct types of complement. Given a subgraph $X \subseteq W$ we can obtain both a largest subgraph disjoint from X and also a smallest subgraph whose union with X gives all of W. These are denoted $\neg X$ and $\lrcorner X$ respectively and can be expressed as $H \ominus (-X)$ and $(-X) \oplus H$ respectively.

To give a semantics for a modal logic where formulae are interpreted as subgraphs we need a notion of a relation on a graph which extends the notion of a relation on a set as used in classical Kripke semantics.

Let W be a set, let $\mathcal{P}_W$ be its power set, and let S be a function such that $S : \mathcal{P}_W \mapsto \mathcal{P}_W$; then

▶ **Definition 5.** S is a union preserving function on $\mathcal{P}_W$ if and only if for any family of indexed set $Z_i \subseteq X$ we have $S(\cup_i(Z_i) = \cup_i S(Z_i)$.

Given a union preserving function S on $\mathcal{P}_W$, it is always possible to define a binary relation $R \subseteq W \times W$ as follows: for any w and $v \in W$, $w\ R\ v$ if and only if $v \in S(\{w\})$. All binary relations on sets come in this way. On the other hand, given a relation $R \subseteq W \times W$, a union preserving function on $\mathcal{P}_W$ can be defined as follows, given $V \subseteq W$

$$S(V) = \{w \in W \mid \exists v \in V \text{ and } v\ R\ w\}\,.$$

Therefore, relations on a set W can be modelled as union-preserving functions on the power set of W. When it comes to the case of a graph, so a set carrying a pre order H, the union preserving functions on the lattice of subgraphs correspond to relations on W that are stable. Given a graph (W, H) we say a relation $R \subseteq W \times W$ is stable with respect to H provided $H\ ;\breve{R}\ ; H \subseteq R$ where ; denotes the composition of relations. The stable relations include the universal relation $U = W \times W$, and the relations Q and H.

Stable relations are closed under composition, H being the identity element, but are not, in general, closed under converse. Denoting the standard converse of R by $\breve{R}$, is not always the case that $H;\breve{R};H \subseteq \breve{R}$. However, for a stable R, it is possible to define a relation, called *left converse*, characterized as the smallest stable relation containing $\breve{R}$.

▶ **Definition 6.** The **left converse** of a stable relation R, is $\circlearrowleft R = H;\breve{R};H$, where $\breve{R}$ is the (ordinary) converse of R.

3.3 Graph-based Modal Logic

Bi-intuitionistic stable tense logics are a group of logics, described in [15], with a Kripke semantics where worlds in a frame are equipped with a pre-order as well as with an accessibility relation which is 'stable' with respect to the pre-order. We do not need the full generality of this setting here, and will give a semantics in a graph $G = (W, H)$. The relation H is easily seen to be reflexive and transitive, so that it is a pre-order. The syntax of the multi-modal version of *BISKT*, called so from [15] because is the system K of this group of logics, is that of classical propositional logic extended with dual negation $\lrcorner$, dual implication $\succ$, and four indexed modalities: $[R]$, $\langle R\rangle$, $\rangle R\langle$, and $]R[$.

The semantics needs, besides a graph $G = (W, H)$, a stable relation for each index R. Such a structure will be called a *BISKT*-model, and often denoted $\mathcal{M}$. Given a valuation, assigning to each propositional variable p a subgraph $[\![p]\!]$, we extend the semantic function $[\![\_]\!]_\nu$ thus (we will omit the subscription 'ν' when no confusion arises):

$$\begin{array}{rclrcl}
[\![\bot]\!]_\nu &=& \varnothing & [\![\top]\!]_\nu &=& W \\
[\![\varphi \vee \psi]\!]_\nu &=& [\![\varphi]\!]_\nu \cup [\![\psi]\!]_\nu & [\![\varphi \wedge \psi]\!]_\nu &=& [\![\varphi]\!]_\nu \cap [\![\psi]\!]_\nu \\
[\![\neg\varphi]\!]_\nu &=& \neg[\![\varphi]\!]_\nu & [\![\lrcorner\,\varphi]\!]_\nu &=& \lrcorner[\![\varphi]\!]_\nu \\
[\![\varphi \to \psi]\!]_\nu &=& H \ominus ((-[\![\varphi]\!]_\nu) \cup [\![\psi]\!]_\nu) & [\![\varphi \succ \psi]\!]_\nu &=& ([\![\varphi]\!]_\nu \cap (-[\![\psi]\!]_\nu)) \oplus H \\
[\![[R]\,\varphi]\!]_\nu &=& R \ominus [\![\varphi]\!]_\nu & [\![\langle R\rangle\,\varphi]\!]_\nu &=& [\![\varphi]\!]_\nu \oplus (\circlearrowleft R) \\
[\![\rangle R\langle\,\varphi]\!]_\nu &=& [\![\varphi]\!]_\nu \oplus R & [\![]R[\,\varphi]\!]_\nu &=& (\circlearrowleft R) \ominus [\![\varphi]\!]_\nu
\end{array}$$

A graph will be indicated by $G = (W, H)$; a valuation function ν, is a function going from formulas in the logic to subgraphs, such that for a formula φ, $\nu(\varphi) = [\![\varphi]\!]_\nu$. The pair G, ν is a *BISKT*-model and we write $G, \nu \vDash \varphi$ when $[\![\varphi]\!]_\nu$ is the whole graph G.

