ON DEPENDENT ELEMENTS IN RINGS

JOSO VUKMAN and IRENA KOSI-ULBL

Received 18 November 2003

Let $R$ be an associative ring. An element $a \in R$ is said to be dependent on a mapping $F : R \to R$ in case $F(x)a = ax$ holds for all $x \in R$. In this paper, elements dependent on certain mappings on prime and semiprime rings are investigated. We prove, for example, that in case we have a semiprime ring $R$, there are no nonzero elements which are dependent on the mapping $\alpha + \beta$, where $\alpha$ and $\beta$ are automorphisms of $R$.

2000 Mathematics Subject Classification: 16E99, 16W10, 13N15, 08A35.

This research has been motivated by the work of Laradji and Thaheem [11]. Throughout, $R$ will represent an associative ring with center $Z(R)$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We will use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring $R$ is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $x \to x^*$ on a ring $R$ is called involution in case $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. An additive mapping $D : R \to R$ is called a derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A derivation $D$ is inner in case there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. An additive mapping $T : R \to R$ is a left centralizer in case $T(xy) = T(x)y$ is fulfilled for all pairs $x, y \in R$. This concept appears naturally in $C^*$-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would simply write that $T : RR \to RR$ is a homomorphism of a right $R$-module $R$ into itself. For any fixed element $a \in R$, the mapping $T(x) = ax, x \in R$, is a left centralizer. In case $R$ has the identity element $T : R \to R$ is a left centralizer if and only if $T$ is of the form $T(x) = ax$, $x \in R$, where $a \in R$ is a fixed element. For a semiprime ring $R$, a mapping $T : R \to R$ is a left centralizer if and only if $T(x) = qx$ holds for all $x \in R$, where $q$ is an element of Martindale right ring of quotients $Q_r$ (see [1, Chapter 2]). An additive mapping $T : R \to R$ is said to be a right centralizer in case $T(xy) = xT(y)$ holds for all pairs $x, y \in R$. In case $R$ has the identity element $T : R \to R$ is both left and right centralizer if and only if $T(x) = ax$, $x \in R$, where $a \in Z(R)$ is a fixed element. In case $R$ is a semiprime ring with extended centroid $C$ a mapping $T : R \to R$ is both left and right centralizer in case $T$ is of the form $T(x) = \lambda x$, $x \in R$, where $\lambda \in C$ is a fixed element (see [1, Theorem 2.3.2]). For results concerning centralizers on prime and semiprime rings, we refer to [18, 19, 20, 21, 22, 23, 24]. Following [11] an element $a \in R$ is said to be an element dependent on a mapping $F : R \to R$ if $F(x)a = ax$ holds for all $x \in R$. A mapping $F : R \to R$ is called a free action in case zero is the only element dependent
on $F$. It is easy to see that in semiprime rings there are no nonzero nilpotent dependent elements (see [11]). This fact will be used throughout the paper without specific references. Dependent elements were implicitly used by Kallman [10] to extend the notion of free action of automorphisms of abelian von Neumann algebras of Murray and von Neumann [14, 17]. They were later on introduced by Choda et al. [8]. Several other authors have studied dependent elements in operator algebras (see [6, 7]). A brief account of dependent elements in $W^*$-algebras has been also appeared in the book of Strătilă [16]. The purpose of this paper is to investigate dependent elements of some mappings related to derivations and automorphisms on prime and semiprime rings.

We will need the following two lemmas.

**Lemma 1** (see [2, Lemma 4]). Let $R$ be a 2-torsion-free semiprime ring and let $a, b \in R$. If, for all $x \in R$, the relation $axb + bxa = 0$ holds, then $axb = bxa = 0$ is fulfilled for all $x \in R$.

**Lemma 2** (see [12, Theorem 1]). Let $R$ be a prime ring with extended centroid $C$ and let $a, b \in R$ be such that $axb = bxa$ holds for all $x \in R$. If $a \neq 0$, then there exists $\lambda \in C$ such that $b = \lambda a$.

