

Application of the Quadrupole Method for Simulation of Passive Thermography

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ABSTRACT

Passive thermography has been shown to be an effective method for in situ and real time nondestructive evaluation (NDE) to measure damage growth in a composite structure during cyclic loading. The heat generation by subsurface flaw results in a measurable thermal profile at the surface. This paper models the heat generation as a planar subsurface source and calculates the resultant temperature profile at the surface using a three dimensional quadrupole. The results of the model are compared to finite difference simulations of the same planar sources.

Keywords: passive thermography, composite, nondestructive evaluation, simulation

1. INTRODUCTION

Active thermography has been shown to be a very effective method for rapid inspection of large carbon fiber reinforced polymer (CFRP) composite structures and is finding increased acceptance. Active thermography uses an external heat source to induce a temperature gradient in the material or structure and measures the variation in thermal response caused by a flaw. Passive thermography has been shown to be a viable inspection technique for hidden defects or damage in structures such as the road or bridge pavement. These thermographic inspections rely on thermal gradients in the material or structure, which are due to conditions (such as daytime heating) to detect the damage.

For monitoring the damage growth during carbon fiber reinforced polymer fiber (CFRP) composite fatigue testing, passive thermography is almost an ideal technique. Capturing the progression of damage only requires an infrared imager monitoring the surface of the composite being tested. The CFRP composite also normally has a high enough surface emissivity not to require surface preparation. During the fatigue testing there are at least two sources of heating. The first is a result of the dilation deformation of the material and is commonly referred to as the thermoelastic effect. For small cyclic strains with sufficiently high frequencies, there is no heat transfer to surrounding environment and the mechanism is adiabatic and the time average of the temperature is equal to the ambient temperature. It has been shown that it is possible to estimate the strain in the material or structure by measuring the cyclic temperature.

A more interesting heat source during fatigue testing is a flaw in the material. The flaw changes the amplitude of the thermoelastic heat generated by changing the load distribution in the vicinity of the flaw. The amplitude change is not typically accompanied by a significant phase change. However, if the cyclic loading results two surfaces rubbing together, friction generates heat at the flaw. There is phase difference between the cyclic heat generation and the cyclic temperature change at the surface, which is a function of the frequency and flaw depth. This paper focuses on simulating the heat diffusion from the internal source to the surface.

While it is desirable to simulate the passive thermography technique, there are few analytical or series solutions for the diffusion of heat in a solid and they are only for simple geometries.¹ Numerical techniques such as finite difference and finite element methods are well suited for such problems,^{2,3} but can be computationally

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intensive. Both commercial⁴⁻⁷ and noncommercial^{8,9} simulation packages have been successfully applied to simulating flash thermography nondestructive evaluation techniques. However, there have been limited attempts to simulate passive thermography in composites, in part since passive thermography is primarily a laboratory technique.

A common method for solving the heat equation is to solve the Laplace transform of the heat equation,¹ then invert the Laplace transform to produce a time domain response. There are a limited number of cases where it is possible to analytically invert the Laplace transform, however, it is a very powerful technique for one dimensional multilayered materials when an analytic solution exists for the Laplace transform that can be numerically inverted. This is often much faster and more accurate than finite difference or finite element techniques for the same configuration, since it is possible to solve for only the times of interest.

Often this is referred to as the thermal quadrupole method, and is applicable for modeling responses in materials and structures in general¹⁰ as well as flash thermography. For flash thermography, it has been used extensively for simulating the thermal response of multilayer systems with and without contact resistances at the interfaces.¹¹⁻¹⁶ It has also been shown to be applicable for three dimensional configurations, in particular for delaminations in composites¹⁷⁻¹⁹ by representing the spatial variation in temperature as a cosine series and solving for the coefficients.

This paper follows a similar approach, except the source is interior to the solid and the surface thermal response is calculated. The method is illustrated by solving for a temperature response at the surface for a given internal source and comparing that to a finite difference solution for the same configuration. While comparisons to measurements is preferred, the exact nature of internal sources is difficult to characterize.

2. QUADRUPOLE METHOD OF SIMULATING THERMAL RESPONSE OF LAYERS

A typical application of the quadrupole method is solving the one dimensional heat equation of multilayer systems. For one dimensional problems, a matrix is used to represent the relationship between the temperature and flux of one surface of a layer to the temperature and flux at another surface (see Fig. 1). If two of the quantities are known (typically the fluxes at the surfaces), then it is possible to solve for the other two. For simulating passive thermography, the heat source is not external to the material, but is within the material. The next subsection illustrates the quadrupole method in one dimension assuming a flux source at an inner surface. This facilitates understanding of the extension of the technique to higher dimensions.

