# Multivector and multivector matrix inverses in real Clifford algebras

Eckhard Hitzer

*College of Liberal Arts, International Christian University, 181-8585 Mitaka, Tokyo, Japan.*

Stephen Sangwine

*School of Computer Science and Electronic Engineering, University of Essex, Colchester, CO4 3SQ, UK.*

**Abstract**

We show how to compute the inverse of multivectors in finite dimensional real Clifford algebras $Cl(p, q)$. For algebras over vector spaces of fewer than six dimensions, we provide explicit formulae for discriminating between divisors of zero and invertible multivectors, and for the computation of the inverse of a general invertible multivector. For algebras over vector spaces of dimension six or higher, we use isomorphisms between algebras, and between multivectors and matrix representations with multivector elements in Clifford algebras of lower dimension. Towards this end we provide explicit details of how to compute several forms of isomorphism that are essential to invert multivectors in arbitrarily chosen algebras. We also discuss briefly the computation of the inverse of matrices of multivectors by adapting an existing textbook algorithm for matrices to the multivector setting, using the previous results to compute the required inverses of individual multivectors.

*Keywords:* Clifford algebra, inverse, multivector matrix algebra
*2000 MSC:* 15A66, 11E88, 15A09, 68-04

## 1. Introduction

For many computations in matrix algebra it is important to invert the entries of a matrix. This equally applies to computations with matrices, which have general Clifford algebra multivectors as their entries (Clifford matrices). We find that, similar to the linear algebra of real matrices, especially provided that individual matrix entries are invertible[1], then non-singular Clifford matrices (with linearly independent row or column vectors of Clifford algebra elements) are also generally invertible. In our current work, we first establish algebraic product formulas for the direct computation of the Clifford product inverses of multivectors in Clifford algebras $Cl(p, q)$, $n = p + q \leq 5$, excluding the case of divisors of zero. In each case a scalar is computed, which if zero (or numerically close to zero), indicates that the multivector is a divisor of zero, otherwise it is not. Then we show how to further employ Clifford algebra matrix isomorphisms to compute the inverses of multivectors of quadratic Clifford algebras $Cl(p, q)$ of any finite dimension. All numeric calculations to verify results in this work, including extensive numerical tests with random multivector matrices, have been carried out with the Clifford Multivector Toolbox for Matlab [12, 13], developed by the authors since 2013 and first released publicly in 2015, and the algorithms developed in this paper are now available publicly in the toolbox.

There has been previous research [14][2], which uses Clifford algebra matrix isomorphisms (which we do not use in the cases $Cl(p, q)$, $p + q < 6$) to deduce part of the results of this paper, but essentially leaves the proofs to the reader by stating as proof for Theorem 5 in [14] only that: *The proof is by direct calculation.* Because of the relevance of the results in [14], we add footnotes with detailed comparisons with our results.

The results presented in [3][3] include even more of our results for vector space dimensions $1 \leq n \leq 5$. But the method employed in [3] is different from ours, in that there extensive tables of products of explicit basis elements ordered by grade are employed. Furthermore special non-standard notation, is introduced, which makes the

*Email addresses:* `hitzer@icu.ac.jp` (Eckhard Hitzer), `sjs@essex.ac.uk` (Stephen Sangwine)

[1]This assumption is most likely to be overly restrictive, since also in linear algebra matrices containing zero entries, continue to be invertible, as long as the matrix as a whole is non-singular, i.e. provided that the determinant is non-zero.

[2]We thank the anonymous reviewers for pointing out reference [14].

[3]Paper (preprint) [3] appeared as a reference in [14].

interpretation of the method difficult. For Clifford algebras over six dimensional vector spaces in Section III.B, page 9, of [3], systematic search of 262144 *possible expressions*, seems to imply the use of computer algebra software, without being specified explicitly in the text or references. One may infer from that, that possibly the results for lower dimensions may also have involved the use of computer algebra software. On the one hand, it is reassuring, that our (independently obtained closed expression pencil and paper) results do not differ from [3], on the other hand, we emphasize, that we only use standard notation, and standard properties of Clifford algebras[4] for the compact manipulation of multivector expressions, without the need to employ explicit multivector definitions in terms of basis blades. Even though at the beginning of each section we specify an orthonormal basis, but only in order to provide a straightforward strategy of implementation of all equations on standard computer algebra platforms with symbolic or numerical Clifford algebra software packages.

The results in Sections 4 to 7 for all $Cl(p,q)$, $n = p + q$, $1 \leq n \leq 4$, have been verified by symbolic computations with the MAPLE 17 package CLIFFORD [1]. The corresponding worksheets are available as supplementary material for download. For $n = 5$, our Mac Book Pro computer with Intel Core i7, 2.3GHz, and 16GB RAM, was not able to complete the symbolic computations within a reasonable time of more than two hours. Therefore we abandoned the symbolic computer algebra verification of the $n = 5$ case.

Beyond the inversion of general multivectors, and of matrices of multivectors, there are still some open questions, e.g., about the role of multivector elements (in vector space dimensions greater than five) in matrices, which are divisors of zero, etc. But our treatment shows what future theoretical research should focus on, in order to resolve remaining special cases.

The paper is structured as follows. Section 2 gives some background on Clifford algebras, standard involutions and similar maps later employed in the paper. Sections 3 to 8 show how to discriminate between divisors of zero and invertible multivectors and to compute the inverse of general (not divisors of zero) multivectors in Clifford algebras $Cl(p,q)$, $n = p + q \leq 5$. Section 9 applies Clifford algebra matrix isomorphisms to compute the inverse of general (not divisors of zero) multivectors in Clifford algebras $Cl(p,q)$ of any finite dimension. Finally, Section 10 briefly discusses the use of multivector inversion for the inversion of Clifford matrices. We also provide appendices for the detailed explanation of some of the isomorphisms used in the present work, where we judged that implementation relevant detail, additional to the description in [9], may be desirable. Note: the online version of this paper contains coloured figures and mathematical symbols.

## 2. Clifford algebras

**Definition 2.1** (Clifford's geometric algebra [4, 9, 5, 7, 13])**.** Let $\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n\}$, with $n = p + q$, $e_k^2 = Q(e_k)1 = \varepsilon_k$, $\varepsilon_k = +1$ for $k = 1, \ldots, p$, $\varepsilon_k = -1$ for $k = p + 1, \ldots, n$, be an *orthonormal base* of the non-degenerate inner product vector space $(\mathbb{R}^{p,q}, Q)$, $Q$ the non-degenerate quadratic form, with a geometric product according to the multiplication rules

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k, l = 1, \ldots n, \tag{2.1}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$. This non-commutative product and the additional axiom of *associativity* generate the $2^n$-dimensional Clifford geometric algebra $Cl(p,q) = Cl(\mathbb{R}^{p,q}) = Cl_{p,q} = \mathcal{G}_{p,q} = \mathbb{R}_{p,q}$ over $\mathbb{R}$. For Euclidean vector spaces ($n = p$) we use $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $Cl(n) = Cl(n,0)$. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$, $e_\emptyset = 1$, the unity in the Clifford algebra, forms a graded (blade) basis of $Cl(p,q)$. The grades $k$ range from 0 for scalars, 1 for vectors, 2 for bivectors, $s$ for $s$-vectors, up to $n$ for pseudoscalars. The quadratic space $(\mathbb{R}^{p,q}, Q)$ is embedded into $Cl(p,q)$ as a subspace, which is identified with the subspace of 1-vectors. All linear combinations of basis elements of grade $k$, $0 \leq k \leq n$, form the subspace $Cl^k(p,q) \subset Cl(p,q)$ of $k$-vectors. The general elements of $Cl(p,q)$ are real linear combinations of basis blades $e_A$, called Clifford numbers, multivectors or hypercomplex numbers.

In general $\langle A \rangle_k$ denotes the grade $k$ part of $A \in Cl(p,q)$. Following [5, 9, 7], the parts of grade 0, $k+s$, $s-k$, and $k - s$, respectively, of the geometric product of a $k$-vector $A_k \in Cl(p,q)$ with an $s$-vector $B_s \in Cl(p,q)$

$$A_k * B_s := \langle A_k B_s \rangle_0, \qquad A_k \wedge B_s := \langle A_k B_s \rangle_{k+s}, \qquad A_k \rfloor B_s := \langle A_k B_s \rangle_{s-k}, \qquad A_k \lfloor B_s := \langle A_k B_s \rangle_{k-s}, \tag{2.2}$$

[4]See, e.g., textbooks like [5] and [9].

are called *scalar product*, *outer product*, *left contraction* and *right contraction* respectively. They are bilinear products mapping a pair of multivectors to a resulting product multivector in the same algebra. The outer product is also associative, the scalar product and the contractions are not.

Every $k$-vector $B$ that can be written as the outer product $B = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots \wedge \boldsymbol{b}_k$ of $k$ vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in \mathbb{R}^{p,q}$ is called a *simple* $k$-vector or *blade*.

Multivectors $M \in Cl(p,q)$ have $k$-vector parts ($0 \leq k \leq n$): scalar part $Sc(M) = \langle M \rangle = \langle M \rangle_0 = M_0 \in \mathbb{R}$, vector part $\langle M \rangle_1 \in \mathbb{R}^{p,q}$, bi-vector part $\langle M \rangle_2 \in \bigwedge^2 \mathbb{R}^{p,q}$, ..., and pseudoscalar part $\langle M \rangle_n \in \bigwedge^n \mathbb{R}^{p,q}$

$$M = \sum_A M_A \boldsymbol{e}_A = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n \,. \tag{2.3}$$

The following *involutions* are of importance. First the *main (grade) involution*

$$\widehat{M} = \sum_{k=0}^{n} (-1)^k \langle M \rangle_k. \tag{2.4}$$

Next *reversion*, which reverses the order of every product of vectors

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{k(k-1)/2} \langle M \rangle_k. \tag{2.5}$$

Moreover we have the *Clifford conjugation*, the composition of main involution and reversion

$$\overline{M} = \widetilde{\widehat{M}} = \widehat{\widetilde{M}} = \sum_{k=0}^{n} (-1)^{k(k+1)/2} \langle M \rangle_k. \tag{2.6}$$

Finally we introduce *grade specific maps*, which change the sign of specified grade parts only

$$m_{\bar{j},\bar{k}}(M) = M - 2(\langle M \rangle_j + \langle M \rangle_k), \quad 0 \leq j, k \leq n, \tag{2.7}$$

which can be easily generalized to any set of indices $0 \leq j_1 < j_2 < ... < j_l \leq n$.

