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Generalized large-scale semigeostrophic approximations for the f-plane primitive equations

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Abstract

We derive a family of balance models for rotating stratified flow in the primitive equation setting. By construction, the models possess conservation laws for energy and potential vorticity and are formally of the same order of accuracy as Hoskins' semigeostrophic equations. Our construction is based on choosing a new coordinate frame for the primitive equation variational principle in such a way that the consistently truncated Lagrangian degenerates. We show that the balance relations so obtained are elliptic when the fluid is stably stratified and certain smallness assumptions are satisfied. Moreover, the potential temperature can be recovered from the potential vorticity via inversion of a non-standard Monge–Ampère problem which is subject to the same ellipticity condition. While the present work is entirely formal, we conjecture, based on a careful rewriting of the equations of motion and a straightforward derivative count, that the Cauchy problem for the balance models is well posed subject to conditions on the initial data. Our family of models includes, in particular, the stratified analog of the L_1 balance model of R. Salmon.

1 Introduction

In 1996, R. Salmon published the last in a series of original papers promoting the use of Hamiltonian approximation methods in the context of balance

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models for geophysical fluid flow [15]. His construction starts out from the f-plane primitive equations in semigeostrophic scaling and derives a simple Hamiltonian balance model, the *large-scale semigeostrophic (LSG) equations.*

Salmon's construction is a two-step process. The first step is a variational analog of the geostrophic momentum approximation—certain terms in the model Lagrangian are replaced by their geostrophic counterparts. Alternatively, this step can be interpreted as imposing a constraint to geostrophic balance in the underlying Hamilton's principle. The second step is a near-identity change of coordinates that brings the resulting equations of motion into a very simple form. Salmon introduced his method first in the context of the shallow water equations [14] where the two steps are clearly separated and explained; the corresponding argument in [15] is less explicit in this regard, but two steps can be performed in sequence there as well.

In the context of the shallow water equations, Salmon named the result of the first step the "L_1 model" referring to the fact that it accurately represents the shallow water Lagrangian up to first order in the Rossby number ε. The L_1 model is globally well posed under mild conditions [6] and is numerically robust, while its further simplification in the second step of Salmon's procedure yields a model which is dynamically ill posed for all we know. Salmon's derivation has been re-interpreted and generalized in [11], the key idea being that Salmon's two steps can be reversed: choosing a new coordinate system first, expanding the Lagrangian in the small parameter, and truncating the result at a chosen order of approximation will *imply* a balance relation so long as the truncated Lagrangian is degenerate. The balance relation may be interpreted as a Dirac constraint. Thus, achieving degeneracy emerges as the key construction principle of balance models and leads to a much greater flexibility in the possible choices of coordinates. In this view, the apparent ill-posedness of Salmon's LSG model emerges as the loss of ellipticity in the potential vorticity inversion when a scalar parameter λ, which controls the change of coordinates, approaches a critical value.

For stratified flow, the situation is less well studied. An early study of Shepherd [16] focused on the *depth invariant temperature (DIT) equations*—the restriction of the LSG equations to a single thermally active layer—which show clear signs of numerical ill-posedness. To our knowledge, the matter has not been pursued since, but by all appearance, the general stratified LSG equations suffer from similar loss of regularity.

The stratified analog of the L_1 model, implicit in Salmon's work, was first written out and solved numerically by Allen *et al.* [2] who also derive a second order correction; cf. [1] for earlier work in the shallow water setting.

They find that the L_1 and L_2 models perform rather well. Here, we ask the following questions: Can the stratified L_1 model can be cast in a potential vorticity formulation? Is it as well behaved as its shallow water counterpart? And is there a stratified analog of the generalized large-scale semigeostrophic models introduced in [11]?

This paper provides, at a formal level, a positive answer to all of these questions. Our construction follows [11] with one crucial difference. For the shallow water equations, geostrophic balance determines the velocity field completely. For the primitive equations, geostrophic balance can only determine the velocity up to a constant of vertical integration which may be a function of horizontal position. Without loss of generality, we can take the vertical mean velocity $\bar{u}$ as this constant of integration and split the velocity field $u = \bar{u} + \hat{u}$ into its vertical average $\bar{u}$ and perturbation field $\hat{u}$. Degeneracy of the Lagrangian can be imposed on the perturbation field only. As in [11], this principle leads to a family of balance models controlled by a parameter λ. The structure of the resulting Euler–Lagrange equations is somewhat more complicated: they split into a transport equation for the relative vorticity which determines the evolution of $\bar{u}$ and a kinematic balance relation which determines $\hat{u}$.

By construction, this family of balance models has a materially advected potential vorticity. The potential temperature can be computed from the potential vorticity as the solution of a Monge–Ampère equation, thereby gaining regularity in Sobolev space. Moreover, the perturbation field $\hat{u}$ is determined by a second order linear balance relation. It turns out that the ellipticity conditions for the two problems coincide: ellipticity requires a stable stratification and sufficient smallness of the Rossby number relative to relative vorticity and to gradients of potential temperature.

With this structure in place, we can identify three interesting special cases for the parameter λ. When $\lambda = \frac{1}{2}$, the transformation reduces to the identity up to the formal order of accuracy of the model and we obtain the stratified analog of Salmon's L_1 model. We expect that this model is well posed at least locally in time. When $\lambda = 0$, the balance relation gains two additional derivatives relative to the general case. This corresponds directly to the shallow water case discussed in [11]. Finally, when $\lambda = -\frac{1}{2}$, we obtain Salmon's LSG equations. The balance relation ceases to be elliptic and well-posedness is expected to fail.

Our work is entirely formal at this point, but we formulate the equations of motion in a way we believe will be useful for proving well-posedness as well as for numerical simulations. A complete proof of well-posedness will require work well beyond the scope of this paper due to the nonlinear nature of the kinematic relations.

For simplicity, we assume a basin with a flat bottom, infinite horizontal extend with decay toward a steady state at infinity, and a constant Coriolis parameter. We believe that it is relatively straightforward to include bottom topography terms and a spatially varying Coriolis parameter, as has been done in the shallow water case [13], although the computation and the resulting equations of motion will be much more complicated. The conservation laws for energy and potential vorticity, however, remain simple even when f is varying in the horizontal spatial variables, so that we provide a brief account of the necessary modifications at the end of Section 6.

The paper is structured as follows. In Section 2, we introduce notation and state the primitive equations in semigeostrophic scaling. Section 3 reviews how the primitive equations arise as the Euler–Poincaré equations from a variational principle and states the leading order thermal wind balance for later use. Section 4 contains the key steps of the derivation via the method of degenerate variational asymptotics. Although it is a routine exercise to derive the resulting equations of motion, they need careful rewriting to fully separate the dynamic from the kinematic part. This is done in Section 5. In Section 6 we derive the expression for the potential vorticity and show that the potential vorticity inversion is given by a non-standard Monge–Ampère equation. The paper closes with a discussion of the full system of balance model equations, three special cases, and brief conclusions.

2 The primitive equations

The starting point of our investigation are the primitive equations (PE), which describe atmospheric and oceanic flows in the Boussinesq and the hydrostatic approximation. Here, we consider the primitive equations in a very simple setting: we neglect dissipation, write the equations on the f-plane, use rigid lid upper boundary conditions, and exclude bottom topography.

Throughout the paper, we adapt the following notation: bold-face letters always denote three-component objects, while regular typeface is used for their horizontal parts; e.g., $\boldsymbol{u} = (u_1, u_2, u_3)^T = (u, u_3)^T$, $\boldsymbol{x} = (x_1, x_2, z)^T = (x, z)^T$, and $\boldsymbol{\nabla} = (\partial_{x_1}, \partial_{x_2}, \partial_z) = (\nabla, \partial_z)$. Further, we write $u^\perp = (-u_2, u_1)^T$ to denote the counter clockwise rotation of u by $\pi/2$ and $\boldsymbol{u}_\mathrm{h} = (u, 0)^T$ to denote the projection of $\boldsymbol{u}$ onto the horizontal coordinate plane.