2.1 Quadrature Method in One Dimension

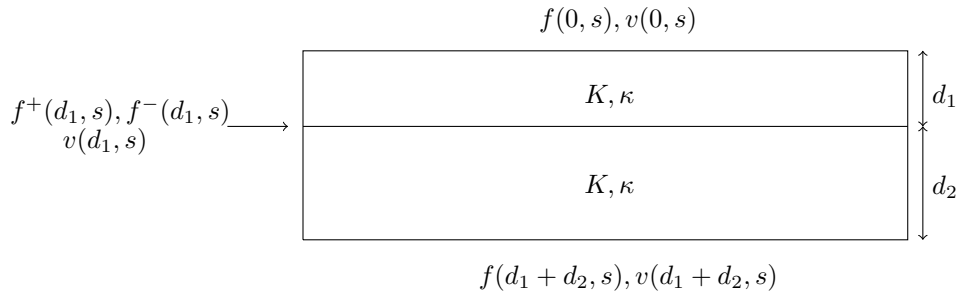


Figure 1. One dimensional configuration for two layer system with a flux at the surface between the two layers.

A one dimensional solution for the Laplace transform of temperature in a homogeneous material is¹

$$v(z, s) = v(0, s) \cosh \left(z \sqrt{\frac{s}{\kappa}} \right) - f(0, s) \frac{\sinh \left(z \sqrt{\frac{s}{\kappa}} \right)}{K \sqrt{\frac{s}{\kappa}}}. \quad (1)$$

where K and κ are the thermal conductivity and diffusivity, respectively, $v(0, s)$ and $f(0, s)$ are the Laplace transform of temperature and flux at $z = 0$, s is the complex frequency variable and z is the vertical location. A similar expression for the Laplace transform of the flux as a function of position is

$$f(z, s) = f(0, s) \cosh\left(z\sqrt{\frac{s}{\kappa}}\right) - v(0, s) \sinh\left(z\sqrt{\frac{s}{\kappa}}\right) K\sqrt{\frac{s}{\kappa}}. \quad (2)$$

A simple way to express both of the equations is with the matrix formula

$$\begin{bmatrix} \cosh(zq) & -\frac{\sinh(zq)}{Kq} \\ -Kq \sinh(zq) & \cosh(zq) \end{bmatrix} \begin{bmatrix} v(0, s) \\ f(0, s) \end{bmatrix} = \begin{bmatrix} v(z, s) \\ f(z, s) \end{bmatrix}, \quad (3)$$

where $q = \sqrt{\frac{s}{\kappa}}$. From this matrix equation, the Laplace transform of the temperature at the two surfaces can be solved for in terms of the flux, and is found to be

$$v(0, s) = \frac{f(0, s) \coth(dq) - f(d, s) \operatorname{csch}(dq)}{Kq} \quad (4)$$

and

$$v(d, s) = \frac{\operatorname{csch}(dq)f(0, s) - f(d, s) \cosh(dq)}{Kq}, \quad (5)$$

where d is the thickness of the layer. For flash heating at the front surface ($z = 0$) and an insulated back surface ($z = d$), the thermal response of the layer can be found by setting $f(0, s) = f_0$, which represents impulse heating and $f(d, s) = 0$, which also can be analytically inverted to give a well known series solution.¹

The advantage of this representation is obvious when solving for the thermal response of a multiple layer system where each layer is represented by a matrix similar to that given in Eq.3. The matrix is then a transfer matrix, and the combined response of multiple layers is obtained by matrix multiplication. For the system shown in Fig. 1, the two layers are separated into two sets of equations given by

$$\begin{bmatrix} \cosh(q_1 d_1) & -\frac{\sinh(q_1 d_1)}{Kq} \\ -Kq \sinh(q_1 d_1) & \cosh(q_1 d_1) \end{bmatrix} \begin{bmatrix} v(0, s) \\ f(0, s) \end{bmatrix} = \begin{bmatrix} v(d_1, s) \\ f^+(d_1, s) \end{bmatrix} \quad (6)$$

and

$$\begin{bmatrix} \cosh(q_2 d_2) & -\frac{\sinh(q_2 d_2)}{Kq} \\ -Kq \sinh(q_2 d_2) & \cosh(q_2 d_2) \end{bmatrix} \begin{bmatrix} v(d_1, s) \\ f^-(d_1, s) \end{bmatrix} = \begin{bmatrix} v(d_1 + d_2, s) \\ f(d_1 + d_2, s) \end{bmatrix}, \quad (7)$$

where d_1 and d_2 are the thicknesses for the first and second layers, $f^+(d_1, s)$ and $f^-(d_1, s)$ are the fluxes from the interface into the upper and lower layers respectively and $q = \sqrt{s/\kappa}$. The boundary conditions between the two layers represented by this matrix equation are the flux discontinuities due to the heat generated at the interface and the temperature is continuous. For no heat generated at the interface, $f^+(d_1, s)$ and $f^-(d_1, s)$ are equal (flux is continuous).