Clifford algebras $Cl(p,q)$, $n = p + q$ are isomorphic to $2^n \times 2^n$ square matrices [9]. Therefore, as in the theory of square matrices [8], the right inverse $x_r$ of a multivector $x \in Cl(p,q)$, if it exists, will necessarily be identical to the left inverse $x_l$, i.e., for every multivector $x \in Cl(p,q)$ not being a divisor of zero, we have

$$x_l x = x x_r = 1, \qquad x_l = x_r. \tag{2.8}$$

## 3. Inverse of real and complex numbers

Every nonzero real number $\alpha \in \mathbb{R}$ has a multiplicative inverse

$$\alpha^{-1} = \frac{1}{\alpha}, \qquad \alpha^{-1}\alpha = \alpha\alpha^{-1} = 1. \tag{3.1}$$

Similarly every nonzero complex number $x = a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$, has an inverse, because the product of $x$ with its complex conjugate $\overline{x}$ is a nonzero positive real number scalar[5]

$$x\overline{x} = (a+ib)\overline{(a+ib)} = (a+ib)(a-ib) = a^2 + ib(-ib) + iba - aib = a^2 + i(-i)b^2 = a^2 + b^2 \in \mathbb{R}_+ \setminus \{0\}. \tag{3.2}$$

Multiplication with the complex conjugate $\overline{x}$ and division by the product scalar $x\overline{x}$ allows us therefore to define the inverse of a non-zero complex number

$$x^{-1} = \frac{\overline{x}}{x\overline{x}}, \qquad x^{-1}x = xx^{-1} = \frac{x\overline{x}}{x\overline{x}} = 1. \tag{3.3}$$

[5] The scalar $x\overline{x}$ is zero iff $x = 0$.

## 4. Inverse of elements of Clifford algebras of one-dimensional vector spaces

We consider the one-dimensional vector spaces $\mathbb{R}^{1,0}$, $\mathbb{R}^{0,1}$ and their Clifford algebras $Cl(1,0)$, $Cl(0,1)$. In both cases there is only one unit basis vector $e_1$

$$\{e_1\}, \qquad e_1^2 = \varepsilon_1 = \pm 1, \tag{4.1}$$

with $\varepsilon_1 = +1$ for $\mathbb{R}^{1,0}$, and $\varepsilon_1 = -1$ for $\mathbb{R}^{0,1}$. A general element $Cl(1,0)$ or $Cl(0,1)$ can be expressed as $x = a + be_1$, $a, b \in \mathbb{R}$. The product of $x$ with its main (grade) involution $\widehat{x} = a + b\widehat{e_1} = a - be_1$ gives a generally non-zero scalar[6]

$$x\widehat{x} = \widehat{x}x = (a + be_1)(a - be_1) = a^2 - be_1be_1 + be_1a - abe_1 = a^2 - \varepsilon_1 b^2 \in \mathbb{R}, \tag{4.2}$$

which is only zero[7] for $\varepsilon_1 = +1$ (i.e. for $Cl(1,0)$) together with $a = \pm b$. In this case $x$ is called a *divisor of zero*. In all other cases we can define the left- and right inverse of a general multivector in $Cl(1,0)$ or $Cl(0,1)$ therefore as

$$x^{-1} = \frac{\widehat{x}}{x\widehat{x}}, \qquad x^{-1}x = xx^{-1} = 1. \tag{4.3}$$

## 5. Inverse of elements of Clifford algebras of two-dimensional vector spaces

We consider the three two-dimensional vector spaces $\mathbb{R}^{2,0}$, $\mathbb{R}^{1,1}$, and $\mathbb{R}^{0,2}$, and their Clifford algebras[8] $Cl(2,0)$, $Cl(1,1)$, and $Cl(0,2)$. In all three cases, the vector basis can be written in the form of two orthonormal vectors each squaring to $\pm 1$.

$$\{e_1, e_2\}, \qquad e_k^2 = \varepsilon_k = \pm 1, \qquad k \in \{1, 2\}. \tag{5.1}$$

A general element of $Cl(2,0)$, $Cl(1,1)$, $Cl(0,2)$ can be expressed as

$$x = a + \vec{v} + \beta e_{12}, \qquad a, \beta \in \mathbb{R}, \qquad e_{12}^2 = -\varepsilon_1\varepsilon_2, \tag{5.2}$$

and $\vec{v} \in \mathbb{R}^{2,0}$, $\mathbb{R}^{1,1}$ or $\mathbb{R}^{0,2}$. $e_{12} = e_1e_2$ is the unit oriented bivector of the respective Clifford algebra. The product of $x$ with its Clifford conjugate $\overline{x}$, which is a composition of reversion and grade involution, gives the real scalar[9] result

$$\begin{aligned} x\overline{x} &= (a + \vec{v} + \beta e_{12})\overline{(a + \vec{v} + \beta e_{12})} \\ &= (a + \vec{v} + \beta e_{12})(a - \vec{v} - \beta e_{12}) \\ &= a^2 - \vec{v}^2 - \beta^2 e_{12}^2 - a\vec{v} + \vec{v}a - a\beta e_{12} + \beta e_{12}a - \vec{v}\beta e_{12} - \beta e_{12}\vec{v} \\ &= a^2 - \vec{v}^2 + \beta^2\varepsilon_1\varepsilon_2 \in \mathbb{R}, \end{aligned} \tag{5.3}$$

where we used the fact that in $Cl(2,0)$, $Cl(1,1)$ and $Cl(0,2)$, vectors and bivectors anticommute, i.e.,

$$-\vec{v}\beta e_{12} - \beta e_{12}\vec{v} = -\beta\vec{v}e_{12} + \beta\vec{v}e_{12} = 0. \tag{5.4}$$

It is easy to check that the product $\overline{x}x$ leads to the same result $\overline{x}x = a^2 - \vec{v}^{\,2} + \beta^2\varepsilon_1\varepsilon_2$. Except for the special cases, that $a^2 - \vec{v}^{\,2} + \beta^2\varepsilon_1\varepsilon_2 = 0$, when such a non-zero $x$ is again a *divisor of zero*, we otherwise have a left- and right inverse[10] for every multivector in $x \in Cl(2,0)$, $Cl(1,1)$ or $Cl(0,2)$

$$x^{-1} = \frac{\overline{x}}{x\overline{x}}, \qquad x^{-1}x = xx^{-1} = 1. \tag{5.5}$$

[6] In Theorem 5 of [14], it is shown, that the scalar $x\widehat{x}$ is the determinant of $x$ in a matrix representation of minimal dimension.

[7] Note, that the case $\varepsilon_1 = -1$ is isomorphic to the above treated case of complex numbers. The case of $\varepsilon_1 = 1$ is that of *hyperbolic numbers*.

[8] Note, that $Cl(0,2)$ is isomorphic to quaternions, and in this case there are no divisors of zero.

[9] In Theorem 5 of [14], it is shown, that the scalar $x\overline{x}$ is the determinant of $x$ in a matrix representation of minimal dimension.

[10] Note that for $Cl(0,2) \cong \mathbb{H}$ with $\vec{v}^{\,2} \leq 0$, $\varepsilon_1\varepsilon_2 = +1$, there are as expected no divisors of zero.

## 6. Inverse of elements of Clifford algebras of three-dimensional vector spaces

We now assume Clifford algebras $Cl(p,q)$ with $p+q=3$, an orthonormal basis $\{e_1,e_2,e_3\}$ and unit vector squares $\{\varepsilon_1=\pm1,\varepsilon_2=\pm1,\varepsilon_3=\pm1\}$, basis bivectors $\{e_{12},e_{23},e_{31}\}$, and central[11] pseudoscalar $i=e_1e_2e_3$, $i^2=-\varepsilon_1\varepsilon_2\varepsilon_3$. A general element $x$ in $Cl(p,q)$ can be expressed as a sum of a scalar $a\in\mathbb{R}$, a vector $\vec{v}\in\mathbb{R}^{p,q}$, a bivector $A\in Cl^2(p,q)$ and a central trivector $\beta i$, $\beta\in\mathbb{R}$, i.e.

$$x=a+\vec{v}+A+\beta i. \tag{6.1}$$

We first compute the product of $x$ with its Clifford conjugate $\overline{x}$ and obtain

$$\begin{aligned}
x\overline{x} &= (a+\vec{v}+A+\beta i)\overline{(a+\vec{v}+A+\beta i)}\\
&= (a+\vec{v}+A+\beta i)(a-\vec{v}-A+\beta i)\\
&= a^2-\vec{v}^2-A^2+\beta i^2-a\vec{v}-aA+a\beta i+\vec{v}a-\vec{v}A+\vec{v}\beta i+Aa-A\vec{v}+A\beta i+\beta ia-\beta i\vec{v}-\beta iA\\
&= a^2-\vec{v}^2-A^2+\beta i^2-(A\vec{v}+\vec{v}A)+2a\beta i\\
&= a^2-\vec{v}^2-A^2+\beta i^2-2\vec{v}\wedge A+2a\beta i\\
&= r_0+ir_3\in\mathbb{R}+i\mathbb{R},
\end{aligned} \tag{6.2}$$

where we used $A\vec{v}+\vec{v}A=2\vec{v}\wedge A$, and where we set the two scalars $r_0=a^2-\vec{v}^2-A^2+\beta i^2\in\mathbb{R}$, $r_3=-2(\vec{v}\wedge A)i^{-1}+2\beta\in\mathbb{R}$. We further multiply $x\overline{x}=r_0+ir_3$ by its reverse to obtain

$$\begin{aligned}
x\overline{x}(x\overline{x})^\sim &= (r_0+ir_3)(r_0+ir_3)^\sim=(r_0+ir_3)(r_0-ir_3)=r_0^2-r_3^2i^2\\
&= (a^2-\vec{v}^2-A^2+\beta i^2)^2-4(-(\vec{v}\wedge A)i^{-1}+2\beta)^2i^2\in\mathbb{R}.
\end{aligned} \tag{6.3}$$

We further observe that the real scalar

$$x\overline{x}(x\overline{x})^\sim=x\overline{x}\,\widetilde{\overline{x}}\,\widetilde{x}=x\overline{x}\,\widehat{x}\,\widetilde{x}\in\mathbb{R} \tag{6.4}$$

is a product of $x\in Cl(p,q)$ times its Clifford conjugate $\overline{x}$ times its main involution $\widehat{x}$ times its reverse $\widetilde{x}$. All $x$ with $x\overline{x}(x\overline{x})^\sim=(a^2-\vec{v}^2-A^2+\beta i^2)^2-4(-(\vec{v}\wedge A)i^{-1}+2\beta)^2i^2=0$ are divisors of zero. In all other cases we can define the right inverse with respect to the geometric product as

$$x_r^{-1}=\frac{\overline{x}\,\widehat{x}\,\widetilde{x}}{x\overline{x}\,\widehat{x}\,\widetilde{x}},\qquad xx_r^{-1}=1. \tag{6.5}$$

We note that the result $x\overline{x}\in\mathbb{R}+i\mathbb{R}$ of (6.2) is *central*[12] in $Cl(p,q)$ with $p+q=3$. Therefore its reverse

$$(x\overline{x})^\sim=\widehat{x}\,\widetilde{x} \tag{6.6}$$

is also central. Furthermore, inspection of (6.2) shows that the central

$$x\overline{x}=\overline{x}x. \tag{6.7}$$

The reverse of (6.7) gives the identity of the two central valued products

$$\widehat{x}\,\widetilde{x}=\widetilde{x}\,\widehat{x}. \tag{6.8}$$

Using the two central products $x\overline{x}$ and $(x\overline{x})^\sim=\widehat{x}\,\widetilde{x}$, and their two symmetries (6.7) and (6.8), we can now re-express the real scalar (6.4) in 16 different ways[13] as products of $x$, $\overline{x}$, $\widehat{x}$, and $\widetilde{x}$

$$\begin{aligned}
x\overline{x}(x\overline{x})^\sim &= x\overline{x}\,\widehat{x}\,\widetilde{x}=x\overline{x}\,\widetilde{x}\,\widehat{x}=\overline{x}\,x\widehat{x}\,\widetilde{x}=\overline{x}\,x\widetilde{x}\,\widehat{x}=\widehat{x}\,\widetilde{x}\,x\overline{x}=\widehat{x}\,\widetilde{x}\,\overline{x}\,x=\widetilde{x}\,\widehat{x}\,x\overline{x}=\widetilde{x}\,\widehat{x}\,\overline{x}\,x\\
&= x\widehat{x}\,\widetilde{x}\,\overline{x}=x\widetilde{x}\,\widehat{x}\,\overline{x}=\overline{x}\,\widehat{x}\,\widetilde{x}\,x=\overline{x}\,\widetilde{x}\,\widehat{x}\,x=\widehat{x}\,x\overline{x}\,\widetilde{x}=\widehat{x}\,\overline{x}\,x\widetilde{x}=\widetilde{x}\,x\overline{x}\,\widehat{x}=\widetilde{x}\,\overline{x}\,x\widehat{x}.
\end{aligned} \tag{6.9}$$

Careful inspection of (6.9) reveals, that $x$ is in four cases on the very left, and in four cases on the very right, respectively. In analogy to (6.5) we therefore find in total four right inverses and four left inverses for $x\in Cl(p,q)$, $p+q=3$, respectively, which are all identical, based on the centrality of the products $x\overline{x}$ and $(x\overline{x})^\sim=\widehat{x}\,\widetilde{x}$, and their two symmetries (6.7) and (6.8).

