TESTING THE EQUALITY OF SEVERAL COVARIANCE FUNCTIONS FOR FUNCTIONAL DATA

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TESTING THE EQUALITY OF SEVERAL COVARIANCE FUNCTIONS FOR FUNCTIONAL DATA

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Guo Jia
1 December 2016
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In this thesis, we focus on the multi-sample equal covariance function (ECF) testing problem and provide various methods for this problem. In Chapter 1, we give a brief introduction to functional data analysis and review some hypothesis testing problems and methods in functional data analysis.

For the multi-sample equal covariance function (ECF) testing problem, Zhang (2013) proposed an $L^2$-norm based test. However, its asymptotic power and finite sample performance have not been studied. In Chapter 2, its asymptotic power is investigated under some mild conditions. It is shown that the $L^2$-norm based test is root-$n$ consistent. In addition, intensive simulation studies demonstrate that in terms of size controlling and power, the $L^2$-norm based test outperforms the dimension-reduction based test proposed by Fremdt et al. (2013) when the functional data are less correlated or when the effective signal information is located in high frequencies. Applications to the orthosis data (Abramovich et al. 2004) and the egg-laying curves of fruit flies (Müller and Stadtmüller 2005) are presented to demonstrate the good performance of the $L^2$-norm based test.
In Chapter 3, we propose a new test for the equality of several covariance functions for functional data. Its test statistic is taken as the supremum value of the sum of the squared differences between the estimated individual covariance functions and the pooled sample covariance function, hoping to obtain a more powerful test than some existing tests for the same testing problem. The asymptotic random expression of this test statistic under the null hypothesis is obtained. To approximate the null distribution of the proposed test statistic, we describe a parametric bootstrap method and a non-parametric bootstrap method. The asymptotic random expression of the proposed test is also studied under a local alternative and it is shown that the proposed test is root-$n$ consistent. Intensive simulation studies are conducted to demonstrate the finite sample performance of the proposed test and it turns out that the proposed test is indeed more powerful than the $L^2$-norm based test studied in Chapter 2 when functional data are highly correlated. We also illustrate the applications of the proposed test by real data examples of Canadian temperature data, nitrogen oxide emission level data and the Berkeley growth data.

In Chapter 4, we propose two new tests for testing the equality of the covariance functions of several functional populations, namely a quasi GPF test and a quasi $F_{\text{max}}$ test. The asymptotic random expressions of the two tests under the null hypothesis are derived. We show that the asymptotic null distribution of the quasi GPF test is a chi-squared-type mixture whose distribution can be well approximated by a simple scaled chi-squared distribution. We also adopt a random permutation method for approximating the null distributions of the quasi GPF and $F_{\text{max}}$ tests. The random permutation method is applicable for both large and finite sample sizes. The asymptotic distributions of the two tests under a local
alternative are investigated and they are shown to be root-$n$ consistent. Simulation studies are presented to demonstrate the finite-sample performance of the new tests against three other tests. It is shown that the new tests are more powerful than the other three tests when the covariance functions at different time points have different scales. An illustrative example of the lifetime of medflies is also presented.
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In recent decades, functional data are commonly encountered in many areas, like biology, ergonomics and economics, and attract increasing attention of people from various fields. Functional data can be easily monitored over a period of time, especially with the development of data collecting technology, and are usually in the form of curves, surfaces, or images, such as growth curves (Ramsay and Silverman 2006), stock market index charts (Aguilera et al. 1999), temperatures measured over time (Canadian Climate Program, 1982), corneal surfaces (Locantore et al. 1999), and brain imaging scans (Viviani et al. 2005, Zhang et al. 2008). It is natural to use a function instead of a scalar or a vector as the basic element for analysis of such a kind of data, and the related study has now become a new research field: functional data analysis (FDA). A number of novel and effective tools have been developed to solve the related problems in functional data analysis in the last two decades. A good survey is given by Ramsay and Silverman (2006), which introduced much basic knowledge in functional data analysis.