Considering a special case similar to the passive thermography configuration, with the flux at the front and back surfaces equal to the convection losses from the surfaces with an ambient temperature of zero and the $f^+(d_1, s)$ and $f^-(d_1, s)$ are set to $f_1 - f_s(s)$ and $f_1 + f_s(s)$, respectively, where f_1 is an unknown corresponding to the heat flux across the surface and $f_s(s)$ is the Laplace transform of $F_s(t)$, the heat source at the interface. Eqs. 6 and 7, then become

$$\begin{bmatrix} \cosh(qd_1) & -\frac{\sinh(qd_1)}{Kq} \\ -Kq \sinh(qd_1) & \cosh(qd_1) \end{bmatrix} \begin{bmatrix} v(0, s) \\ -h v(0, s) \end{bmatrix} = \begin{bmatrix} v(d_1, s) \\ f_1 - f_s(s) \end{bmatrix} \quad (8)$$

and

$$\begin{bmatrix} \cosh(qd_2) & \frac{-\sinh(qd_2)}{Kq} \\ -Kq \sinh(qd_2) & \cosh(qd_2) \end{bmatrix} \begin{bmatrix} v(d_1, s) \\ f_1 + f_s(s) \end{bmatrix} = \begin{bmatrix} v(d_1 + d_2, s) \\ h v(d_1 + d_2, s) \end{bmatrix}, \quad (9)$$

where h is the heat transfer coefficient. If $f_s(s)$ is given, it is possible to solve this set of equations for temperature at either the front or back surface.

If the thermal properties of the two layers are the same, with a thermal conductivity of K and a thermal diffusivity of κ , then Laplace transform for $v(0, s)$ is

$$v(0, s) = 2 \frac{f_s(s)}{K} \frac{\kappa \left(\sqrt{\frac{s}{\kappa}} \cosh(d_2 \sqrt{\frac{s}{\kappa}}) + \frac{h}{K} \sinh(d_2 \sqrt{\frac{s}{\kappa}}) \right)}{2 \frac{h}{K} \sqrt{\frac{s}{\kappa}} \cosh((d_1 + d_2) \sqrt{\frac{s}{\kappa}}) + \left(s + \left(\frac{h}{K} \right)^2 \kappa \right) \sinh((d_1 + d_2) \sqrt{\frac{s}{\kappa}})} \quad (10)$$

which can be inverted numerically to calculate the temperature as a function of time. Since there are instabilities in the numerical inverse Laplace transform of a sinusoidal function, the numerical inversion is performed for an impulse function. The time response for this solution is convolved with the time dependence of the source. The time response for $F_s(t) = 1 - \cos(2\pi t)$ for $t > 0$ is shown in Fig. 2. The two layers are both 0.1 cm thick,

