Graded Frobenius cluster categories

Jan E. Grabowski†
Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, United Kingdom

Matthew Pressland‡
Max-Planck-Institut für Mathematatik, Vivatsgasse 7, 53111 Bonn, Germany

30th September 2016

Abstract

Recently the first author studied multi-gradings for generalised cluster categories, these being 2-Calabi–Yau triangulated categories with a choice of cluster-tilting object. The grading on the category corresponds to a grading on the cluster algebra without coefficients categorified by the cluster category and hence knowledge of one of these structures can help us study the other.

In this work, we extend the above to certain Frobenius categories that categorify cluster algebras with coefficients. We interpret the grading K-theoretically and prove similar results to the triangulated case, in particular obtaining that degrees are additive on exact sequences.

We show that the categories of Buan, Iyama, Reiten and Scott, some of which were used by Geiß, Leclerc and Schröer to categorify cells in partial flag varieties, and those of Jensen, King and Su, categorifying Grassmannians, are examples of graded Frobenius cluster categories.

MSC (2010): 13F60 (Primary), 18E30, 16G70 (Secondary)

1 Introduction

Gradings for cluster algebras have been introduced in various ways by a number of authors and for a number of purposes. The evolution of the notion started with the foundational work of Fomin and Zelevinsky [FZ02], who consider $\mathbb{Z}^n$-gradings where n is precisely the rank of the cluster algebra.

We continue here the study of gradings in cluster algebra theory. In [Gra15] the first author examined the natural starting case of finite type cluster algebras without coefficients. A complete classification of the gradings that occur was given and it was observed that the gradings so obtained were all *balanced*, that is, there exist bijections between the set of variables of degree $\underline{d}$ and those of degree $-\underline{d}$.

This phenomenon was explained by means of graded generalised cluster categories, where—following [DG14]—by generalised cluster category we mean a 2-Calabi–Yau triangulated category $\mathcal{C}$ with a basic cluster-tilting object T. The definition made in [Gra15] associates an integer vector (the multi-degree) to an object in the category in such a way that the vectors are additive on

†Email: j.grabowski@lancaster.ac.uk. Website: http://www.maths.lancs.ac.uk/~grabowsj/
‡Email: mdpressland@bath.edu. Website: http://guests.mpim-bonn.mpg.de/mdp33/

distinguished triangles and transform naturally under mutation. This is done via the key fact that every object in a generalised cluster category has a well-defined associated integer vector-valued datum called the index with respect to T; in order to satisfy the aforementioned two properties, degrees are necessarily linear functions of the index.

The categorical approach has the advantage that it encapsulates the global cluster combinatorics, or more accurately the set of indices does. Another consequence is an explanation for the observed balanced gradings in finite type: the auto-equivalence of the cluster category given by the shift functor induces an automorphism of the set of cluster variables that reverses signs of degrees. Hence any cluster algebra admitting a (triangulated) cluster categorification necessarily has all its gradings being balanced, for example finite type or, more generally, acyclic cluster algebras having no coefficients.

Our main goal is to provide a version of the above in the Frobenius, i.e. exact category, setting, similarly to the triangulated one. A Frobenius category is exact with enough projective objects and enough injective objects, and these classes coincide. From work of Fu and Keller [FK10] and the second author [Pre15], we have a definition of a Frobenius cluster category and objects in such a category also have indices.

From this we may proceed along similar lines to [Gra15] to define gradings and degrees, except that we elect to work in a more basis-free way by working K-theoretically and with the associated Euler form. We prove the foundational properties of gradings for Frobenius cluster categories: that degrees are compatible with taking the cluster character, that they are additive on exact sequences and that they are compatible with mutation.

Furthermore, we prove an analogue of a result of Palu [Pal09] in which we show that the space of gradings for a graded Frobenius cluster category $\mathcal{E}$ is closely related to the Grothendieck group, namely that the former is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{E}), \mathbb{Z})$. This enables one to show that some categorical datum is a grading by seeing that it respects exact sequences, and conversely that from the cluster algebra categorified by $\mathcal{E}$ we may deduce information about $\mathrm{K}_0(\mathcal{E})$. We exhibit this on examples, notably the categories of Buan, Iyama, Reiten and Scott [BIRS09] corresponding to Weyl group elements, also studied by Geiß, Leclerc and Schröer [GLS08] in the context of categorifying cells in partial flag varieties.

The homogeneous coordinate rings of Grassmannians are an example of particular importance in this area. They admit a graded cluster algebra structure but beyond the small number of cases when this structure is of finite type, little is known about the cluster variables. A first step towards a better understanding is to describe how the degrees of the cluster variables are distributed: are the degrees unbounded? does every natural number occur as a degree? are there finitely many or infinitely many variables in each occurring degree? By using the Frobenius cluster categorification of Jensen, King and Su [JKS16] and the grading framework here, we can hope to begin to examine these questions.

Acknowledgements

The work herein was begun during a research visit of the second author to Lancaster University, funded by EPSRC grant EP/M001148/1.

The second author would like to thank Paul Balmer, Andrew Hubery, Alastair King and Sondre Kvamme for useful conversations, and acknowledge financial support from Sonderforschungsbereich 701 at Universität Bielefeld, and from the Max-Planck-Gesellschaft.

2 Preliminaries

The construction of a cluster algebra of geometric type from an initial seed $(\underline{x}, B)$, due to Fomin and Zelevinsky [FZ02], is now well-known. Here $\underline{x}$ is a transcendence base for a certain field of fractions of a polynomial ring and B is a skew-symmetrizable integer matrix; in the skew-symmetric case B is often replaced by the quiver $Q = Q(B)$ it defines in the natural way.

We refer the reader who is unfamiliar with this construction to the survey of Keller [Kel10] and the books of Marsh [Mar14] and of Gekhtman, Shapiro and Vainshtein [GSV10] for an introduction to the topic and summaries of the main related results in this area.

We set some notation for later use. For k a mutable index, set

$$\begin{aligned} \underline{b}_k^+ &= -\underline{\boldsymbol{e}}_k + \sum_{b_{ik}>0} b_{ik}\underline{\boldsymbol{e}}_i \qquad \text{and} \\ \underline{b}_k^- &= -\underline{\boldsymbol{e}}_k - \sum_{b_{ik}<0} b_{ik}\underline{\boldsymbol{e}}_i, \end{aligned}$$

where the vector $\underline{\boldsymbol{e}}_i \in \mathbb{Z}^r$ (r being the number of rows of B) is the ith standard basis vector. Note that the kth row of B may be recovered as $B_k = \underline{b}_k^+ - \underline{b}_k^-$.

Then given a cluster $\underline{x} = (x_1, \dots, x_r)$ and exchange matrix B, the exchange relation for mutation in the direction k is given by

$$x_k' = \underline{x}^{\underline{b}_k^+} + \underline{x}^{\underline{b}_k^-},$$

where for $\underline{a} = (a_1, \dots, a_r)$ we set

$$\underline{x}^{\underline{a}} = \prod_{i=1}^{r} x_i^{a_i}.$$

Later we will briefly discuss cluster algebras with coefficients (also called frozen variables). That is, we designate some of the elements of the initial cluster to be mutable (i.e. we are allowed to mutate these) and the remainder to be non-mutable. We will also talk about the corresponding indices for the variables as being mutable or not. The rank of the cluster algebra is the number of *mutable* variables in a cluster; we will refer to the total number of variables, mutable and not, as the cardinality of the cluster.

We will retain the usual convention that B will be a matrix with rows indexed by the initial cluster variables and columns indexed by the mutable initial cluster variables. We use the notation B^t for the transpose of B.

Throughout, for simplicity, we will assume that all algebras and categories are defined over $\mathbb{C}$. All modules are left modules.

2.1 Multi-graded seeds, cluster algebras and cluster categories

The natural definition for a multi-graded seed is as follows.

Definition 2.1. A multi-graded seed is a triple $(\underline{x}, B, G)$ such that

(a) $(\underline{x} = (x_1, \dots, x_r), B)$ is a seed of cardinality r, and

(b) G is an $r \times m$ integer matrix such that $B^t G = 0$.

From now on, we use the term "graded" to encompass multi-graded; if the context is unclear, we will say $\mathbb{Z}^m$-graded.

The above data defines $\underline{\deg}_G(x_i) = G_i$ (the ith row of G) and this can be extended to rational expressions in the generators x_i in the obvious way. We can also mutate our grading, and repeated mutation propagates a grading on an initial seed to every cluster variable and hence to the associated cluster algebra, as condition (b) in the definition of a grading ensures that every exchange relation is homogeneous. Hence we obtain the following well-known result, given in various forms in the literature.

Proposition 2.2. *The cluster algebra $\mathcal{A}(\underline{x}, B, G)$ associated to an initial graded seed $(\underline{x}, B, G)$, with G an $r \times m$ integer matrix, is a $\mathbb{Z}^m$-graded algebra. Every cluster variable of $\mathcal{A}(\underline{x}, B, G)$ is homogeneous with respect to this grading.* □

We refer the reader to [Gra15] for a more detailed discussion of the above and further results regarding the existence of gradings, relationships between gradings and a study of gradings for cluster algebras of finite type with no coefficients.

2.2 Graded triangulated cluster categories

Our interest here is in generalising the categorical parts of [Gra15]. In order to motivate what will follow for the Frobenius setting, we give the key definitions and statements from the triangulated case, without proofs as these may be found in [Gra15].

Definition 2.3 ([DG14])**.** Let $\mathcal{C}$ be a triangulated 2-Calabi–Yau category with suspension functor Σ and let $T \in \mathcal{C}$ be a basic cluster-tilting object. We will call the pair $(\mathcal{C}, T)$ a generalised cluster category.