The use of morphology in connection with modal logic for spatial reasoning by Bloch [2] is in a classical setting. In *BISKT*, unlike the classical case, box and diamond modalities are not mutually interdefinable. Working in discrete space the bi-intuitionistic logic is essential to express spatial relations as we will see in the next section.

4 Expressing qualitative relations

In this section we express in a direct way, in *BISKT*, some qualitative spatial relationships between graphs. Then, we compare these expressions with their correspondents found in [3] and [10]. The first relation we analyse is the one of connection[1].

4.1 A Cech Closure for Graphs

We have seen that connection in Galton's sense cannot always be expressed as one of the connections in the framework of Cohn and Varzi by using a topological closure. However, a

[1] The reader must be aware of the fact that the notion of connection *between* graphs, which we refer to, is not the same as the notion of connectivity *of* a graph. In the latter case a graph is connected if between any two nodes there is always an edge connecting them. In our context, two subgraphs, that are connected in the sense expressed by one of the forthcoming relations of connection, are not necessarily connected subgraphs themselves.

weaker notion of closure does have the right properties. An operator satisfying Kuratowski axioms $K1$, $K2$ and $K4$ but not necessarily $K3$ is known as a Cech closure (see definition from [14, p.657]).

In $BISKT$ the following formulas are tautologies:

- $\lrcorner\neg\bot \leftrightarrow \bot$,
- $\varphi \rightarrow \lrcorner\neg\varphi$,
- $\lrcorner\neg(\varphi \vee \psi) \leftrightarrow \lrcorner\neg\varphi \vee \lrcorner\neg\psi$.

However, the following formula is not a tautology:

- $\lrcorner\neg(\lrcorner\neg\varphi) \rightarrow \lrcorner\neg\varphi$.

Interpreting $\bot$ as the empty subgraph $\varnothing$, the formulas φ and ψ on the respective subgraphs $[\![\varphi]\!]$ and $[\![\psi]\!]$, we have that the operator on graphs '$\lrcorner\neg$' satisfies

K1. $\lrcorner\neg[\![\bot]\!] = \varnothing$,
K2. $[\![\varphi]\!] \subseteq \lrcorner\neg[\![\varphi]\!]$,
K4 $\lrcorner\neg([\![\varphi]\!] \cup [\![\psi]\!]) = \lrcorner\neg[\![\varphi]\!] \cup \lrcorner\neg[\![\psi]\!]$,
but not necessarily
K3. $\lrcorner\neg(\lrcorner\neg[\![\varphi]\!]) \subseteq \lrcorner\neg[\![\varphi]\!]$.

Given the above tautologies we can define a Cech closure c on a graph G by

$$c(K) = \lrcorner\neg K$$

for any $K \subseteq G$.

4.2 Connection expressed modally

In this section we show that Cech closure operator defined earlier, is expressible by a modality in *BISKT*, for a suitable choice of a stable accessibility relation R on W.

Consider the relation Q introduced in Section 3.2. When it is taken as the stable relation, then the Cech closure of a subgraph, can be expressed by a modality indexed by Q.

▶ **Theorem 7.** *Let Q be the stable relation introduced above. Consider the graph G. For any formula φ, the following holds:*

$$\lrcorner\neg[\![\varphi]\!] = [\![\langle Q\rangle\,\varphi]\!]\,.$$

Proof. We sketch the idea of the proof in Figure 2, showing how $\circlearrowleft Q$ works. When dilation by $\circlearrowleft Q$ is applied to a node, it takes the node itself and all the nodes one-edge-away from it. When dilation by $\circlearrowleft Q$ is applied to an edge, it produces the edge itself , the edges one-node-apart from it, and all the nodes incident with these edges. Since, in **BISKT**, the smallest subgraph including a node is the node itself, and the smallest subgraph including an edge is the edge plus the nodes incident to it, $\lrcorner\neg$ acts extending any subgraph with all the nodes one-edge away and all the edges incident to them, which means dilating the subgraph by $\circlearrowleft Q$. ◀

By taking R to be the universal relation U on W, that is $U = W \times W$ we can interpret "somewhere φ" by $\langle U\rangle\,\varphi$ and "everywhere φ" by $[U]\,\varphi$, or equivalently just φ. The three notions of connection expressed by Cohn and Varzi [3] depend on being able to express that a subset is non-empty. We can handle this within our modal logic as $\langle U\rangle(\varphi)$ holds if and only if $[\![\varphi]\!] \neq \varnothing$. Thus, for any graph G and any valuation ν, given two formula x and y, and given the Cech closure c introduced above we have that