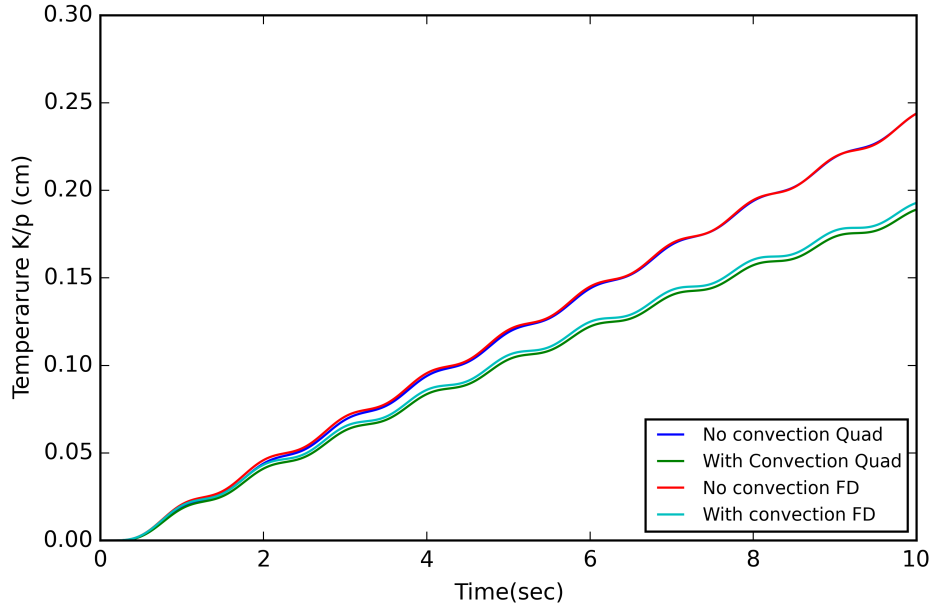


Figure 2. The thermal response calculated from the quadrapole and finite difference methods for a $F_s(t) = 1 - \cos(2\pi t)$ excitation 0.1 cm below the surface with and without convection.

with a thermal diffusivity of $0.005 \text{ cm}^2/\text{sec}$. Included in the figure are responses for no convection loss ($h = 0$), convection loss of $h/K = 1.0/\text{cm}$ and the finite difference solution for both cases. There is reasonable agreement between the finite difference solution and the quadrapole solutions.

For no convection and an impulse heat flux, it is possible to analytically invert the $v(0, s)$ (Eq.10) to give a series solution for the thermal response of a one dimension layer. After convolving that solution with $\cos(\omega t)$, the surface temperature for a flux source with a time dependence of $\cos(\omega t)$ at d_1 is found to be

$$v_{\cos}(0, t) = \frac{p\kappa}{K(d_1 + d_2)} (s_1(t) + \cos(\omega t)s_2 + \sin(\omega t)s_3) \quad (11)$$

where p is the power amplitude of the source and s_1 , s_2 and s_3 are given by the equations

$$s_1(t) = -2 \sum_{n=1}^{\infty} \frac{(d_1 + d_2)^2 \cos\left(\frac{\pi n d_1}{(d_1 + d_2)}\right) e^{-\frac{\pi^2 \kappa n^2 t}{(d_1 + d_2)^2}}}{(d_1 + d_2)^4 \omega^2 + \pi^4 \kappa^2 n^4}, \quad (12)$$

$$s_2 = 2 \sum_{n=1}^{\infty} \frac{(d_1 + d_2)^2 n^2 \pi^2 \kappa \cos\left(\frac{\pi n d_1}{(d_1 + d_2)}\right)}{(d_1 + d_2)^4 \omega^2 + \pi^4 \kappa^2 n^4} \quad (13)$$

and

$$s_3 = \frac{1}{\omega} + 2 \sum_{n=1}^{\infty} \frac{(d_1 + d_2)^4 \omega \cos\left(\frac{\pi n d_1}{(d_1 + d_2)}\right)}{(d_1 + d_2)^4 \omega^2 + \pi^4 \kappa^2 n^4}. \quad (14)$$

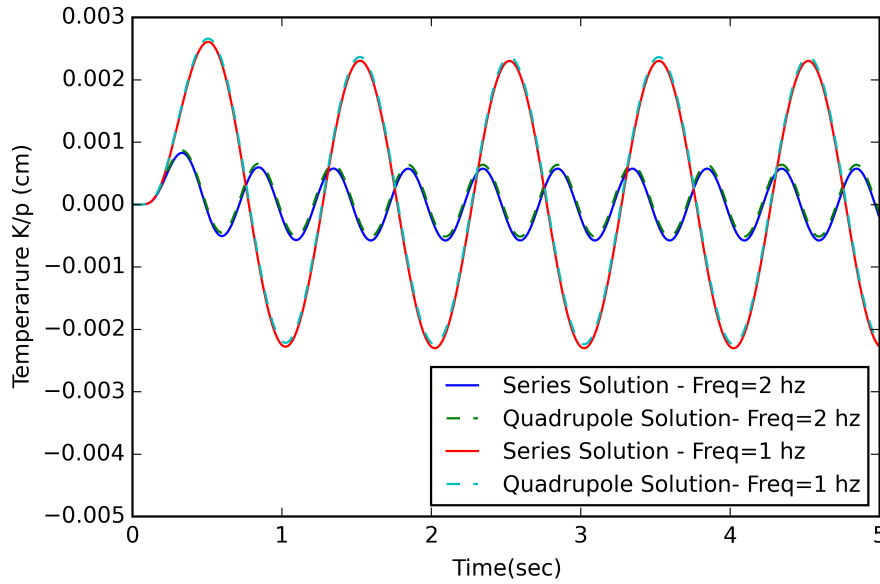


Figure 3. The thermal response calculated from the quadrupole and series solution for a cosine excitation 0.1 cm below the surface.