Write $T = T_1 \oplus \cdots \oplus T_r$. Setting $\Lambda = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, the functor[1] $F = \mathcal{C}(T, -)\colon \mathcal{C} \to \operatorname{mod}\Lambda$ induces an equivalence $\mathcal{C}/\operatorname{add}(\Sigma T) \to \operatorname{mod}\Lambda$. We may also define an exchange matrix associated to T by

$$(B_T)_{ij} = \dim \operatorname{Ext}^1_\Lambda(S_i, S_j) - \dim \operatorname{Ext}^1_\Lambda(S_j, S_i).$$

Here the $S_i = FT_i/\operatorname{rad} FT_i$, $i = 1, \dots, r$ are the simple Λ-modules. Thus, if the Gabriel quiver of the algebra Λ has no loops or 2-cycles, B_T is its corresponding skew-symmetric matrix.

For each $X \in \mathcal{C}$ there exists a distinguished triangle

$$\bigoplus_{i=1}^{r} T_i^{m(i,X)} \to \bigoplus_{i=1}^{r} T_i^{p(i,X)} \to X \to \Sigma\left(\bigoplus_{i=1}^{r} T_i^{m(i,X)}\right)$$

Define the index of X with respect to T, denoted $\underline{\mathrm{ind}}_T(X)$, to be the integer vector with $\underline{\mathrm{ind}}_T(X)_i = p(i,X) - m(i,X)$. By [Pal08, §2.1], $\underline{\mathrm{ind}}_T(X)$ is well-defined and we have a cluster character

$$\begin{aligned} C^T_? \colon \operatorname{Obj}(\mathcal{C}) &\to \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}] \\ X &\mapsto \underline{x}^{\underline{\mathrm{ind}}_T(X)} \sum_{\underline{e}} \chi(\operatorname{Gr}_{\underline{e}}(F\Sigma X))\underline{x}^{B_T \cdot \underline{e}} \end{aligned}$$

Here $\operatorname{Gr}_{\underline{e}}(F\Sigma X)$ is the quiver Grassmannian of Λ-submodules of $F\Sigma X$ of dimension vector $\underline{e}$ and χ is the topological Euler characteristic. We also use the same monomial notation $\underline{x}^{\underline{a}}$ as previously.

[1]This functor is replaced by $E = F\Sigma$ in [DG14], [Gra15]; we use F here, as in [FK10], for greater compatibility with the Frobenius case.

We recall that for any cluster-tilting object U of $\mathcal{C}$ such that the quiver of $\mathrm{End}_{\mathcal{C}}(U)^{\mathrm{op}}$ has no loops or 2-cycles, and for any indecomposable summand U_k of U, there exists a unique indecomposable object $U_k^* \ncong U_k$ such that $U^* = (U/U_k) \oplus U_k^*$ is again cluster-tilting, and there are non-split triangles

$$U_k^* \to M \to U_k \to \Sigma U_k^* \qquad \text{and} \qquad U_k \to M' \to U_k^* \to \Sigma U_k$$

with $M, M' \in \mathrm{add}(U/U_k)$. In the generality of our setting, this is due to Iyama and Yoshino [IY08].

The natural definition of a graded generalised cluster category is then the following.

Definition 2.4 ([Gra15, Definition 5.2])**.** Let $(\mathcal{C}, T)$ be a generalised cluster category and let G be an $r \times m$ integer matrix such that $B_T G = 0$. We call the tuple $(\mathcal{C}, T, G)$ a graded generalised cluster category.

Note that B_T is skew-symmetric, so we may suppress taking the transpose in the equation $B_T G = 0$.

Definition 2.5 ([Gra15, Definition 5.3])**.** Let $(\mathcal{C}, T, G)$ be a graded generalised cluster category. For any $X \in \mathcal{C}$, we define $\underline{\deg}_G(X) = \underline{\mathrm{ind}}_T(X)G$.

The main results about graded generalised cluster categories are summarised in the following Proposition, the most significant of these being (ii).

Proposition 2.6 ([Gra15, §5])**.** *Let $(\mathcal{C}, T, G)$ be a graded generalised cluster category.*

(i) For all $X \in \mathcal{C}$, $\underline{\deg}_G(X)$ is equal to the degree of $C_X^T \in \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ where the latter is $\mathbb{Z}^m$-graded by $\underline{\deg}_G(x_i) = G_i$ (the ith row of G).

(ii) For any distinguished triangle $X \to Y \to Z \to \Sigma X$ of $\mathcal{C}$, we have

$$\underline{\deg}_G(Y) = \underline{\deg}_G(X) + \underline{\deg}_G(Z).$$

(iii) The degree $\underline{\deg}_G$ is compatible with mutation in the sense that for every cluster-tilting object U of $\mathcal{C}$ with indecomposable summand U_k we have

$$\underline{\deg}_G(U_k^*) = \underline{\deg}_G(M) - \underline{\deg}_G(U_k) = \underline{\deg}_G(M') - \underline{\deg}_G(U_k),$$

where U_k^*, M and M' are as in the above description of mutation in $\mathcal{C}$.

(iv) The space of gradings for a generalised cluster category $(\mathcal{C}, T)$ may be identified with the Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ as a triangulated category.

(v) For each $X \in \mathcal{C}$, $\underline{\deg}_G(\Sigma X) = -\underline{\deg}_G(X)$. That is, for each $\underline{d} \in \mathbb{Z}^d$, the shift automorphism Σ on $\mathcal{C}$ induces a bijection between the objects of $\mathcal{C}$ of degree $\underline{d}$ and those of degree $-\underline{d}$. □

Part (iii) of the preceding proposition shows how to mutate the data of G when mutating the cluster-tilting object T, to obtain a new matrix compatible with the exchange matrix of the new cluster-tilting object, and defining the same grading on the cluster algebra.

However, we may obtain an even stronger conclusion from part (iv), since this provides a "base-point free" definition of a grading, depending only on the category $\mathcal{C}$ and not on the cluster-tilting object T. Read differently, this shows that if $(\mathcal{C}, T, G)$ is a graded generalised cluster category, then for any cluster-tilting object $T' \in \mathcal{C}$, there is a unique matrix G' such that $(\mathcal{C}, T', G')$ is a graded generalised cluster category and $\underline{\deg}_G(X) = \underline{\deg}_{G'}(X)$ for all $X \in \mathcal{C}$. We will explain this in more detail below in the case of Frobenius categories.

3 Graded Frobenius cluster categories

In this section, we provide the main technical underpinnings for the Frobenius version of the above theory, in which we consider exact categories rather than triangulated ones. Background on exact categories, and homological algebra in them, can be found in Bühler's survey [Büh10].

An exact category $\mathcal{E}$ is called a Frobenius category if it has enough projective objects and enough injective objects, and these two classes of objects coincide. A typical example of such a category is the category of modules over a self-injective algebra. More generally, if B is a Noetherian algebra with finite left and right injective dimension as a module over itself (otherwise known as an Iwanaga–Gorenstein algebra), the category

$$\mathrm{GP}(B) = \{X \in \operatorname{mod} B : \operatorname{Ext}^i_B(X, B) = 0,\ i > 0\},$$

is Frobenius [Buc86]. (Here $\mathrm{GP}(B)$ is equipped with the exact structure in which the exact sequences are precisely those that are exact when considered in the abelian category $\operatorname{mod} B$.) The initials "GP" are chosen for "Gorenstein projective".

Given a Frobenius category $\mathcal{E}$, its stable category $\underline{\mathcal{E}}$ is formed by taking the quotient of $\mathcal{E}$ by the ideal of morphisms factoring through a projective-injective object. By a famous result of Happel [Hap88, Theorem 2.6], $\underline{\mathcal{E}}$ is a triangulated category with shift functor Ω^{-1}, where $\Omega^{-1}X$ is defined by the existence of an exact sequence

$$0 \to X \to Q \to \Omega^{-1}X \to 0$$

in which Q is injective. The distinguished triangles of $\underline{\mathcal{E}}$ are isomorphic to those of the form

$$X \to Y \to Z \to \Omega^{-1}X$$

where

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence in $\mathcal{E}$.

Definition 3.1. A Frobenius category $\mathcal{E}$ is stably 2-Calabi–Yau if the stable category $\underline{\mathcal{E}}$ is Hom-finite and there is a functorial duality

$$\mathrm{D}\operatorname{Ext}^1_{\mathcal{E}}(X, Y) = \operatorname{Ext}^1_{\mathcal{E}}(Y, X)$$

for all $X, Y \in \mathcal{E}$.

Remark 3.2. The above definition is somewhat slick—it is equivalent to requiring that $\underline{\mathcal{E}}$ is 2-Calabi–Yau as a triangulated category (that is, that $\underline{\mathcal{E}}$ is Hom-finite and Ω^{-2} is a Serre functor), as one might expect.

Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category with cluster-tilting objects, and assume that for each cluster-tilting object $T \in \mathcal{E}$, the quiver of $\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ has no loops or 2-cycles incident with the vertices corresponding to non-projective summands of T. In this case, the cluster-tilting objects of $\mathcal{E}$ satisfy a mutation property similar to that in the triangulated case; see, for example, [BIRS09, Theorem II.1.6]. For any cluster-tilting object U of $\mathcal{E}$ and for any U_k a non-projective indecomposable summand of U, there exists a unique indecomposable object $U_k^* \not\cong U_k$ such that $U^* = (U/U_k) \oplus U_k^*$ is again cluster-tilting and there exist non-split sequences

$$0 \to U_k^* \to M \to U_k \to 0 \qquad \text{and} \qquad 0 \to U_k \to M' \to U_k^* \to 0$$

with $M, M' \in \operatorname{add}(U/U_k)$.