Because of the limitations of measurement instruments, for most of the time, data can be observed only at discrete time points although their underlying trend is a continuous curve. In the past, this type of data were usually treated as multivariate data and were dealt with by the classical multivariate data analysis (MDA) tools. In fact, if the sampling time points are not equally divided or the sampling time points are different for different subjects, the MDA methods may not work. In another case, when the number of sampling time points is larger than the number of subjects, the classical MDA methods also fail because the dimension is too large compared to the sample size, and in fact this problem is now another popular statistical research area called high-dimensional data analysis. Therefore, to overcome the difficulties caused by discreteness, some smoothing techniques are needed to reconstruct the observations as functions. There is rich literature on statistical smoothing techniques. Fan and Gijbels (1996) discussed the local polynomial kernel (LPK) smoothing technique; Eubank (1999) reviewed the regression spline smoothing method and Ruppert et al. (2003) studied the penalized regression spline smoothing. See also Zhang (2013) (Ch. 2 and 3) for an
overview of these smoothing methods and their applications to the reconstruction of functional data.

After smoothing the data, much work can be done on functional data analysis, especially on the estimation and inference problems related to mean and covariance functions. As the most basic characteristic of data, mean captures the overall trend and covariance can reflect the overall variation due to randomness. In multivariate statistics, the mean is a vector and the covariance is a matrix, while in functional data analysis, the mean and covariance are functions. Because of their importance, much effort has been made on studying the mean and covariance functions and many functional versions of classical statistical tools and methods have been developed to provide a more informative way of exploring them. Among these, functional principal component analysis (FPCA), which is an extension of the multivariate principal component analysis, is a very important tool for facilitating more sophisticated analyses of functional data. Since the data are treated as functions in functional data analysis, their randomness is described using stochastic processes — the functional analogy to multivariate distributions. The foundation of FPCA is the Karhunen–Loève Theorem (Wahba 1990, P.3), which states that many proper stochastic processes can be expanded as a linear combination of basis functions with some random univariate coefficients. Building on this theorem, Ramsay and Silverman (2006) introduced more details about the functional principal component analysis. Benko et al. (2009) extended one-sample FPCA to common functional principal components for two-sample problems.

When dealing with more than one groups of functional data, it is of interest to compare the distributions of different groups. And in most cases, we would like to check the equality of mean functions and covariance functions and this is known
as the functional hypothesis testing problems. On the basis of FPCA, hypothesis testing on mean and covariance functions becomes an important sub-field in functional data analysis. A broad perspective of testing procedures in functional hypothesis testing can be found, for example, in Horváth and Kokoszka (2012) and Zhang (2013). The following subsection will discuss a number of methodologies for first and second order comparisons in functional hypothesis testing.

1.1. Hypothesis Testing in Functional Data Analysis

1.1.1. Equal-mean Function Testing. Numerous testing procedures on two-sample inference for the first-order property have been proposed. Let $y_{i1}(t), y_{i2}(t), \ldots, y_{in_i}(t), i = 1, 2, t \in T$ be the two groups of functional samples defined over a given finite time period $T$, where $n_i, i = 1, 2$ are the associated sample sizes. Suppose they are i.i.d. (independent and identically distributed) and follow stochastic processes with the mean functions $\mu_i(t), i = 1, 2 t \in T$ and the covariance functions $\gamma_i(s,t), i = 1, 2, s,t \in T$, respectively. The two-sample mean testing problem is

$$H_0 : \mu_1(t) = \mu_2(t) \text{ versus } H_1 : \mu_1(t) \neq \mu_2(t). \quad (1.1.1)$$

The general and direct testing procedure is the pointwise $t$-test proposed by Ramsay and Silverman (2006). For any $t \in T$, let $\bar{y}_i(t) = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}(t)$ and $\hat{\gamma}_i(t) = (n_i - 1)^{-1} \sum_{j=1}^{n_i} (y_{ij}(t) - \bar{y}_i)^2, i = 1, 2$ be the estimators of the group mean functions and variance functions. Since the two variance functions may not be the same, Ramsay and Silverman (2006) proposed two versions of the pointwise $t$-test statistics:

$$T_n(t) = \frac{\bar{y}_1(t) - \bar{y}_2(t)}{\sqrt{\hat{\gamma}_1(t)/n_1 + \hat{\gamma}_2(t)/n_2}} \quad \text{when } \gamma_1(t) \neq \gamma_2(t)$$
1.1 Hypothesis Testing in Functional Data Analysis

and

\[ T_n(t) = \frac{\bar{y}_1(t) - \bar{y}_2(t)}{\sqrt{\hat{\gamma}^2(t)(1/n_1 + 1/n_2)}} \quad \text{when } \gamma_1(t) = \gamma_2(t), \]

where \( \hat{\gamma}(t) = [(n_1 - 1)\hat{\gamma}_1(t) + (n_2 - 1)\hat{\gamma}_2(t)]/(n_1 + n_2 - 2) \).