- $G, \nu \vDash \langle U\rangle(x \wedge y)$ iff $([\![x]\!]_\nu \cap [\![y]\!]_\nu) \neq \varnothing$,
- $G, \nu \vDash \langle U\rangle((\langle Q\rangle\, x \wedge y) \vee (x \wedge \langle Q\rangle\, y))$ iff $((c([\![x]\!]_\nu) \cap [\![y]\!]_\nu) \cup (x \cap c([\![y]\!]_\nu))) \neq \varnothing$,
- $G, \nu \vDash \langle U\rangle(\langle Q\rangle\, x \wedge \langle Q\rangle\, y)$ iff $c([\![x]\!]_\nu) \cap c([\![y]\!]_\nu) \neq \varnothing$.

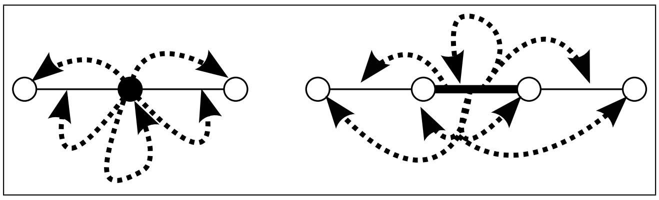

Figure 2 Effect of $\circlearrowright Q$ on one node and on one edge.

This gives us three definitions of connection for regions in discrete space, analogous to C_1, C_2, C_3 defined by Cohn and Varzi for regions in continuous space:

1. $C_1([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \langle U\rangle(x \wedge y)$.
2. $C_2([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \langle U\rangle((\langle Q\rangle x \wedge y) \vee (x \wedge \langle Q\rangle y))$.
3. $C_3([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \langle U\rangle(\langle Q\rangle x \wedge \langle Q\rangle y)$.

4.3 Qualitative Relations Modally

Other qualitative spatial relations can be expressed in a direct way in this modal logic. We consider here the notions of non empty part, general part, proper part, tangential proper part, non-tangential proper part, and external connection. We index them notationally with $\Diamond$ in order to make the distinction with their mereotopological correspondents.

- $NEP_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \langle U\rangle\, x \wedge x \rightarrow y$
- $GP_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash x \rightarrow y$
- $PP_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash x \rightarrow y \wedge \langle U\rangle(\neg x \wedge y)$
- $O_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \langle U\rangle(x \wedge y)$
- $EC_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \neg(x \wedge y) \wedge \langle U\rangle(\langle Q\rangle x \wedge y) \vee \langle U\rangle(x \wedge \langle Q\rangle y)$
- $TP_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash x \rightarrow y \wedge \langle U\rangle(\langle Q\rangle x \wedge \neg y)$
- $NTP_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \langle Q\rangle x \rightarrow y$
- $EQ_\Diamond([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash x \leftrightarrow y$
- $DC([\![x]\!]_\nu, [\![y]\!]_\nu)$ iff $G, \nu \vDash \neg((\langle Q\rangle x \wedge y) \vee (x \wedge \langle Q\rangle y))$

The notion of parthood comprises three different relations: the general notion of part ($GP_\Diamond$), the one restricted to those parts x of y different from the empty graph ($NEP_\Diamond$), and the notion of proper part ($PP_\Diamond$). A separate section will be dedicated to the notion of boundary graph, since, as we shall see, a variety of possible definitions arises in *BISKT*.

In the next sections we will compare our definitions with those given in [3, 10]. In order to do so, we introduce the following lemmas

▶ **Lemma 8.** *Given a world $w \in W$, if for some $v \in W$ $v \circlearrowright Q\, w$, then $w \circlearrowright Q\, v$ or there exists a world u such that $w \circlearrowright Q\, u$ and $u \in v \oplus H$.*

Proof. Four cases need to be addressed. (i) v and w are both nodes. The case $v = w$ is trivial. So suppose $v \neq w$. $v \circlearrowright Q\, w$ holds. Then there is an edge u incident with both v and w such that $v\ H\ u\ \breve{Q}\ w\ H\ w$. Therefore $w\ H\ w\ \breve{Q}\ u\ H\ v$. (ii) v and w are both edges. The case $v = w$ is trivial. So suppose $v \neq w$. Then there exists a node u incident with both v and w such that $v\ H\ u\ \breve{Q}\ w\ H\ w$. So $w\ H\ u\ \breve{Q}\ v\ H\ v$. (iii) v is a node and w is an edge. If $v \circlearrowright Q\, w$ then the only possibility is $v\ H\ v\ \breve{Q}\ w\ H\ w$. Therefore $w\ H\ v\ \breve{Q}\ w\ H\ v$. (iv) Suppose v is an edge and w is a node. Then (iv.i) w is a node incident to v: $v\ H\ w\ \breve{Q}\ v\ H\ w$. In this case $w\ H\ w\ \breve{Q}\ v\ H\ v$; or (iv.ii) there is a node u such that $u \in v \oplus H$ and an edge k, such that $v\ H\ u\ \breve{Q}\ k\ H\ w$. Therefore $w\ H\ w\ \breve{Q}\ k\ H\ u$, and $u \in v \oplus H$. ◀