Our first result has been motivated by Posner’s first theorem [15] which states that the compositum of two nonzero derivations on a 2-torsion-free prime ring cannot be a derivation.

**Theorem 3.** Let $R$ be a semiprime ring and let $D$ and $G$ be derivations of $R$ into itself. In this case the mapping $x \rightarrow D^2(x) + G(x)$ is a free action.

**Proof.** We have the relation

$$F(x)a = ax, \quad x \in R, \quad (1)$$

where $F(x)$ stands for $D^2(x) + G(x)$. A routine calculation shows that the relation

$$F(xy) = F(x)y + xF(y) + 2D(x)D(y) \quad (2)$$

holds for all pairs $x, y \in R$. Putting $xa$ for $x$ in (1) and using (2) we obtain $F(x)a^2 + xF(a)a + 2D(x)D(a)a = axa, \quad x \in R$, which reduces because of (1) to

$$2D(x)D(a)a + xa^2 = 0, \quad x \in R. \quad (3)$$

Putting, in the above relation, $yx$ for $x$ and applying (3) we obtain $2D(y)xD(a)a = 0, \quad x, y \in R$, whence it follows, putting $D(x)$ for $x$, that

$$2D(y)D(x)D(a)a = 0, \quad x, y \in R. \quad (4)$$

Multiplying relation (3) from the left by $D(y)$ and applying the above relation we obtain $D(y)xa^2 = 0, \quad x, y \in R$, which gives, for $x = D(a)$ and $y = a$,

$$D(a)^2a^2 = 0. \quad (5)$$
Multiplying relation (3) from the right by \( a \), putting \( x = a \) in (3), and applying the above relation we obtain \( a^4 = 0 \), which means that also \( a = 0 \). The proof of the theorem is complete.

For our next result, we need the concept of the so-called generalized derivations introduced by Brešar in [3]. An additive mapping \( F : R \to R \), where \( R \) is an arbitrary ring, is called a generalized derivation in case \( F(xy) = F(x)y + xD(y) \) holds for all pairs \( x, y \in R \), where \( D : R \to R \) is a derivation. It is easy to see that \( F \) is a generalized derivation if and only if \( F \) is of the form \( F = D + T \), where \( D \) is a derivation and \( T \) a left centralizer. For some results concerning generalized derivations, we refer the reader to [9].

**Theorem 4.** Let \( F : R \to R \) be a generalized derivation, where \( R \) is a semiprime ring, and let \( a \in R \) be an element dependent on \( F \). In this case \( a \in Z(R) \).

**Proof.** We have the relation

\[
F(x)a = ax, \quad x \in R. \tag{6}
\]

Let \( x \) be \( xy \) in the above relation. Then we have

\[
(F(x)y + xD(y))a = axy, \quad x, y \in R. \tag{7}
\]

Using the fact that \( F \) can be written in the form \( F = D + T \), where \( T \) is a left centralizer, we can replace \( D(y)a \) by \( F(y)a - T(y)a \) in (7), which gives, because of (6),

\[
F(x)y a + [x, a]y - xT(y)a = 0, \quad x, y \in R. \tag{8}
\]

Let \( y \) be \( yF(x) \) in (8). We have

\[
F(x)yF(x)a + [x, a]yF(x) - xT(y)F(x)a = 0, \quad x, y \in R, \tag{9}
\]

which reduces, according to (6), to

\[
F(x)yax + [x, a]yF(x) - xT(y)ax = 0, \quad x, y \in R. \tag{10}
\]

Right multiplication of (8) by \( x \) gives

\[
F(x)yax + [x, a]yx - xT(y)ax = 0, \quad x, y \in R. \tag{11}
\]

Subtracting (11) from (10) we arrive at

\[
[x, a]y(F(x) - x) = 0, \quad x, y \in R. \tag{12}
\]

Right multiplication of the above relation by \( a \) gives, because of (6), \( [x, a]y[x, a] = 0 \), \( x, y \in R \), whence it follows that \( [x, a] = 0, x \in R \). The proof of the theorem is complete.