Then by classical results, if the data are Gaussian, the first statistic above can be approximated by a \( t \)-distribution and the second one follows a \( t \)-distribution with degrees of freedom \( n_1 + n_2 - 2 \) for any given time point \( t \in \mathcal{T} \). It is obvious that the pointwise testing procedure has some drawbacks: it only tests the problem at individual time point, and cannot give an overall testing result.

To overcome this drawback, Zhang et al. (2010b) proposed the \( UCL^2 \) test which is an \( L^2 \)-norm based test for testing the equality of mean functions of two Gaussian processes with possibly unequal covariance functions. Let \( f(t) \) be any \( L^2 \)-integrable function over \( \mathcal{T} \) and \( ||f|| \) denote the \( L^2 \)-norm of \( f(t) \), i.e., \( ||f|| = [\int_T f^2(t)dt]^{1/2} \).

Then their test statistic is defined as:

\[ T_n = ||z(t)||^2 = n \int_T [\bar{y}_1(t) - \bar{y}_2(t)]^2 dt, \quad (1.1.2) \]

where \( n = n_1 + n_2 \). The test statistic summarizes the pointwise testing results over the whole interval \( \mathcal{T} \) via the \( L^2 \)-norm of \( z(t) = \sqrt{n}[\bar{y}_1(t) - \bar{y}_2(t)] \).

Under the null hypothesis, \( T_n \) asymptotically follows a \( \chi^2 \)-type mixture. Based on a Welch-type method via matching two cumulants, the null is approximately a scaled \( \chi^2 \)-distribution. Then under the local alternatives and certain regularity conditions, the asymptotic power of (1.1.2) is investigated and the proposed test is shown to be \( \sqrt{n} \)-consistent.

Zhang et al. (2010b) also compared the \( UCL^2 \) test with two existing testing procedures assuming a common covariance function: the F-type test (Shen and Faraway 2004) and the \( L^2 \)-norm based test (Zhang and Chen 2007). If the equal
covariance functions assumption is not satisfied and the two sample sizes are unequal, the $UCL^2$ outperforms these two methods.

The disadvantage of the $UCL^2$ method is that it assumes the two functional samples follow the Gaussian process. However, even though the samples are not Gaussian, the distribution of $z(t)$ is still asymptotically Gaussian based on the central limit theorem which guarantees all asymptotic results introduced in Zhang et al. (2010b).

For the same problem, Zhang et al. (2010a) proposed another testing method — bootstrap based test. Firstly, resample

$$y_{11}^*(t), y_{12}^*(t), \ldots, y_{1n_1}^*(t) \text{ from } y_{11}(t), y_{12}(t), \ldots, y_{1n_1}(t),$$

$$y_{21}^*(t), y_{22}^*(t), \ldots, y_{2n_2}^*(t) \text{ from } y_{21}(t), y_{22}(t), \ldots, y_{2n_2}(t).$$

Then similar to the statistic (1.1.2), the new statistic is

$$T_n^* = n||\hat{e}^*_1(t) - \hat{e}^*_2(t)||^2 = n \int_T [\hat{e}^*_1(t) - \hat{e}^*_2(t)]^2 dt,$$

where $\hat{e}^*_i(t) = \bar{y}_i^*(t) - \bar{y}_i(t), \ i = 1, 2$ and $\bar{y}_i^*(t), \ i = 1, 2$ denote the sample mean functions of the two bootstrapped samples. However, the bootstrap method is time-consuming since we need to resample and calculate the test statistic many times.