The non-dimensionalized primitive equations then read

$$\varepsilon \left(\partial_t u + \boldsymbol{u} \cdot \boldsymbol{\nabla} u\right) + u^\perp = \frac{\varepsilon}{\mathrm{Fr}^2} \nabla \phi \,, \tag{1a}$$

$$\partial_z \phi = \theta \,, \tag{1b}$$

$$\partial_t \theta + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = 0 \,, \tag{1c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \tag{1d}$$

where $\boldsymbol{u}$ denotes the full three-dimensional fluid velocity. For the sake of brevity, we will refer to ϕ and θ as geopotential and potential temperature, respectively, with the understanding that for the oceanic flows the respective physical quantities are pressure and buoyancy. We remark that the atmospheric primitive equations take the form (1) in pressure coordinates.

The physical regime is determined by the Rossby number $\varepsilon = U_0/(fL_0)$ and the Froude number $\mathrm{Fr} = U_0/(NH_0)$, where U_0, L_0, F_0, H_0 are characteristic values for horizontal velocity, horizontal length scale, and vertical length scale, respectively; f denotes the Coriolis parameter which we assume constant, and N is the Brunt–Väisälä frequency. In what follows, we assume *semigeostrophic scaling* where $\mathrm{Fr}^2 = \varepsilon \ll 1$.

For simplicity, we study (1) in the strip $\Omega = \mathbb{R}^2 \times [0, -H]$ of constant height H with a solid bottom boundary and a rigid lid upper boundary, so that

$$u_3 = 0 \qquad \text{for } z = 0 \text{ and } z = -H \,, \tag{2a}$$

$$\phi = 0 \qquad \text{for } z = 0 \,. \tag{2b}$$

In the horizontal, we impose a sufficient rate of decay at infinity so that we can freely integrate by parts in the horizontal variables without incurring boundary terms. Thus, we are in exactly the setting in which Hoskins first introduced a transformation to semigeostrophic coordinates [10].

We note that we could endow the balance models derived below with periodic lateral boundary conditions. However, in the periodic setting, the Coriolis parameter is not an exact scalar field (it cannot be written as the two-dimensional curl of a vector potential), so that our derivation does not literally apply; see [12] for a discussion of this issue in the shallow water context.

3 Variational principle for the primitive equations

The primitive equations (1) can be derived from a variational principle as follows. Let $\mathfrak{g}$ denote the Lie algebra of vector fields on Ω satisfying the incompressibility condition (1d) and boundary conditions (2), and let $\boldsymbol{\eta}$ denote the flow of a time dependent vector field $\boldsymbol{u} \in \mathfrak{g}$, i.e.,

$$\dot{\boldsymbol{\eta}}(\boldsymbol{a}, t) = \boldsymbol{u}(\boldsymbol{\eta}(\boldsymbol{a}, t), t) \qquad \text{with} \qquad \boldsymbol{\eta}(\boldsymbol{a}, 0) = \boldsymbol{a} \,. \tag{3}$$

Here and in the following, the letter $\boldsymbol{a}$ is used for Lagrangian label coordinates, while $\boldsymbol{x} = \boldsymbol{\eta}(\boldsymbol{a}, t)$ denotes the corresponding Eulerian position at time t. As $\boldsymbol{u}$ is divergence free, $\boldsymbol{\eta}$ is volume preserving. In the following, we shall write $\dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta}$ for short. Correspondingly, advection of potential temperature (1c) is equivalent to

$$\theta \circ \boldsymbol{\eta} = \theta_0 \,, \tag{4}$$

where θ_0 is the given initial distribution of potential temperature.

Throughout the article, we use the letters L and ℓ (with appropriate subscripts as necessary) to distinguish Lagrangians expressed in Lagrangian and Eulerian quantities, respectively. With this notation in place, the primitive equation Lagrangian reads

$$L_{\mathrm{PE}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}; \theta_0) = \int_\Omega R \circ \boldsymbol{\eta} \cdot \dot{\eta} + \frac{\varepsilon}{2} \, |\dot{\eta}|^2 + \theta_0 \, \eta_3 \, \mathrm{d}\boldsymbol{a} \,, \tag{5}$$

where $\boldsymbol{\nabla} \times \boldsymbol{R} = \boldsymbol{e}_z$, the unit vector in z-direction. In other words, its horizontal part R is a two-dimensional vector potential for the Coriolis parameter which, in non-dimensionalized variables, equals one. We note that L_{PE} can be expressed in terms of purely Eulerian quantities as

$$L_{\mathrm{PE}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}; \theta_0) = \int_\Omega R \cdot u + \frac{\varepsilon}{2} \, |u|^2 + \theta z \, \mathrm{d}\boldsymbol{x} \equiv \ell_{\mathrm{PE}}(\boldsymbol{u}, \theta) \,. \tag{6}$$

More generally, L_{PE} is invariant under compositions of the flow map with arbitrary volume and domain preserving maps. This is known as the *particle relabelling symmetry*, which implies a conservation law for the potential vorticity, further discussed in Section 6 below.

Given a Lagrangian L satisfying the relabelling symmetry $L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}; \theta_0) = \ell(\boldsymbol{u}, \theta)$, the Euler–Poincaré theorem for continua ([8] or [9, Theorem 17.8]) asserts that the following are equivalent.

(i) $\boldsymbol{\eta}$ satisfies the *variational principle*

$$\delta \int_{t_1}^{t_2} L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}; \theta_0) \, \mathrm{d}t = 0 \tag{7}$$

with respect to variations of the flow map $\delta \boldsymbol{\eta} = \boldsymbol{w} \circ \boldsymbol{\eta}$ where $\boldsymbol{w}$ is a curve in $\mathfrak{g}$ vanishing at the temporal end points.

(ii) $\boldsymbol{u}$ and θ satisfy the *reduced variational principle*

$$\delta \int_{t_1}^{t_2} \ell(\boldsymbol{u}, \theta) \, \mathrm{d}t = 0 \,, \tag{8}$$

where the variations $\delta \boldsymbol{u}$ and $\delta\theta$ are subject to the Lin constraints

$$\delta \boldsymbol{u} = \dot{\boldsymbol{w}} + \boldsymbol{\nabla} \boldsymbol{w}\, \boldsymbol{u} - \boldsymbol{\nabla} \boldsymbol{u}\, \boldsymbol{w} = \dot{\boldsymbol{w}} + [\boldsymbol{u}, \boldsymbol{w}] \,, \tag{9a}$$

$$\delta\theta + \boldsymbol{w} \cdot \boldsymbol{\nabla}\theta = 0 \,, \tag{9b}$$

with $\boldsymbol{w}$ as in (i).