A comparison of the solutions is shown in Fig. 3. Shown are the results for a two layer system, where the thermal properties of the two layers are the same, the thermal diffusivity is $0.005 \text{ cm}^2/\text{sec}$, p/K is 1 and $d_1=0.1\text{cm}$ and $d_2=0.1\text{cm}$. The solutions are shown for both a frequency of 1 and 2 hertz. The series solution is represented as a solid line and the quadrupole solution is represented as a dash line. As can be seen from the figure, the solutions are nearly identical for both frequencies.

For $t \gg (d_1 + d_2)^2/(\pi^2 \kappa)$, the $s_1(t)$ is approximately zero, s_2 and s_3 give the amplitude and phase of the long term sinusoidal solution. The amplitude for different depths is shown in Fig. 4. The depth steps are each 0.01 cm which is approximately a single ply thickness in a composite. As would be expected, the amplitude significantly drops as the frequency increases. As can be seen in the figure, there is a dramatic decrease in the amplitude as the depth of the source increases.

The phase change is shown in Fig. 5. The calculated phase goes from $-\pi$ to π , however, to show the continuous variation in the phase, when the phase decreases by more than π , 2π is added. As can be seen for the figure, there is almost a linear relationship between the phase and the depth of the source. As is expected from a simple model, the rate of change increases as the frequency changes.

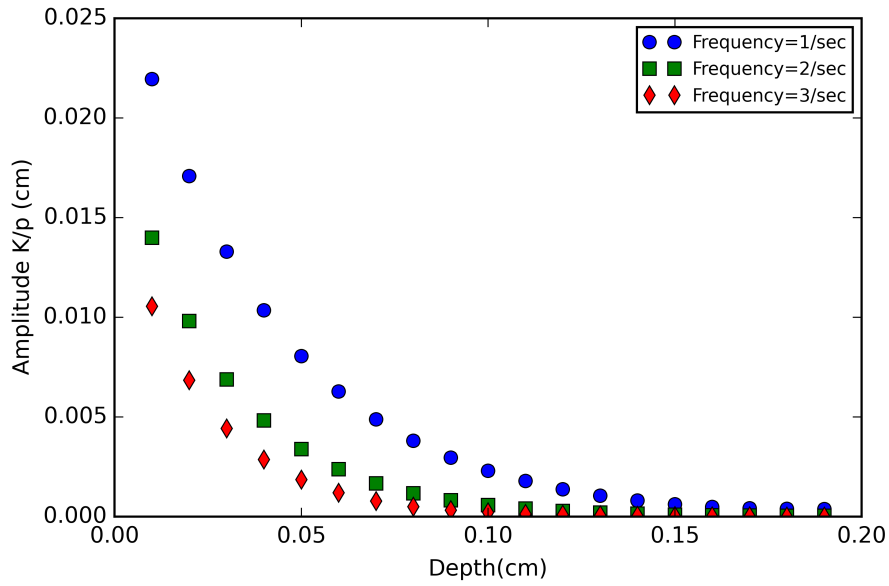


Figure 4. The amplitude of thermal response at different depths below the surface.

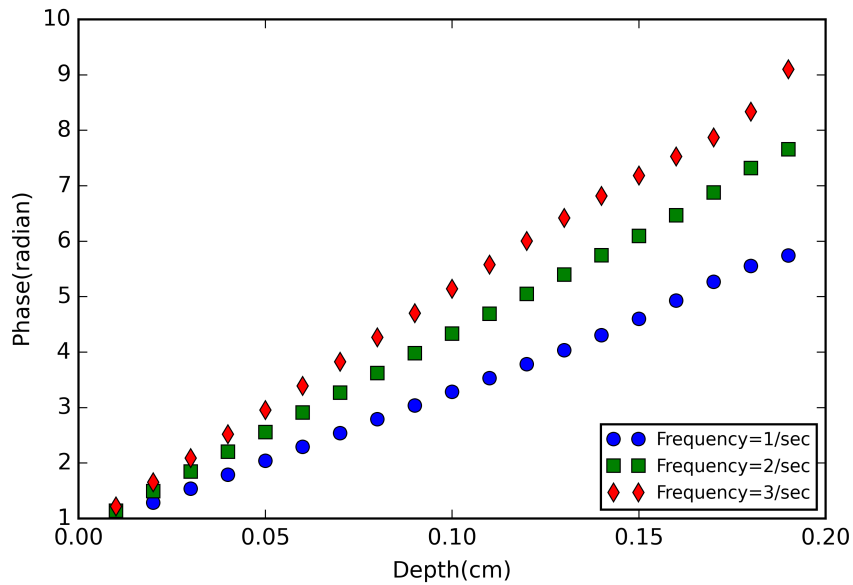


Figure 5. The phase of thermal response at different depths below the surface.