[11] This means that the unit trivector $i$ in the algebras $Cl(p,q)$ with $p+q=3$ commutes with all other elements of the respective algebra.

[12] This interesting remark can be found in [14].

[13] The identity $x\overline{x}\,\widehat{x}\,\widetilde{x}=x\widetilde{x}\,\widehat{x}\,\overline{x}$ can also be found in Theorem 5 of [14].

**Theorem 6.1.** *There are four identical left and right inverses*[14] *for $x \in Cl(p,q)$, $p+q=3$, iff the scalar*[15] *of (6.9) is non-zero, i.e. iff $x\overline{x}\widehat{x}\widetilde{x} \neq 0$,*

$$x_r^{-1} = x_l^{-1} = \frac{\overline{x}\widehat{x}\widetilde{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\overline{x}\widetilde{x}\widehat{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\widehat{x}\widetilde{x}\overline{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\widetilde{x}\widehat{x}\overline{x}}{x\overline{x}\widehat{x}\widetilde{x}}. \tag{6.10}$$

As a secondary result, we can similarly read off from (6.9) four forms of left and right inverses for $\overline{x}$, $\widehat{x}$, and $\widetilde{x}$, respectively.

**Corollary 6.2.** *There are four identical (left and right) inverses for each of the three involutions of $x \in Cl(p,q)$, $p+q=3$, i.e. for $\overline{x}$, $\widehat{x}$, and $\widetilde{x}$, respectively, iff the scalar of (6.9) is non-zero, i.e. iff $x\overline{x}\widehat{x}\widetilde{x} \neq 0$ :*

$$\overline{x}_r^{-1} = \overline{x}_l^{-1} = \frac{x\widehat{x}\widetilde{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{x\widetilde{x}\widehat{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\widehat{x}\widetilde{x}x}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\widetilde{x}\widehat{x}x}{x\overline{x}\widehat{x}\widetilde{x}}, \tag{6.11}$$

$$\widehat{x}_r^{-1} = \widehat{x}_l^{-1} = \frac{\widetilde{x}x\overline{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\widetilde{x}\overline{x}x}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{x\overline{x}\widetilde{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\overline{x}x\widetilde{x}}{x\overline{x}\widehat{x}\widetilde{x}}, \tag{6.12}$$

*and*

$$\widetilde{x}_r^{-1} = \widetilde{x}_l^{-1} = \frac{\widehat{x}x\overline{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\widehat{x}\overline{x}x}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{x\overline{x}\widehat{x}}{x\overline{x}\widehat{x}\widetilde{x}} = \frac{\overline{x}x\widehat{x}}{x\overline{x}\widehat{x}\widetilde{x}}. \tag{6.13}$$

*Remark* 6.3. Note, that these inverses for $\overline{x}$, $\widehat{x}$, and $\widetilde{x}$, respectively, can also be derived directly from (6.10) using the reverse, main (grade) involution, and Clifford conjugation, together with applying the centrality of the products $x\overline{x}$ and $(x\overline{x})^\sim = \widehat{x}\widetilde{x}$, and their two symmetries (6.7) and (6.8).

*Remark* 6.4. Numerical tests with random multivectors in the four Clifford algebras $Cl(p,q)$, $p+q=3$, confirm that the above right inverses $x_r$ and left inverses $x_l$ agree, which is also expected from the general theory, see (2.8).

Since sixteen permutations of the product of $x$, $\overline{x}$, $\widehat{x}$, and $\widetilde{x}$ give the same scalar (6.9), we may ask the question of what the other eight permutations ($4! - 16 = 8$) give. We now show, that the eight other product permutations result in eight distinct non-scalar values.

We begin with

$$x\widehat{x}\overline{x}\widetilde{x} = x\widehat{x}\,(x\widehat{x})^\sim, \tag{6.14}$$

where

$$\begin{aligned} x\widehat{x} &= (a + \vec{v} + A + \beta i)(a + \vec{v} + A + \beta i)^\wedge \\ &= (a + \vec{v} + A + \beta i)(a - \vec{v} + A - \beta i) \\ &= a^2 - \vec{v}^{\,2} + A^2 - \beta i^2 - a\vec{v} + aA - a\beta i + \vec{v}a + \vec{v}A - \vec{v}\beta i + Aa - A\vec{v} - A\beta i + \beta i a - \beta i\vec{v} + \beta i A \\ &= a^2 - \vec{v}^{\,2} + A^2 - \beta i^2 + (\vec{v}A - A\vec{v}) + 2aA - 2\beta\vec{v}i \\ &= a^2 - \vec{v}^{\,2} + A^2 - \beta i^2 + 2\vec{v}\cdot A + 2aA - 2\beta\vec{v}i \\ &= s_0 + \vec{s}_1 + s_2, \end{aligned} \tag{6.15}$$

with scalar $s_0 = a^2 - \vec{v}^{\,2} + A^2 - \beta i^2$, vector $\vec{s}_1 = 2(\vec{v}\cdot A)$, and bivector $s_2 = 2aA - 2\beta\vec{v}i$. We therefore obtain

$$\begin{aligned} x\widehat{x}\overline{x}\widetilde{x} &= x\widehat{x}\,(x\widehat{x})^\sim = (s_0 + \vec{s}_1 + s_2)(s_0 + \vec{s}_1 + s_2)^\sim = (s_0 + \vec{s}_1 + s_2)(s_0 + \vec{s}_1 - s_2) \\ &= s_0^2 + \vec{s}_1^{\,2} - s_2^2 + 2s_0\vec{s}_1 - (\vec{s}_1 s_2 - s_2\vec{s}_1) = s_0^2 + \vec{s}_1^{\,2} - s_2^2 + 2s_0\vec{s}_1 - 2\vec{s}_1\cdot s_2 = t_0 + \vec{t}_1, \end{aligned} \tag{6.16}$$

with scalar $t_0 = s_0^2 + \vec{s}_1^{\,2} - s_2^2$, and vector $\vec{t}_1 = 2s_0\vec{s}_1 - 2\vec{s}_1\cdot s_2$. Next, we have

$$\widehat{x}x\widetilde{x}\overline{x} = (x\widehat{x}\overline{x}\widetilde{x})^\wedge = t_0 - \vec{t}_1. \tag{6.17}$$

Then, we compute

$$x\widetilde{x}\overline{x}\widehat{x} = x\widetilde{x}\widehat{(\overline{x}x)}, \tag{6.18}$$

[14] The first and the last form in (6.10) can also be found in Theorem 6 of [14].

[15] In [14], it is shown, that the scalar $x\overline{x}\widehat{x}\widetilde{x}$ of our (6.9) is the determinant of $x$ in a matrix representation of minimal dimension.

with the first factor

$$
\begin{aligned}
x\widetilde{x} &= (a+\vec{v}+A+\beta i)(a+\vec{v}+A+\beta i)^{\sim} \\
&= (a+\vec{v}+A+\beta i)(a+\vec{v}-A-\beta i) \\
&= a^2+\vec{v}^{\,2}-A^2-\beta i^2+a\vec{v}-aA-a\beta i+\vec{v}a-\vec{v}A-\vec{v}\beta i+Aa+A\vec{v}-A\beta i+\beta ia+\beta i\vec{v}-\beta iA \\
&= a^2+\vec{v}^{\,2}-A^2-\beta i^2+2a\vec{v}-(\vec{v}A-A\vec{v})-2\beta Ai \\
&= a^2+\vec{v}^{\,2}-A^2-\beta i^2+2a\vec{v}-2\vec{v}\cdot A-2\beta Ai \\
&= z_0+\vec{z}_1,
\end{aligned}
\tag{6.19}
$$

with scalar $z_0 = a^2+\vec{v}^{\,2}-A^2-\beta i^2$ and vector $\vec{z}_1 = 2a\vec{v}-2\vec{v}\cdot A-2\beta Ai$. Analogously, we compute

$$
\widetilde{x}\,x = z_0+\vec{z}_1{}', \qquad \vec{z}_1{}' = 2a\vec{v}+2\vec{v}\cdot A-2\beta Ai \neq \vec{z}_1. \tag{6.20}
$$

We therefore obtain

$$
\begin{aligned}
x\widetilde{x}\,\overline{x}\,\widehat{x} = x\widetilde{x}\widehat{(\widetilde{x}\,x)} &= (z_0+\vec{z}_1)\widehat{(z_0+\vec{z}_1{}')} = (z_0+\vec{z}_1)(z_0-\vec{z}_1{}') \\
&= z_0^2-\vec{z}_1\cdot\vec{z}_1{}'+z_0(\vec{z}_1-\vec{z}_1{}')-\vec{z}_1\wedge\vec{z}_1{}' = u_0+\vec{u}_1+u_2,
\end{aligned}
\tag{6.21}
$$

with scalar $u = z_0^2-\vec{z}_1\cdot\vec{z}_1{}'$, vector $\vec{u}_1 = z_0(\vec{z}_1-\vec{z}_1{}')$ and bivector $u_2 = -\vec{z}_1\wedge\vec{z}_1{}'$. We further have

$$
\widehat{x}\,\overline{x}\,\widetilde{x}\,x = (x\widetilde{x}\,\overline{x}\,\widehat{x})^{\wedge} = u_0-\vec{u}_1+u_2, \tag{6.22}
$$

and

$$
\overline{x}\,\widehat{x}\,x\widetilde{x} = (x\widetilde{x}\,\overline{x}\,\widehat{x})^{\sim} = u_0+\vec{u}_1-u_2, \tag{6.23}
$$

and

$$
\widetilde{x}\,x\widehat{x}\,\overline{x} = \overline{(x\widetilde{x}\,\overline{x}\,\widehat{x})} = u_0-\vec{u}_1-u_2. \tag{6.24}
$$

Furthermore, we apply (6.15) to compute

$$
\begin{aligned}
\overline{x}\,\widetilde{x}\,x\widehat{x} = (x\widehat{x})^{\sim}x\widehat{x} &= (s_0+\vec{s}_1+s_2)^{\sim}(s_0+\vec{s}_1+s_2) = (s_0+\vec{s}_1-s_2)(s_0+\vec{s}_1+s_2) \\
&= s_0^2+\vec{s}_1^{\,2}-s_2^2+2s_0\vec{s}_1+\vec{s}_1s_2-s_2\vec{s}_1 = s_0^2+\vec{s}_1^{\,2}-s_2^2+2s_0\vec{s}_1+2\vec{s}_1\cdot s_2 = t_0+\vec{t}_1{}',
\end{aligned}
\tag{6.25}
$$

with vector $\vec{t}_1{}' = +2s_0\vec{s}_1+2\vec{s}_1\cdot s_2 \neq \vec{t}_1$. Therefore $\overline{x}\,\widetilde{x}\,x\widehat{x} \neq x\widehat{x}\,\overline{x}\,\widetilde{x}$. We finally compute

$$
\widetilde{x}\,\overline{x}\,\widehat{x}\,x = \widehat{(\overline{x}\,\widetilde{x}\,x\widehat{x})} = t_0-\vec{t}_1{}' \neq \widehat{x}\,x\widetilde{x}\,\overline{x}, \tag{6.26}
$$

because $\vec{t}_1{}' \neq \vec{t}_1$.