▶ **Lemma 9.** *Given a BISKT-model G, ν, and two formulas x and y representing two subgraphs $[\![x]\!]$ and $[\![y]\!]$*

$$G, \nu \vDash \langle U \rangle(\langle Q \rangle x \wedge y) \iff G, \nu \vDash \langle U \rangle(x \wedge \langle Q \rangle y)$$

Proof. Given a graph G, a valuation ν and a world w, we can give the semantics clause for $\langle Q \rangle \varphi$ as follows

$$w \vDash \langle Q \rangle \varphi \text{ iff for some } v, (v \smile Q\ w) \text{ and } v \vDash \varphi\,.$$

For the left-to-right direction: assume that $G, \nu \vDash \langle U \rangle(\langle Q \rangle x \wedge y)$. Then, for all $w \in W$ exists a $v \in W$ such that $(w\ U\ v)$ and $v \vDash \langle Q \rangle x \wedge y$. This means that $v \vDash y$, and for some u such that $(u \smile Q\ v)$, $u \vDash x$. For lemma 1, or $v \smile Q\ u$, so that $u \vDash x \wedge \langle Q \rangle y$, or there is a $j \in v \oplus H$ such that $j \smile Q\ u$. So $j \vDash y$ because $j \in v \oplus H$, and $j \vDash \langle Q \rangle x$. So $j \vDash x \wedge \langle Q \rangle y$. Therefore, under the initial assumption, $G, \nu \vDash \langle U \rangle(x \wedge \langle Q \rangle y)$ holds. The right-to-left direction works in analogous way. ◀

Given this lemma, the notion of connection C_2 can be shortened to $\langle U \rangle(\langle Q \rangle x \wedge y)$. When this holds for x and y, the corresponding subgraphs are C_2-connected.

▶ **Lemma 10.** *Given a graph G and a valuation function, and x, y propositional variables, ν, if $G, \nu \vDash \langle U \rangle x$ and $G, \nu \vDash x \to y$ then $G, \nu \vDash \langle U \rangle y$.*

Proof. $G, \nu \vDash \langle U \rangle x$ iff for all $w \in v$, there is a $v \in W$ such that $v \vDash x$. $G, \nu \vDash x \to y$ iff for all $w \in W$, for all $u \in W$ such that $w\ H\ u$, if $u \vDash x$ then $u \vDash y$. Take v: $v \vDash \langle U \rangle x$ and $v \vDash x$. Also, $v\ H\ v$. Then $v \vDash y$. Therefore, under the initila assumptions, somewhere in the graph y must hold: $G, \nu \vDash \langle U \rangle y$. ◀

4.4 Overlapping

Overlapping regions are defined by Cohn and Varzi [3] as $O(X, Y) \equiv \exists Z(P(Z, X) \& P(Z, Y))$, where the predicate of parthood is restricted just to non-empty regions. We show that

▶ **Theorem 11.** *Let G be a graph and ν a valuation. Given x, y and z propositional variables, the following holds*

$$G, \nu \vDash \langle U \rangle(x \wedge y)$$

iff there is a subgraph K of G such that for any valuation ν' which agrees with ν on x and y, and where $[\![z]\!]_{\nu'} = K$

$$G, \nu' \vDash \langle U \rangle z \wedge z \to x \text{ and } G, \nu' \vDash \langle U \rangle z \wedge z \to y\,.$$

Proof. Suppose $G, \nu \vDash \langle U \rangle(x \wedge y)$. Then, for all $w \in W$, $w \vDash \langle U \rangle(x \wedge y)$. So for all $w \in W$ there is a world $v \in W$ such that $v \vDash x$ and $v \vDash y$. This means that $v \in [\![x]\!]_\nu$ and $v \in [\![y]\!]_\nu$. So $v \in [\![x]\!]_\nu \cap [\![y]\!]_\nu$. Then, for all the valuations ν' that agree with ν on x and y, there is a subgraph $[\![z]\!]_{\nu'} = K = v \oplus H$ such that $K, \nu' \vDash z$, because $K = [\![z]\!]_{\nu'}$, and $K, \nu' \vDash z \to x$, because $[\![z]\!]_{\nu'} \subseteq [\![x]\!]_{\nu'}$, and $K, \nu' \vDash z \to y$, because $[\![z]\!]_{\nu'} \subseteq [\![y]\!]_{\nu'}$. Therefore $G, \nu' \vDash \langle U \rangle z \wedge z \to x$ and $G, \nu' \vDash \langle U \rangle z \wedge z \to y$.