2.2 Quadrupole Method in Two Dimensions

Assuming a homogeneous layer with finite lateral dimensions, as is shown in Fig. 6, with no heat flow across the vertical edges at $x = 0$ and $x = L$, a solution to the Laplace transform of the heat equation in the layer can

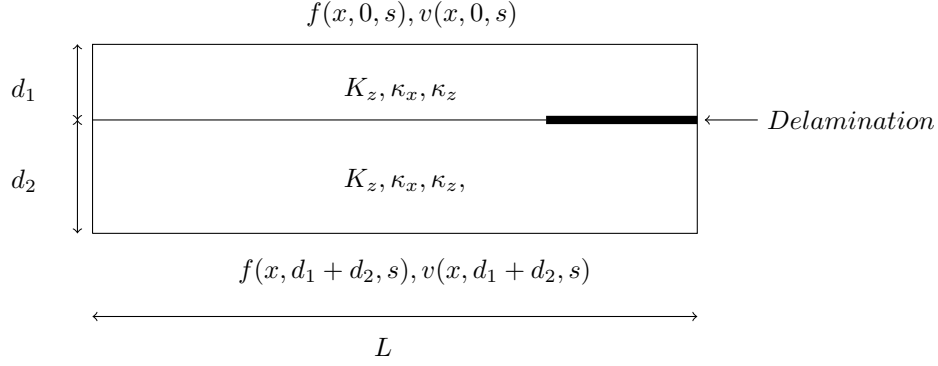


Figure 6. Two dimensional configuration for two layer system represented with delamination that total blocks flux from one layer to the other layer.

be represented by the expression

$$v(x, z, s) = \sum_{m=0}^{N-1} a_m \tilde{v}_m(z, s) \cos\left(\frac{\pi x m}{L}\right), \quad (15)$$

where L is the lateral width of the layer and $a_m = 1$ if $m = 0$ or $m = N - 1$, otherwise $a_m = 2$ and the cosine series coefficient is $\tilde{v}_m(z, s)$ is given by

$$\tilde{v}_m(z, s) = \frac{\sum_{n=0}^{N-1} a_n v(x_n, z, s) \cos\left(\frac{\pi x_n m}{L}\right)}{2N - 2} = \frac{\sum_{n=0}^{N-1} a_n v_n(z, s) \cos\left(\frac{\pi m n}{N-1}\right)}{2N - 2}, \quad (16)$$

where the temperature is defined at a set of N evenly spaced locations given by $x_n = nL/(N - 1)$. The flux in the layer is given by a similar expression

$$f(x, z, s) = \sum_{m=0}^{N-1} a_m \tilde{f}_m(z, s) \cos\left(\frac{\pi x m}{L}\right), \quad (17)$$

where $\tilde{f}_m(z, s)$ is also given by Eq. 16, if f is substituted for v . Eq. 15 is a solution to the Laplace transform of the heat equation if the cosine series coefficients for the temperature have a z dependence and is given by

$$\tilde{v}_m(z, s) = \tilde{v}_m(0, s) \cosh(zq_m) - \tilde{f}_m(0, s) \frac{\sinh(zq_m)}{K_z q_m}, \quad (18)$$

where q_m is given by

$$q_m = \sqrt{\frac{s}{\kappa_z} + \frac{\kappa_x}{\kappa_z} \left(\frac{\pi m}{L}\right)^2}. \quad (19)$$