We therefore summarize, that the eight product permutations $x\widehat{x}\,\overline{x}\,\widetilde{x}$, $\widehat{x}\,x\widetilde{x}\,\overline{x}$, $x\widetilde{x}\,\overline{x}\,\widehat{x}$, $\widehat{x}\,\overline{x}\,\widetilde{x}\,x$, $\overline{x}\,\widehat{x}\,x\widetilde{x}$, $\widetilde{x}\,x\widehat{x}\,\overline{x}$, $\overline{x}\,\widetilde{x}\,x\widehat{x}$, and $\widetilde{x}\,\overline{x}\,\widehat{x}\,x$, are generally eight distinct (pair wise different) non-scalar combinations of scalars, vectors and bivectors.

## 7. Inverse of elements of Clifford algebras of four-dimensional vector spaces

We now assume Clifford algebras $Cl(p,q)$ with $p+q=4$, an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ with unit vector squares $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4 \in \{-1,+1\}$, basis bivectors $\{e_{12},e_{23},e_{31},e_{14},e_{24},e_{34}\}$, and trivectors $e_1e_2e_3$, $e_1e_2e_4$, $e_2e_3e_4$, $e_1e_3e_4$ (duals of the four basis vectors), and pseudoscalar $i = e_1e_2e_3e_4$, $i^2 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$. A general element $x$ in $Cl(p,q)$ can be expressed as a sum of a scalar $\alpha \in \mathbb{R}$, a vector $\vec{a} \in \mathbb{R}^{p,q}$, a bivector $A \in Cl^2(p,q)$, a trivector $\vec{b}i \in Cl^3(p,q)$, and a pseudoscalar $\beta i$, i.e.

$$
x = \alpha+\vec{a}+A+\vec{b}i+\beta i. \tag{7.1}
$$

We first compute the product of $x$ with its Clifford conjugate $\overline{x}$ and obtain

$$
\begin{aligned}
x\overline{x} &= (\alpha + \vec{a} + A + \vec{b}i + \beta i)\overline{(\alpha + \vec{a} + A + \vec{b}i + \beta i)} \\
&= (\alpha + \vec{a} + A + \vec{b}i + \beta i)(\alpha - \vec{a} - A + \vec{b}i + \beta i) \\
&= \alpha^2 - \vec{a}^2 - A^2 + \vec{b}i\vec{b}i + \beta^2 i^2 - \alpha\vec{a} - \alpha A + \alpha\vec{b}i + \alpha\beta i + \vec{a}\alpha + \vec{a}A + \vec{a}\vec{b}i + \vec{a}\beta i \\
&\quad + A\alpha - A\vec{a} + A\vec{b}i + A\beta i + \vec{b}i\alpha - \vec{b}i\vec{a} - \vec{b}iA + \vec{b}i\beta i + \beta i\alpha - \beta i\vec{a} - \beta iA + \beta i\vec{b}i \\
&= \alpha^2 - \vec{a}^2 - A^2 + \vec{b}i\vec{b}i + \beta^2 i^2 - (\vec{a}A + A\vec{a}) + (\vec{a}\vec{b} + \vec{b}\vec{a})i + 2\beta\vec{a}i \\
&= \alpha^2 - \vec{a}^2 - \langle A^2\rangle_0 + (-\vec{b}^2 + \beta^2)i^2 - 2\vec{a}\wedge A + 2\beta\vec{a}i - 2\vec{b}\cdot Ai + 2\vec{a}\cdot\vec{b}i + 2\alpha\beta i - \langle A^2\rangle_4 \\
&= r_0 + r_3 + r_4,
\end{aligned}
\tag{7.2}
$$

with scalar $r_0 = \alpha^2 - \vec{a}^2 - \langle A^2\rangle_0 + (-\vec{b}^2 + \beta^2)i^2 \in \mathbb{R}$, trivector $r_3 = -2\vec{a}\wedge A + 2\beta\vec{a}i - 2\vec{b}\cdot Ai \in Cl^3(p,q)$, and pseudoscalar $r_4 = 2\vec{a}\cdot\vec{b}i + 2\alpha\beta i - \langle A^2\rangle_4 \in Cl^4(p,q)$. For the expression reduction in (7.2) we freely use that

$$
\vec{a}i = -i\vec{a}, \qquad \vec{b}i = -i\vec{b}, \qquad Ai = iA. \tag{7.3}
$$

Now we define a special map, which negates the sign of the components of grade three and grade four of a multivector, but preserves the sign of all other grade parts

$$
m_{\bar{3},\bar{4}}(x) = \alpha + \vec{a} + A - \vec{b}i - \beta i \tag{7.4}
$$

With the application of the map $m_{\bar{3},\bar{4}}$ to $x\overline{x}$ we can moreover write

$$
x\overline{x}\, m_{\bar{3},\bar{4}}(x\overline{x}) = (r_0 + r_3 + r_4)(r_0 - r_3 - r_4) = r_0^2 - r_3^2 - r_4^2 \in \mathbb{R}, \tag{7.5}
$$

where we used $r_3 r_4 = -r_4 r_3$. We finally obtain the real scalar[16]

$$
x\overline{x}\, m_{\bar{3},\bar{4}}(x\overline{x}) = [\alpha^2 - \vec{a}^2 - \langle A^2\rangle_0 + (-\vec{b}^2 + \beta^2)i^2]^2 - [-2\vec{a}\wedge A + 2\beta\vec{a}i - 2\vec{b}\cdot Ai]^2 - [2\vec{a}\cdot\vec{b}i + 2\alpha\beta i - \langle A^2\rangle_4]^2 \in \mathbb{R}. \tag{7.6}
$$

If $x\overline{x}\, m_{\bar{3},\bar{4}}(x\overline{x})$ is zero, then $x$ is a divisor of zero. In all other cases, the right inverse[17] of $x \in Cl(p,q)$ can therefore be defined as

$$
x_r^{-1} = \frac{\overline{x}\, m_{\bar{3},\bar{4}}(x\overline{x})}{x\overline{x}\, m_{\bar{3},\bar{4}}(x\overline{x})}, \qquad x x_r^{-1} = 1. \tag{7.7}
$$

We note that similarly to the above derivation of the right inverse, it is possible to derive a left inverse. The result is

$$
x_l^{-1} = \frac{m_{\bar{3},\bar{4}}(\overline{x}x)\,\overline{x}}{m_{\bar{3},\bar{4}}(\overline{x}x)\,\overline{x}x}, \qquad x_l^{-1}x = 1, \tag{7.8}
$$

where we can verify, that the real scalar in the denominator is the same as in (7.5), which also appears in the denominator of (7.7)

$$
x\overline{x}\, m_{\bar{3},\bar{4}}(x\overline{x}) = (r_0 + r_3 + r_4)(r_0 - r_3 - r_4) = (r_0 - r_3 - r_4)(r_0 + r_3 + r_4) = m_{\bar{3},\bar{4}}(\overline{x}x)\,\overline{x}x. \tag{7.9}
$$

*Remark* 7.1. Numerical tests with random multivectors in all five Clifford algebras $Cl(p,q)$, $p+q=4$, confirm that right inverse $x_r$ and left inverse $x_l$ agree, even though algebraically, this is not obvious from equations (7.7) and (7.8), but it is again in agreement with the general theory, see (2.8).

[16] We note that the scalar of (7.6) agrees with the second determinant expression of $x$ (in a matrix representation of minimal dimension) for $n=4$ in Theorem 5 of [14], because $m_{\bar{3},\bar{4}}(x\overline{x}) = [(x\overline{x})^{\wedge}]^{\triangledown} = (\widehat{x\overline{x}})^{\triangledown}$, where [14] on page 12 defines $x^{\triangledown}$ as changing the sign of the grade four part of $x$.

[17] Note that the inverse in (7.7) agrees with the second expression for the inverse given in Theorem 6 of [14], because $m_{\bar{3},\bar{4}}(x\overline{x}) = (\widehat{x\overline{x}})^{\triangledown}$.

## 8. Inverse of elements of Clifford algebras of five-dimensional vector spaces

We now assume Clifford algebras $Cl(p,q)$ with $p+q=5$, an orthonormal basis $\{e_1,e_2,e_3,e_4,e_5\}$ and unit vector squares $e_k^2=\varepsilon_k=\pm 1, 1\leq k\leq 5$. The pseudoscalar five-vector is given by $i=e_1e_2e_3e_4e_5$, and is central, i.e. it commutes with every multivector in $Cl(p,q)$. A general multivector element $x\in Cl(p,q)$, is given by

$$x=\alpha+\vec{a}+A+Bi+\vec{b}i+\beta i, \tag{8.1}$$

with real scalars $\alpha,\beta\in\mathbb{R}$, vectors $\vec{a},\vec{b}\in\mathbb{R}^{p,q}$, bivectors $A,B\in Cl^2(p,q)$, trivector $Bi\in Cl^3(p,q)$, 4-vector $\vec{b}i\in Cl^4(p,q)$ and pseudoscalar $\beta i$.