Unlike the above work which considered only independent samples, functional time series, for which the samples exhibit temporal dependence, are also studied by some authors. Hörmann and Kokoszka (2010) introduced functional data analysis for samples with weakly dependence structure. Horváth et al. (2013) focused on the estimation of the mean function and the two-sample problem for functional time series. Horváth et al. (2014) discussed the problem of testing stationarity of functional time series.
A natural extension of the functional two-sample problem is the functional one-way ANOVA (analysis of variance) problem of more than two functional samples. Let $SP(\mu_i, \gamma)$ denote a stochastic process whose mean function is $\mu_i(t), t \in \mathcal{T}$ and covariance function is $\gamma(s,t), s,t \in \mathcal{T}$ where $\mathcal{T}$ is defined as before. Suppose we have $k$ groups of functional samples as follows:

$$y_{i1}(t), y_{i2}(t), \cdots, y_{im_i}(t) \overset{i.i.d.}{\sim} SP(\mu_i, \gamma), \quad i = 1, 2, \cdots, k;$$

We wonder whether the $k$ mean functions are equal and the null hypothesis is defined as follows:

$$H_0 : \mu_1(t) \equiv \mu_2(t) \equiv \cdots \equiv \mu_k(t), \quad t \in \mathcal{T}.$$ 

To solve the above one-way ANOVA problem, Ramsay and Silverman (2006) proposed the pointwise $F$-test. Their test statistic is simply an extension of the classic $F$-test to the functional data case. This pointwise test has many limitations as stated before. To overcome these limitations, Fan and Lin (1998) proposed a HANOVA test to test the differences between multiple groups of curves based on the adaptive Neyman test and wavelet thresholding techniques. However, Fan and Lin (1998) only considered the discrete sample case. Inspired by Fan and Lin (1998)’s work, Cuevas et al. (2004) gave a functional ANOVA test. But it is computationally intensive to calculate the null distribution of the ANOVA test since they used the Monte Carlo method. Zhang and Liang (2013) then proposed a GPF test via globalizing the pointwise $F$-test. They used the above-mentioned Welch-type method to approximate the null distribution and hence save a lot of computation time.

Unlike the aforementioned GPF test which uses the integral of the pointwise $F$-test over $\mathcal{T}$, the $F_{\text{max}}$ test proposed by Cheng et al. (2012) used the supremum of
Chapter 1. Introduction

the pointwise $F$-test over $\mathcal{T}$. To approximate the null distribution of the $F_{\text{max}}$ test statistic, Cheng et al. (2012) suggested a PB (parametric bootstrap) and a NPB (non-parametric bootstrap) method. Since the PB method only works well for large sample sizes and needs intensive calculation, the NPB method is preferred for it is applicable under more general conditions. Lately, Górecki and Smaga (2015) gave a comparison of the existing eleven methodologies for functional one-way ANOVA problem via intensive simulation studies. And it is found that the GPF test proposed in Zhang and Liang (2013) and the $L^2$-norm based global test proposed in Zhang and Chen (2007) perform best.

1.1.2. Equal-covariance Function Testing. When testing the equality of two mean functions, we usually assume that the two covariance functions are the same. This assumption is commonly assumed in the equal-mean function testing problems. So it is of interest to check whether this assumption holds or not. For example, to use the methods proposed by Shen and Faraway (2004) and Zhang and Chen (2007), firstly we need to check the equality of the covariance functions. The above problem can be formally described as follows.

Suppose we have two independent functional samples following Gaussian process with mean functions $\mu_i(t)$, $i = 1, 2$ and covariance functions $\gamma_i(s, t)$, $i = 1, 2$ respectively:

$$y_{i1}(t), y_{i2}(t), \cdots, y_{in_i}(t) \sim \text{GP}(\mu_i, \gamma_i), \ i = 1, 2. \quad (1.1.3)$$

The testing problem is:

$$H_0 : \gamma_1(s, t) = \gamma_2(s, t) \quad \text{versus} \quad H_1 : \gamma_1(s, t) \neq \gamma_2(s, t), \ s, t \in \mathcal{T}. \quad (1.1.4)$$

Let $\hat{\gamma}_i$ denote the estimators of $\gamma_i$, $i = 1, 2$. For any fixed $s, t \in \mathcal{T}$, we can easily conduct the pointwise test of (1.1.4) using the pointwise difference between two
1.1 Hypothesis Testing in Functional Data Analysis

estimated covariance functions as the statistic. Alternatively, Zhang and Sun (2010) proposed an $L^2$-norm based global test via summarizing the information in pointwise test. Their $L^2$-norm based test statistic is:

$$T_n = (n_1 + n_2) \int_T \int_T (\hat{\gamma}_1(s,t) - \hat{\gamma}_2(s,t))^2 dsdt.$$ 

Under certain conditions, the asymptotic expression of the test statistic is a $\chi^2$-type mixture and the proposed test is shown to enjoy good asymptotic powers.