This is similar to the one dimensional equation for the z dependence of the Laplace transform (Eq. 1) and the z dependence of flux similar to Eq. 2. The matrix equations for the upper and lower layers are similar to Eq. 8 and Eq. 9. The two matrix equations for each spatial frequency term are

$$\begin{bmatrix} \cosh(q_m d_1) & \frac{-\sinh(q_m d_1)}{K_z q_m} \\ -K_z q_m \sinh(q_m d_1) & \cosh(q_m d_1) \end{bmatrix} \begin{bmatrix} \tilde{v}_m(0, s) \\ -h \tilde{v}_m(0, s) \end{bmatrix} = \begin{bmatrix} \tilde{v}_m(d_1, s) \\ \tilde{f}_{m1} - \tilde{f}_{ms}(s) \end{bmatrix} \quad (20)$$

and

$$\begin{bmatrix} \cosh(q_m d_2) & \frac{-\sinh(q_m d_2)}{K_z q_m} \\ -K_z q_m \sinh(q_m d_2) & \cosh(q_m d_2) \end{bmatrix} \begin{bmatrix} \tilde{v}_m(d_1, s) \\ \tilde{f}_{m1} + \tilde{f}_{ms}(s) \end{bmatrix} = \begin{bmatrix} \tilde{v}_m(d_1 + d_2, s) \\ h \tilde{v}_m(d_1 + d_2, s) \end{bmatrix}, \quad (21)$$

where $\tilde{f}_{ms}(s)$ is the Laplace transform of the cosine series coefficient the source flux $F_s(x)$. There is an assumption in this equation that there is no discontinuity in the temperature between the upper and lower layers or that there is no contact resistance. The surfaces of the delamination are considered to be in contact enough to generate friction and contact resistance of air gaps less than 10 microns is small, therefore is a reasonable approximation. The Laplace transform of the cosine coefficient is determined from the equations to be

$$\tilde{v}_m(0, s) = 2 \frac{\tilde{f}_{ms}(s)}{K_z} \frac{q_m \left(\sqrt{\frac{s}{q_m}} \cosh \left(d_2 \sqrt{\frac{s}{q_m}} \right) + \frac{h}{K_z} \sinh \left(d_2 \sqrt{\frac{s}{q_m}} \right) \right)}{2 \frac{h}{K_z} \sqrt{\frac{s}{q_m}} \cosh \left((d_1 + d_2) \sqrt{\frac{s}{q_m}} \right) + \left(s + \left(\frac{h}{K_z} \right)^2 q_m \right) \sinh \left((d_1 + d_2) \sqrt{\frac{s}{q_m}} \right)} \quad (22)$$

By substituting Eq. 22 into Eq. 15, it is possible to find $v(x, 0, s)$. This can be numerically inverted to give the time response for the surface temperature for a spatially varying subsurface impulse source. This result is convolved with $\cos(\omega t)$, to find the time response for a sinusoidal source. An example of the amplitude and phase for times significantly greater than $(d_1 + d_2)^2 / (\pi^2 \kappa)$ are of shown in Fig. 7. Phase fluctuations of more than π are removed by either adding or subtracting 2π to maintain a continuous phase profile. The simulation was for a source 0.1 cm below the surface in a 0.2 cm thick block, with in-plane and surface normal thermal diffusivities both set to $0.005 \text{ cm}^2/\text{sec}$. A 1 hz source extends from 0.5 cm to 1.0 cm. The convection coefficient is set to zero. Included in the figure are the results of a finite difference simulation for the same material properties and a convection coefficient of zero. There is reasonably good agreement between the finite difference solution and the quadrupole solution. The poorest agreement is in the phase for areas greater than 0.25 cm from the end of the source, where the signal amplitude is close to zero. The phase is relatively constant from 0.5 cm to 1.0 cm, however, the amplitude is only relatively constant from 0.6 cm to 1.0 cm. This indicates the phase is a better parameter for characterizing the size of the source.