**Corollary 5.** Let \( R \) be a semiprime ring and let \( a, b \in R \) be fixed elements. Suppose that \( c \in R \) is an element dependent on the mapping \( x \mapsto ax + xb \). In this case \( c \in Z(R) \).
Proof. A special case of Theorem 4, since it is easy to see that the mapping \( x \mapsto ax + xb \) is a generalized derivation.

In the theory of operator algebras the mappings \( x \mapsto ax + xb \), which we met in the above corollary, are considered as an important class of the so-called elementary operators (i.e., mappings of the form \( x \mapsto \sum_{i=1}^{n} a_i x b_i \)). We refer the reader to [13] for a good account of this theory.

Theorem 6. Let \( R \) be a noncommutative prime ring with extended centroid \( C \) and let \( a, b \in R \) be fixed elements. Suppose that \( c \in R \) is an element dependent on the mapping \( x \mapsto axb \). In this case the following statements hold:

1. \( bc \in Z(R) \);
2. \( abc = c \);
3. \( c = \lambda a \) for some \( \lambda \in C \).

Proof. We will assume that \( a \neq 0 \) and \( b \neq 0 \) since there is nothing to prove in case \( a = 0 \) or \( b = 0 \). We have

\[(axb)c = cx, \quad x \in R.\]  

(13)

Let \( x \) be \( xy \) in (13). Then

\[(axyb)c = cxy, \quad x, y \in R.\]  

(14)

According to (13) one can replace \( cx \) by \( (axb)c \) in the above relation. Then we have

\[ax[bc, y] = 0, \quad x, y \in R,\]  

(15)

which gives \( bc \in Z(R) \), which makes it possible to rewrite relation (13) in the form

\[(abc - c)x = 0, \quad x \in R,\]  

(16)

whence it follows that

\[abc = c.\]  

(17)

Putting \( xa \) for \( x \) in relation (13) we obtain, because of (17),

\[axc = cxa, \quad x \in R,\]  

(18)

whence it follows, according to Lemma 2, that \( c = \lambda a \) for some \( \lambda \in C \). The proof of the theorem is complete.

Corollary 7. Let \( R \) be a noncommutative prime ring with the identity element and extended centroid \( C \) and let \( \alpha(x) = axa^{-1}, \ x \in R, \) be an inner automorphism of \( R \). An element \( b \in R \) is an element dependent on \( \alpha \) if and only if \( b = \lambda a \) for some \( \lambda \in C \).

Proof. According to Theorem 6 any element dependent on \( \alpha \) is of the form \( \lambda a \) for some \( \lambda \in C \). It is trivial to see that any element of the form \( \lambda a \), where \( \lambda \in C \), is an element dependent on \( \alpha \).
We proceed to our next result.

**Theorem 8.** Let $R$ be a noncommutative 2-torsion-free prime ring and let $a, b \in R$ be fixed elements. Suppose that $c \in R$ is an element dependent on the mapping $x \mapsto axb + bxa$. In this case the following statements hold:

1. $ac \in Z(R)$ and $bc \in Z(R)$;
2. $(ab + ba)c = c$;
3. $c^2 \in Z(R)$.

**Proof.** Similarly, as in the proof of Theorem 6, we will assume that $a \neq 0$ and $b \neq 0$. We have the relation

\[(axb + bxa)c = cx, \quad x \in R.\]  

(19)

Let $x$ be $xy$ in the above relation. Then we have

\[(axyb + bxya)c = cxy, \quad x, y \in R.\]  

(20)

Right multiplication of relation (19) by $y$ gives

\[(axb + bxa)c = cxy, \quad x, y \in R.\]  

(21)

Subtracting (21) from (20) we arrive at

\[ax[y, bc] + bx[y, ac] = 0, \quad x, y \in R.\]  

(22)

Putting $cx$ for $x$ in the above relation we arrive at

\[acx[y, bc] + bcx[y, ac] = 0, \quad x, y \in R.\]  

(23)

Now, multiplying the above relation first from the left by $y$, then putting $yx$ for $x$ in (23), and finally subtracting the relations so obtained from one another, we arrive at

\[[y, ac]x[y, bc] + [y, bc]x[y, ac] = 0, \quad x, y \in R.\]  