In addition, Zhang and Sun (2010) proposed three methods to estimate the unknown critical values of the proposed test. The first method is direct simulation: they repeatedly and independently generate $\chi^2$ samples and compute the above-mentioned $\chi^2$-type mixture to approximate the null distribution of $T_n$. Then the upper $100\alpha$-th percentile of the null distribution is the desired critical value. This method is simple but computationally intensive. To overcome this difficulty, they studied the second method: $\chi^2$-approximation method which is similar to the previously mentioned Welch-type method but via matching three cumulants. The first two methods need to estimate the eigenvalues of a big covariance matrix which is very challenging in computation because of the huge computer memory required. They also proposed a bootstrap method to avoid this problem but it is still time-consuming.

Motivated by the need of comparison of two DNA minicircle groups, Panaretos et al. (2010) provided a testing procedure dealing with the second-order comparison of Gaussian functional observations. Based on their work, Fremdt et al. (2013) studied a non-parametric test for comparing the equality of the covariance structures in two functional samples via projecting the observations onto a suitably chosen finite-dimensional space, applicable to both Gaussian and non-Gaussian observations. The main idea of the above two tests is dimension reduction, but a
suitable finite-dimensional space is not easy to select and the dimension reduction approach they use may cause a loss of information.

Previous studies usually concentrated on testing the equality of mean functions of two or several populations, or the equality of covariance functions in two samples. Little work has been done for the multi-sample equal covariance function (ECF) testing problem although it is encountered frequently in many areas, like the homogeneous one-way and two-way ANOVA. It is very useful to test whether the ECF assumption holds for the functional samples included in a functional ANOVA model.

Our work aims to study how to test if several functional samples have the same covariance function. The main purpose is to provide a number of simple methods for the ECF testing problem, i.e., the $L^2$-norm based tests, the supremum norm based test and the quasi $F$-type tests. In addition, we also study the associated properties of these methods for better understanding and application. The methods provided in this thesis are easy to interpret and offer a better understanding on the comparison of covariance functions. It is understood that sometimes there may be weak dependence within the samples. This case is more complicated and not central to this study, hence in this thesis we only focus on i.i.d. samples from a stochastic process.

The thesis is organized as follows. In Chapter 2, we will introduce the $L^2$-norm based tests for the ECF problem. In Chapter 3, we will discuss the supremum norm based test which is an alternative way to globalize the difference between the covariance functions. In Chapter 4, two quasi $F$-type tests are proposed and studied.
CHAPTER 2

An $L^2$-norm Based Test

2.1. Introduction

Let $y_{i1}(t), y_{i2}(t), \ldots, y_{in_i}(t), i = 1, 2, \ldots, k$ be $k$ independent functional samples defined over a given finite time period $\mathcal{T} = [a, b], -\infty < a < b < \infty$, which satisfy

$$
y_{ij}(t) = \eta_i(t) + v_{ij}(t), \quad j = 1, 2, \ldots, n_i,
$$

$$
v_{i1}(t), v_{i2}(t), \ldots, v_{in_i}(t) \sim i.i.d. \text{SP}(0, \gamma_i); \quad i = 1, 2, \ldots, k, \quad (2.1.1)
$$

where $\eta_1(t), \eta_2(t), \ldots, \eta_k(t)$ model the unknown group mean functions of the $k$ samples, $v_{ij}(t), j = 1, 2, \ldots, n_i; \quad i = 1, 2, \ldots, k$ represent the subject-effect functions, which follow a stochastic process with mean function $0$ and covariance functions $\gamma_i(s, t), \quad i = 1, 2, \ldots, k$ respectively. It is of interest to test the equality of the $k$ covariance functions:

$$
H_0 : \gamma_1(s, t) \equiv \gamma_2(s, t) \equiv \cdots \equiv \gamma_k(s, t), \quad \text{for } s, t \in \mathcal{T}. \quad (2.1.2)
$$
The above problem is known as the multi-sample equal covariance function (ECF) testing problem for functional data, which is an extension of the two-sample ECF testing problem.