(iii) $\boldsymbol{m}$ and θ satisfy the *Euler–Poincaré equation*

$$\int_\Omega (\partial_t + \mathcal{L}_{\boldsymbol{u}}) \boldsymbol{m} \cdot \boldsymbol{w} + \frac{\delta \ell}{\delta \theta} \mathcal{L}_{\boldsymbol{w}} \theta \, \mathrm{d}\boldsymbol{x} = 0 \tag{10}$$

for every $\boldsymbol{w} \in \mathfrak{g}$, where $\mathcal{L}$ denotes the Lie derivative and $\boldsymbol{m}$ is the momentum one-form

$$\boldsymbol{m} = \frac{\delta \ell}{\delta \boldsymbol{u}} \,. \tag{11}$$

In the language of vector fields in a region of $\mathbb{R}^3$, the Euler–Poincaré equation (10) reads

$$\int_\Omega \Big(\partial_t \boldsymbol{m} + (\boldsymbol{\nabla} \times \boldsymbol{m}) \times \boldsymbol{u} + \boldsymbol{\nabla}(\boldsymbol{m} \cdot \boldsymbol{u}) + \frac{\delta \ell}{\delta \theta} \boldsymbol{\nabla}\theta \Big) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = 0 \tag{12}$$

for every $\boldsymbol{w} \in \mathfrak{g}$. Due to the Hodge decomposition, the term in parentheses must be a gradient, i.e.,

$$\partial_t \boldsymbol{m} + (\boldsymbol{\nabla} \times \boldsymbol{m}) \times \boldsymbol{u} + \frac{\delta \ell}{\delta \theta} \boldsymbol{\nabla}\theta = \boldsymbol{\nabla}\tilde{\phi} \,. \tag{13}$$

Noting that

$$\frac{\delta \ell_{\mathrm{PE}}}{\delta \boldsymbol{u}} = \boldsymbol{R} + \varepsilon\, \boldsymbol{u}_{\mathrm{h}} \qquad \text{and} \qquad \frac{\delta \ell_{PE}}{\delta \theta} = z \,, \tag{14}$$

re-defining the geopotential $\phi = \tilde{\phi} - z\theta + \frac{1}{2} \varepsilon\, |u|^2$, and using the vector identity

$$(\boldsymbol{\nabla} \times \boldsymbol{u}_{\mathrm{h}}) \times \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}_{\mathrm{h}} - \tfrac{1}{2} \boldsymbol{\nabla} |u|^2 \,, \tag{15}$$

we can write the Euler–Poincaré equation for L_{PE} as

$$\varepsilon\, (\partial_t \boldsymbol{u}_{\mathrm{h}} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}_{\mathrm{h}}) + \boldsymbol{e}_z \times \boldsymbol{u} - \boldsymbol{e}_z\, \theta = \boldsymbol{\nabla}\phi \,. \tag{16}$$

This expression encodes the horizontal momentum equation (1a) and the hydrostatic equation (1b).

We remark that traditional approach to variational derivation of the primitive equations treats the geopotential as a Lagrange multiplier responsible for enforcing the incompressibility constraint, cf. [15]. Here, we build the constraint into the definition of the configuration space. The gradient of the geopotential then appears naturally due to the fact that the L^2 pairing with divergence free vector fields determines a vector field only up to a gradient. Both approaches, of course, lead to identical equations of motion.

4 Derivation of the balance model Lagrangian

We observe that any notion of balance entails restricting the motion to a submanifold of the phase space. Therefore, when a balance model arises from a variational principle, its Lagrangian is necessarily degenerate. Our general strategy for the derivation of approximate balance models is to find a near-identity configuration space transformation such that the original Lagrangian becomes degenerate when truncated at the desired order of an expansion in the small parameter. The truncated degenerate Lagrangian then determines the balance model variational principle. Its degeneracy implies a Dirac constraint on the system, which is precisely the balance relation.

To implement this strategy, we largely follow the construction in [11]. We consider the primitive equation flow map as a family of maps $\boldsymbol{\eta}_\varepsilon$ parameterized by ε. They relate to a balance model flow map $\boldsymbol{\eta}$ via a change of coordinates generated by a vector field $\boldsymbol{v}_\varepsilon \in \mathfrak{g}$, so that

$$\boldsymbol{\eta}_\varepsilon' = \boldsymbol{v}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \qquad \text{with} \qquad \boldsymbol{\eta}_0 = \boldsymbol{\eta} \,, \tag{17}$$

where $\boldsymbol{\eta}_\varepsilon'$ denotes the derivative of $\boldsymbol{\eta}_\varepsilon$ with respect to ε. At this point, we have free choice of $\boldsymbol{v}_\varepsilon$. To obtain a balance model, we seek an expression for $\boldsymbol{v}_\varepsilon$ that renders the PE Lagrangian degenerate. Generally, it is not possible to do this exactly, but we can proceed iteratively in a formal power series expansion of the Lagrangian with respect to ε. Here, we pursue the construction to order $O(\varepsilon)$ only, which is the setting used by Salmon [14, 15]. In principle, higher order balance models are possible, but the resulting expressions are extremely complicated.

At the leading order in ε, the Euler–Poincaré equation (16) reads

$$\begin{pmatrix} u^\perp \\ -\theta \end{pmatrix} = \boldsymbol{\nabla}\phi \,. \tag{18}$$

Taking the curl and noting that, for a general vector field $(v, f)^T$ in three dimensions,

$$\boldsymbol{\nabla} \times \begin{pmatrix} v \\ f \end{pmatrix} = \begin{pmatrix} \partial_z v^\perp \\ \nabla^\perp \cdot v \end{pmatrix} - \begin{pmatrix} \nabla^\perp f \\ 0 \end{pmatrix} \,, \tag{19}$$

we obtain the thermal wind relation

$$\partial_z u = \nabla^\perp \theta \tag{20}$$

in the horizontal components and the divergence free condition $\nabla \cdot u = 0$ in the vertical component.

The thermal wind relation specifies the horizontal velocity field only up to a constant of integration, which may be a function of the horizontal space variables x. Without loss of generality, we may assume that this constant of integration is the mean horizontal velocity, which we write $\bar{u} = \bar{u}(x)$, while the thermal wind relation specifies the deviation from the mean, denoted $\hat{u} = \hat{u}(\boldsymbol{x})$. Writing $u = \bar{u} + \hat{u}$ so that $\partial_z u = \partial_z \hat{u}$ and noting that, by definition,

$$0 = \int_{-H}^{0} \hat{u} \, \mathrm{d}z = H \, \hat{u}(-H) - \int_{-H}^{0} z \, \partial_z \hat{u} \, \mathrm{d}z \,, \tag{21}$$

we find that the vertically mean free thermal wind is given by

$$\hat{u}(z) = \hat{u}(-H) + \int_{-H}^{z} \partial_z \hat{u} \, \mathrm{d}z' = \int_{-H}^{0} \frac{z}{H} \, \nabla^\perp \theta \, \mathrm{d}z + \int_{-H}^{z} \nabla^\perp \theta \, \mathrm{d}z' \,. \tag{22}$$

As we remarked earlier, balance in the stratified setting only constrains the mean free component $\hat{u}$ of the horizontal velocity, while the vertical mean $\bar{u}$ remains entirely unconstrained. Thus, the principle that the balance model Lagrangian must be degenerate applies only to the contribution from $\hat{u}$.

To implement this construction, we substitute $\boldsymbol{\eta}_\varepsilon$ for $\boldsymbol{\eta}$ in the primitive equation Lagrangian (5), write $\boldsymbol{\eta}' \equiv \boldsymbol{\eta}'_\varepsilon\big|_{\varepsilon=0}$, and expand in powers of ε, so that

$$L_{\mathrm{PE}} = \int_\Omega R \circ \boldsymbol{\eta} \cdot \dot{\eta} + \theta_0 \, \eta_3 \, \mathrm{d}\boldsymbol{a} + \varepsilon \int_\Omega (-\eta' \cdot \dot{\eta}^\perp) + \frac{1}{2} \, |\dot{\eta}|^2 + \theta_0 \, \eta_3' \, \mathrm{d}\boldsymbol{a} + O(\varepsilon^2) \,. \tag{23}$$

We now change to Eulerian variables, split the transformed velocity field into its vertical mean and its mean free component via $u = \bar{u} + \hat{u}$, and note that

$$\int_\Omega |u|^2 \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\bar{u}|^2 + 2 \, \bar{u} \cdot \hat{u} + |\hat{u}|^2 \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\bar{u}|^2 + |\hat{u}|^2 \, \mathrm{d}\boldsymbol{x} \tag{24}$$

because the cross term drops out via the mean free condition on $\hat{u}$, so that

$$\ell_{\mathrm{PE}} = \int_\Omega R \cdot u + \theta \, z \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega (v^\perp + \tfrac{1}{2} \, \bar{u}) \cdot \bar{u} + (v^\perp + \tfrac{1}{2} \, \hat{u}) \cdot \hat{u} + \theta \, v_3 \, \mathrm{d}\boldsymbol{x} + O(\varepsilon^2) \tag{25}$$

where $\boldsymbol{v} = \boldsymbol{v}_\varepsilon\big|_{\varepsilon=0}$.