(24)

Suppose that $ac \notin Z(R)$. In this case we have $[y, ac] \neq 0$ for some $y \in R$. Then it follows from relation (24) and Lemma 1 that $[y, bc] = 0$, which reduces relation (22) to $bx[y, ac] = 0, \quad x, y \in R$, which means (recall that $b$ is different from zero) that $[y, ac] = 0$, contrary to the assumption. We have therefore $ac \in Z(R)$. Now relation (22) reduces to $ax[y, bc] = 0, \quad x, y \in R$, whence it follows that $bc \in Z(R)$. Since $ac$ and $bc$ are in $Z(R)$, one can write relation (19) in the form $((ab + ba)c - c)x = 0, \quad x \in R$, which gives

\[(ab + ba)c = c.\]  

(25)

Putting $x = c$ in relation (19) we obtain

\[2(ac)(bc) = c^2.\]  

(26)

Since $ac$ and $bc$ are both in $Z(R)$ it follows from the above relation that $c^2 \in Z(R)$. The proof of the theorem is complete. \qed
**Theorem 9.** Let $R$ be a noncommutative 2-torsion-free prime ring with extended centroid $C$ and let $a, b \in R$ be fixed elements. In this case the mapping $x \mapsto axb - bxa$ is a free action.

**Proof.** Again we assume that $a \neq 0$ and $b \neq 0$. Besides, we will also assume that $a$ and $b$ are $C$-independent, otherwise the mapping $x \mapsto axb - bxa$ would be zero. We have the relation

$$(axb - bxa)c = cx, \quad x \in R. \quad (27)$$

Let $x$ be $xy$ in the above relation. Then we have

$$(axyb - bxya)c = cxy, \quad x, y \in R. \quad (28)$$

Right multiplication of relation (27) by $y$ gives

$$(axb - bxa)cy = cxy, \quad x, y \in R. \quad (29)$$

Subtracting (29) from (28) we arrive at

$$ax[y, bc] - bx[y, ac] = 0, \quad x, y \in R. \quad (30)$$

Putting $cx$ for $x$ in the above relation we arrive at

$$acx[y, bc] - bcx[y, ac] = 0, \quad x, y \in R. \quad (31)$$

Now, multiplying first the above relation from the left by $y$, then putting $yx$ for $x$ in (31), and finally subtracting the relations so obtained from one another, we arrive at

$$[y, ac]x[y, bc] - [y, bc]x[y, ac] = 0, \quad x, y \in R. \quad (32)$$

Suppose that $ac \notin Z(R)$. In this case there exists $y \in R$ such that $[y, ac] \neq 0$. Now it follows from the above relation and from Lemma 2 that

$$[y, bc] = \lambda_y[y, ac] \quad (33)$$

holds for some $\lambda_y \in C$. According to (33) one can replace $[y, bc]$ by $\lambda_y[y, ac]$ in (30), which gives

$$(b - \lambda_ya)x[y, ac] = 0, \quad x \in R. \quad (34)$$

Since $[y, ac] \neq 0$ it follows from the above relation that $b = \lambda_ya$, contrary to the assumption that $a$ and $b$ are $C$-independent. We have therefore proved that $ac \in Z(R)$. Using this fact relation (30) reduces to

$$ax[y, bc] = 0, \quad x, y \in R, \quad (35)$$

whence it follows (recall that $a \neq 0$) that $bc \in Z(R)$. Since $ac$ and $bc$ are both in $Z(R)$, one can rewrite relation (27) in the form $((ab - ba)c - c)x = 0, x \in R$, which gives

$$(ab - ba)c = c. \quad (36)$$
Putting $x = c$ in relation (27) and using the fact that $bc$ is in $Z(R)$, we obtain

$$a[b,c]c = -c^2. \tag{37}$$

From relation (36) one obtains, using the fact that $ac \in Z(R)$,

$$a[b,c] = c. \tag{38}$$

Right multiplication of the above relation by $c$ gives

$$a[b,c]c = c^2. \tag{39}$$

Comparing relations (37) and (39) one obtains $c^2 = 0$, since $R$ is 2-torsion-free. Now it follows that $c = 0$, which completes the proof of the theorem. \qed

**Theorem 10.** Let $R$ be a semiprime ring and let $\alpha$ and $\beta$ be automorphisms of $R$. In this case the mapping $\alpha + \beta$ is a free action.