As we all know, testing the equality of mean functions is a widely discussed problem in the literature; see, e.g., Ramsay and Silverman (2006), Zhang et al. (2010a) for testing two-sample mean functions and Ramsay and Silverman (2006), Cuevas et al. (2004), Zhang and Chen (2007) and Zhang and Liang (2013) for testing functional one-way ANOVA problem. Besides testing the equality of the mean functions of one or several functional populations, some novel and effective methods have also been proposed for testing the equality of two covariance functions. For example, for testing the equality of the covariance functions of two functional populations, Zhang and Sun (2010) proposed an $L^2$-norm based test while Fremdt et al. (2013) studied a dimension-reduction based test which is an extension of the work of Panaretos et al. (2010) to the non-Gaussian case. The testing procedure of Fremdt et al. (2013) was obtained via projecting the observations onto a suitably chosen finite-dimensional space. However, little work has been done for testing the equality of several covariance functions. This multi-sample ECF testing problem is encountered frequently in many areas. For example, in the functional one-way and two-way ANOVA, we usually assume that the covariance functions of different samples are the same. However, in real data analysis, this assumption may not be true and a formal test may be needed before applying the previously-mentioned testing procedures for the functional one-way or two-way ANOVA. For this multi-sample ECF testing problem, Zhang (2013) (Ch. 10) described an $L^2$-norm based test, which is simple to implement and easy to interpret. However, its asymptotic power has not been studied. In addition, no simulation results are given to demonstrate its finite-sample performance. In this chapter, we present a further study on this
2.2 Review of the $L^2$-norm Based Test

$L^2$-norm based test via studying its asymptotic power. As a result, we show that it is a root-n consistent test. We also demonstrate its finite-sample performance via comparing it with Fremdt et al. (2013)’s dimension-reduction based test through intensive simulations. We found that when the functional data are less correlated or when the effective signal information is located in high frequencies, the $L^2$-norm based test is more powerful than the afore-mentioned dimension-reduction based test.

This chapter is organized as follows. Section 2.2 reviews the $L^2$-norm based tests in Zhang (2013) (Ch. 10). The main results of the asymptotic power are presented in Section 2.3. Section 2.4 illustrates the simulation studies and Section 2.5 states two applications of the $L^2$-norm based test. Theoretical proofs can be found in Section 2.6.

### 2.2. Review of the $L^2$-norm Based Test

In this section, we review the $L^2$-norm based test proposed in Zhang (2013) for the $k$-sample ECF testing problem. For further study, we firstly give some useful estimations. Given the $k$ samples, the group mean functions $\eta_i(t), \quad i = 1, 2, \cdots, k$ and the covariance functions $\gamma_i(s, t), \quad i = 1, 2, \cdots, k$ can be unbiasedly estimated by

\[
\hat{\eta}_i(t) = \bar{y}_i(t) = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}(t), \quad i = 1, 2, \cdots, k,
\]

\[
\hat{\gamma}_i(s, t) = (n_i - 1)^{-1} \sum_{j=1}^{n_i} [y_{ij}(s) - \bar{y}_i(s)][y_{ij}(t) - \bar{y}_i(t)], \quad i = 1, 2, \cdots, k,
\]  

(2.2.1)

where $\hat{\gamma}_i(s, t), \quad i = 1, 2, \cdots, k$ are independent and $E\hat{\gamma}_i(s, t) = \gamma_i(s, t), \quad i = 1, 2, \cdots, k$. Then the estimated subject-effect functions can be written as

\[
\hat{v}_{ij}(t) = y_{ij}(t) - \bar{y}_i(t), \quad j = 1, 2, \cdots, n_i; \quad i = 1, 2, \cdots, k.
\]  

(2.2.2)
Chapter 2. An $L^2$-norm Based Test

Under the null hypothesis (2.1.2), the common covariance function $\gamma(s,t)$ of the $k$ samples can be estimated by the following pooled sample covariance function

$$
\hat{\gamma}(s,t) = \sum_{i=1}^{k} (n_i - 1) \hat{\gamma}_i(s,t)/(n-k),
$$

(2.2.3)

where $\hat{\gamma}_i(s,t), i = 1,2,\ldots, k$ are given in (2.2.1).

For further investigation, the following assumptions are imposed.