To ensure that the $O(\varepsilon)$ contribution is affine, hence degenerate in the variable $\hat{u}$, our choice of v must cancel the term $|\hat{u}|^2$. This suggests that the horizontal components of the transformation vector field $\boldsymbol{v}$ should be of the form

$$v = \tfrac{1}{2} \, \hat{u}^\perp + F(\theta) \,. \tag{26}$$

The third component of $\boldsymbol{v}$ is then determined by the condition that $\boldsymbol{v}$ is divergence free and tangent to the top and bottom boundaries.

As in [11], motivated by dimensional considerations, we specialize to the one-parameter family of transformation vector fields

$$\boldsymbol{v} = \frac{1}{2} \begin{pmatrix} \hat{u}^\perp \\ \nabla^\perp \cdot \hat{U} \end{pmatrix} - \lambda \begin{pmatrix} g^\perp \\ \nabla^\perp \cdot G \end{pmatrix} , \tag{27}$$

where g is the thermal wind defined through an expression of the form (22), namely

$$g = \nabla^\perp \Theta \tag{28}$$

with

$$\Theta = \int_{-H}^{0} \frac{z}{H} \, \theta \, \mathrm{d}z + \int_{-H}^{z} \theta \, \mathrm{d}z' , \tag{29}$$

and where

$$G = - \int_{z}^{0} g \, \mathrm{d}z' \qquad \text{and} \qquad \hat{U} = - \int_{z}^{0} \hat{u} \, \mathrm{d}z' \tag{30}$$

are the vertical antiderivatives of g and $\hat{u}$, respectively. By construction, $\boldsymbol{v}$ is a divergence free field, $v_3(0) = v_3(-H) = 0$, and v has zero vertical mean.

The case $\lambda = \frac{1}{2}$ is special: whenever u satisfies the thermal wind relation, then $\boldsymbol{v} = 0$ so that $O(\varepsilon)$ deviations from thermal wind correspond to a transformation vector field that is $O(\varepsilon)$ small. Hence, the transformation for $\lambda = \frac{1}{2}$ is overall an $O(\varepsilon^2)$ change of variables so that quantities in physical and computational coordinates may be identified up to errors that are beyond the order of accuracy of the approximation.

We further remark that Salmon's [15] transformation corresponds to the choice $\lambda = -\frac{1}{2}$ up to terms of higher order, as the leading order balance implies that

$$\boldsymbol{v}_{\text{Salmon}} = \begin{pmatrix} g^\perp \\ \nabla^\perp \cdot G \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \hat{u}^\perp \\ \nabla^\perp \cdot \hat{U} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} g^\perp \\ \nabla^\perp \cdot G \end{pmatrix} + O(\varepsilon) \, . \tag{31}$$

Returning to the general case, we plug the transformation (27) back into (25), drop all higher order terms, and note that the integral of $v^\perp \cdot \bar{u}$ is zero as v has zero vertical mean. This yields the balance model reduced Lagrangian

$$\ell_{\text{BM}} = \int_\Omega R \cdot u + \theta \, z \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega \tfrac{1}{2} \, |\bar{u}|^2 + \lambda \, g \cdot \hat{u} + \theta \, \nabla^\perp \cdot (\tfrac{1}{2} \, \hat{U} - \lambda \, G) \, \mathrm{d}\boldsymbol{x} \, . \tag{32}$$

To simplify this expression, we note that

$$\int_\Omega \theta\, \nabla^\perp \cdot \hat{U} \,\mathrm{d}\boldsymbol{x} = -\int_\Omega \hat{U} \cdot \nabla^\perp \theta \,\mathrm{d}\boldsymbol{x} = -\int_\Omega \hat{U} \cdot \partial_z g \,\mathrm{d}\boldsymbol{x} = \int_\Omega \hat{u} \cdot g \,\mathrm{d}\boldsymbol{x}\,, \tag{33}$$

where, in the first equality, we integrated by parts in the horizontal variables, while in the last equality, we integrated by parts in z using the fact that $\hat{U}$ vanishes on the top and bottom boundaries by construction. Similarly,

$$\int_\Omega \theta\, \nabla^\perp \cdot G \,\mathrm{d}\boldsymbol{x} = \int_\Omega |g|^2 \,\mathrm{d}\boldsymbol{x}\,. \tag{34}$$

Altogether, the balance model reduced Lagrangian is given by

$$\begin{aligned} \ell_{\mathrm{BM}} &= \int_\Omega R \cdot u + \theta\, z \,\mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega \tfrac{1}{2}\, |\bar{u}|^2 + (\lambda + \tfrac{1}{2})\, g \cdot \hat{u} - \lambda\, |g|^2 \,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega R \cdot u + \theta\, z \,\mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega \tfrac{1}{2}\, u \cdot (\bar{u} + g) + \lambda\, g \cdot (\hat{u} - g) \,\mathrm{d}\boldsymbol{x} \\ &\equiv l_0 + \varepsilon\, l_1\,. \end{aligned} \tag{35}$$

5 Balance Euler–Poincaré equations

Taking the variation of l_1 from (35) and noting that products of mean-free with vertically averaged functions vanish, we find that

$$\begin{aligned} \delta l_1 &= \int_\Omega \bigl(\bar{u} + (\lambda + \tfrac{1}{2})\, g\bigr) \cdot \delta u + \bigl((\tfrac{1}{2} + \lambda)\, \hat{u} - 2\lambda\, g\bigr) \cdot \delta g \,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega p \cdot \delta u + b \cdot \delta g \,\mathrm{d}\boldsymbol{x}\,, \end{aligned} \tag{36}$$

where, for future reference,

$$p = \bar{u} + \nu\, g \qquad \text{and} \qquad b = \nu\, \hat{u} - 2\lambda\, g \tag{37}$$

with $\nu = \lambda + \frac{1}{2}$. Further, we shall write $B = \nu\, \hat{U} - 2\lambda\, G$ to denote the vertical antiderivative of b.