In a sense, this is the same mutation property as holds in the 2-Calabi–Yau triangulated category $\underline{\mathcal{E}}$; when U is thought of as an object of $\underline{\mathcal{E}}$, its indecomposable summands are precisely the non-projective summands of U in $\mathcal{E}$. Mutating such a summand gives an indecomposable object $\underline{U_k^*}$ in $\underline{\mathcal{E}}$. Each object of $\mathcal{E}$ isomorphic to $\underline{U_k^*}$ in $\underline{\mathcal{E}}$ is of the form $U_k^* \oplus P$ for some projective object P and an indecomposable non-projective object U_k^*. Moreover, the summand U_k^* is determined up to isomorphism in $\mathcal{E}$ by the isomorphism class of $\underline{U_k^*}$ in $\underline{\mathcal{E}}$, and provides our desired object of $\mathcal{E}$. The two non-split sequences relating U_k and U_k^* may then be obtained by lifting the corresponding non-split triangles in $\underline{\mathcal{E}}$.

Fu and Keller [FK10] give the following definition of a cluster character on a stably 2-Calabi–Yau Frobenius category.

Definition 3.3 ([FK10, Definition 3.1])**.** Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category, and let R be a commutative ring. A cluster character on $\mathcal{E}$ is a map φ on the set of objects of $\mathcal{E}$, taking values in R, such that

(i) if $M \cong M'$ then $\varphi_M = \varphi_{M'}$,

(ii) $\varphi_{M\oplus N} = \varphi_M \varphi_N$, and

(iii) if $\dim \operatorname{Ext}^1_{\mathcal{E}}(M,N) = 1$ (equivalently, $\dim \operatorname{Ext}^1_{\mathcal{E}}(N,M) = 1$), and

$$0 \to M \to X \to N \to 0,$$
$$0 \to N \to Y \to M \to 0$$

are non-split sequences, then

$$\varphi_M \varphi_N = \varphi_X + \varphi_Y.$$

Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category, and assume there exists a cluster-tilting object $T \in \mathcal{E}$. Assume without loss of generality that T is basic, and let $T = \bigoplus_{i=1}^n T_i$ be a decomposition of T into pairwise non-isomorphic indecomposable summands. We number the summands so that T_i is projective if and only if $r < i \leqslant n$. Let $\Lambda = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$, and $\underline{\Lambda} = \operatorname{End}_{\underline{\mathcal{E}}}(T)^{\mathrm{op}} = \Lambda/\Lambda e \Lambda$, where e is the idempotent given by projection onto a maximal projective-injective summand of T.

We assume that Λ is Noetherian, as with this assumption the forms discussed below will be well-defined. The examples that concern us later will have Noetherian Λ, but we acknowledge that this assumption is somewhat unsatisfactory, given that it is often difficult to establish.

Fu and Keller [FK10] show that such a T determines a cluster character on $\mathcal{E}$, as we now explain; while the results of [FK10] are stated in the case that $\mathcal{E}$ is Hom-finite, the assumption that Λ is Noetherian is sufficient providing one is careful to appropriately distinguish between the two Grothendieck groups $\mathrm{K}_0(\operatorname{mod}\Lambda)$ and $\mathrm{K}_0(\operatorname{fd}\Lambda)$ of finitely generated and finite dimensional Λ-modules respectively.

We write

$$F = \operatorname{Hom}_{\mathcal{E}}(T,-)\colon \mathcal{E} \to \operatorname{mod}\Lambda,$$
$$E = \operatorname{Ext}^1_{\mathcal{E}}(T,-)\colon \mathcal{E} \to \operatorname{mod}\Lambda.$$

Note that E may also be expressed as $\operatorname{Hom}_{\underline{\mathcal{E}}}(T, \Omega^{-1}(-))$, meaning it takes values in $\operatorname{mod}\underline{\Lambda}$. For $M \in \operatorname{mod}\Lambda$ and $N \in \operatorname{fd}\Lambda$, we write

$$<M,N>_1 = \dim \operatorname{Hom}_\Lambda(M,N) - \dim \operatorname{Ext}^1_\Lambda(M,N),$$
$$<M,N>_3 = \dim \operatorname{Hom}_\Lambda(M,N) - \dim \operatorname{Ext}^1_\Lambda(M,N) + \dim \operatorname{Ext}^2_\Lambda(M,N) - \dim \operatorname{Ext}^3_\Lambda(M,N).$$

The algebra $\underline{\Lambda} = \operatorname{End}_{\underline{\mathcal{E}}}(T)^{\mathrm{op}}$ is finite dimensional since $\underline{\mathcal{E}}$ is Hom-finite, so $\operatorname{mod}\underline{\Lambda} \subseteq \operatorname{fd}\Lambda$. Fu and Keller show [FK10, Proposition 3.2] that if $M \in \operatorname{mod}\underline{\Lambda}$, then $< M, N >_3$ depends only on the dimension vector $(\dim \operatorname{Hom}_\Lambda(P_i, M))_{i=1}^n$, where the $P_i = FT_i$ are a complete set of indecomposable projective Λ-modules. Thus if $v \in \mathbb{Z}^r$, we define

$$< v, N >_3 := < M, N >_3$$

for any $M \in \operatorname{mod}\underline{\Lambda}$ with dimension vector v.

Let $R = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of Laurent polynomials in $x_1, \ldots, x_n$. Define a map $X \to C_X^T$ on objects of $\mathcal{E}$, taking values in R, via the formula

$$C_X^T = \prod_{i=1}^n x_i^{< FX, S_i >_1} \sum_{v \in \mathbb{Z}^r} \chi(\operatorname{Gr}_v(EX)) \prod_{i=1}^n x_i^{- < v, S_i >_3}.$$

Here, as before, $\operatorname{Gr}_v(EX)$ denotes the projective variety of submodules of EX with dimension vector v, and $\chi(\operatorname{Gr}_v(EX))$ denotes its Euler characteristic. The modules $S_i = FT_i / \operatorname{rad} FT_i$ are the simple tops of the projective modules P_i. By [FK10, Theorem 3.3], the map $X \mapsto C_X^T$ is a cluster character, with the property that $C_{T_i}^T = x_i$.

The cluster-tilting object T also determines an index and coindex for each object $X \in \mathcal{E}$. To see that these quantities are well-defined we will use the following lemma, the proof of which is included for the convenience of the reader.

Lemma 3.4. *Let $\mathcal{E}$ be an exact category, and let $M, T \in \mathcal{E}$.*

(i) If there exists an admissible monomorphism $M \to T'$ for $T' \in \operatorname{add} T$, then any left $\operatorname{add} T$*-approximation of M is an admissible monomorphism.*

(ii) If there exists an admissible epimorphism $T' \to M$ for $T' \in \operatorname{add} T$, then any right $\operatorname{add} T$*-approximation of M is an admissible epimorphism.*

Proof: We prove only (i), as (ii) is dual. Pick an admissible monomorphism $i \colon M \to T'$ with $T' \in \operatorname{add} T$ and a left $\operatorname{add} T$-approximation $f \colon M \to L$. Consider the pushout square

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ T' & \xrightarrow{g} & X \end{array}$$

As f is a left $\operatorname{add} T$-approximation, there is a map $h \colon L \to T'$ such that the square

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle h} \\ T' & \xrightarrow{1} & T' \end{array}$$

commutes, and so by the universal property of pushouts, there is $g' \colon X \to T'$ such that $g'g = 1$. Thus g is a split monomorphism, fitting into an exact sequence

$$0 \longrightarrow T' \xrightarrow{g} X \xrightarrow{\pi} C \longrightarrow 0.$$

It then follows, again by the universal property of pushouts, that $\pi i'$ is a cokernel of f. Since $i' f = gi$ is the composition of two admissible monomorphisms, f is itself an admissible monomorphism by the obscure axiom [Kel90, A.1], [Büh10, Proposition 2.16]. $\square$

Given an object $X \in \mathcal{E}$, we may pick a minimal right $\operatorname{add} T$-approximation $R_X \to X$, where R_X is determined up to isomorphism by X and the existence of such a morphism. Let $P \to X$ be a projective cover of X, which exists since $\mathcal{E}$ has enough projectives; this is an admissible epimorphism by definition, and $P \in \operatorname{add} T$ since T is cluster-tilting. Thus by Lemma 3.4, the approximation $R_X \to X$ is an admissible epimorphism, and so there is an exact sequence

$$0 \to K_X \to R_X \to X \to 0$$

in $\mathcal{E}$. Since T is cluster-tilting, $K_X \in \operatorname{add} T$, and we define $\underline{\operatorname{ind}}_T(X) = [R_X] - [K_X] \in \mathrm{K}_0(\operatorname{add} T)$. Dually, any minimal left $\operatorname{add} T$-approximation $X \to L_X$ fits into an exact sequence

$$0 \to X \to L_X \to C_X \to 0,$$

and we define $\underline{\operatorname{coind}}_T(X) = [L_X] - [C_X] \in \mathrm{K}_0(\operatorname{add} T)$. It is crucial here that we define $\underline{\operatorname{ind}}_T(X)$ and $\underline{\operatorname{coind}}_T(X)$ in $\mathrm{K}_0(\operatorname{add} T)$, where they are usually distinct, rather than in $\mathrm{K}_0(\mathcal{E})$, where they are both equal to $[X]$.