On the other hand, suppose $[\![z]\!]_{\nu'} = K$, and $G, \nu' \vDash \langle U \rangle z \wedge z \to x$ and $G, \nu' \vDash \langle U \rangle z \wedge z \to y$. But then $G, \nu' \vDash \langle U \rangle z \wedge z \to (x \wedge y)$, since, given p, q, r propositional variables, $(p \to q) \wedge (p \to r) \leftrightarrow p \to (q \wedge r)$ is a tautology in BISKT. But then for lemma 4, $G, \nu' \vDash \langle U \rangle(x \wedge y)$ and since ν' and ν agree on x and y, $G, \nu \vDash \langle U \rangle(x \wedge y)$. ◀

4.5 Tangential Part

Cohn and Varzi give the following definition of Tangential part: $TP(X,Y) \equiv P(X,Y)$ & $\exists Z(C(X,Z) \& - O(Z,Y))$, where '$-$', is the symbol for classical negation. We show that our relation $TP_{\Diamond}([\![x]\!],[\![y]\!])$ gives the same entailments on subgraphs.

▶ **Theorem 12.** *Let G be a graph and ν a valuation. Given x, y and z propositional variables, the following holds*

$$G, \nu \models (x \to y) \wedge \langle U\rangle(\langle Q\rangle\, x \wedge \neg y)$$

iff there is a subgraph K of G such that for any valuation ν' which agrees with ν on x and y, and where $[\![z]\!]_{\nu'} = K$

$$G, \nu' \models x \to y \text{ and } G, \nu' \models \langle U\rangle(\langle Q\rangle\, x \wedge z) \text{ and } G, \nu' \not\models z \wedge y\,.$$

Proof. Assume $G, \nu \models (x \to y) \wedge \langle U\rangle(\langle Q\rangle\, x \wedge \neg y)$. Then, for all $w \in W$ in the graph $w \in W$, $w \models (x \to y)$. Then, since ν and ν' agree on x and y, also $G, \nu' \models (x \to y)$. From the assumption it follows also that $G, \nu \models \langle U\rangle(\langle Q\rangle\, x \wedge \neg y)$. Again since ν and ν' agree on x and y, also $G, \nu' \models \langle U\rangle(\langle Q\rangle\, x \wedge \neg y)$. Take as $[\![z]\!]_{\nu'} = K$ the subgraph $[\![\neg y]\!]_{\nu'}$. There exists such a subgraph such that $\langle U\rangle(\langle Q\rangle\, x \wedge z)$. Suppose $G, \nu' \models z \wedge y$. That means that $G, \nu' \models \neg y \wedge y$. But this is impossible since in *BISKT*, for any propositional variable p, $(p \wedge \neg p) \to \bot$ is a tautology. Therefore, for the chosen subgraph $[\![z]\!]_{\nu'}$, necessarily $G, \nu' \not\models z \wedge y$.

For the other direction, assume $G, \nu' \models x \to y$. Since ν' and ν agree on x and y, $G, \nu \models x \to y$. Suppose there exists some $[\![z]\!]_{\nu'} = K$, subgraph of G, and G, $\nu' \models \langle U\rangle(\langle Q\rangle\, x \wedge z)$ and $G, \nu' \not\models z \wedge y$. That means that i) for all $w \in W$, there is the $v \in W$ such that $v \models \langle Q\rangle\, x$ and $v \models z$; and ii) it does not exist any $t \in W$ such that $t \models z \wedge y$. So, for all $t \in W$, if $t \models z$ then $t \not\models y$. But, since $[\![z]\!]_{\nu'} = K$, for all $k \in W'$ such that $(W', H) = K$, $k \models z$. So, also, for all those k, $k \not\models y$. That means that for all m such that vHm, $m \not\models y$, and then, $v \models \neg y$. Therefore, $v \models \langle Q\rangle\, x$ and $v \models \neg y$. That means that, considering the valuation ν, $G, \nu \models \langle U\rangle(\langle Q\rangle\, x \wedge \neg y)$. ◀

4.6 Non-tangential part

The spatial relation of non-tangential parthood is defined by Cohn and Varzi as follows, for any two regions X and Y: $NTP(X,Y) \equiv P(X,Y)$ & $\forall Z(C(X,Y) \to O(Z,X))$. We show that just one direction of this entailment holds in *BISKT*.