Taking the variation of (28), we have

$$\delta g = \int_{-H}^{0} \frac{z}{H}\, \nabla^\perp \delta\theta \,\mathrm{d}z + \int_{-H}^{z} \nabla^\perp \delta\theta \,\mathrm{d}z'\,. \tag{38}$$

When integrated against the vertically mean-free vector field b, the contribution from the first term vanishes, so that, reversing the order of the z and the z' integration, we obtain

$$\int_\Omega b \cdot \delta g \, \mathrm{d}\boldsymbol{x} = \int_\Omega \nabla^\perp \delta\theta \cdot \int_z^0 b \, \mathrm{d}z' \, \mathrm{d}\boldsymbol{x} = \int_\Omega \delta\theta \, \nabla^\perp \cdot B \, \mathrm{d}\boldsymbol{x} \,. \tag{39}$$

Combining (35), (36), and (39), we find that

$$\boldsymbol{m}_{\mathrm{BM}} \equiv \frac{\delta \ell_{\mathrm{BM}}}{\delta \boldsymbol{u}} = \boldsymbol{R} + \varepsilon \, \boldsymbol{p}_{\mathrm{h}} \qquad \text{and} \qquad \frac{\delta \ell_{\mathrm{BM}}}{\delta \theta} = \nabla^\perp \cdot B \,, \tag{40}$$

so that the Euler–Poincaré equation (13) for L_{BM} reads

$$\boldsymbol{e}_z \times \boldsymbol{u} - \theta \, \boldsymbol{e}_z + \varepsilon \left(\partial_t \boldsymbol{p}_{\mathrm{h}} + (\boldsymbol{\nabla} \times \boldsymbol{p}_{\mathrm{h}}) \times \boldsymbol{u} + \boldsymbol{\nabla} \theta \, \nabla^\perp \cdot B \right) = \boldsymbol{\nabla} \phi \,. \tag{41}$$

Introducing the relative vorticity

$$\zeta = \nabla^\perp \cdot p = \nabla^\perp \cdot \bar{u} + \nu \, \Delta \Theta \tag{42}$$

and separating the horizontal and vertical component equations, we write (41) in the form

$$\begin{pmatrix} u^\perp \\ -\theta \end{pmatrix} + \varepsilon \begin{pmatrix} \partial_t p + u^\perp \zeta + u_3 \, \partial_z p \\ -u \cdot \partial_z p \end{pmatrix} + \varepsilon \, \boldsymbol{\nabla} \theta \, \nabla^\perp \cdot B = \boldsymbol{\nabla} \phi \,. \tag{43}$$

Notice that incompressibility implies $\partial_z u_3 = -\nabla \cdot u$, so that

$$u_3 = \int_z^0 \nabla \cdot u \, \mathrm{d}z' = -\nabla \cdot \hat{U} \,, \tag{44}$$

where the second equality is due to $u_3(-H) = 0$, which implies that $\nabla \cdot \bar{u} = 0$. Since g and G are horizontally divergence free, we can write

$$u_3 = -\nu^{-1} \, \nabla \cdot B \tag{45}$$

so long as $\nu \neq 0$. Then, taking the curl of (43), noting that $\partial_z p = \nu \, \nabla^\perp \theta$ and $\partial_z \zeta = \nu \, \Delta \theta$, using (19), and applying the vector identity

$$\boldsymbol{\nabla} \times (f \boldsymbol{\nabla} \theta) = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} \theta = \begin{pmatrix} \partial_z f \, \nabla^\perp \theta - \partial_z \theta \, \nabla^\perp f \\ \nabla^\perp f \cdot \nabla \theta \end{pmatrix} \,, \tag{46}$$

we rewrite the Euler–Poincaré equation in the form

$$
\begin{pmatrix} \nabla^\perp\theta - \partial_z u \\ \nabla\cdot u \end{pmatrix} + \varepsilon \begin{pmatrix} -\nu\,\nabla\dot\theta - \partial_z u\,\zeta - \nu\,u\,\Delta\theta + \nu\,\nabla\cdot u\,\nabla\theta - \nu\,u_3\,\partial_z\nabla\theta) \\ \partial_t\zeta + \nabla\cdot(u\zeta) - \nabla\cdot(\nabla\theta\,\nabla\cdot B) \end{pmatrix}
$$
$$
+ \nu\varepsilon \begin{pmatrix} \nabla^\perp(u\cdot\nabla^\perp\theta) \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \nabla^\perp\theta\,\nabla^\perp\cdot b - \partial_z\theta\,\nabla^\perp\nabla^\perp\cdot B \\ \nabla\theta\cdot\nabla^\perp\nabla^\perp\cdot B \end{pmatrix} = 0\,. \tag{47}
$$

The vertical component of (47) describes the evolution of relative vorticity ζ. Rearranging terms, we can write it in the form

$$
\begin{aligned}
\partial_t\zeta + \nabla\cdot(\zeta u) &= -\varepsilon^{-1}\,\nabla\cdot\hat u + \nabla\cdot(\nabla\theta\,\nabla\cdot B) - \nabla\theta\cdot\nabla^\perp\nabla^\perp\cdot B \\
&\equiv F_1(\nabla\cdot\hat u, \nabla\theta, \nabla\nabla\theta, \nabla B, \nabla\nabla B)\,.
\end{aligned} \tag{48}
$$

Since, by (54), $\nabla\cdot\hat u$ is formally an $O(\varepsilon)$ quantity, all terms on the right hand side are $O(1)$ so that ζ evolves on the slow time scale.

Setting $\omega = \nabla^\perp\cdot\bar u$ and taking the vertical average of (48), we obtain

$$
\partial_t\omega + \bar u\cdot\nabla\omega = \overline{\nabla\cdot(\nabla\theta\,\nabla\cdot B - \nu\,\hat u\,\Delta\Theta) - \nabla\theta\cdot\nabla^\perp\nabla^\perp\cdot B} \equiv F_2\,. \tag{49}
$$

The horizontal component equation of the Euler–Poincaré equation (47), on the other hand, is entirely kinematic and will lead to an equation for $\hat U$, the balance relation. To make this relation more explicit, we note that the advection equation for θ implies

$$
\partial_t\nabla\theta = -\nabla(\boldsymbol{u}\cdot\boldsymbol{\nabla}\theta) = -\nabla(u\cdot\nabla\theta) - \nabla(u_3\,\partial_z\theta)\,. \tag{50}
$$

Then, after rearrangement of terms, the horizontal part of (47) reads

$$
\begin{aligned}
&\nabla^\perp\theta - \partial_z u + \varepsilon\left(-\zeta\,\partial_z u + \nabla^\perp\theta\,\nabla^\perp\cdot b - \partial_z\theta\,(\nabla^\perp\nabla^\perp\cdot B + \nabla\nabla\cdot B)\right) \\
&\quad + \nu\varepsilon\left(\nabla(u\cdot\nabla\theta) - u\,\Delta\theta + \nabla\cdot u\,\nabla\theta + \nabla^\perp(u\cdot\nabla^\perp\theta)\right) = 0\,.
\end{aligned} \tag{51}
$$

Now we use the general vector identifies

$$
\nabla\nabla\cdot B + \nabla^\perp\nabla^\perp\cdot B = \Delta B\,, \tag{52}
$$
$$
\nabla(u\cdot\nabla\theta) - u\,\Delta\theta + \nabla\cdot u\,\nabla\theta + \nabla^\perp(u\cdot\nabla^\perp\theta) = 2\,\mathrm{Def}\,u\,\nabla\theta\,, \tag{53}
$$

where $\mathrm{Def}\,u = \frac{1}{2}\,(\nabla u + (\nabla u)^T)$, to obtain

$$
(1+\varepsilon\,\zeta)\,\partial_z\hat u - (1+\varepsilon\,\nabla^\perp\cdot b)\,\nabla^\perp\theta + \varepsilon\,\partial_z\theta\,\Delta B = 2\varepsilon\nu\,\mathrm{Def}\,u\,\nabla\theta\,. \tag{54}
$$

Recalling the definition of b and using the identity

$$
(\nabla^\perp\cdot\hat u)J + 2\,\mathrm{Def}\,\hat u = 2\,\nabla\hat u\,, \tag{55}
$$

where J denotes the standard 2×2 symplectic matrix, we obtain

$$(1+\varepsilon\,\zeta)\,\partial_z^2 \hat{U} + \varepsilon\,\partial_z\theta\,\Delta B - 2\varepsilon\nu\,\nabla\partial_z \hat{U}\,\nabla\theta = (1-2\varepsilon\lambda\,\Delta\Theta)\,\nabla^\perp\theta + 2\varepsilon\nu\,\mathrm{Def}\,\bar{u}\,\nabla\theta\,. \tag{56}$$

We note that the balance relation (56) together with the advection of potential temperature and vorticity transport equation (49) imply (48).