We will perform the computation in three steps:

1. $w=x\overline{x}$,
2. $y=w\widetilde{w}$,
3. $z=y\,m_{\overline{1}\overline{4}}(y)$,

where the map $m_{\overline{1}\overline{4}}()$ negates only the signs of the vector and 4-vector parts. As our first step we compute

$$\begin{aligned}
w=x\overline{x}&=(\alpha+\vec{a}+A+Bi+\vec{b}i+\beta i)\overline{(\alpha+\vec{a}+A+Bi+\vec{b}i+\beta i)}\\
&=(\alpha+\vec{a}+A+Bi+\vec{b}i+\beta i)(\alpha-\vec{a}-A+Bi+\vec{b}i-\beta i)\\
&=\alpha^2-\vec{a}^{\,2}-A^2+BiBi+\vec{b}i\vec{b}i-\beta^2i^2-\alpha\vec{a}-\alpha A+\alpha Bi+\alpha\vec{b}i-\alpha\beta i+\vec{a}\alpha-\vec{a}A+\vec{a}Bi+\vec{a}\vec{b}i-\vec{a}\beta i\\
&\quad+A\alpha-A\vec{a}+ABi+A\vec{b}i-A\beta i+\vec{b}i\alpha-\vec{b}iveca+\vec{b}iA+\vec{b}iBi-\vec{b}i\beta i\\
&\quad+Bi\alpha-Bi\vec{a}-BiA+Bi\vec{b}i-Bi\beta i+\beta i\alpha-\beta i\vec{a}-\beta iA+\beta iBi-\beta i\vec{b}i\\
&=\alpha^2-\vec{a}^{\,2}-\langle A^2\rangle_0+\langle B^2\rangle_0i^2+\vec{b}^{\,2}i^2-\beta^2i^2+2\alpha Bi-2\vec{a}\wedge A+2(\vec{a}\wedge\vec{b})i+(AB-BA)i-2\beta Ai+2(\vec{b}\wedge B)i^2\\
&\quad+2\alpha\vec{b}i+2(\vec{a}\cdot B)i-2\beta\vec{a}i-\langle A^2\rangle_4-2(\vec{b}\cdot A)i+\langle B^2\rangle_4i^2\\
&=r_0+r_3+r_4,
\end{aligned} \tag{8.2}$$

where real scalar $r_0\in\mathbb{R}$, trivector $r_3\in Cl^3(p,q)$, and 4-vector $r_4\in Cl^4(p,q)$ are

$$\begin{aligned}
r_0&=\alpha^2-\vec{a}^{\,2}-\langle A^2\rangle_0+\langle B^2\rangle_0i^2+\vec{b}^{\,2}i^2-\beta^2i^2,\\
r_3&=2\alpha Bi-2\vec{a}\wedge A+2(\vec{a}\wedge\vec{b})i+(AB-BA)i-2\beta Ai+2(\vec{b}\wedge B)i^2,\\
r_4&=2\alpha\vec{b}i+2(\vec{a}\cdot B)i-2\beta\vec{a}i-\langle A^2\rangle_4-2(\vec{b}\cdot A)i+\langle B^2\rangle_4i^2.
\end{aligned} \tag{8.3}$$

Applying duality we redefine the result with the help of a bivector $R_3=r_3i^{-1}$, and a vector $\vec{R}_4=r_4i^{-1}$ to

$$w=x\overline{x}=r_0+r_3+r_4=r_0+R_3i+\vec{R}_4i. \tag{8.4}$$

Secondly, we compute

$$\begin{aligned}
y=w\widetilde{w}&=(r_0+R_3i+\vec{R}_4i)(r_0+R_3i+\vec{R}_4i)^{\sim}\\
&=(r_0+R_3i+\vec{R}_4i)(r_0-R_3i+\vec{R}_4i)\\
&=r_0^2-R_3^2i^2+\vec{R}_4^2i^2-r_0R_3i+r_0\vec{R}_4i+R_3ir_0+R_3\vec{R}_4i^2+\vec{R}_4ir_0-\vec{R}_4R_3i^2\\
&=r_0^2-\langle R_3^2\rangle_0i^2+\vec{R}_4^2i^2-2(\vec{R}_4\cdot R_3)i^2+2r_0\vec{R}_4i-\langle R_3^2\rangle_4i^2\\
&=s_0+\vec{s}_1+\vec{s}_4i,
\end{aligned} \tag{8.5}$$

with real scalar

$$s_0=r_0^2-\langle R_3^2\rangle_0i^2+\vec{R}_4^2i^2\in\mathbb{R}, \tag{8.6}$$

and vector

$$\vec{s}_1=-2(\vec{R}_4\cdot R_3)i^2\in\mathbb{R}^{p,q}, \tag{8.7}$$

and 4-vector

$$\vec{s}_4i=2r_0\vec{R}_4i-\langle R_3^2\rangle_4i^2\in Cl^4(p,q), \tag{8.8}$$

such that $\vec{s}_4$ is indeed a vector

$$\vec{s}_4 = 2r_0\vec{R}_4 - \langle R_3^2\rangle_4 i = 2r_0\vec{R}_4 - (R_3 \wedge R_3)i \in \mathbb{R}^{p,q}. \tag{8.9}$$

Thirdly, we compute

$$\begin{aligned} z &= y\, m_{\bar{1}\bar{4}}(y) = (s_0 + \vec{s}_1 + \vec{s}_4 i)\, m_{\bar{1}\bar{4}}(s_0 + \vec{s}_1 + \vec{s}_4 i) \\ &= (s_0 + \vec{s}_1 + \vec{s}_4 i)(s_0 - \vec{s}_1 - \vec{s}_4 i) \\ &= s_0^2 - \vec{s}_1^{\,2} - \vec{s}_4^{\,2} i^2 - \vec{s}_1\vec{s}_4 i - \vec{s}_4 i \vec{s}_1 \\ &= s_0^2 - \vec{s}_1^{\,2} - \vec{s}_4^{\,2} i^2 - 2(\vec{s}_1 \cdot \vec{s}_4)i. \end{aligned} \tag{8.10}$$

The remaining computation consists in proving that for the above values of $\vec{s}_1$ and $\vec{s}_4$ we have $\vec{s}_1 \cdot \vec{s}_4 = 0$. We compute

$$\vec{s}_1 \cdot \vec{s}_4 = (-2(\vec{R}_4 \cdot R_3)i^2) \cdot (2r_0\vec{R}_4 - \langle R_3^2\rangle_4 i) = -4r_0 i^2[(\vec{R}_4 \cdot R_3) \cdot \vec{R}_4] + 2i^2\langle(\vec{R}_4 \cdot R_3)\langle R_3^2\rangle_4 i\rangle_0. \tag{8.11}$$

For the first term above we have

$$(\vec{R}_4 \cdot R_3) \cdot \vec{R}_4 = \vec{R}_4 \cdot (\vec{R}_4 \cdot R_3) = (\vec{R}_4 \wedge \vec{R}_4) \cdot R_3 = 0. \tag{8.12}$$

It remains for us to analyze the expression

$$\langle(\vec{R}_4 \cdot R_3))\langle R_3^2\rangle_4 i\rangle_0 = \langle(\vec{R}_4 \cdot R_3)\langle R_3^2\rangle_4\rangle_5 i. \tag{8.13}$$

We assume, that $\vec{R}_4^2 \neq 0$, i.e. that $\vec{R}_4$ is not isotropic. Then we can split the bivector $R_3 = R_{3||} + R_{3\perp}$ into a part $R_{3||}$ parallel (containing the projection of $\vec{R}_4$ onto $R_3$), and a part $R_{3\perp}$ completely orthogonal to $\vec{R}_4$ :

$$\begin{aligned} R_{3||} &= (R_3 \cdot \vec{R}_4)\vec{R}_4^{-1}, \qquad R_{3\perp} = (R_3 \wedge \vec{R}_4)\vec{R}_4^{-1}, \\ R_{3||} + R_{3\perp} &= (R_3 \cdot \vec{R}_4 + R_3 \wedge \vec{R}_4)\vec{R}_4^{-1} = R_3\vec{R}_4\vec{R}_4^{-1} = R_3. \end{aligned} \tag{8.14}$$

The part $R_{3\perp}$ is a bivector in the four-dimensional hyperplane subspace $S_4 \subset \mathbb{R}^{p,q}$ defined by the dual $\vec{R}_4^* = \vec{R}_4 i^{-1}$ of $\vec{R}_4$. In this hyperplane $S_4$ the vector $\vec{t} = R_3 \cdot \vec{R}_4 \in S_4$ is orthogonal to $\vec{R}_4$, and we can perform now a further split of $R_{3\perp}$ with respect to $\vec{t}$. This splits the bivector $R_{3\perp}$ into one part $R_{3\perp||}$ containing the projection of $\vec{t}$ onto $R_{3\perp}$ and another part $R_{3\perp\perp}$ perpendicular to both $\vec{R}_4$ and $\vec{t}$. $R_{3\perp\perp}$ is therefore a bivector orthogonal to the plane spanned by $\vec{R}_4$ and $\vec{t}$, it is hence contained in the three-dimensional subspace $S_3 \subset S_4 \subset \mathbb{R}^{p,q}$ given by the dual $(\vec{R}_4\vec{t}\,)^* = (\vec{R}_4\vec{t}\,)i^{-1}$ of the bivector $\vec{R}_4\vec{t}$. $R_{3\perp\perp}$ can therefore be expressed as the geometric product of two orthogonal vectors $\vec{c}, \vec{d} \in S_3$, i.e. $R_{3\perp\perp} = (R_{3\perp} \wedge \vec{t}\,)\vec{t}^{-1} = \vec{c}\vec{d}$. This means

$$R_3 = R_{3||} + R_{3\perp} = R_{3||} + R_{3\perp||} + R_{3\perp\perp} = \vec{t}\vec{R}_4^{-1} + (R_{3\perp} \cdot \vec{t}\,)\vec{t}^{-1} + (R_{3\perp} \wedge \vec{t}\,)\vec{t}^{-1} = \vec{t}\vec{R}_4^{-1} + (R_{3\perp} \cdot \vec{t}\,)\vec{t}^{-1} + \vec{c}\vec{d}, \tag{8.15}$$

which is a linear combination of three simple bivectors. We observe that the inner product $R_3 \cdot \vec{R}_4 = \vec{t}$ is the first vector factor in the above decomposition of $R_3$. We now return to the pseudoscalar expression (8.13) subject of our investigation

$$\langle(\vec{R}_4 \cdot R_3)\langle R_3^2\rangle_4\rangle_5 = \langle\vec{t}(R_3 \wedge R_3)\rangle_5. \tag{8.16}$$

We now compute $R_3 \wedge R_3$ and obtain

$$\begin{aligned} R_3 \wedge R_3 &= [\vec{t}\vec{R}_4^{-1} + (R_{3\perp} \cdot \vec{t})\vec{t}^{-1} + \vec{c}\vec{d}] \wedge [\vec{t}\vec{R}_4^{-1} + (R_{3\perp} \cdot \vec{t})\vec{t}^{-1} + \vec{c}\vec{d}] = 2[\vec{t}\vec{R}_4^{-1}\vec{c}\vec{d} + (R_{3\perp} \cdot \vec{t})\vec{t}^{-1}\vec{c}\vec{d}] \\ &= 2[\vec{t} \wedge \vec{R}_4^{-1} \wedge \vec{c} \wedge \vec{d} + (R_{3\perp} \cdot \vec{t}) \wedge \vec{t}^{-1} \wedge \vec{c} \wedge \vec{d}], \end{aligned} \tag{8.17}$$

where the terms

$$(\vec{t}\vec{R}_4^{-1}) \wedge ((R_{3\perp} \cdot \vec{t})\vec{t}^{-1}) = (\vec{t} \wedge \vec{R}_4^{-1}) \wedge ((R_{3\perp} \cdot \vec{t}) \wedge \vec{t}^{-1}) \tag{8.18}$$

have vanished, because they both contain contain $\vec{t}$ as a factor, when we remember that $\vec{t}^{-1} = \vec{t}/\vec{t}^{\,2}$ is only a rescaled version of $\vec{t}$. In the end result of $R_3 \wedge R_3$ in (8.17) each term has the vector $\vec{t}$ as an outer product factor, therefore

$$\langle\vec{t}(R_3 \wedge R_3)\rangle_5 = \vec{t} \wedge (R_3 \wedge R_3) = 0 \tag{8.19}$$

must be zero. This proves that, if $\vec{R}_4^2 \neq 0$, the second term on the right side of the second equality in (8.11) is zero as well, and therefore $\vec{s}_1 \cdot \vec{s}_4 = 0$.