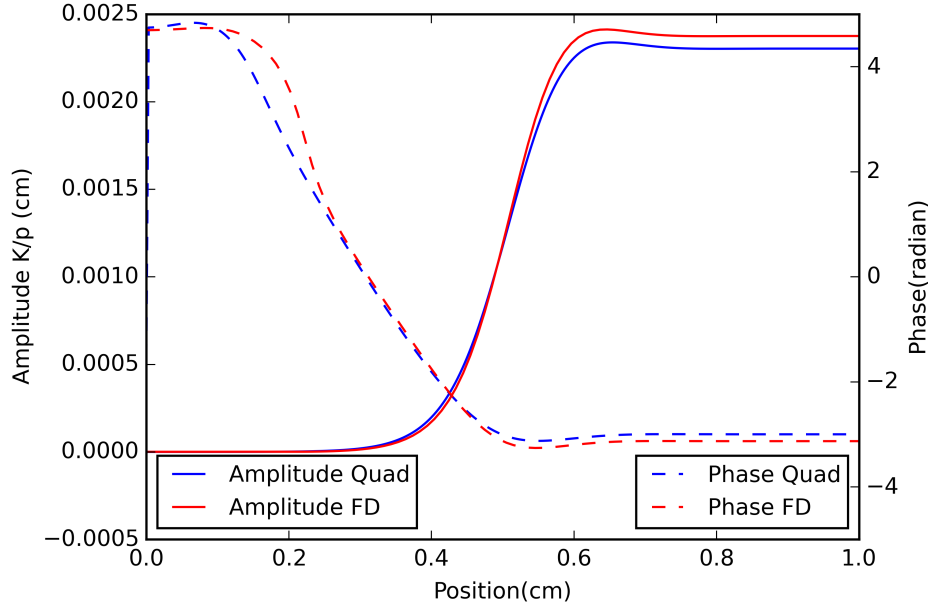


Figure 7. The amplitude and phase of thermal response for a source that extends from 0.5 cm to 1.0 cm, 0.1 cm below the surface. Quadrupole solutions are labeled as Quad and finite difference solutions are labeled FD.

Since the model assumes no contact resistance at the interface, it is simple to include multiple sources at different depths. Results are shown in Fig. 8, where the phase and amplitude are plotted for three 2 hz sources, the first 0.03 cm below the surface from 0 cm to 1 cm, the second from 1 cm to 2 cm, 0.02 cm below the surface and the third from 2 cm to 3 cm, 0.01 cm below the surface. The block was still 0.2 cm thick, however, the thermal

diffusivity in the in-plane direction is $0.05 \text{ cm}^2/\text{sec}$ and the surface normal thermal diffusivity is $0.005 \text{ cm}^2/\text{sec}$. At the horizontal center of the sources, the phase and amplitude are the same as seen in Fig. 4 and Fig. 5 for the appropriate depths of the source. The amplitude profile does not as clearly define the edge of the source as the phases as was the case illustrated in Fig. 7. It is also notable that the deeper the flaw, the more "blurring" of the edge.

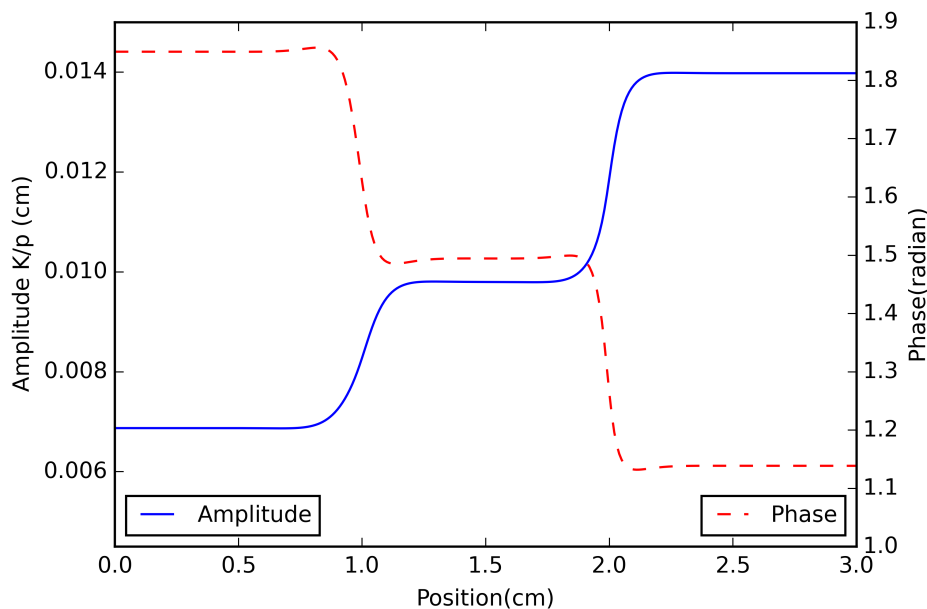


Figure 8. The amplitude and phase of thermal response for three different sources at three different locations from 0.01 cm to 0.03 cm below the surface.

3. CONCLUSION

A method is presented for calculating the time dependence of temperature at the surface from a time dependent subsurface spatially varying flux. The model is compared to a finite difference measurement of the same configuration. The agreement between the two techniques is reasonable.

A series solution for the one dimensional configuration with a subsurface cosine source is also given. The series solution is in excellent agreement with the quadrupole method. The series solution provides an approximate time for when the solution becomes totally sinusoidal.

Future effort will focus on the straight forward process of extending the simulation to three dimensions. It is also possible to extend the formulation to multiple layers, which have different thermal properties. The simulation outputs will also be compared to experimental results. This is not straight forward, since characteristics of internal heat sources are difficult to determine independently.

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