We also associate to T the exchange matrix B_T given by the first r columns of the antisymmetrisation of the incidence matrix of the quiver of Λ. By definition, B_T has entries

$$(B_T)_{ij} = \dim \operatorname{Ext}^1_\Lambda(S_i, S_j) - \dim \operatorname{Ext}^1_\Lambda(S_j, S_i)$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant r$.

Remark 3.5. By a result of Keller and Reiten [KR07, §4] (see also [Pre15, Theorem 3.4]), $\operatorname{mod} \Lambda$ has enough 3-Calabi–Yau symmetry for us to deduce that $\dim \operatorname{Ext}^k_\Lambda(S_i, S_j) = \dim \operatorname{Ext}^{3-k}_\Lambda(S_j, S_i)$ when $1 \leqslant j \leqslant r$. It follows that

$$(-B_T)_{ij} = \langle S_i, S_j \rangle_3,$$

so the matrix of $\langle -, - \rangle_3$, when restricted to the span of the simple modules in the first entry and the span of the first r simple modules in the second entry, is given by $-B_T$.

Definition 3.6 (cf. [Pre15, Definition 3.3])**.** A Frobenius category $\mathcal{E}$ is a Frobenius cluster category if it is Krull–Schmidt, stably 2-Calabi–Yau and satisfies $\operatorname{gldim}(\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}) \leqslant 3$ for all cluster-tilting objects $T \in \mathcal{E}$, of which there is at least one.

Note that a Frobenius cluster category $\mathcal{E}$ need not be Hom-finite, but the stable category $\underline{\mathcal{E}}$ must be, since this is part of the definition of 2-Calabi–Yau.

Let $\mathcal{E}$ be a Frobenius cluster category. Let $T = \bigoplus_{i=1}^n T_i \in \mathcal{E}$ be a basic cluster-tilting object, where each T_i is indecomposable and is projective-injective if and only if $i > r$, let $\Lambda = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ be its endomorphism algebra, and let $\underline{\Lambda} = \operatorname{End}_{\underline{\mathcal{E}}}(T)^{\mathrm{op}}$ be its stable endomorphism algebra. We continue to write $F = \mathcal{E}(T, -) = \operatorname{Hom}_{\mathcal{E}}(T, -) \colon \mathcal{E} \to \operatorname{mod} \Lambda$ and $E = \operatorname{Ext}^1_{\mathcal{E}}(T, -) \colon \mathcal{E} \to \operatorname{mod} \underline{\Lambda}$. Since $\underline{\mathcal{E}}$ is Hom-finite, $\underline{\Lambda}$ is a finite dimensional algebra.

The Krull–Schmidt property for $\mathcal{E}$ is equivalent to $\mathcal{E}$ being idempotent complete and having the property that the endomorphism algebra A of any object is a semiperfect ring [Kra15, Corollary 4.4], meaning there are a complete set $\{e_i : i \in I\}$ of pairwise orthogonal idempotents of A such that $e_i A e_i$ is local for each $i \in I$. For many representation-theoretic purposes, semiperfect $\mathbb{K}$-algebras behave in much the same way as finite dimensional ones; for example, if A is semiperfect then the quotient $A/\operatorname{rad} A$ is semi-simple, and its idempotents lift to A. For more background on semiperfect rings, see, for example, Anderson and Fuller [AF74, Chapter 27].

For us, a key property of a semiperfect ring A is that the A-modules $Ae_i/\operatorname{rad} Ae_i$ (respectively, their projective covers Ae_i) form a complete set of finite-dimensional simple A-modules (respectively indecomposable projective A-modules) up to isomorphism [AF74, Proposition 27.10]. As we will require this later, we include being Krull–Schmidt in our definition of a Frobenius cluster category, noting that other work in this area—notably [Pre15]—requires only idempotent completeness.

Since Λ is Noetherian and $\operatorname{gldim}\Lambda \leqslant 3$, the Euler form

$$<M, N>_e = \sum_{i\geqslant 0}(-1)^i \dim \operatorname{Ext}^i_\Lambda(M, N)$$

is well-defined as a map $\mathrm{K}_0(\operatorname{mod}\Lambda) \times \mathrm{K}_0(\operatorname{fd}\Lambda) \to \mathbb{Z}$, and coincides with the form $< -, - >_3$ introduced earlier.

One can show by via taking projective resolutions that the classes $[P_i]$ of indecomposable projective Λ-modules span $\mathrm{K}_0(\operatorname{mod}\Lambda)$. Moreover, since $< P_i, S_j >_e = \delta_{ij}$, any $x \in \mathrm{K}_0(\operatorname{mod}\Lambda)$ has a unique expression

$$x = \sum_{i=1}^{n} < x, S_i >_e [P_i]$$

as a linear combination of the $[P_i]$, and so these classes in fact freely generate $\mathrm{K}_0(\operatorname{mod}\Lambda)$.

Recall from the definition of the index that if $X \in \mathcal{E}$, there is an exact sequence

$$0 \to K_X \to R_X \to X \to 0$$

in which K_X and R_X lie in $\operatorname{add} T$. Since E vanishes on $\operatorname{add} T$, the functor F takes the above sequence to a projective resolution

$$0 \to FK_X \to FR_X \to FX \to 0$$

of FX in $\operatorname{mod}\Lambda$. Thus FX has projective dimension at most 1, and so $< FX, - >_1 = < FX, - >_e$. We can therefore rewrite the cluster character of X as

$$C^T_X = \prod_{i=1}^{n} x_i^{< FX, S_i >_e} \sum_{v\in\mathbb{Z}^r} \chi(\operatorname{Gr}_v(EX)) \prod_{i=1}^{n} x_i^{-< v, S_i >_e}.$$

We now proceed to defining gradings for Frobenius cluster categories. We can follow the same approach as in the triangulated case, using the index. However, by [FK10], we have the following expansion of the index in terms of the classes of the indecomposable summands of T:

$$\underline{\operatorname{ind}}_T(X) = \sum_{i=1}^{n} < FX, S_i >_e [T_i] \in \mathrm{K}_0(\operatorname{add} T).$$

Since $\operatorname{Ext}^1_\Lambda(T, T) = 0$, there are no non-split exact sequences in $\operatorname{add} T$, and so $\mathrm{K}_0(\operatorname{add} T)$ is freely generated by the $[T_i]$. For the same reason, the functor F is exact when restricted to $\operatorname{add} T$, and so induces a map $F_*\colon K_0(\operatorname{add} T) \to K_0(\operatorname{mod}\Lambda)$, which takes $[T_i]$ to $[P_i]$, and so is an isomorphism. Applying this isomorphism to the above formula, we obtain $F_*(\underline{\operatorname{ind}}_T(X)) = \sum < FX, S_i >_e [P_i] = [FX]$.

From this we see that if we wish to work concretely with matrix and vector entries, the index can be computed explicitly. For the general theory, however, the equivalent K-theoretic expression is cleaner and so we shall phrase our definition of grading in those terms, the above observation showing us that this is equivalent to the approach in [Gra15].

Thus we arrive at the following definition of a graded Frobenius cluster category, exactly analogous to Definitions 2.4 and 2.5 in the triangulated case.

Definition 3.7. Let $\mathcal{E}$ be a Frobenius cluster category and T a cluster-tilting object of $\mathcal{E}$. Write $\Lambda = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$. We say that $G = (G_1, \dots, G_m) \in \mathrm{K}_0(\operatorname{fd}\Lambda)^m$ is a grading for $\mathcal{E}$ if $<M, G_j>_e = 0$ for all $M \in \operatorname{mod}\underline{\Lambda}$ and all $1 \leqslant j \leqslant m$. We call $(\mathcal{E}, T, G)$ a graded Frobenius cluster category.

Definition 3.8. Let $(\mathcal{E}, T, G)$ be a graded Frobenius cluster category. Define $\underline{\deg}_G\colon \mathcal{E} \to \mathbb{Z}^m$ by $\underline{\deg}_G(X) = (<FX, G_j>_e)_{j=1}^m =: <FX, G>_e$.

We record some straightforward consequences of the above definitions.

Remark 3.9.

(i) By definition, the components G_j of a grading lie in a subspace of $K_0(\operatorname{fd}\Lambda)$, namely that orthogonal to $K_0(\operatorname{mod}\underline{\Lambda})$ with respect to the form $<\ ,\ >_e$.

(ii) Let $G_j = \sum_{i=1}^n G_{ij}[S_i]$. If we write $\underline{G}$ for the matrix with entries G_{ij}, the grading condition is equivalent to requiring $B_T^t \underline{G} = 0$, by Remark 3.5 and the assumption that $\underline{\Lambda}$ is finite dimensional.

(iii) Since $FT_i = P_i$ and $<P_i, S_j>_e = \delta_{ij}$, we may compute

$$(\underline{\deg}_G(T_i))_j = <FT_i, G_j>_e = G_{ij},$$

as expected.

The K-theoretic phrasing of the above definition leads us to the following observation.

Lemma 3.10. *Let $\mathcal{E}$ be Hom-finite, let $T \in \mathcal{E}$ be a cluster-tilting object with endomorphism algebra Λ and let $V \in \mathcal{E}$ be projective-injective. Write $F = \operatorname{Hom}_{\mathcal{E}}(T, -)$. Then $[FV] \in \mathrm{K}_0(\operatorname{fd}\Lambda)$ is a grading for $\mathcal{E}$, and $\underline{\deg}_{[FV]}(X) = \dim \operatorname{Hom}_{\mathcal{E}}(X, V)$.*

Proof: Letting $M \in \operatorname{mod}\underline{\Lambda}$, we need to check that $<M, FV>_e = 0$. By the internal Calabi–Yau property of $\operatorname{mod}\Lambda$ (see Remark 3.5), we may instead check that $<FV, M>_e = 0$. Firstly, $\operatorname{Ext}^i_\Lambda(FV, M) = 0$ for $i > 0$ since FV is projective.