**Proof.** We have the relation

$$(\alpha(x) + \beta(x))a = ax, \quad x \in R. \tag{40}$$

Let $x$ be $xy$ in the above relation. Then

$$(\alpha(x)\alpha(y) + \beta(x)\beta(y))a = axy, \quad x,y \in R. \tag{41}$$

Replacing first $ax$ by $(\alpha(x) + \beta(x))a$ in the above relation and then $ay$ by $(\alpha(y) + \beta(y))a$, we arrive at

$$(\alpha(x)\alpha(y) + \beta(x)\beta(y))a = (\alpha(x) + \beta(x))(\alpha(y) + \beta(y))a, \quad x,y \in R, \tag{42}$$

which reduces to

$$\alpha(x)\beta(y)a + \beta(x)\alpha(y)a = 0, \quad x,y \in R. \tag{43}$$

The substitution $zx$ for $x$ in the above relation gives

$$\alpha(z)\alpha(x)\beta(y)a + \beta(z)\beta(x)\alpha(y)a = 0, \quad x,y,z \in R. \tag{44}$$

Left multiplication of (43) by $\alpha(z)$ gives

$$\alpha(z)\alpha(x)\beta(y)a + \alpha(z)\beta(x)\alpha(y)a = 0, \quad x,y,z \in R. \tag{45}$$

Subtracting (44) from (45), we arrive at $(\alpha(z) - \beta(z))\beta(x)\alpha(y)a = 0, x,y,z \in R$. We therefore have

$$(\alpha(z) - \beta(z))xya = 0, \quad x,y,z \in R. \tag{46}$$

Putting $x = a$ and $y(\alpha(z) - \beta(z))$ for $y$ in the above relation, we obtain $(\alpha(z) - \beta(z))ax(\alpha(z) - \beta(z))a = 0, x,z \in R$, whence it follows that

$$\alpha(z)a = \beta(z)a, \quad z \in R. \tag{47}$$
According to (47) one can replace $\beta(y)a$ by $\alpha(y)a$ in (43), which gives $(\alpha(x) + \beta(x))\alpha(y)a = 0$, $x, y \in R$. We therefore have

$$(\alpha(x) + \beta(x))\alpha(y)a = 0, \quad x, y \in R.$$  \hspace{1cm} (48)

Putting $y = a$ in the above relation and replacing $(\alpha(x) + \beta(x))a$ by $ax$, we obtain $axa = 0$, $x \in R$, which gives $a = 0$. The proof of the theorem is complete. \hfill $\Box$

The following question arises: what can be proved in case we have $\alpha - \beta$ instead of $\alpha + \beta$ in the above theorem? The mapping $\alpha - \beta$ is a special case of the so-called $(\alpha, \beta)$-derivations. An additive mapping $D : R \to R$, where $R$ is an arbitrary ring, is an $(\alpha, \beta)$-derivation if $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ holds for all pairs $x, y \in R$, where $\alpha$ and $\beta$ are automorphisms of $R$. For results concerning $(\alpha, \beta)$-derivations, we refer the reader to [4, 5].

**Theorem 11.** Let $R$ be a semiprime ring and let $D : R \to R$ be an $(\alpha, \beta)$-derivation. In this case $D$ is a free action.