*Remark* 8.1. We have not yet found a corresponding argument for evaluating (8.13) in the special case of $\vec{R}_4^2 = 0$.

We found therefore that

$$z = y\, m_{\overline{1}\overline{4}}(y) = s_0^2 - \vec{s}_1^{\,2} - \vec{s}_4^{\,2} i^2 \in \mathbb{R}, \tag{8.20}$$

is a real scalar. Successively inserting the expressions for $y$ and then for $w$ in terms of $x$ into $z$ we get the real scalar

$$z = y\, m_{\overline{1}\overline{4}}(y) = w\widetilde{w}\, m_{\overline{1}\overline{4}}(w\widetilde{w}) = x\overline{x}\widetilde{(x\overline{x})}\, m_{\overline{1}\overline{4}}(x\overline{x}\widetilde{(x\overline{x})}) = x\overline{x}\,\widehat{x}\,\widetilde{x}\, m_{\overline{1}\overline{4}}(x\overline{x}\,\widehat{x}\,\widetilde{x}) \in \mathbb{R}. \tag{8.21}$$

If the above scalar $z$ is zero, then $x$ is a divisor of zero. In all other cases we can define a right inverse[18] for $x$ as

$$x_r^{-1} = \frac{\overline{x}\,\widehat{x}\,\widetilde{x}\, m_{\overline{1}\overline{4}}(x\overline{x}\,\widehat{x}\,\widetilde{x})}{x\overline{x}\,\widehat{x}\,\widetilde{x}\, m_{\overline{1}\overline{4}}(x\overline{x}\,\widehat{x}\,\widetilde{x})}, \qquad x x_r^{-1} = 1. \tag{8.22}$$

In analogy to the above derivation the alternative three step strategy

1. $w' = \overline{x}x$,
2. $y' = \widetilde{w'}w'$,
3. $z' = m_{\overline{1}\overline{4}}(y')\, y'$,

leads to a scalar

$$z' = m_{\overline{1}\overline{4}}(\widetilde{x}\,\widehat{x}\,\overline{x}x)\,\widetilde{x}\,\widehat{x}\,\overline{x}x \in \mathbb{R} \tag{8.23}$$

For $z' = 0$, $x$ is a divisor of zero. In all other cases we can define a left inverse for $x \in Cl(p,q)$, $p+q=5$, as

$$x_l^{-1} = \frac{m_{\overline{1}\overline{4}}(\widetilde{x}\,\widehat{x}\,\overline{x}x)\,\widetilde{x}\,\widehat{x}\,\overline{x}}{m_{\overline{1}\overline{4}}(\widetilde{x}\,\widehat{x}\,\overline{x}x)\,\widetilde{x}\,\widehat{x}\,\overline{x}x}, \qquad x_l^{-1}x = 1. \tag{8.24}$$

*Remark* 8.2. Numerical tests with random multivectors in all six Clifford algebras $Cl(p,q)$, $p+q=5$, confirm that the scalars $z$ of (8.21) and $z'$ of (8.23) agree, and furthermore that the right inverse $x_r$ and the left inverse $x_l$ agree as well. Algebraically, this is not obvious from equations (8.21), (8.23), (8.22) and (8.24), but the identity $x_l = x_r$ is in agreement with the general theory, see (2.8).

## 9. Computations of inverses of multivectors in $Cl(p,q)$, $p+q>5$

Figure 1 shows a table of all Clifford algebras $Cl(p,q)$, $p,q \leq 9$. The inverses of multivectors in Clifford algebras with $p+q \leq 5$ have been derived algebraically in Sections 3 to 8. In this section we show how to use Clifford algebra to matrix isomorphisms, combined with the results of Sections 3 to 8, to compute the inverse of every multivector in any Clifford algebra, assuming that the multivector is not a zero-divisor. Four types of isomorphisms, described in chapters 16.3 and 16.4 of [9] will be employed

$$Cl(p,q) \cong Cl(p-4,q+4), \qquad p \geq 4, \tag{9.1}$$
$$Cl(p,q) \cong Cl(p+4,q-4), \qquad q \geq 4, \tag{9.2}$$
$$Cl(p+1,q+1) \cong \mathrm{Mat}(2, Cl(p,q)), \tag{9.3}$$
$$\mathrm{Mat}(2, Cl(p,q)) \cong Cl(p+1,q+1), \tag{9.4}$$

The pair of isomorphisms (9.2) and (9.1) maps between Clifford algebras of the same dimension by changing the signature. Isomorphism (9.1) is the inverse of (9.2). Isomorphism (9.3) maps a Clifford algebra of dimension $2^n$ to an algebra of $2\times 2$ matrices with entries from a Clifford algebra of dimension $2^{n-2}$. Isomorphism (9.4) is the inverse of (9.3). The appendices provide relevant computational details for the isomorphisms (9.1) to (9.4).

[18] Theorem 6 of [14] states a different expression for the inverse, which could principally be tested for equality both by random values for $x$ and by explicit symbolic computation with symbolic computer algebra software. Note that (8.22) is shown by us explicitly in full generality by arguing step by step with the properties of multivectors in $Cl(p,q)$, $p+q=5$, whereas Theorem 5 of [14] essentially leaves the proof to the reader by stating only: *The proof is by direct calculation.*

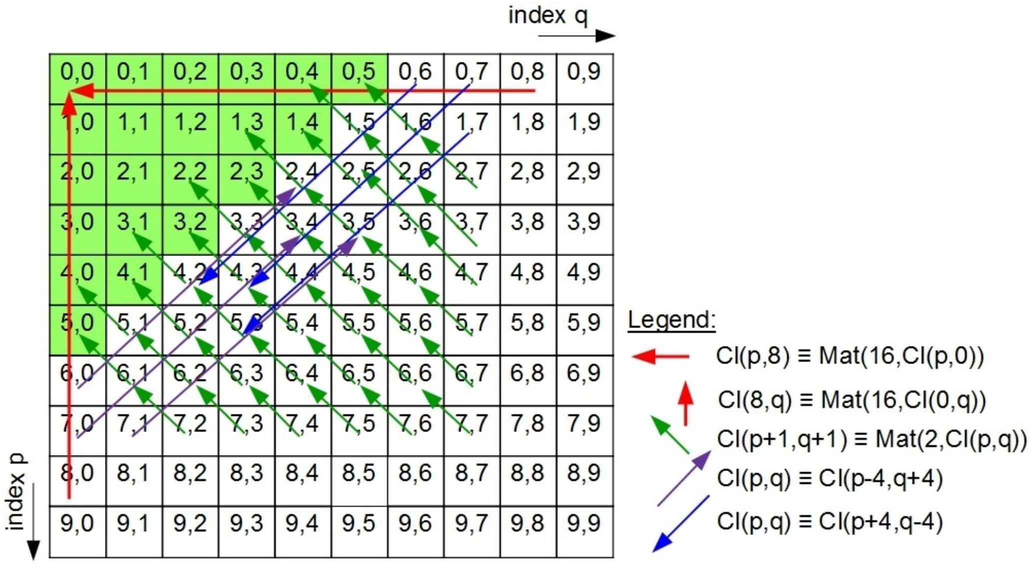


Figure 1: Table of Clifford algebras with $p, q \leq 9$ and isomorphisms. The first index $p$ increases vertically from the top left corner, and the second index $q$ horizontally from the top left corner.

*Computation of the inverse of a multivector in any Clifford algebra*

- For computing the inverse of a multivector in a Clifford algebra with $n = p + q \leq 5$ we apply the results of Sections 3 to 8. See the green fields in Fig. 1.

- For Clifford algebras $Cl(p, q)$ with $p+q > 5$, $p < 8$, $q < 8$, and $(p, q) \notin \{(6, 0), (7, 0), (7, 1), (0, 6), (0, 7), (1, 7)\}$ we simply apply the isomorphism (9.3) between one and five times (green arrows in Fig. 1), in order to iteratively change to a matrix algebra with Clifford algebra entries in $Cl(p, q)$, $p+q \leq 5$. Then we compute the inverse of the multivector in this hybrid representation, applying the results of Sections 3 to 8, and finally apply the isomorphism (9.4) recursively to return to the pure multivector representation of the inverse in the original Clifford algebra $Cl(p, q)$.

- For Clifford algebras $Cl(p, q)$ with $(p, q) \in \{(6, 0), (7, 0), (7, 1)\}$ we first apply the isomorphism (9.1) once (violet arrows in Fig. 1), and then the Clifford algebra to block matrix isomorphism (9.3) once or twice (green arrows in Fig. 1) to reach a matrix representation with entries in $Cl(p, q)$, $p + q \leq 5$. Then results of Sections 3 to 8 are applied to compute the inverse in the hybrid representation, and the isomorphism (9.4) is applied once or twice, and then (9.2), in order change the representation of the inverse back to the original Clifford algebra $Cl(p, q)$.

- For Clifford algebras $Cl(p, q)$ with $(p, q) \in \{(0, 6), (0, 7), (1, 7)\}$ we first apply the isomorphism (9.2) once (blue arrows in Fig. 1), and then the Clifford algebra to block matrix isomorphism (9.3) once or twice (green arrows in Fig. 1) to reach a matrix representation with entries in $Cl(p, q)$, $p + q \leq 5$. Then results of Sections 3 to 8 are applied to compute the inverse in the hybrid representation, and the isomorphism (9.4) is applied once or twice, and then (9.1), in order change the representation of the inverse back to the original Clifford algebra $Cl(p, q)$.

- For Clifford algebras with $p \geq 8$ or $q \geq 8$ we apply the isomorphisms $Cl(p, q) \cong \mathrm{Mat}(16, Cl(p - 8, q)$, respectively $Cl(p, q) \cong \mathrm{Mat}(16, Cl(p, q - 8)$ (see vertical and horizontal red arrows in Fig. 1), which in turn are combinations of the isomorphism $Cl(p, q) \cong Cl(p - 4, q + 4)$ and four times (9.3), or of the isomorphism $Cl(p, q) \cong Cl(p+4, q-4)$ and four times (9.3), respectively. Then we compute the inverse of the multivector by using the methods described in the previous items, and use the opposite isomorphisms $\mathrm{Mat}(16, Cl(p - 8, q) \cong Cl(p, q)$, respectively $\mathrm{Mat}(16, Cl(p, q - 8) \cong Cl(p, q)$, to return to the original Clifford algebra representation.