Recall from above that there is an idempotent $e \in \Lambda$, given by projecting onto a maximal projective summand of T, such that $\underline{\Lambda} = \Lambda/\Lambda e\Lambda$. Using this, $FV \in \operatorname{add}\Lambda e$ by the definition of e, and $\operatorname{Hom}_\Lambda(\Lambda e, M) = eM = 0$ since M is a $\underline{\Lambda}$-module. Hence $\operatorname{Hom}_\Lambda(FV, M) = 0$ also, so that $<FV, M>_e = <M, FV>_e = 0$ as required.

By definition, $\underline{\deg}_{[FV]}(X) = \dim \operatorname{Hom}_\Lambda(FX, FV)$ for $X \in \mathcal{E}$. Since T is cluster-tilting, there is a short exact sequence

$$0 \to T_1 \to T_0 \to X \to 0$$

with $T_0, T_1 \in \operatorname{add} T$, to which we may apply $\operatorname{Hom}_{\mathcal{E}}(-, V)$ to obtain the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{E}}(X, V) \to \operatorname{Hom}_{\mathcal{E}}(T_0, V) \to \operatorname{Hom}_{\mathcal{E}}(T_1, V).$$

Alternatively, we can apply $\operatorname{Hom}_\Lambda(F-, FV)$ to obtain the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(FX, FV) \to \operatorname{Hom}_\Lambda(FT_0, FV) \to \operatorname{Hom}_\Lambda(FT_1, FV).$$

Since F restricts to an equivalence on $\operatorname{add} T$, and $V \in \operatorname{add} T$ since it is projective-injective, the right-hand maps in these two exact sequences are isomorphic, yielding an isomorphism $\operatorname{Hom}_{\mathcal{E}}(X, V) \cong \operatorname{Hom}_\Lambda(FX, FV)$ of their kernels, from which the result follows. $\square$

This gives us a family of gradings canonically associated to any Hom-finite Frobenius cluster category; note that in fact we only need $FV = \mathrm{Hom}_{\mathcal{E}}(T, V) \in \mathrm{fd}\,\Lambda$, so for some specific Hom-infinite $\mathcal{E}$ and specific V and T the result may still hold.

We will give some more examples of gradings later but first give the main results regarding graded Frobenius cluster categories, analogous to those in Proposition 2.6 for the triangulated case. We treat the straightforward parts first.

Proposition 3.11. *Let $(\mathcal{E}, T, G)$ be a graded Frobenius cluster category.*

(i) For all $X \in \mathcal{E}$, $\underline{\deg}_G(X)$ is equal to the degree of $C_X^T \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where the latter is $\mathbb{Z}^m$-graded by $\underline{\deg}_G(x_j) = \underline{G}_j$ (the jth column of $\underline{G}$).

(ii) For any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$, we have

$$\underline{\deg}_G(Y) = \underline{\deg}_G(X) + \underline{\deg}_G(Z).$$

(iii) The degree $\underline{\deg}_G$ is compatible with mutation in the sense that for every cluster-tilting object U of $\mathcal{E}$ with indecomposable summand U_k we have

$$\underline{\deg}_G(U_k^*) = \underline{\deg}_G(M) - \underline{\deg}_G(U_k) = \underline{\deg}_G(M') - \underline{\deg}_G(U_k),$$

where U_k^*, M and M' are as in the above description of mutation in $\mathcal{E}$. It follows that $\underline{\deg}_G(M) = \underline{\deg}_G(M')$, which is the categorical version of the claim that all exchange relations in a graded cluster algebra are homogeneous.

Proof:

(i) As usual, for $v \in \mathbb{Z}^n$ we write $\underline{x}^v = \prod_{i=1}^n x_i^{v_i}$. Then if $\underline{\deg}_G x_j = \underline{G}_j$, we have

$$(\underline{\deg}\underline{x}^v)_j = \sum_{i=1}^{n} v_i G_{ij} = <\sum_{i=1}^{n} v_i [P_i], G_j>_e$$

Each term of C_X^T may be written in the form form $\lambda \underline{x}^v$, where

$$v_i = <FX, S_i>_e - <M, S_i>_e$$

for some $M \in \mathrm{mod}\,\underline{\Lambda}$, and λ is a constant. It follows that

$$\sum_{i=1}^{n} v_i [P_i] = [FX] - [M],$$

so the degree of $\underline{x}^v$ is

$$<FX, G>_e - <M, G>_e = <FX, G>_e = \underline{\deg}_G X,$$

since $<M, G>_e = 0$ by the definition of a grading. In particular, this is independent of M, so C_X^T is homogeneous of degree $\underline{\deg}_G X$.

(ii) Applying F to the exact sequence $0 \to X \to Y \to Z \to 0$ and truncating gives an exact sequence

$$0 \to FX \to FY \to FZ \to M \to 0$$

for some $M \subseteq EX$. In particular, $M \in \operatorname{mod} \underline{\Lambda}$. In $\mathrm{K}_0(\operatorname{mod} \Lambda)$, we have

$$[FX] + [FZ] = [FY] + [M],$$

so applying $<-, G>_e$ gives

$$\underline{\deg}_G(X) + \underline{\deg}_G(Z) = \underline{\deg}_G(Y) + <M, G>_e = \underline{\deg}_G(Y)$$

since $M \in \operatorname{mod} \underline{\Lambda}$.

(iii) This follows directly from (ii) applied to the exchange sequences

$$0 \to U_k^* \to M \to U_k \to 0 \qquad \text{and} \qquad 0 \to U_k \to M' \to U_k^* \to 0. \qquad \square$$

The following theorem is the analogue of Proposition 2.6(iv), concerning the relationship between gradings and the Grothendieck group of a graded Frobenius cluster category. We restrict (for the rest of the section) to the case of $\mathbb{Z}$ gradings to simplify the notation, but since the results apply equally well to the components of $\mathbb{Z}^m$-gradings, it is straightforward to extend them to the multi-graded setting.

Theorem 3.12. *Let $\mathcal{E}$ be a Frobenius cluster category with cluster tilting object T, and let $\Lambda = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$. Then the space of $\mathbb{Z}$-gradings of $\mathcal{E}$, defined above as a subspace of $\mathrm{K}_0(\operatorname{fd} \Lambda)$, is isomorphic to $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{E}), \mathbb{Z})$, via the map $G \mapsto \underline{\deg}_G$.*

Proof: Let $\mathcal{H}^b(\operatorname{add}_{\mathcal{E}} T)$ denote the bounded homotopy category of complexes with terms in $\operatorname{add}_{\mathcal{E}} T$, and let $\mathcal{H}^b_{\mathcal{E}\text{-ac}}(\operatorname{add}_{\mathcal{E}} T)$ denote the full subcategory of $\mathcal{E}$-acyclic complexes. By work of Palu [Pal09, Lemma 2], there is an exact sequence

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}\text{-ac}}(\operatorname{add}_{\mathcal{E}} T) \longrightarrow \mathcal{H}^b(\operatorname{add}_{\mathcal{E}} T) \longrightarrow \mathcal{D}^b(\mathcal{E}) \longrightarrow 0,$$

of triangulated categories, to which we may apply the right exact functor K_0 to obtain

$$\mathrm{K}_0(\mathcal{H}^b_{\mathcal{E}\text{-ac}}(\operatorname{add}_{\mathcal{E}} T)) \longrightarrow \mathrm{K}_0(\mathcal{H}^b(\operatorname{add}_{\mathcal{E}} T)) \longrightarrow \mathrm{K}_0(\mathcal{D}^b(\mathcal{E})) \longrightarrow 0.$$

By [Pal09, Proof of Lemma 9], there is a natural isomorphism $\mathrm{K}_0(\mathcal{H}^b_{\mathcal{E}\text{-ac}}(\operatorname{add}_{\mathcal{E}} T)) \xrightarrow{\sim} \mathrm{K}_0(\operatorname{mod} \underline{\Lambda})$. Moreover, since T is cluster-tilting, there are no non-split exact sequences in $\operatorname{add} T$, and so $\mathrm{K}_0(\operatorname{add} T)$ is freely generated by the indecomposable summands of T. Thus taking the alternating sum of terms gives an isomorphism $\mathrm{K}_0(\mathcal{H}^b(\operatorname{add}_{\mathcal{E}} T)) \xrightarrow{\sim} \mathrm{K}_0(\operatorname{add}_{\mathcal{E}} T)$ [Ros11].

These isomorphisms induce a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(\mathcal{H}^b_{\mathcal{E}\text{-ac}}(\operatorname{add}_{\mathcal{E}} T)) \arrow[r] \arrow[d] & \mathrm{K}_0(\mathcal{H}^b(\operatorname{add}_{\mathcal{E}} T)) \arrow[r] \arrow[d] & \mathrm{K}_0(\mathcal{D}^b(\mathcal{E})) \arrow[r] \arrow[d] & 0 \\
\mathrm{K}_0(\operatorname{mod} \underline{\Lambda}) \arrow[r, "\varphi"] & \mathrm{K}_0(\operatorname{add}_{\mathcal{E}} T) \arrow[r] & \mathrm{K}_0(\mathcal{E}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since the two leftmost vertical maps are isomorphisms, the induced map $\mathrm{K}_0(\mathcal{D}^b(\mathcal{E})) \to \mathrm{K}_0(\mathcal{E})$, which is again given by taking the alternating sum of terms, is also an isomorphism.