When $\nu \neq 0$, it is better to write out an equation for B, so that the third order horizontal derivative on Θ implicit in the second term of (56) appears on the domain side rather than on the range side of an elliptic operator. Our final expression for the horizontal balance relation then reads

$$\begin{aligned}
&(1+\varepsilon\,\zeta)\,\partial_z^2 B + \varepsilon\nu\,\partial_z\theta\,\Delta B - 2\varepsilon\nu\,\nabla\partial_z B\,\nabla\theta \\
&\quad = \Big(\nu - 2\lambda - 2\lambda\varepsilon\,\zeta - 2\varepsilon\lambda\nu\,\Delta\Theta\Big)\,\nabla^\perp\theta + 2\varepsilon\nu^2\,\mathrm{Def}\,\bar{u}\,\nabla\theta + 4\varepsilon\lambda\nu\,\nabla g\,\nabla\theta \\
&\quad \equiv F_3(\nabla^2\Theta, \nabla\theta, \nabla u)\,.
\end{aligned} \tag{57}$$

For fixed u and θ, equation (57) is a linear second order PDE in B. It is elliptic as long as the matrix

$$\Lambda(\zeta,\theta) = \begin{pmatrix} \varepsilon\nu\,\partial_z\theta & 0 & -\varepsilon\nu\,\partial_{x_1}\theta \\ 0 & \varepsilon\nu\,\partial_z\theta & -\varepsilon\nu\,\partial_{x_2}\theta \\ -\varepsilon\nu\,\partial_{x_1}\theta & -\varepsilon\nu\,\partial_{x_2}\theta & 1+\varepsilon\,\zeta \end{pmatrix} \tag{58}$$

is positive definite, or, equivalently,

$$\nu\,\partial_z\theta > 0 \qquad \text{and} \qquad 1 + \varepsilon\left(\zeta - \frac{\nu|\nabla\theta|^2}{\partial_z\theta}\right) > 0\,. \tag{59}$$

Moreover, the operator on the left-hand side of (57) is uniformly elliptic whenever the bounds (59) hold uniformly in space, i.e.,

$$\inf_{\boldsymbol{x}\in\Omega} \nu\,\partial_z\theta \geq \mu_1 > 0 \qquad \text{and} \qquad \inf_{\boldsymbol{x}\in\Omega} 1 + \varepsilon\left(\zeta - \frac{\nu|\nabla\theta|^2}{\partial_z\theta}\right) \geq \mu_2 > 0\,. \tag{60}$$

Thus, assuming $\nu > 0$, uniform ellipticity holds so long as the fluid is stably stratified and provided ζ and horizontal gradient of θ are not too large.

6 Conservation of energy and potential vorticity

As the balance model Lagrangian is invariant under time translation, the model possesses a conserved energy of the form

$$H_{\mathrm{BM}} = \int_\Omega \frac{\delta \ell_{\mathrm{BM}}}{\delta \boldsymbol{u}} \cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} - \ell_{\mathrm{BM}}(\boldsymbol{u},\theta) = \int_\Omega \varepsilon\left(\tfrac{1}{2}\,|\bar{u}|^2 + \lambda\,|\nabla^\perp\Theta|^2\right) - \theta\,z\,\mathrm{d}\boldsymbol{x}\,. \tag{61}$$

The symmetry of the balance model Lagrangian under particle relabeling leads to a material conservation law for the balance model *potential vorticity*. It can be derived geometrically as follows. First note that the abstract Euler–Poincaré equation (10) can be written as

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\boldsymbol{m} + \frac{\delta \ell}{\delta \theta}\, \mathrm{d}\theta = \mathrm{d}\tilde{\phi}\,, \tag{62}$$

where d denotes the exterior derivative. Taking the exterior (wedge) product between the exterior derivative of (62) and $\mathrm{d}\theta$, we find that

$$\begin{aligned} 0 &= \mathrm{d}\big((\partial_t + \mathcal{L}_{\boldsymbol{u}})\boldsymbol{m} + \frac{\delta \ell}{\delta \theta}\, \mathrm{d}\theta - \mathrm{d}\tilde{\phi}\big) \wedge \mathrm{d}\theta \\ &= (\partial_t + \mathcal{L}_{\boldsymbol{u}})(\mathrm{d}\boldsymbol{m} \wedge \mathrm{d}\theta) - \mathrm{d}\boldsymbol{m} \wedge \mathrm{d}(\partial_t + \mathcal{L}_{\boldsymbol{u}})\theta \\ &= (\partial_t + \mathcal{L}_{\boldsymbol{u}})(\mathrm{d}\boldsymbol{m} \wedge \mathrm{d}\theta)\,, \end{aligned} \tag{63}$$

where we used the commutativity of Lie and exterior derivatives in the second equality and the advection of θ in the third equality. In three dimensions, we can identify this conservation law with material advection of the scalar quantity

$$q = *(\mathrm{d}\boldsymbol{m} \wedge \mathrm{d}\theta)\,, \tag{64}$$

where $*$ denotes the Hodge dual operator. Indeed, writing $\mu = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}z$ to denote the canonical volume form on Ω, we have $\mathrm{d}\boldsymbol{m} \wedge \mathrm{d}\theta = q\mu$, so that

$$0 = (\partial_t + \mathcal{L}_{\boldsymbol{u}})(q\mu) = \mu\,(\partial_t + \mathcal{L}_{\boldsymbol{u}})q + q\,\mathcal{L}_{\boldsymbol{u}}\mu\,. \tag{65}$$

Since the flow is volume preserving and μ is non-degenerate, this proves that $\partial_t q + \mathcal{L}_{\boldsymbol{u}} q = 0$, i.e., q is conserved on fluid particles.

In the language of vector calculus, expression (64) for the potential vorticity reads

$$q = (\boldsymbol{\nabla} \times \boldsymbol{m}) \cdot \boldsymbol{\nabla}\theta\,. \tag{66}$$

In fact, the derivation above corresponds to taking the inner product of the curl of the Euler–Poincaré equations (13) with $\boldsymbol{\nabla}\theta$ and manipulating correspondingly; the advantage of the abstract approach is that commuting exterior and Lie derivative in traditional notation is not linked to any intrinsic operation, thus requires tedious verification.

To write out the explicit balance model potential vorticity, note that

$$\boldsymbol{\nabla} \times \begin{pmatrix} R \\ 0 \end{pmatrix} = \boldsymbol{e}_z\,, \quad \boldsymbol{\nabla} \times \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \quad \boldsymbol{\nabla} \times \begin{pmatrix} g \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_z g^{\perp} \\ \nabla^{\perp} \cdot g \end{pmatrix}, \tag{67}$$

where once again $\omega = \nabla^\perp \cdot \bar{u}$, and insert g from (28) into the expression (40) for $\boldsymbol{m}_{\rm BM}$, so that

$$q = \partial_z^2 \Theta \, (1 + \varepsilon \, \omega + \varepsilon \nu \, \Delta\Theta) - \varepsilon \nu \, |\nabla \partial_z \Theta|^2 \equiv F(D^2\Theta; \omega) \, . \tag{68}$$