## 10. Computations of matrix inverses for matrices with multivector elements

We conclude with some remarks on the computation of the inverse of a matrix with multivector elements (a Clifford matrix). We have found that, provided we have the ability to compute the inverse of a general multivector, as described in earlier sections of this paper, and subject to the proviso that divisors of zero within a matrix may cause numerical problems, it is possible to compute the inverse of a matrix of multivectors by adapting a standard block-structured recursive algorithm, as for example given in [8, §0.7.3, p. 18]. The role of the divisors of zero as matrix elements, and their influence on the invertibility of such Clifford matrices should be studied further in the future, in order to establish a comprehensive theory of Clifford matrix inversion.


### Acknowledgements

EH thanks *God* for his wonderful creation: *They speak of the glorious splendor of your majesty – and I will meditate on your wonderful works.* [11]. EH thanks his *dear family* for always supporting his research activities. EH thanks SJS for the excellent scientific cooperation. Both authors thank the Programme Committee of the London Mathematical Society which provided a Scheme 4 (Research in Pairs) Grant Number 41408 which funded a 2-week visit by SJS to the International Christian University in Tokyo to work with EH in March 2015, during which about half of the work reported in this paper was completed, as well as an initial public release of the Clifford Toolbox for Matlab which made this work possible. EH would like to ask readers to respect the Creative Peace License [6]. We thank R. Abłamowicz for assisting us in personal communication when using his MAPLE package CLIFFORD for the symbolic verification computation for the results of Sections 4 to 7.


### References


[1] Abłamowicz R., Fauser B. *CLIFFORD with Bigebra – A Maple Package for Computations with Clifford and Grassmann Algebras.* `http://math.tntech.edu/rafal/` (© 1996-2017).

[2] Dorst L., et al. *Geometric Algebra for Computer Science.* San Francisco: The Morgan Kaufmann Series in Computer Graphics; 2007.

[3] Dadbeh P. *Inverse and Determinant in 0 to 5 Dimensional Clifford Algebra.* `arXiv:1104.0067v2`.

[4] Falcao M. I., Malonek H. R. Generalized Exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions. AIP Conference Proceedings 2007;936:738–741.

[5] Hestenes D., Sobczyk G. Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics. Heidelberg: Springer; 1984.

[6] Hitzer E. *Creative Peace License*, `http://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/`

[7] Hitzer E. Introduction to Clifford's Geometric Algebra. Journal of the Society of Instrument and Control Engineers April 2012;51(4):338–350. Available as preprint: `http://arxiv.org/abs/1306.1660v1` .

[8] Horn R. A., Johnson C. R. *Matrix Analysis*, Cambridge University Press, 1985.

[9] Lounesto P. *Clifford algebras and spinors*, 2nd ed., CUP, 2001.

[10] El Mir G., Saint-Jean C., Berthier M. Conformal Geometry for Viewpoint Change Representation, Adv. Appl. Clifford Algebras, June 2014, Volume 24, Issue 2; 443–463. DOI: 10.1007/s00006-013-0431-3 .

[11] New International Version of the Bible, Psalm 145, verse 5, `http://www.biblegateway.com/`

[12] Sangwine S. J., Hitzer E. Clifford Multivector Toolbox (for MATLAB) 2015–2016, Software library available at: `http://clifford-multivector-toolbox.sourceforge.net/`

[13] Sangwine S. J., Hitzer E. Clifford Multivector Toolbox (for MATLAB). Adv. Appl. Clifford Algebras Online First 19 April 2016; 1–20. DOI: 10.1007/s00006-016-0666-x .

[14] Shirokov D.S. *Concepts of trace, determinant and inverse of Clifford algebra elements.* arXiv:1108.5447v1.

## Appendix A. Isomorphism from $Cl(p,q)$ to $Cl(p-4,q+4)$

We refer the reader for the basic definition of the isomorphism to [9, § 16.4]. The purpose of this appendix is to provide all necessary details for the application of the isomorphisms $Cl(p,q) \cong Cl(p-4,q+4)$, and in Appendix B for the inverse isomorphism $Cl(p,q) \cong Cl(p+4,q-4)$, to each basis blade of $Cl(p,q)$, and a strategy for targeted implementation.

We start with the orthonormal basis $\{e_1, \ldots e_{p-4}, e_{p-4+1}, \ldots, e_{p-4+4}, e_{p+1}, \ldots, e_{p+q}\}$ of the vector space $\mathbb{R}^{p,q}$. We define the four-dimensional Euclidean subspace $V_B = \text{span}[e_{p-4+1}, \ldots, e_{p-4+4}]$, and the $q$-dimensional anti-Euclidean subspace $V_C = \text{span}[e_{p+1}, \ldots e_{p+q}]$, such that $\mathbb{R}^{p,q} = \mathbb{R}^{p-4} \cup V_B \cup V_C$. We denote the Clifford subalgebras generated with the Clifford product over the subspaces $V_B$ and $V_C$: $Cl(V_B)$ and $Cl(V_C)$. A general basis blade $M \in Cl(p,q)$ can then be factored into the following three parts

$$M = ABC, \qquad \text{with} \qquad A \in Cl(p-4,0), \qquad B = Cl(V_B), \qquad C = Cl(V_C). \tag{A.1}$$

For ease of notation we define the following convenient vector labels

$$\underline{e}_1 = e_{p-4+1} = e_{p-3}, \qquad \underline{e}_2 = e_{p-4+2} = e_{p-2}, \qquad \underline{e}_3 = e_{p-4+3} = e_{p-1}, \qquad \underline{e}_4 = e_{p-4+4} = e_p. \tag{A.2}$$

The isomorphism $Cl(p,q)$ to $Cl(p-4,q+4)$ works in three steps, which are also relevant for implementation:

1. Factorize each basis blade of $Cl(p,q)$ according to (A.1).
2. Use only the central factor $B$, which itself is a basis blade in $Cl(V_B) \cong Cl(4,0)$, and map it isomorphically to $Cl(0,4)$ with the following the map provided in (A.11), which expresses the key isomorphism $Cl(4,0) \cong Cl(0,4)$. Note that every factor $B$ is a product of zero, one, two, three or four different vectors in lexical order from the four-dimensional vector subspace basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4\}$. We further denote the quadvector (4-vector) $h = \underline{e}_1\underline{e}_2\underline{e}_3\underline{e}_4$. The isomorphism map is constructed based on

$$\begin{aligned}
\underline{e}'_1 &= \underline{e}_1 h = \underline{e}_1\underline{e}_1\underline{e}_2\underline{e}_3\underline{e}_4 = \underline{e}_2\underline{e}_3\underline{e}_4, \\
\underline{e}'_2 &= \underline{e}_2 h = \underline{e}_2\underline{e}_1\underline{e}_2\underline{e}_3\underline{e}_4 = -\underline{e}_1\underline{e}_3\underline{e}_4, \\
\underline{e}'_3 &= \underline{e}_3 h = \underline{e}_3\underline{e}_1\underline{e}_2\underline{e}_3\underline{e}_4 = \underline{e}_1\underline{e}_2\underline{e}_4, \\
\underline{e}'_4 &= \underline{e}_4 h = \underline{e}_4\underline{e}_1\underline{e}_2\underline{e}_3\underline{e}_4 = -\underline{e}_1\underline{e}_2\underline{e}_3.
\end{aligned} \tag{A.3}$$

This already defines the isomorphic mapping of all trivectors in $Cl(V_B)$ to the four basis vectors $\underline{e}'_i$, $1 \le i \le 4$, of $Cl(0,4) \cong Cl(V'_B)$. And it allows us to compute the mappings of vectors, bivectors and the quadvector $h$ as well. The square of each vector $\underline{e}'_i$, $1 \le i \le 4$, is $-1$:

$$\underline{e}'_i\underline{e}'_i = \underline{e}_i h \underline{e}_i h = -\underline{e}_i\underline{e}_i hh = -h^2 = -1. \tag{A.4}$$

We now compute the maps of bivectors from

$$\underline{e}'_i\underline{e}'_j = \underline{e}_i h \underline{e}_j h = -\underline{e}_i\underline{e}_j hh = -\underline{e}_i\underline{e}_j. \tag{A.5}$$

This in turn allows to compute the maps of trivectors from

$$\underline{e}'_1\underline{e}'_2\underline{e}'_3 = \underline{e}_1 h(-\underline{e}_2\underline{e}_3) = -\underline{e}_1\underline{e}_2\underline{e}_3 h = \underline{e}_4, \tag{A.6}$$

and similarly

$$\underline{e}'_1\underline{e}'_2\underline{e}'_4 = -\underline{e}_1\underline{e}_2\underline{e}_4 h = -\underline{e}_1\underline{e}_2\underline{e}_4\underline{e}_{1234} = \underline{e}_1\underline{e}_2\underline{e}_{1234}\underline{e}_4 = -\underline{e}_{34}\underline{e}_4 = -\underline{e}_3, \tag{A.7}$$

and

$$\underline{e}'_1\underline{e}'_3\underline{e}'_4 = -\underline{e}_1\underline{e}_3\underline{e}_4 h = -\underline{e}_1\underline{e}_3\underline{e}_4\underline{e}_{1234} = \underline{e}_1\underline{e}_{12} = \underline{e}_2, \tag{A.8}$$

and

$$\underline{e}'_2\underline{e}'_3\underline{e}'_4 = -\underline{e}_2\underline{e}_3\underline{e}_4 h = -\underline{e}_2\underline{e}_3\underline{e}_4\underline{e}_{1234} = \underline{e}_1\underline{e}_2\underline{e}_3\underline{e}_4\underline{e}_{234} = -\underline{e}_1. \tag{A.9}$$

And finally the unit quadvector maps to the unit quadvector

$$\underline{e}'_1\underline{e}'_2\underline{e}'_3\underline{e}'_4 = (-\underline{e}_1\underline{e}_2)(-\underline{e}_3\underline{e}_4) = \underline{e}_1\underline{e}_2\underline{e}_3\underline{e}_4 = h. \tag{A.10}$$

We summarize these results for the isomorphism $Cl(V_B) \cong Cl(4,0)$ to $Cl(0,4) \cong Cl(V_B')$

$$
\begin{aligned}
1 &\to 1 \\
\underline{e}_1 &\to -\underline{e}'_{234} \\
\underline{e}_2 &\to \underline{e}'_{134} \\
\underline{e}_3 &\to -\underline{e}'_{124} \\
\underline{e}_4 &\to \underline{e}'_{123} \\
\underline{e}_{12} &\to -\underline{e}'_{12} \\
\underline{e}_{13} &\to -\underline{e}'_{13} \\
\underline{e}_{14} &\to -\underline{e}'_{14} \\
\underline{e}_{23} &\to -\underline{e}'_{23} \\
\underline{e}_{24} &\to -\underline{e}'_{24} \\
\underline{e}_{34} &\to -\underline{e}'_{34} \\
\underline{e}_{123} &\to -\underline{e}'_{4} \\
\underline{e}_{124} &\to \underline{e}'_{3} \\
\underline{e}_{134} &\to -\underline{e}'_{2} \\
\underline{e}_{234} &\to \underline{e}'_{1} \\
\underline{e}_{1234} &\to \underline{e}'_{1234}
\end{aligned}
\tag{A.11}
$$