**Proof.** We have the relation

$$D(x)a = ax, \quad x \in R.$$  \hspace{1cm} (49)

Putting $xy$ for $x$ in the above relation we obtain

$$D(x)\alpha(y)a + \beta(x)D(y)a = axy, \quad x, y \in R.$$  \hspace{1cm} (50)

According to (49) one can replace $D(y)a$ by $ay$ in the above relation, which gives

$$D(x)\alpha(y)a + (\beta(x)a - ax)y = 0, \quad x, y \in R.$$  \hspace{1cm} (51)

Putting $yz$ for $y$ in (51) we obtain

$$D(x)\alpha(y)\alpha(z)a + (\beta(x)a - ax)yz = 0, \quad x, y, z \in R.$$  \hspace{1cm} (52)

On the other hand, right multiplication of (51) by $z$ gives

$$D(x)\alpha(y)az + (\beta(x)a - ax)yz = 0, \quad x, y, z \in R.$$  \hspace{1cm} (53)

Subtracting (53) from (52) we obtain $D(x)\alpha(y)(\alpha(z)a - az) = 0$, $x, y, z \in R$. In other words, we have

$$D(x)y(\alpha(z)a - az) = 0, \quad x, y, z \in R.$$  \hspace{1cm} (54)

The substitution $ay$ for $y$ in the above relation gives, because of (49),

$$axy(\alpha(z)a - az) = 0, \quad x, y, z \in R.$$  \hspace{1cm} (55)

Putting $zx$ for $x$ in the above relation we obtain

$$azxy(\alpha(z)a - az) = 0, \quad x, y, z \in R.$$  \hspace{1cm} (56)
Left multiplication of (55) by $\alpha(z)$ gives

$$\alpha(z)axy(\alpha(z)a - az) = 0, \; x, y, z \in R. \quad (57)$$

Subtracting (56) from (57) and multiplying the relation so obtained from the right-hand side by $x$, we arrive at

$$(\alpha(z)a - az)xy(\alpha(z)a - az)x = 0, \; x, y, z \in R, \quad (58)$$

which gives first

$$(\alpha(z)a - az)x = 0, \; x, z \in R, \quad (59)$$

and then

$$\alpha(z)a - az = 0, \; z \in R. \quad (60)$$

Putting $D(x)a$ instead of $ax$ in (50), and $ay$ for $D(y)a$, we obtain $D(x)(\alpha(y)a - ay) + \beta(x)ay = 0, x, y \in R$, which reduces because of (60) to $\beta(x)ay = 0, x, y \in R$, whence it follows that $a = 0$. The proof of the theorem is complete.

**Corollary 12.** Let $R$ be a semiprime ring and let $\alpha$ and $\beta$ be automorphisms of $R$. In this case the mappings $\alpha - \beta$ and $a\alpha - \beta a$, where $a \in R$ is a fixed element, are free actions on $R$.

**Proof.** According to Theorem 11 there is nothing to prove, since the mappings $\alpha - \beta$ and $a\alpha - \beta a$ are $(\alpha, \beta)$-derivations.

**Corollary 13.** Let $R$ be a semiprime ring, let $D : R \to R$ be a derivation, and let $\alpha$ be an automorphism of $R$. In this case the mappings $x \mapsto D(\alpha(x))$, $x \mapsto \alpha(D(x))$, $x \mapsto D(\alpha(x)) + \beta(D(x))$, and $x \mapsto D(\alpha(x)) - \alpha(D(x))$ are free actions.

**Proof.** A special case of Theorem 11, since all mappings are $(\alpha, \alpha)$-derivations.

For our next result we need the following lemma.

**Lemma 14** (see [24, Lemma 1.3]). Let $R$ be a semiprime ring and let $a \in R$ be a fixed element. If $a[x, y] = 0$ holds for all pairs $x, y \in R$, then there exists an ideal $I$ of $R$ such that $a \in I \subset Z(R)$.

**Proposition 15.** Let $R$ be a semiprime ring and let $\alpha : R \to R$ be an antiautomorphism. Suppose $a \in R$ is an element dependent on $\alpha$. In this case there exists an ideal $I$ of $R$ such that $a \in I \subset Z(R)$. In case $R$ is a prime ring, then either $\alpha$ is a free action or $\alpha$ is the identity mapping and $R$ is commutative.