This expression is a fully nonlinear second order equation for Θ. The standard solvability condition for equations of this type is uniform ellipticity of F (see, e.g., [4, 18]). Ellipticity for a nonlinear equation is essentially defined as ellipticity of its linearization. To be precise, we follow the definition given in [18]. Let $\mathcal{U}$ be an open subset $\mathbb{R}^n$, $\mathcal{S}^n$ the space of symmetric $n \times n$ matrices, and F a real-valued function on $\mathcal{U} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ that is C^1 in its last argument. Then the operator

$$F(x, \Theta, D\Theta, D^2\Theta) \tag{69}$$

is *elliptic* on some subset $\Gamma \subset \mathcal{U} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ provided the matrix

$$[F_{ij}] = \frac{\partial F}{\partial M}(x, \alpha, \beta, M) \equiv \left[\frac{\partial F(x, \alpha, \beta, M)}{\partial M_{ij}} \right] \tag{70}$$

is positive definite for all $(x, \alpha, \beta, M) \in \Gamma$. Furthermore, F is *uniformly elliptic* on Γ provided there exist positive functions λ_1, λ_2 on Γ and a constant μ such that for any $\xi \in \mathbb{R}^n$,

$$\lambda_1 \, |\xi|^2 \leq F_{ij} \, \xi_i \, \xi_j \leq \lambda_2 \, |\xi|^2 \qquad \text{with } \lambda_2/\lambda_1 \leq \mu \, . \tag{71}$$

When F is defined by (68), we find that

$$\frac{\partial F}{\partial D^2\Theta} = \Lambda(\zeta, \theta) \, . \tag{72}$$

Thus, the ellipticity conditions for the nonlinear equation (68) and for the balance relation (57) coincide and are both given by (59). Moreover, if condition (60) is satisfied and, in addition, there is a uniform upper bound of the form

$$1 + \varepsilon \, \zeta + \varepsilon \nu \, \partial_z \theta \leq \mu_3 \, , \tag{73}$$

then F is uniformly elliptic.

In fact, (68) has the form of a generalized Monge–Ampère equation. To expose this structure, we formally set $|\nabla| = \sqrt{-\Delta}$ and define the generalized Hessian

$$\operatorname{Hess} \Theta = \begin{pmatrix} \Delta\Theta & \partial_z |\nabla| \Theta \\ \partial_z |\nabla| \Theta & \partial_z^2 \Theta \end{pmatrix} \, , \tag{74}$$

so that

$$q = \partial_z^2 \Theta \, (1 + \varepsilon \, \omega) + \varepsilon \nu \, \det \operatorname{Hess} \Theta \, . \tag{75}$$

Finally, defining

$$\Psi = \frac{1}{\sqrt{\varepsilon \nu}} \, \Delta^{-1} (1 + \varepsilon \, \omega) + \sqrt{\varepsilon \nu} \, \Theta \, , \tag{76}$$

we have

$$q = \det \operatorname{Hess} \Psi \, . \tag{77}$$

Thus, the potential vorticity inversion relation is of Monge–Ampère type, albeit in a non-standard form. For Monge–Ampère equations coming from a standard Hessian, Cafferelli [3] proved $W^{2,p}$ regularity provided the right hand side is sufficiently close to a constant; also see [7]. We expect that similar results hold for (77), where Caffarelli's condition would impose a restriction on the deviation from linear stratification. We remark that the passage from (68) to (77) is not possible for periodic lateral boundary conditions as the Laplacian is not invertible on constants; however, this is not an issue as everything can be done using (68) directly.

In the context of classical semigeostrophic theory, Cullen and Purser [5] have shown that for the system to be dynamically stable, the matrix $\operatorname{Hess} \Psi$ defining the PV has to be positive definite, which is equivalent to a certain convexity condition for Ψ. We expect this condition to be applicable here as well. However, we stress that the nature of our results is different from classical semigeostrophic theory, which asserts existence of possibly discontinuous front-type solutions under only natural stability conditions for the front, see Shutts and Cullen [17]. Here we need to impose, additionally, a smallness assumption on gradients of θ which is more restrictive, but appears to allow a stronger notion of solutions.

When the Coriolis parameter $f = \nabla^\perp \cdot R$ varies horizontally in space, the balance model equations become rather cumbersome and shall not be pursued further in this paper. However, we note that the conservation laws remain simple even in this case and outline the principal changes compared to the case of non-varying f.

When $f = f(x)$ but non-dimensionalized as before, the thermal wind relation becomes

$$\partial_z u = \frac{1}{f} \, \nabla^\perp \theta \, , \tag{78}$$

so that we define the thermal wind as g/f with g is still given by (28). Proceeding as in Section 4, we use the transformation vector field

$$\boldsymbol{v} = \frac{1}{2} \begin{pmatrix} \hat{u}^\perp \\ \nabla^\perp \cdot \hat{U} \end{pmatrix} - \frac{\lambda}{f} \begin{pmatrix} g^\perp \\ \nabla^\perp \cdot G \end{pmatrix} \, , \tag{79}$$

which yields the balance model Lagrangian

$$\ell_{\rm BM} = \int_\Omega R \cdot u + \theta \, z \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega \frac{1}{2} \, u \cdot (\bar{u} + g) + \frac{\lambda}{f} \, g \cdot (\hat{u} - g) \, \mathrm{d}\boldsymbol{x}. \tag{80}$$

Therefore, (40) becomes

$$\boldsymbol{m}_{\rm BM} \equiv \frac{\delta \ell_{\rm BM}}{\delta \boldsymbol{u}} = \boldsymbol{R} + \varepsilon \, \boldsymbol{p}_{\rm h} \,, \tag{81}$$

where

$$p = \bar{u} + \nu_f \, g \qquad \text{with} \qquad \nu_f = \frac{1}{2} + \frac{\lambda}{f} \,. \tag{82}$$

Substituting the expressions for $\ell_{\rm BM}$ and $\boldsymbol{m}_{\rm BM}$ into the general formulas of the balance model energy (61) and potential vorticity (66), we obtain, respectively,

$$H_{\rm BM} = \int_\Omega \varepsilon \left(\frac{1}{2} \, |\bar{u}|^2 + \frac{\lambda}{f} \, |\nabla^\perp \Theta|^2 \right) - \theta \, z \, \mathrm{d}\boldsymbol{x} \tag{83}$$

and

$$q = \partial_z^2 \Theta \, (f + \varepsilon \, \omega + \varepsilon \, \nu_f \, \Delta\Theta) - \varepsilon \, \nu_f \, |\nabla \partial_z \Theta|^2 \,. \tag{84}$$

PV inversion is given by the generalized Monge-Ampère problem

$$q = \partial_z^2 \Theta \, (f + \varepsilon \, \omega) + \varepsilon \, \nu_f \, \det \operatorname{Hess} \Theta \,. \tag{85}$$

or, equivalently,

$$\nu_f \, q = \det \operatorname{Hess} \Psi \,, \tag{86}$$

where

$$\Psi = \frac{1}{\sqrt{\varepsilon}} \, \Delta^{-1} (f + \varepsilon \, \omega) + \sqrt{\varepsilon} \, \Theta \,. \tag{87}$$

Hence, potential vorticity satisfies an equation of the same type as in the case of constant Coriolis parameter provided ν_f and f are bounded away from zero. In particular, the invertibility conditions do not change when f is a small perturbations of a constant.

We remark that a spatially variable Coriolis parameter in a shallow water setting leads to similarly simple balance model energy and potential vorticity [13].