3. Finally the basis blades in the algebra $Cl(p-4, q+4)$ are expressed by application of (A.11) to the blade factors $B$, resulting in isomorphic blade factors $B'$:

$$
M \to M' = AB'C \in Cl(p-4, q+4). \tag{A.12}
$$

## Appendix B. Isomorphism from $Cl(p,q)$ to $Cl(p+4,q-4)$

The approach is very similar to that described in Appendix A. We start with the orthonormal basis $\{e_1, \ldots, e_p, e_{p+1}, e_{p+2}, e_{p+3}, e_{p+4}, e_{p+4+1}, \ldots, e_{p+q}\}$ of the vector space $\mathbb{R}^{p,q}$. We define the four-dimensional anti-Euclidean subspace $V_D = \text{span}[e_{p+1}, \ldots, e_{p+4}]$, and the $q-4$-dimensional anti-Euclidean subspace $V_C = \text{span}[e_{p+4+1}, \ldots, e_{p+q}]$, such that $\mathbb{R}^{p,q} = \mathbb{R}^p \cup V_D \cup V_C$. We denote the Clifford subalgebras generated with the Clifford product over the subspaces $V_D$ and $V_C$: $Cl(V_D)$ and $Cl(V_C)$. A general basis blade $M \in Cl(p,q)$ can then be factored into the following three parts

$$
M = ADC, \qquad \text{with} \qquad A \in Cl(p,0), \qquad D = Cl(V_D), \qquad C = Cl(V_C). \tag{B.1}
$$

For ease of notation we define the following convenient vector labels

$$
\underline{e}_1 = e_{p+1}, \qquad \underline{e}_2 = e_{p+2}, \qquad \underline{e}_3 = e_{p+3}, \qquad \underline{e}_4 = e_{p+4}. \tag{B.2}
$$

To the basis blades $D$ of the anti-Euclidean Clifford algebra $Cl(V_D) \cong Cl(0,4)$ we apply the isomorphism $Cl(0,4) \cong Cl(4,0)$ to obtain the Euclidean Clifford algebra $Cl(V'_D) \cong Cl(4,0)$

$$
\begin{aligned}
1 &\to 1\\
\underline{e}_1 &\to \underline{e}'_{234}\\
\underline{e}_2 &\to -\underline{e}'_{134}\\
\underline{e}_3 &\to \underline{e}'_{124}\\
\underline{e}_4 &\to -\underline{e}'_{123}\\
\underline{e}_{12} &\to -\underline{e}'_{12}\\
\underline{e}_{13} &\to -\underline{e}'_{13}\\
\underline{e}_{14} &\to -\underline{e}'_{14}\\
\underline{e}_{23} &\to -\underline{e}'_{23}\\
\underline{e}_{24} &\to -\underline{e}'_{24}\\
\underline{e}_{34} &\to -\underline{e}'_{34}\\
\underline{e}_{123} &\to \underline{e}'_{4}\\
\underline{e}_{124} &\to -\underline{e}'_{3}\\
\underline{e}_{134} &\to \underline{e}'_{2}\\
\underline{e}_{234} &\to -\underline{e}'_{1}\\
\underline{e}_{1234} &\to \underline{e}'_{1234}
\end{aligned}
\tag{B.3}
$$

*Remark* Appendix B.1. The blocks of four magenta and four cyan mappings between vectors and trivectors in equations (A.11) and (B.3) correspond to (are inverses of) each other.

Finally the basis blades in the algebra $Cl(p+4, q-4)$ are expressed by application of (B.3) to the basis blade factors $D$, resulting in isomorphic blade factors $D'$:

$$M \to M' = AD'C \in Cl(p+4, q-4). \tag{B.4}$$

## Appendix C. Isomorphism from $Cl(p+1, q+1)$ to Mat$(2, Cl(p,q))$

We follow the definition of the isomorphism $Cl(p+1, q+1) \cong \mathrm{Mat}(2, Cl(p,q)$ given in [9], section 16.3, pp. 214 and 215. The orthonormal basis vectors of $Cl(p+1, q+1)$ are mapped to

$$e_i \to \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}, \qquad i \in \{1, \ldots, p, p+2, \ldots, p+1+q\}, \tag{C.1}$$

$$e_{p+1} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_{p+1+q+1} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{C.2}$$

This means that, after excluding $e_{p+1}$ and $e_{p+1+q+1}$, all blades $B_1 \in Cl(p+1, q+1)$ formed from products of basis vectors map to

$$B_1 \to \begin{pmatrix} B_1 & 0 \\ 0 & \widehat{B_1} \end{pmatrix}, \tag{C.3}$$

where $\widehat{B}$ is the main (grade) involution of $Cl(p+1, q+1)$. The same applies to all linear combinations of these blades, and therefore to all multivectors $m_1 \in Cl(p+1, q+1)$, which exclude the vector factors $e_{p+1}$ and $e_{p+1+q+1}$ in their basis blades. These multivectors $m_1$ are therefore elements of a subalgebra of $Cl(p+1, q+1)$, which is isomorphic to $Cl(p,q)$ over the space spanned by the subspace basis $\{e_1, \ldots, e_p, e_{p+2}, \ldots, e_{p+1+q}\}$. Indeed general multivectors in $Cl(p+1, q+1)$ can be written as a linear combination of four elements from the subalgebra isomorphic to $Cl(p,q)$

$$m = m_1 1 + m_2\, e_{p+1} + m_3\, e_{p+1+q+1} + m_4\, e_{p+1,p+1+q+1}, \qquad m_1, m_2, m_3, m_4 \in Cl(p,q). \tag{C.4}$$

The multivectors $m_1, m_2, m_3, m_4$ are mapped according to (C.3). The four-dimensional blade basis $\{1, e_{p+1}, e_{p+1+q+1}, e_{p+1,p+1+q+1}\}$ of the subalgebra of $Cl(p+1, q+1)$ generated by $\{e_{p+1}, e_{p+1+q+1}\}$, which is isomorphic to $Cl(1,1)$, is mapped following (C.2). This means that the products map to

$$1 = e_{p+1}^2 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_{p+1,p+1+q+1} = e_{p+1}\, e_{p+1+q+1} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.5}$$

Putting everything together gives the isomorphic representation of $Cl(p+1, q+1)$ in terms of $2 \times 2$ matrices with entries in $Cl(p, q)$

$$\begin{aligned} m \to M &= \begin{pmatrix} m_1 & 0 \\ 0 & \widehat{m_1} \end{pmatrix} 1 + \begin{pmatrix} m_2 & 0 \\ 0 & \widehat{m_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} m_3 & 0 \\ 0 & \widehat{m_3} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} m_4 & 0 \\ 0 & \widehat{m_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} m_1 + m_4 & m_2 - m_3 \\ \widehat{m_2} + \widehat{m_3} & \widehat{m_1} - \widehat{m_4} \end{pmatrix}. \end{aligned} \tag{C.6}$$

For an implementation of this isomorphism we only need equations for extracting the subalgebra factors $m_1, m_2, m_3, m_4$ from a general multivector $m \in Cl(p+1, q+1)$. For this purpose we will use left and right contractions (symbols $\rfloor, \lfloor$ , see [2, 7] ) with the following six blades

$$\begin{aligned} b_1 &= e_{1\ldots p,p+2\ldots p+1+q}, \\ b_2 &= e_{1\ldots p+1+q}, \\ b_3 &= e_{p+1}, \\ b_4 &= e_{1\ldots p,p+2\ldots p+1+q+1}, \\ b_5 &= e_{p+1+q+1}, \\ b_6 &= b_3 b_5 = e_{p+2,p+1+q+1}, \end{aligned} \tag{C.7}$$

with vectors $b_3, b_5$, bivector $b_6$, $(p+q)$-vector $b_1$, and pseudo-vectors $b_2, b_4$ of grade $p+q+1$. The extraction equations for $m_1, m_2, m_3, m_4$ are then

$$\begin{aligned} m_1 &= (m \rfloor b_1) b_1^{-1}, \\ m_2 &= \left( \{(m - m_1) \rfloor b_2 \}\, b_2^{-1} \right) \lfloor b_3^{-1}, \\ m_3 &= \left( \{(m - m_1 - m_2) \rfloor b_4 \}\, b_4^{-1} \right) \lfloor b_5^{-1}, \\ m_4 &= (m - m_1 - m_2 - m_3) \lfloor\, b_6^{-1}, \end{aligned} \tag{C.8}$$

where $(m \rfloor b_1) b_1^{-1}$ is a typical projection operation of all blade parts of $m$ wholly contained in the subalgebra of $Cl(p+1, q+1)$, isomorphic to $Cl(p, q)$.

*Remark* Appendix C.1. The isomorphism $Cl(p+1, q+1) \cong \mathrm{Mat}(Cl(p,q))$ is also of great relevance to conformal geometric algebra [2, 7], because it shows how to compute the representation of conformal geometric algebra multivectors in $Cl(p+1, q+1)$ with the help linear combinations (C.4), or with $2 \times 2$ matrices of the corresponding geometric algebra multivectors in $Cl(p, q)$ as in (C.6). Furthermore, replacing the vectors $e_{p+1}$ and $e_{p+1+q+1}$ by any pair of orthonormal vectors with positive and negative square, respectively, allows to decompose the algebra $Cl(p+1, q+1)$ with a free steerable choice of plane subalgebra $Cl(1,1)$ (representing the origin-infinity plane in conformal geometric algebra). The steerability of the decomposition in a slightly different way, has already been shown to be of great importance in camera object geometry [10].

## Appendix D. Isomorphism from $\mathrm{Mat}(2, Cl(p,q))$ to $Cl(p+1, q+1)$

This isomorphism is the inverse of the one described in Appendix C. We can therefore assume to be given a $2 \times 2$ matrix $M$ with multivector entries from $Cl(p,q)$, in the form of (C.6). We only need to extract from the matrix of (C.6) the four entities $m_1, m_2, m_3, m_4$ and recombine them according to (C.4) with the basis blades $\{1, e'_{p+1}, e'_{p+1+q+1}, e'_{p+1,p+1+q+1}\}$ of $Cl(1,1)$.

The extraction of $m_1, m_2, m_3, m_4$ from the $2 \times 2$ matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathrm{Mat}(2, Cl(p,q)), \tag{D.1}$$

with multivector entries from $Cl(p,q)$ is achieved by

$$m_1 = \tfrac{1}{2}(M_{11} + \widehat{M_{22}}), \quad m_2 = \tfrac{1}{2}(M_{12} + \widehat{M_{21}}), \quad m_3 = \tfrac{1}{2}(-M_{12} + \widehat{M_{22}}), \quad m_4 = \tfrac{1}{2}(M_{11} - \widehat{M_{22}}). \tag{D.2}$$

The final representation in $Cl(p+1,q+1)$ is then given by

$$M \to m = m_1 1 + m_2\, e'_{p+1} + m_3\, e'_{p+1+q+1} + m_4\, e'_{p+1,p+1+q+1}, \tag{D.3}$$

where the two additionally inserted orthonormal basis vectors $\{e'_{p+1}, e'_{p+1+q+1}\}$ square to $e'^2_{p+1} = 1$, and $e'^2_{p+1+q+1} = -1$. So the new basis is further related to the old basis by

$$e'_k = e_k \ (1 \le k \le p), \qquad e'_{p+1+l} = e_{p+l} \ (1 \le l \le q). \tag{D.4}$$