**Assumption A**

1. The $k$ samples are Gaussian with $\text{tr}(\gamma_i) < \infty$ and $\eta_i(t) \in L^2(T)$, $i = 1,2,\ldots,k$, where $L^2(T)$ denotes the Hilbert space formed by all the squared integrable functions over $T$ with the inner-product defined as $<f,g> = \int_T f(t)g(t)dt$, $f(t), g(t) \in L^2(T)$.

2. As $n_{\min} = \min_{i=1}^k n_i \to \infty$, the $k$ sample sizes satisfy $n_i/n \to \tau_i$, $i = 1,2,\ldots,k$ such that $\tau_1, \tau_2, \ldots, \tau_k \in (0,1)$ where $n = \sum_{i=1}^k n_i$ denotes the total sample size.

3. The variance functions are uniformly bounded, that is, $\rho_i = \sup_{t \in T} \gamma_i(t,t) < \infty$, $i = 1,2,\ldots,k$.

It is easy to note that $\hat{\gamma}_i(s,t) - \hat{\gamma}(s,t)$ measures the difference between the $i$-th sample covariance function (2.2.1) and the pooled sample covariance function (2.2.3), which should be small when the null hypothesis holds. Based on this, Zhang (2013) proposed the following so-called $L^2$-norm based test statistic for the $k$-sample ECF testing problem (2.1.2):

$$
T_n = \sum_{i=1}^{k} (n_i - 1) \int_T \int_T [\hat{\gamma}_i(s,t) - \hat{\gamma}(s,t)]^2 dsdt,
$$

(2.2.4)

which summarizes all the squared differences between the $k$ sample covariance functions and the pooled sample covariance function. Therefore, when the null hypothesis holds, $T_n$ will be small and otherwise large.
2.2 Review of the $L^2$-norm Based Test

Lemma 2.4 in the Appendix states that the test statistic $T_n$ is asymptotically a $\chi^2$-type mixture. Therefore, the null distribution of $T_n$ can be approximated by the well-known Welch-Satterthwaite $\chi^2$-approximation. By this method, Zhang (2013) approximated the null distribution of $T_n$ using that of a random variable $R \sim \beta \chi^2_d$.

The parameters $\beta$ and $d$ are determined via matching the first two moments of $T_n$ and $R$, which are given by

$$
\begin{align*}
\beta &= \frac{\text{tr}(\varpi^{\otimes 2})}{\text{tr}(\varpi)}, \\
(d, \kappa) &= \frac{\text{tr}^2(\varpi)}{\text{tr}(\varpi^{\otimes 2})},
\end{align*}
$$

(2.2.6)

where $\varpi[(s_1, t_1), (s_2, t_2)]$ denotes the covariance function of $\sqrt{n-k}[\hat{\gamma}(s, t) - \gamma(s, t)]$, and

$$
\begin{align*}
\text{tr}(\varpi) &= \int_T \int_T \varpi[(s, t), (s, t)] dsdt, \\
\text{tr}(\varpi^{\otimes 2}) &= \int_T \int_T \int_T \int_T \varpi^2[(s_1, t_1), (s_2, t_2)] ds_1 dt_1 ds_2 dt_2.
\end{align*}
$$

Under the Gaussian assumption A1, it is easy to verify that

$$
\begin{align*}
\varpi[(s_1, t_1), (s_2, t_2)] &= \gamma(s_1, s_2)\gamma(t_1, t_2) + \gamma(s_1, t_2)\gamma(s_2, t_1), \\
\text{tr}(\varpi) &= \text{tr}^2(\gamma) + \text{tr}(\gamma^{\otimes 2}), \\
\text{tr}(\varpi^{\otimes 2}) &= 2\text{tr}^2(\gamma^{\otimes 2}) + 2\text{tr}(\gamma^{\otimes 4}),
\end{align*}
$$

(2.2.7)

where $\text{tr}(\gamma) = \int_T \gamma(t, t) dt$, $\text{tr}(\gamma^{\otimes 2}) = \int_T \int_T \gamma^2(s, t) dsdt$ and

$$
\text{tr}(\gamma^{\otimes 4}) = \int_T \int_T \int_T \int_T \gamma(t, u_1)\gamma(u_1, u_2)\gamma(u_2, u_3)\gamma(u_3, t) du_1 du_2 du_3 dt.
$$