**Proof.** We have the relation

$$\alpha(x)a = ax, \; x \in R. \quad (61)$$
Putting $xy$ for $y$ in (61) and using (61) we obtain

$$axy = \alpha(xy)a = \alpha(y)\alpha(x)a = \alpha(y)ax = ayx.$$  \hfill (62)

We therefore have

$$a[x,y] = 0, \quad x, y \in R. \quad \hfill (63)$$

From (63) and Lemma 14 it follows that there exists an ideal $I$ of $R$ such that $a \in I \subset Z(R)$, which completes the first part of the proof. The fact that $a \in Z(R)$ makes it possible to rewrite relation (61) in the form $(\alpha(x) - x)a = 0, \quad x \in R$, whence it follows that

$$(\alpha(x) - x)ya = 0, \quad x, y \in R. \quad \hfill (64)$$

In case $R$ is a prime ring it follows from the above relation that either $a = 0$ or $\alpha(x) = x$ for all $x \in R$, which completes the proof of the theorem. \hfill \Box

**Proposition 16.** Let $R$ be a semiprime $\ast$-ring. Suppose that $a \in R$ is dependent on the involution. In this case there exists an ideal $I$ of $R$ such that $a \in I \subset Z(R)$ and $a^\ast = a$. In case $R$ is a prime ring, then either the involution is a free action or the involution is the identity mapping and $R$ is commutative.

**Proof.** Since all the assumptions of Proposition 15 are fulfilled, it remains to prove that $a^\ast = a$. Putting

$$x^\ast a = ax, \quad x \in R, \quad \hfill (65)$$

and $x = a$ in the relation we obtain $a^2 = a^\ast a$, which can be written in the form

$$(a - a^\ast)a = 0. \quad \hfill (66)$$

From the above relation we obtain, using the fact that $a \in Z(R)$,

$$0 = (a(a-a^\ast))^\ast = (a-a^\ast)a^\ast. \quad \hfill (67)$$

Thus we have

$$(a-a^\ast)a^\ast = 0. \quad \hfill (68)$$

Right multiplication of (66) by $x$ gives

$$(a-a^\ast)xa = 0, \quad x \in R, \quad \hfill (69)$$

since $a \in Z(R)$. Similarly, from (68) one obtains (note that also $a^\ast \in Z(R)$)

$$(a-a^\ast)xa^\ast = 0, \quad x \in R. \quad \hfill (70)$$
Subtracting (70) from (69), we obtain

\[(a - a^*)x(a - a^*) = 0, \quad x \in R, \tag{71}\]

whence it follows that \(a^* = a\), which completes the proof.

**Theorem 17.** Let \(R\) be a semiprime ring and let \(\alpha\) be an antiautomorphism of \(R\). In this case the mapping \(x \mapsto \alpha(x) + x\) is a free action.

**Proof.** We have \((\alpha(x) + x)a = ax, \quad x \in R\), which can be written in the form

\[\alpha(x)a = D(x), \quad x \in R, \tag{72}\]

where \(D(x)\) stands for \([a,x]\). Putting \(xy\) for \(x\) in the above relation we obtain

\[\alpha(y)\alpha(x)a = D(x)y + xD(y), \quad x,y \in R. \tag{73}\]

According to (72) one can replace \(\alpha(x)a\) by \(D(x)\) in the above relation, which gives

\[\alpha(y)D(x) = D(x)y + xD(y), \quad x,y \in R. \tag{74}\]

Putting \(x = a\) in the above relation (note that \(D(a) = 0\)) one obtains

\[aD(y) = 0, \quad y \in R. \tag{75}\]

According to (75), left multiplication of relation (72) by \(a\) reduces it to \(a\alpha(x)a = 0, \quad x \in R\), whence it follows that \(a = 0\) by semiprimeness of \(R\), which completes the proof.

**Corollary 18.** Let \(R\) be a semiprime *-ring. The mapping \(x \mapsto x^* + x\) is a free action.

**Acknowledgment.** This work was supported by the Research Council of Slovenia.

**References**


Joso Vukman: Department of Mathematics, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia

E-mail address: joso.vukman@uni-mb.si

Irena Kosi-Ulbl: Department of Mathematics, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia

E-mail address: irena.kosi@uni-mb.si