▶ **Theorem 13.** *Let G be a graph and ν a valuation. Given x, y and z propositional variables, the following holds*

$$\text{if } G, \nu \models \langle Q\rangle(x) \to y\,,$$

then, for any valuation ν' which agrees with ν on x and y

$$G, \nu' \models x \to y$$

and for all the subgraph K of G such that $[\![z]\!]_{\nu'} = K$

$$\text{if } G, \nu' \models \langle U\rangle(\langle Q\rangle\, x \wedge z) \text{ then } G, \nu' \models \langle U\rangle(z \wedge y)\,.$$

Proof. Assume $G, \nu \models \langle Q\rangle\, x \to y$. Then, for all $w \in W$, if $w \models \langle Q\rangle\, x$ then $w \models y$, or, if $w \in [\![\langle Q\rangle\, x]\!]_{\nu}$ then $w \in [\![y]\!]_{\nu}$. But '$\lrcorner\neg$' is a Cech closure, and by theorem 1, $\lrcorner\neg[\![x]\!]_{\nu} = [\![\langle Q\rangle\, x]\!]_{\nu}$.

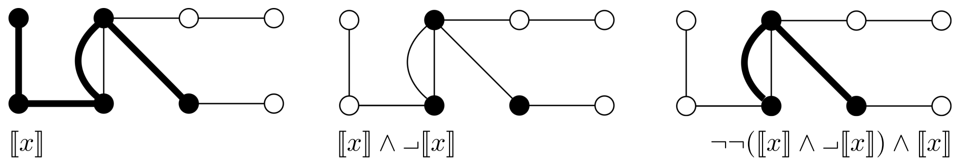

Figure 3 The Boundaries of $[\![x]\!]$.

So if $w \in [\![x]\!]_\nu$ then $w \in [\![\langle Q\rangle\, x]\!]_\nu$, and then $w \in [\![x]\!]_\nu$ implies that $w \in [\![y]\!]_\nu$. So, for all the valuations ν' agreeing with ν on x and y, we have that $G, \nu' \vDash x \to y$. Suppose that for some subgraph K such that $[\![z]\!]_{\nu'} = K$, i) $G, \nu' \vDash \langle U\rangle(\langle Q\rangle\, x \wedge z)$, and ii) $G, \nu' \nvDash \langle U\rangle(x \wedge y)$. So for i) there exists some $v \in W$ such that $v \vDash \langle Q\rangle\, x$ and $v \vDash z$. Since $[\![z]\!]_{\nu'} = K$, $v \in K$, and $K, \nu' \vDash \langle Q\rangle\, x$ and $K, \nu' \vDash z$. For ii) for all $w \in W$, if $w \vDash z$ then $w \nvDash y$. So for all the $k \in K$ $k \nvDash y$, that implies that for all the m such that vHm, $m \nvDash y$. That implies that $v \vDash \neg y$. Therefore $G, \nu' \vDash \langle U\rangle(\langle Q\rangle\, x \wedge \neg y)$. That means x is both non-tangential and tangential part of y. However, in *BISKT* this formula is a tautology: $(\langle Q\rangle\, x \to y) \to \neg(\langle Q\rangle\, x \wedge \neg y)$. So, under the assumption that $G, \nu \vDash \langle Q\rangle\, x \to y$, it cannot be the case that $G, \nu' \vDash \langle U\rangle(\langle Q\rangle\, x \wedge \neg y)$, because $\neg(\langle Q\rangle\, x \wedge \neg y)$ must hold everywhere in the graph. This also means that if a $[\![x]\!]$ is a non tangential part of $[\![y]\!]$, it cannot be also its tangential part. Therefore, under the initial assumption, if a subgraph $[\![z]\!]_{\nu'}$ is such that $G, \nu' \vDash \langle U\rangle(\langle Q\rangle\, x \wedge z$, we must have $G, \nu' \vDash \langle U\rangle(z \wedge y)$.

The other direction of the entailment does not hold. Consider the example of a graph G composed of a node n and an edge e going from n to n itself. Consider $[\![x]\!]_{\nu'} = [\![y]\!]_{\nu'} = n$. We have that $G, \nu' \vDash x \to y$. The possible subgraphs K such that $[\![z]\!]_{\nu'} = K$ are $[\![x]\!]_{\nu'}$, $[\![y]\!]_{\nu'}$ and the whole graph G. All of them are C_2-connected to $[\![x]\!]_{\nu'}$ and overlap $[\![y]\!]_{\nu'}$. Anyway, the closure of $[\![x]\!]_{\nu'}$ is $[\![\langle Q\rangle\, x]\!]_{\nu'} = G$, and G is not part of $[\![y]\!]_{\nu'}$. Therefore $[\![x]\!]_{\nu'}$ is not non-tangential part of $[\![y]\!]_{\nu'}$, and $G, \nu \vDash \langle Q\rangle\, x \to y$ does not hold. ◀

4.7 Boundary and Boundary part

Galton uses a notion of boundary graph already found in [8], that is, in our notation

$$B([\![x]\!]) = [\![x \wedge \neg x]\!]\,.$$

However this is not the only notion of boundary possible in our setting, that is different from Galton's one because, in a *BISKT*-graph, multiple edges may occur between a pair of nodes. We examine, in this section the notion of boundary-graph and the spatial relation of boundary part that can be expressed in *BISKT*.