We claim that the map φ in the above diagram is given by composing the map from $\mathrm{K}_0(\operatorname{mod} \underline{\Lambda})$ to $\mathrm{K}_0(\operatorname{mod} \Lambda)$ induced by the inclusion of categories with the inverse of the isomorphism $F_* \colon \mathrm{K}_0(\operatorname{add}_{\mathcal{E}} T) \xrightarrow{\sim} \mathrm{K}_0(\operatorname{mod} \Lambda)$. Since $\underline{\Lambda}$ is finite dimensional, the Grothendieck group

$\mathrm{K}_0(\operatorname{mod}\underline{\Lambda})$ is spanned by the classes of the simple $\underline{\Lambda}$-modules S_k for $1 \leqslant k \leqslant r$, so it suffices to check that φ acts on these classes as claimed. Let

$$0 \to U_k^* \to Y_k \to U_k \to 0 \qquad \text{and} \qquad 0 \to U_k \to X_k \to U_k^* \to 0$$

be the exchange sequences associated to the summand U_k of T. Then $S_k \cong \operatorname{Ext}^1_{\mathcal{E}}(T, U_k)$ and there is an exact sequence

$$0 \to FU_k \to FX_k \to FY_k \to FU_k \to S_k \to 0.$$

From this we see that $[S_k] = [FX_k] - [FY_k] = F_*([X_k] - [Y_k])$ in $\mathrm{K}_0(\operatorname{mod}\Lambda)$, and so we want to show that $\varphi[S_k] = [X_k] - [Y_k]$. On the other hand, $[S_k]$ is the image of the class of the $\mathcal{E}$-acyclic complex

$$\cdots \to 0 \to U_k \to X_k \to Y_k \to U_k \to 0 \to \cdots$$

under Palu's isomorphism $\mathrm{K}_0(\mathcal{H}^b_{\mathcal{E}\text{-ac}}(\operatorname{add}_{\mathcal{E}} T)) \xrightarrow{\sim} \mathrm{K}_0(\operatorname{mod}\underline{\Lambda})$ (cf. [Pal09, Proof of Theorem 10]), and the image $\varphi[S_k]$ of this complex in $\mathrm{K}_0(\operatorname{add}_{\mathcal{E}} T)$ is $[X_k] - [Y_k]$, as we wanted.

Now applying $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the exact sequence

$$\mathrm{K}_0(\operatorname{mod}\underline{\Lambda}) \xrightarrow{\varphi} \mathrm{K}_0(\operatorname{add}_{\mathcal{E}} T) \longrightarrow \mathrm{K}_0(\mathcal{E}) \longrightarrow 0$$

shows that $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{E}), \mathbb{Z})$ is isomorphic to the kernel of $\varphi^t = \operatorname{Hom}_{\mathbb{Z}}(\varphi, \mathbb{Z})$, which we will show coincides with the space of gradings. Indeed, we may identify $\mathrm{K}_0(\operatorname{add}_{\mathcal{E}} T)$ with $\mathrm{K}_0(\operatorname{mod}\Lambda)$ via F_*, and then use the Euler form to identify $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\operatorname{mod}\Lambda), \mathbb{Z})$ with $\mathrm{K}_0(\operatorname{fd}\Lambda)$. Under this identification, we have $\varphi^t G = \langle -, G \rangle_e|_{\mathrm{K}_0(\operatorname{mod}\underline{\Lambda})}$, and so $G \in \ker \varphi^t$ if and only if it is a grading.

The claim that the isomorphism is given explicitly by $G \mapsto \underline{\deg}_G = \langle F(-), G \rangle_e$ can be seen by diagram chasing. □

The significance of this theorem is that, as in the triangulated case, it provides a basis-free method to identify gradings on Frobenius cluster categories and the cluster algebras they categorify. In the latter context, basis-free essentially means free of the choice of a particular cluster.

Specifically, as explained in more detail below, to establish that some categorical datum gives a grading, one only needs to check that that it respects exact sequences. This is potentially significantly easier than checking the vanishing of the product $B_T^t \underline{G}$ where B_T is given in terms of dimensions of Ext-spaces over the endomorphism algebra Λ of some cluster-tilting object T.

On the other hand, given some knowledge of the cluster algebra being categorified—in particular, knowing a cluster—one can use the above theorem to deduce information about the Grothendieck group of the Frobenius cluster category.

As promised at the end of Section 2, we can use Theorem 3.12 to see how the grading in a graded Frobenius cluster category is independent of the cluster-tilting object. Precisely, let $(\mathcal{E}, T, G)$ be a graded Frobenius cluster category, and let $\underline{\deg}_G$ be the corresponding function on $\mathrm{K}_0(\mathcal{E})$. Let $T' = \bigoplus_{i=1}^n T_i'$ be another cluster-tilting object, with $\Lambda' = \operatorname{End}_{\mathcal{E}}(T')^{\mathrm{op}}$, and denote the simple Λ'-modules by S_i' for $1 \leqslant i \leqslant n$. Using the inverse of the isomorphism of Theorem 3.12, we see that if G' in $\mathrm{K}_0(\operatorname{fd}\Lambda')$ is given by

$$G' = \sum_{i=1}^n \underline{\deg}_G(T_i')[S_i'],$$

then $(\mathcal{E}, T', G')$ is a graded Frobenius cluster category with $\underline{\deg}_G = \underline{\deg}_{G'}$, as one should expect. Note that this statement is not dependent on the existence of a sequence of mutations from T to T', which is not known to exist in general.

As was remarked about the triangulated case in [Gra15], these observations highlight how the categorification of a cluster algebra is able to see global properties, whereas the algebraic combinatorial mutation process is local.

The following example shows the theorem in action, although again we need the additional assumption of Hom-finiteness of $\mathcal{E}$.

Lemma 3.13. *Assume that $\mathcal{E}$ is Hom-finite and let P be a projective-injective object. Then* $\dim \mathrm{Hom}_{\mathcal{E}}(P, -)$ *and* $\dim \mathrm{Hom}_{\mathcal{E}}(-, P)$ *define gradings for $\mathcal{E}$.*

Proof: Since P is projective and injective, both $\mathrm{Hom}_{\mathcal{E}}(P, -)$ and $\mathrm{Hom}_{\mathcal{E}}(-, P)$ are exact functors, and so in each case taking the dimension yields a function in $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{E}), \mathbb{Z})$. Then the result follows immediately from Theorem 3.12. $\square$

In sufficiently nice cases, applying this result with a complete set of indecomposable projectives will yield that the dimension vector of a module is a (multi-)grading.

However, we remark that some care may be needed regarding which algebra we measure "dimension vector" over. If $\mathcal{E} \subset \mathrm{mod}\, \Pi$ for some algebra Π (as in most examples), then we may consider the Π-dimension vector of $X \in \mathcal{E}$, defined in the usual way. On the other hand, any Hom-finite Frobenius cluster category $\mathcal{E}$ is equivalent to $\mathrm{GP}(B) \subset \mathrm{mod}\, B$ for B the opposite endomorphism algebra of a basic projective generator $P = \bigoplus_{i=1}^{n} P_i$ of $\mathcal{E}$, by [KIWY15, Theorem 2.7]. Re-interpreting all of the objects of $\mathcal{E}$ as B-modules, the projective-injectives will now be precisely the projective B-modules, and $(\dim \mathrm{Hom}_{\mathcal{E}}(P_i, X))$ is the B-dimension vector of X (tautologically, since the equivalence $\mathcal{E} \to \mathrm{GP}(B)$ takes X to $\mathrm{Hom}_{\mathcal{E}}(P, X)$). Note that B may not be the same as the algebra Π from which $\mathcal{E}$ originated, and the B-dimension vector of a module may differ from the Π-dimension vector.

Given a complete set of projectives, it is natural to ask whether the associated grading might be standard, as defined in [Gra15]; we briefly recall this definition and some related facts.

Definition 3.14. Let $(\underline{x}, B)$ be a seed. We call a multi-grading G whose columns are a basis for the kernel of B a standard multi-grading, and call $(\underline{x}, B, G)$ a standard graded seed.

It is straightforward to see, from rank considerations, that mutation preserves the property of being standard. Moreover, as shown in [Gra15], if $(\underline{x}, B, G)$ is a standard graded seed and H is any grading for $(\underline{x}, B)$, then there exists an integer matrix $M = M(G, H)$ such that for any cluster variable y in $\mathcal{A}(\underline{x}, B, H)$ we have

$$\underline{\deg}_H(y) = \underline{\deg}_G(y) M,$$

where on the right-hand side we regard y as a cluster variable of $\mathcal{A}(\underline{x}, B, G)$ in the obvious way.

That is, to describe the degree of a cluster variable of a graded cluster algebra $\mathcal{A}(\underline{x}, B, H)$, it suffices to know its degree with respect to some standard grading G and the matrix $M = M(G, H)$ transforming G to H. In particular, to understand the distribution of the degrees of cluster variables, it suffices to know this for standard gradings.

Since the statement applies in the particular case when G and H are both standard, we see that from one choice of basis for the kernel of B, we obtain complete information. For if we chose a second basis, the change of basis matrix tells us how to transform the degrees. Hence up to a change of basis, there is essentially only one standard grading for each seed.

Then, depending on the particular Frobenius cluster category at hand, if we have knowledge of the rank of the exchange matrix, we may be able to examine categorical data such as the number of projective-injective modules or dimension vectors and hence try to find a basis for the space of gradings.