7 Balance model evolution and special cases

We are now ready to collect the complete set of balance model equations. It comprises the two transport equations

$$\partial_t \theta + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = 0 \,, \tag{88a}$$

$$\partial_t \omega + \bar{u} \cdot \nabla \omega = F_2 \,, \tag{88b}$$

the two kinematic linear second order equations

$$(1 + \varepsilon\, \zeta)\, \partial_z^2 B + \varepsilon \nu \, \partial_z \theta \, \Delta B - 2 \varepsilon \nu \, \nabla \partial_z B \, \nabla \theta = F_3(\nabla^2 \Theta, \nabla \theta, \nabla u) \,, \tag{88c}$$

$$\Delta \psi = \omega \,, \tag{88d}$$

and a number of direct relations between the variables,

$$\Theta = \int_{-H}^{0} \frac{z}{H} \, \theta \, \mathrm{d}z + \int_{-H}^{z} \theta \, \mathrm{d}z' \,, \tag{88e}$$

$$\bar{u} = \nabla^\perp \psi \,, \tag{88f}$$

$$\hat{u} = \nu^{-1} \left(\partial_z B + 2 \lambda \, \nabla^\perp \Theta \right) , \tag{88g}$$

$$u_3 = -\nu^{-1} \, \nabla \cdot B \,, \tag{88h}$$

$$u = \bar{u} + \hat{u} \,. \tag{88i}$$

The balance relation (88c) is elliptic provided $\nu > 0$, the fluid is stably stratified so that $\partial_z \theta > 0$, the vorticity ζ and horizontal gradient $\nabla \theta$ are not too large, and ε is sufficiently small; it is complemented with homogeneous Dirichlet conditions on the vertical boundaries. The nonlinearities F_2 and F_3 are defined in (49) and (57), respectively.

We conjecture that this system is well-posed at least locally in time, which is supported by a simple count of derivatives. Suppose that initially outside of a compact domain of sufficiently large radius the fluid is at rest, horizontal gradients of θ vanish, and, furthermore, that both θ and Θ are of Sobolev class H^{s+1}_{loc} and $\boldsymbol{u} \in H^s$ with s large enough so that H^{s-1} is a topological algebra. Then, initially, $\zeta = \omega + \nu \, \Delta \Theta \in H^{s-1}$ and, by elliptic regularity, $B \in H^{s+1}$ from the balance relation (88c).

We now need to assure that these regularity classes are maintained for at least a short interval of time. This is not obvious from the two advection equations (88a) and (88b), which each seem to be one derivative short. However, we note that ζ also satisfies the evolution equation (48), which implies that class H^{s-1} is maintained provided that θ and B remain at the indicated level of the hierarchy. Via the balance relation (88c), the condition on B is satisfied provided θ, Θ, and u remain at the indicated level of the hierarchy.

To guarantee H^{s+1}-regularity of Θ, we note that $q \in H^{s-1}$ initially, so by PV advection with the H^s vector field $\boldsymbol{u}$, it remains in this class for a short interval of time. We then use elliptic regularity of the Monge–Ampère problem (68) directly to conclude that $\Theta \in H^{s+1}$.

Achieving H^{s+1}-regularity of θ is more subtle. We need to assume, in addition, that initially $q \in H^s$, this better regularity class is also maintained over short intervals of time provided $\boldsymbol{u} \in H^s$. Now differentiate (68) with respect to z to find that

$$\partial_z^2\theta \, (1 + \varepsilon \, \omega + \varepsilon\nu \, \Delta\Theta) + \partial_z\theta \, (1 + \varepsilon\nu \, \Delta\theta) - 2\varepsilon\nu \, \nabla\theta \cdot \nabla\partial_z\theta = \partial_z q \, . \tag{89}$$

Then, provided $\partial_z\theta > 0$ and $1 + \varepsilon \, \omega + \varepsilon\nu \, \Delta\Theta > 0$, the required $\theta \in H^{s+1}$ follows by linear elliptic regularity. Further, the positivity conditions are maintained by continuity for at least a short interval of time.

The argument above is not a full proof, but rather a plausibility check which demonstrates that we have brought the equations into a form that is consistent with standard strategies of proof. A full proof of local well-posedness will require a careful study of the solvability conditions for the PV inversion and balance relation.

Finally, it is not clear whether solutions to (88) have a chance to persist for all times subject only to conditions on the initial data. As written, it is not given that the ellipticity condition (59) will persist. Hence, we believe that an argument toward global well-posedness necessarily requires additional structural insight. In this context, we note that the balance model Hamiltonian (61) provides a global *a priori* bound on $\nabla\Theta$ in L^2.

We now discuss three special choices for the model parameter λ.

Case $\lambda = \frac{1}{2}$: stratified L_1 dynamics

The case $\lambda = \frac{1}{2}$ is special because then the transformation vector field $\boldsymbol{v} = O(\varepsilon)$, hence the transformation is an identity to order $O(\varepsilon^2)$. With this choice, we obtain the "missing" stratified L_1 model. In this case $\nu = 1$, $B = \hat{U} - G$, and the balance relation (88c) simplifies to

$$(1 + \varepsilon \, \zeta) \, \partial_z^2 B + \varepsilon \, \partial_z\theta \, \Delta B - 2\varepsilon \, \nabla\partial_z B \nabla\theta = 2\varepsilon \, (\nabla(\bar{u} + g))^T \nabla\theta \, . \tag{90}$$

We remark that the L_1 model could have been derived within Salmon's [15] procedure simply replacing $\hat{u}$ in the primitive equation Lagrangian by g. However, as we saw in Section 5, the bare Euler–Poincaré equations are not immediately useful. The new insight we suggest here is that we should not try to "simplify" the L_1 model by further approximation, but rather understand it as an already fine model by leveraging the conservation of potential vorticity, implicit in its derivation, as a dynamic variable.

Case $\lambda = 0$: more regularity for $\hat{U}$

In the special case when $\lambda = 0$ so that $\nu = \frac{1}{2}$, $b = \frac{1}{2}\,\hat{u}$ and $B = \frac{1}{2}\,\hat{U}$, the balance relation (88c) reads

$$(1 + \varepsilon\,\zeta)\,\partial_z^2 \hat{U} + \tfrac{\varepsilon}{2}\,\partial_z\theta\,\Delta\hat{U} - \varepsilon\,\nabla\partial_z\hat{U}\,\nabla\theta = \nabla^\perp\theta + \varepsilon\,J\;\mathrm{Def}\,\bar{u}\,\nabla^\perp\theta\,. \tag{91}$$

We note that in this case, the right hand side of the balance relation contains only first horizontal derivatives, so $\hat{U}$ is more regular than in the generic case. Whether this leads to a different functional setting for the entire system, as in [6] for the shallow water analog, or is helpful for a numerical implementation remains open.

Case $\lambda = -\frac{1}{2}$: Salmon's LSG model

Finally, when $\lambda = -\frac{1}{2}$ so that $\nu = 0$ and $B = G$, (54) reads

$$(1 + \varepsilon\,\nabla^\perp \cdot \bar{u})\,\partial_z^2 \hat{U} = (1 + \varepsilon\,\Delta\Theta)\,\nabla^\perp\theta - \varepsilon\,\partial_z\theta\,\Delta G\,. \tag{92}$$

It is clear that this balance relation is not elliptic, thus $\hat{u}$ is expected to lose three horizontal derivatives with respect to Θ and one horizontal derivative with respect to q. Hence, we cannot expect that the regularity class of Θ persists for any time $t > 0$; the problem appears to be ill posed.

8 Conclusions

In this paper, we have shown that the method of degenerate variational asymptotics applies to a stratified fluid described by the primitive equations in semigeostrophic scaling. The construction is in many respects parallel to the well-explored case of rapidly rotating shallow water.

Specifically, we derived a family of balance model for stratified flow which conserve energy and potential vorticity. We rewrote the model equations in a robust form, emphasizing the importance of advection of potential vorticity, which appears to be a crucial ingredient for the proof of local well-posedness. It is worth noting that both balance relation and PV inversion are elliptic subject to identical physically reasonable conditions.

Subject to stable stratification and a small data assumption, the models are formally in the same regularity class as their shallow water counterparts. We believe techniques we had used to prove well-posedness for balance models of shallow water [12, 6] can be adapted to the stratified case, hence expect existence and uniqueness of solutions for as long as stable stratification and smallness of data assumptions persist.

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