Consider the subgraph $[\![x]\!]$, in bold in Figure 3, with its underlying graph. The subgraph $([\![x \wedge \neg x]\!])$ corresponds to the nodes incident with the edges which are not in $[\![x]\!]$. However, it is reasonable to ask that also the edges incident with these nodes, and which belong to $[\![x]\!]$, are considered as part of the boundary of $[\![x]\!]$. We can define another notion of graph boundary as

$$B_\Diamond([\![x]\!]) = [\![\neg\neg(x \wedge \neg x) \wedge x]\!]$$

shown in Figure 3.

Cohn and Varzi's definition of the spatial relationship of Boundary part is

$$BP(X, Y) \equiv \forall Z(P(Z, X) \to (TP(Z, Y)))\,.$$

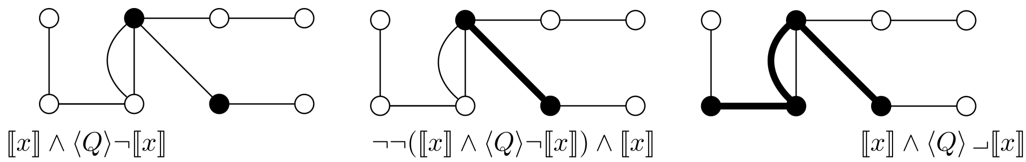

Figure 4 The Boundaries of $[\![x]\!]$.

A region X is part of the boundary of a region Y if for any of its parts Z, Z is a tangential part of Y. This spatial relationship is based on the intuition that the boundary of a region is connected with the *outside* of the region. Translating this definition in our language, the expected definition of Boundary part is the following (we assume any boundary part to be not empty)

$$BP_{\Diamond}([\![x]\!]_\nu, [\![y]\!]_\nu) \text{ iff for all } [\![z]\!]_\nu, \text{ if } NEP_{\Diamond}([\![z]\!]_\nu, [\![x]\!]_\nu) \text{ then } (TP_{\Diamond}([\![z]\!]_\nu, [\![y]\!]_\nu))\,.$$

We want to explore whether the definition of graph boundary $B([\![x]\!])$ given above is coherent with the spatial relation of Boundary part.

We notice the following:

(i) The definition of boundary part does not hold when $[\![\neg y]\!]$ is empty. Take the example of a graph composed of two nodes n_1, n_2 and two edges e_1, e_2 incident with them. Consider the subgraph $[\![x]\!] = \{n_1, e_1, n_2\}$. Here $B([\![x]\!]) = [\![x]\!]$ and $[\![\neg x]\!] = \varnothing$. For any $[\![z]\!]$ part of $[\![x]\!]$, $[\![z]\!] \subseteq [\![x]\!]$, the closure of $[\![z]\!]$ is the whole underlying graph, and the intersection of the whole graph with the empty set is empty. Therefore $G, \nu \nvDash \langle U \rangle (\langle Q \rangle z \wedge \neg x)$ is contradictory. This last consideration gives the hint that a better notion of boundary graph is "what leads *outside* of the graph" where the *outside* is $[\![\neg x]\!]$. Therefore, another sensible definition of boundary of x may be " everything which is connected to $[\![\neg x]\!]$".

(ii) If we adopt $B_{\Diamond}([\![x]\!]) = [\![\neg\neg(x \wedge \lrcorner\, x) \wedge x]\!]$ as definition of graph boundary, the notion of boundary part does not hold. Consider, again, the graph $[\![x]\!]$ in figure3. Its negation $\neg x$ is composed of all the nodes not in $[\![x]\!]$, plus the edges incident with them. According to the notion of Boundaty graph, every subgraph which is part of $[\![\neg\neg(x \wedge \lrcorner\, x) \wedge x]\!]$ is such that its closure overlaps $[\![\neg x]\!]$. It is easy to see that this is not true. Just for two of the three nodes are such that their closure overlap $[\![\neg]\!]$.

We put forward other two definitions of graph boundary:

$$B_{\Diamond}^{\neg,N}([\![x]\!]) = [\![x \wedge \langle Q \rangle \neg x]\!] \text{ and } B_{\Diamond}^{\neg}(x) = [\![\neg\neg(x \wedge \langle Q \rangle \neg x) \wedge x]\!]\,.$$

These new definitions single out the boundary subgraphs which are connected with $[\![\neg x]\!]$, and which support the definition of boundary part of [3]. The former considers just the nodes adjacent with edges adjacent with $[\![\neg x]\!]$, the latter adds also the edges between those nodes as shown in Figure 4. Eventually, another possible notions of boundary part is

$$B^{\lrcorner}([\![x]\!]) = [\![x \wedge \langle Q \rangle \lrcorner\, x]\!]$$

shown also in Figure 4.