For example, for a basic cluster-tilting object T in $\mathcal{E}$ a Hom-finite Frobenius cluster category, we have $n-r$ projective-injective summands in T: if the exchange matrix B_T has full rank, a basis for the space of gradings has size $n-r$ so that, via Lemma 3.10, a canonical standard grading is given by the set $\{[FT_i] \mid i > r\}$, which is linearly independent since it is a subset of the basis of projectives for $\mathrm{K}_0(\mathrm{fd}\,\Lambda) = \mathrm{K}_0(\mathrm{mod}\,\Lambda)$.

From knowledge of this standard grading, we then obtain any other grading by means of some linear transformation. In the next section, we do this for two important examples.

4 Examples of graded Frobenius cluster categories

4.1 Frobenius cluster categories associated to partial flag varieties

Let $\mathfrak{g}$ be the Kac–Moody algebra associated to a symmetric generalised Cartan matrix. Let Δ be the associated Dynkin graph and pick an orientation $\vec{\Delta}$. Let Q be the quiver obtained from $\vec{\Delta}$ by adding an arrow $\alpha^*\colon j \to i$ for each arrow $\alpha\colon i \to j$ of $\vec{\Delta}$. Then the preprojective algebra of Δ is

$$\Pi = \mathbb{C}Q / \sum_{\alpha \in \vec{\Delta}} [\alpha, \alpha^*],$$

which is, up to isomorphism, independent of the choice of orientation $\vec{\Delta}$.

For each $w \in W$, the Weyl group of $\mathfrak{g}$, Buan, Iyama, Reiten and Scott [BIRS09] have introduced a category $\mathcal{C}_w$; the following version of its construction follows [GLS11], and is dual to the original.

Assume w has finite length and set $l(w) = n$; we do this for consistency with the notation used above but note that other authors (notably [GLS11], [GLS13]) use r and their n is our $n-r$.

Set $\hat{I}_i$ to be the indecomposable injective Π-module with socle S_i, the 1-dimensional simple module supported at the vertex i of Q.

Given a module W in $\mathrm{mod}\,\Pi$, we define

- $\mathrm{soc}_{(l)}(W) := \sum_{\substack{U \leqslant W \\ U \cong S_l}} U$ and
- $\mathrm{soc}_{(l_1, l_2, \ldots, l_s)}(W) := W_s$ where the chain of submodules $0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_s \subseteq W$ is such that $W_p / W_{p-1} \cong \mathrm{soc}_{(l_p)}(W/W_{p-1})$.

Let $\mathbf{i} = (i_n, \ldots, i_1)$ be a reduced expression for w. Then for $1 \leqslant s \leqslant n$, we define $V_{\mathbf{i},s} := \mathrm{soc}_{(i_s, i_{s-1}, \ldots, i_1)}(\hat{I}_{i_s})$. Set $V_{\mathbf{i}} = \bigoplus_{s=1}^{n} V_{\mathbf{i},s}$ and let I be the subset of $\{1, \ldots, n\}$ such that the modules $V_{\mathbf{i},i}$ for $i \in I$ are $\mathcal{C}_w$-projective-injective. Set $I_{\mathbf{i}} = \bigoplus_{i \in I} V_{\mathbf{i},i}$ and $n - r = |I|$. Note that this is also the number of distinct simple reflections appearing in $\mathbf{i}$.

Define

$$\mathcal{C}_{\mathbf{i}} = \mathrm{Fac}(V_{\mathbf{i}}) \subseteq \mathrm{nil}\,\Pi.$$

That is, $\mathcal{C}_{\mathbf{i}}$ is the full subcategory of $\mathrm{mod}\,\Pi$ consisting of quotient modules of a direct sum of a finite number of copies of $V_{\mathbf{i}}$.

Then $\mathcal{C}_{\mathbf{i}}$ and $I_{\mathbf{i}}$ are independent of the choice of reduced expression $\mathbf{i}$ (although $V_{\mathbf{i}}$ is not), so that we may write $\mathcal{C}_w := \mathcal{C}_{\mathbf{i}}$ and $I_w := I_{\mathbf{i}}$. It is shown in [BIRS09] that $\mathcal{C}_w$ is a stably 2-Calabi–Yau

Frobenius category. Moreover $\mathcal{C}_w$ has cluster-tilting objects: $V_{\mathbf{i}}$ is one such. Indeed, cluster-tilting objects are maximal rigid, and vice versa. The indecomposable $\mathcal{C}_w$-projective-injective modules are precisely the indecomposable summands of I_w, and $\mathcal{C}_w = \mathrm{Fac}(I_w)$.

Furthermore, it is also shown in [GLS11, Proposition 2.19] that the global dimension condition of Definition 3.6 also holds, leaving only the Krull–Schmidt condition. By [Kra15, Corollary 4.4], we should check that the endomorphism algebras of objects of $\mathcal{C}_w$ are semiperfect, and that this category is idempotent complete. The first of these properties holds since $\mathcal{C}_w$ is Hom-finite. The second follows from the fact that $\mathcal{C}_w$ is a full subcategory of the idempotent complete category $\mathrm{mod}(\Pi/\operatorname{Ann} I_w)$, and that if M is an object of $\mathrm{Fac}(I_w)$, then so are all direct summands of M.

We conclude that $\mathcal{C}_w$ is a Frobenius cluster category, in the sense of Definition 3.6.

Let $\Lambda = \mathrm{End}_{\mathcal{C}_w}(V_{\mathbf{i}})^{\mathrm{op}}$ and $F = \mathrm{Hom}_{\mathcal{C}_w}(V_{\mathbf{i}}, -)$. Then, as above, the modules $P_s := FV_{\mathbf{i},s}$ for $1 \leqslant s \leqslant n$ are the indecomposable projective Λ-modules and the tops of these, S_s, are the simple Λ-modules. Recall that the exchange matrix obtained from the quiver of Λ, which we shall call $B_{\mathbf{i}}$, has entries

$$(B_{\mathbf{i}})_{ij} = \dim \mathrm{Ext}^1_\Lambda(S_i, S_j) - \dim \mathrm{Ext}^1_\Lambda(S_j, S_i)$$

for $1 \leqslant i \leqslant n$ and $j \notin I$, so that the r columns of $B_{\mathbf{i}}$ correspond to to the mutable summands $V_{\mathbf{i},j}$, $j \notin I$, of $V_{\mathbf{i}}$.

Let $L_{\mathbf{i}}$ be the $n \times n$ matrix with entries

$$(L_{\mathbf{i}})_{jk} = \dim \mathrm{Hom}_\Pi(V_{\mathbf{i},j}, V_{\mathbf{i},k}) - \dim \mathrm{Hom}_\Pi(V_{\mathbf{i},k}, V_{\mathbf{i},j}).$$

By [GLS13, Proposition 10.1] we have

$$\sum_{l=1}^{n} (B_{\mathbf{i}})_{lk} (L_{\mathbf{i}})_{lj} = 2\delta_{jk},$$

and hence the matrix $B_{\mathbf{i}}$ has maximal rank, namely r.

It follows that there exists some standard grading $G_{\mathbf{i}} = (G_1, \ldots, G_{n-r}) \in \mathrm{K}_0(\mathrm{mod}\,\Lambda)^{n-r}$ for $\mathcal{C}_w$ and $(\mathcal{C}_w, V_{\mathbf{i}}, G_{\mathbf{i}})$ is a graded Frobenius cluster category. As discussed above, such a standard grading can be used to construct all other gradings, so our goal is to identify one.

We have additional structure on $\mathcal{C}_w$ that we may make use of. Namely, $\mathcal{C}_w$ is Hom-finite and we may apply Lemma 3.10 with respect to the $\mathcal{C}_w$-projective-injective modules $V_{\mathbf{i},i}$ that are the indecomposable summands of $I_{\mathbf{i}}$.

The resulting grading $[FV_{\mathbf{i},i}]$, $i \in I$, is standard, since its $n-r$ components are a subset of the basis of projectives for $\mathrm{K}_0(\mathrm{mod}\,\Lambda)$, and so in particular are linearly independent. By Theorem 3.12, the existence of this standard grading implies that the Grothendieck group $\mathrm{K}_0(\mathcal{C}_w)$ has rank $n-r$.

We wish to understand this standard grading more explicitly. Note that the objects of $\mathcal{C}_w$ are Π-modules and we may consider dimension vectors with respect to the Π-projective modules.

Then we notice that in fact the grading by $([FV_{\mathbf{i},i}])_{i \in I}$ is equal to the Π-dimension vector grading in the case at hand. This is because, by Lemma 3.10, the degree of X with respect to $[FV_{\mathbf{i},i}]$ is $\dim \mathrm{Hom}_\Pi(X, V_{\mathbf{i},i})$, and each $V_{\mathbf{i},i}$ is both a submodule and a minimal right $\mathcal{C}_w$-approximation of an indecomposable injective $\hat{I}_i$ for Π, so $\mathrm{Hom}_\Pi(X, V_{\mathbf{i},i}) = \mathrm{Hom}_\Pi(X, \hat{I}_i)$, the dimensions of the latter giving the Π-dimension vector of X.

In [GLS11, Corollary 9.2], Geiß, Leclerc and Schröer have shown that

$$\underline{\dim}_\Pi V_{\mathbf{i},k} = \omega_{i_k} - s_{i_1} s_{i_2} \cdots s_{i_k}(\omega_{i_k})$$

for all $1 \leqslant k \leqslant n$, where the ω_j are the fundamental weights for $\mathfrak{g}$ and the s_j the Coxeter generators for W. This enables us to construct the above grading purely combinatorially.