5 Conclusions and Further Work

We have examined discrete space from the viewpoint of a modal logic based on relations on graphs, rather than on sets, as the accessibility relations. This has enabled us to bring together for the first time the approach to spatial reasoning using a modal logic based in mathematical morphology proposed by Bloch, with the mereotopological analysis of discrete space developed by Galton.

We have shown that the general framework of Cohn and Varzi can be generalized to accommodate discrete spatial relationships, but that closure operators which satisfy all of the Kuratowski axioms cannot be used to describe the notion of connection in some discrete spaces. By adopting the less restrictive version of closure due to Cech we have been able to realize the connection described by Galton as a C_2 connection in the framework of Cohn and Varzi.

The specific form of closure needed can be expressed as the negation, $\neg$, and dual negation, $\lrcorner$, in the logic BISKT. The combination of the semantic counterparts of these operations to express the idea of extending a subgraph by one step along the links is by no means new. This was already noted at COSIT 1997 by Stell and Worboys [16] citing the work of Reyes and Zolfaghari [11]. However, in the present work we have been able to express this closure as a modality using the stable relation Q which describes the incidence structure of the graph. Reyes and Zolfaghari [11] view this closure in a modal setting quite different from our use of stable relations on graphs. By working within the context of the BISKT logic we have been able to express not only connection itself, but other spatial relations including non-tangential parthood and a variety of notions of boundary.

In our formulation stating that two regions are connected is expressible through a formula in our logic. This depends on being able to express non-emptiness, which we achieve through the universal modality $\langle U \rangle$, "somewhere". In the setting of Bloch [2] a relation such as tangential part is expressed not just by a formula holding but by one formula holding and another being consistent. Expressing mereotopological relations entirely within our logic can be expected to facilitate the use of automated reasoning tools for modal logics, such as in [15], for spatial reasoning. We will explore this in further work, as well as extending our analysis to a wider range of relationships and examining these with notions of uncertainty and vagueness for discrete spatial regions.

References

1 P. Blackburn, J. F. van Benthem, and F. Wolter. *Handbook of modal logic*, volume 3. Elsevier, 2006.

2 I. Bloch. Modal logics based on mathematical morphology for qualitative spatial reasoning. *Journal of Applied Non-Classical Logics*, 12(3–4):399–423, 2002.

3 A. G. Cohn and A. C. Varzi. Mereotopological connection. *Journal of Philosophical Logic*, 32:357–390, 2003.

4 J. Cousty, L. Najman, F. Dias, and J. Serra. Morphological filtering on graphs. *Computer Vision and Image Understanding*, 117:370–385, 2013.

5 M. J. Egenhofer and J. Herring. Categorizing binary topological relations between regions, lines, and points in geographic databases. *Department of Surveying Engineering, , University of Maine, Orono, ME*, 9(94-1):76, 1991.

6 A. Galton. The mereotopology of discrete space. In C. Freksa and D. Mark, editors, *COSIT'99 proceedings*, volume 1661 of *LNCS*, pages 251–266. Springer, 1999.

7 Antony Galton. Discrete mereotopology. In *Mereology and the Sciences*, pages 293–321. Springer, 2014.

8 F. W. Lawvere. Intrinsic co-heyting boundaries and the leibniz rule in certain toposes. In *Category Theory*, pages 279–281. Springer, 1991.

9 L. Najman and H. Talbot. *Mathematical Morphology. From theory to applications.* Wiley, 2010.

10 D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In B. Nebel, C. Rich, and W. Swartout, editors, *Principles of Knowledge Representation and Reasoning. Proceedings of the Third International Conference (KR92)*, pages 165–176. Morgan Kaufmann, 1992.

11 G. E. Reyes and H. Zolfaghari. Bi-Heyting algebras, toposes and modalities. *Journal of Philosophical Logic*, 25:25–43, 1996.

12 Azriel Rosenfeld. Digital topology. *American Mathematical Monthly*, pages 621–630, 1979.

13 P. Simons. *Parts. A Study in Ontology.* Clarendon Press, Oxford, 1987.

14 M. B. Smyth. Topology. In *Handbook of Logic in Computer Science, Vol 1*, pages 641–761. Clarendon Press, Oxford, 1992.

15 J. G. Stell, R. A. Schmidt, and D. Rydeheard. A bi-intuitionistic modal logic: Foundations and automation. *J. Logical and Algebraic Methods in Programming*, 85(4):500–519, 2016.

16 J. G. Stell and M. F. Worboys. The algebraic structure of sets of regions. In S. C. Hirtle and A. U. Frank, editors, *Spatial Information Theory, International Conference COSIT'97, Proceedings*, volume 1329 of *LNCS*, pages 163–174. Springer, 1997.

17 D. Vakarelov, G. Dimov, I. Düntsch, and B. Bennett. A proximity approach to some region-based theories of space. *Journal of Applied Non-Classical Logics*, 12:527–559, 2002.