Example 4.1. We consider the following seed associated to $\mathfrak{g}$ of type A_5 with

$$\mathbf{i} = (3, 2, 1, 4, 3, 2, 5, 4, 3),$$

as given in [GLS13, Example 12.11]. The modules $V_i := V_{\mathbf{i},i}$, in terms of the usual representation illustrating their composition factors as Π-modules, are

$$V_1 = \begin{array}{ccccc} & & 3 & & \end{array} \qquad V_2 = \begin{array}{ccccc} & & 3 & & \\ & & & 4 & \end{array} \qquad V_3 = \begin{array}{ccccc} & & 3 & & \\ & & & 4 & \\ & & & & 5 \end{array}$$

$$V_4 = \begin{array}{ccccc} & & 3 & & \\ & 2 & & & \end{array} \qquad V_5 = \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ & & 3 & & \end{array} \qquad V_6 = \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ & & 3 & & 5 \\ & & & 4 & \end{array}$$

$$V_7 = \begin{array}{ccccc} & & 3 & & \\ & 2 & & & \\ 1 & & & & \end{array} \qquad V_8 = \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 1 & & 3 & & \\ & 2 & & & \end{array} \qquad V_9 = \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 1 & & 3 & & 5 \\ & 2 & & 4 & \\ & & 3 & & \end{array}$$

The exchange quiver for this seed is

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & \boxed{V_3} \\
V_4 \arrow[r] \arrow[d] & V_5 \arrow[r] \arrow[d] \arrow[ul] & \boxed{V_6} \arrow[ul] \\
\boxed{V_7} & \boxed{V_8} \arrow[ul] & \boxed{V_9} \arrow[ul]
\end{tikzcd}$$

It is straightforward to see that Π-dimension vectors yield a grading: for example, looking at the vertex corresponding to V_1, the sums of the dimension vectors of incoming and outgoing arrows are $[0, 1, 2, 1, 0]$ and $[0, 1, 1, 0, 0] + [0, 0, 1, 1, 0]$ respectively.

4.2 Grassmannian cluster categories

Let Π be the preprojective algebra of type A_{n-1}, with vertices numbered sequentially, and let Q_k be the injective module at the k-th vertex. In [GLS08], Geiß, Leclerc and Schröer show that the category Sub Q_k of submodules of direct sums of copies of Q_k "almost" categorifies the cluster algebra structure on the homogeneous coordinate ring of the Grassmannian of k-planes in $\mathbb{C}^n$, but is missing a single indecomposable projective object corresponding to one of the frozen variables of this cluster algebra. The category $\operatorname{Sub} Q_k$ is in fact dual to one of the categories $\mathcal{C}_w$ introduced in the previous section, for $\Delta = \mathsf{A}_{n-1}$ and w a particular Weyl group element depending on k, so it is a Frobenius cluster category in the same way.

Jensen, King and Su [JKS16] complete the categorification via the category $\mathrm{CM}(A)$ of maximal Cohen–Macaulay modules for a Gorenstein order A (depending on k and n) over $Z = \mathbb{C}[[t]]$.

One description of A is as follows. Let Δ be the graph (of affine type A_{n-1}) with vertex set given by the cyclic group $\mathbb{Z}_n$, and edges between vertices i and $i+1$ for all i. Let Π be the completion of the preprojective algebra on Δ with respect to the arrow ideal. Write x for the sum of "clockwise" arrows $i \to i+1$, and y for the sum of "anti-clockwise" arrows $i \to i-1$. Then we have

$$A = \Pi/\langle x^k - y^{n-k}\rangle.$$

In this description, Z may be identified with the centre $\mathbb{C}[[xy]]$ of A.

Jensen, King and Su also show [JKS16, Theorem 4.5] that there is an exact functor $\pi\colon \mathrm{CM}(A) \to \mathrm{Sub}\ Q_k$, corresponding to the quotient by the ideal generated by P_n, and that for any $N \in \mathrm{Sub}\ Q_k$, there is a unique (up to isomorphism) minimal M in $\mathrm{CM}(A)$ with $\pi M \cong N$ and M having no summand isomorphic to P_n. Such an M satisfies $\mathrm{rk}(M) = \dim \mathrm{soc}\ \pi M$, where $\mathrm{rk}(M)$ is the rank of each vertex component of M, thought of as a Z-module.

We now show that $\mathrm{CM}(A)$ is again a Frobenius cluster category. Properties of the algebra A mean that an A-module is maximal Cohen–Macaulay if and only if it is free and finitely generated as a Z-module. Since Z is a principal ideal domain, and hence Noetherian, any submodule of a free and finitely generated Z-module is also free and finitely generated, and so $\mathrm{CM}(A)$ is closed under subobjects. In particular, $\mathrm{CM}(A)$ is closed under kernels of epimorphisms. Moreover [JKS16, Corollary 3.7], $A \in \mathrm{CM}(A)$, and so $\Omega(\mathrm{mod}\, A) \subseteq \mathrm{CM}(A)$.

As a Z-module, any object $M \in \mathrm{CM}(A)$ is isomorphic to Z^k for some k, so we have that $\mathrm{End}_Z(M)^{\mathrm{op}} \cong Z^{k^2}$ is a finitely generated Z-module. Since Z is Noetherian, the algebra $\mathrm{End}_A(M)^{\mathrm{op}} \subseteq \mathrm{End}_Z(M)^{\mathrm{op}}$ is also finitely generated as a Z-module. Thus $\mathrm{End}_A(M)^{\mathrm{op}}$ is Noetherian, as it is finitely generated as a module over the commutative Noetherian ring Z. We may now apply [Pre15, Proposition 3.6] to see that any cluster-tilting object $T \in \mathrm{CM}(A)$ satisfies $\mathrm{gldim}\,\mathrm{End}_A(T)^{\mathrm{op}} \leqslant 3$. Moreover [JKS16, Corollary 4.6], $\underline{\mathrm{CM}}(A) = \underline{\mathrm{Sub}}\, Q_k$, so $\underline{\mathrm{CM}}(A)$ is 2-Calabi–Yau, and $\mathrm{CM}(A)$ is a Frobenius cluster category.

Unlike $\mathrm{Sub}\ Q_k$ and the $\mathcal{C}_w$, the category $\mathrm{CM}(A)$ is not Hom-finite. However, as already observed, the endomorphism algebras of its objects are Noetherian, so we may apply our general theory to this example.

In their study of the category $\mathrm{CM}(A)$, Jensen, King and Su show the following. Let

$$\mathbb{Z}^n(k) = \{x \in \mathbb{Z}^n \mid k \text{ divides} \textstyle\sum_i x_i\}$$

with basis $\alpha_1, \ldots, \alpha_{n-1}, \beta_{[n]}$, where the $\alpha_j = e_{j+1} - e_j$ are the negative simple roots for $\mathrm{GL}_n(\mathbb{C})$ and $\beta_{[n]} = e_1 + \cdots + e_k$ is the highest weight for the representation $\bigwedge^k(\mathbb{C}^n)$.

Then by [JKS16, §8] we have that $\mathrm{K}_0(\mathrm{CM}(A)) \cong \mathrm{K}_0(A) \cong \mathbb{Z}^n(k)$; let $G\colon \mathrm{K}_0(\mathrm{CM}(A)) \to \mathbb{Z}^n(k)$ denote the composition of these isomorphisms. The $\mathrm{GL}_n(\mathbb{C})$-weight of the cluster character of $M \in \mathrm{CM}(A)$ (called $\widetilde{\psi}_M$ in [JKS16]) is given by the coefficients in an expression for $G[M] \in \mathbb{Z}^n(k)$ in terms of the basis of $\mathbb{Z}^n(k)$ given above [JKS16, Proposition 9.3], and thus this weight defines a group homomorphism $\mathrm{K}_0(\mathrm{CM}(A)) \to \mathbb{Z}^n$.

Said in the language of this paper, $\mathrm{CM}(A)$ is a graded Frobenius cluster category with respect to $\mathrm{GL}_n(\mathbb{C})$-weight, this giving a standard multi-grading.

Let $\delta\colon \mathbb{Z}^n(k) \to \mathbb{Z}$ be the (linear) function $\delta(x) = \frac{1}{k}\sum_i x_i$. By the linearity of gradings, composing G with δ yields a $\mathbb{Z}$-grading on $\mathrm{CM}(A)$ also. Explicitly, $\delta(x)$ is the coefficient of $\beta_{[n]}$, and is also equal to the dimension of the socle of πM, which is equal to $\mathrm{rk}(M)$, which is equal to the degree of the cluster character of $M \in \mathrm{CM}(A)$ as a homogeneous polynomial in the Plücker coordinates of the Grassmannian.

It is well known that the cluster structure on the Grassmannian is graded with respect to either the $\mathrm{GL}_n(\mathbb{C})$-weight (also called the content of a minor, and, by extension, of a product of minors) or the natural grading associated to the Plücker embedding. The results of [JKS16] show that these gradings are indeed naturally reflected in the categorification of that cluster structure. This opens the possibility of attacking some questions on, for example, the number of cluster variables of a given degree by examining rigid indecomposable modules in $\mathrm{CM}(A)$ of the corresponding rank, say. We hope to return to this application in the future.

Of course, one can also argue directly that $\mathrm{rk}(M)$ yields a grading on $\mathrm{CM}(A)$, considering it as a function on $\mathrm{K}_0(\mathrm{CM}(A))$. Note that the socle dimension of πM is not a grading on Sub Q_k, but rather it is the datum within Sub Q_k that specifies how one should lift πM to M (see [JKS16, §2] for an illustration of this). As described in the previous section, Sub Q_k (in its guise as one of the $\mathcal{C}_w$) does admit gradings, such as the grading describing the degree of the cluster character of $\pi M \in$ Sub Q_k (called $\psi_{\pi M}$ in [JKS16]) with respect to the standard matrix generators.

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