To conduct the $L^2$-norm based test, we need to estimate the parameters $\beta$ and $d$ based on the data. There are two methods for estimating the parameters $\beta$ and $\kappa$, one is the naive method, and the other is the bias-reduced method. Let $\hat{\beta}$ and $\hat{\kappa}$ denote the estimators of $\beta$ and $\kappa$. The naive estimators of $\beta$ and $\kappa$ are obtained via replacing $\text{tr}(\varpi), \text{tr}^2(\varpi)$ and $\text{tr}(\varpi^{\otimes 2})$ in (2.2.6) respectively with their
Chapter 2. An $L^2$-norm Based Test

naive estimators $\text{tr}(\hat{\varpi}), \text{tr}^2(\hat{\varpi})$ and $\text{tr}(\hat{\varpi} \otimes^2)$:

$$\hat{\beta} = \frac{\text{tr}(\hat{\varpi} \otimes^2)}{\text{tr}(\hat{\varpi})}, \quad \hat{\kappa} = \frac{\text{tr}^2(\hat{\varpi})}{\text{tr}(\hat{\varpi} \otimes^2)}, \quad (2.2.8)$$

where under the Gaussian assumption A1 and based on (2.2.7), we have

$$\hat{\varpi}[(s_1, t_1), (s_2, t_2)] = \hat{\gamma}(s_1, s_2)\hat{\gamma}(t_1, t_2) + \hat{\gamma}(s_1, t_2)\hat{\gamma}(s_2, t_1),$$

$$\text{tr}(\hat{\varpi}) = \text{tr}^2(\hat{\gamma}) + \text{tr}(\hat{\gamma} \otimes^2), \quad \text{tr}(\hat{\varpi} \otimes^2) = 2\text{tr}^2(\hat{\gamma} \otimes^2) + 2\text{tr}(\hat{\gamma} \otimes^4). \quad (2.2.9)$$

The bias-reduced estimators of $\beta$ and $\kappa$ are obtained via replacing $\text{tr}(\hat{\varpi}), \text{tr}^2(\hat{\varpi})$ and $\text{tr}(\hat{\varpi} \otimes^2)$ in (2.2.6) respectively with their bias-reduced estimators $\hat{\text{tr}}(\hat{\varpi}), \hat{\text{tr}}^2(\hat{\varpi})$ and $\hat{\text{tr}}(\hat{\varpi} \otimes^2)$:

$$\hat{\beta} = \frac{\hat{\text{tr}}(\hat{\varpi} \otimes^2)}{\hat{\text{tr}}(\hat{\varpi})}, \quad \hat{\kappa} = \frac{\hat{\text{tr}}^2(\hat{\varpi})}{\hat{\text{tr}}(\hat{\varpi} \otimes^2)}, \quad (2.2.10)$$

where under the Gaussian assumption A1 and based on (2.2.7), we have

$$\hat{\text{tr}}(\hat{\varpi}) = \hat{\text{tr}}^2(\hat{\gamma}) + \hat{\text{tr}}(\hat{\gamma} \otimes^2), \quad \text{tr}(\hat{\varpi} \otimes^2) = 2\hat{\text{tr}}^2(\hat{\gamma} \otimes^2) + 2\hat{\text{tr}}(\hat{\gamma} \otimes^4). \quad (2.2.11)$$

with

$$\hat{\text{tr}}^2(\hat{\gamma}) = \frac{(n-k)(n-k+1)}{(n-k-1)(n-k+2)} \left[ \text{tr}^2(\hat{\gamma}) - \frac{2\text{tr}(\hat{\gamma} \otimes^2)}{n-k+1} \right],$$

$$\hat{\text{tr}}(\hat{\gamma} \otimes^2) = \frac{(n-k)^2}{(n-k-1)(n-k+2)} \left[ \text{tr}(\hat{\gamma} \otimes^2) - \frac{\text{tr}^2(\hat{\gamma})}{n-k} \right]. \quad (2.2.12)$$

Note that under the Gaussian assumption A1, $\hat{\text{tr}}^2(\hat{\gamma})$ and $\hat{\text{tr}}(\hat{\gamma} \otimes^2)$ are the unbiased estimators of $\text{tr}^2(\gamma)$ and $\text{tr}(\gamma \otimes^2)$ respectively and when the data are not Gaussian, they may be asymptotically unbiased under some further assumptions. Notice also that in the expression (2.2.11), the unbiased estimator of $\text{tr}(\gamma \otimes^4)$ is not incorporated since it is quite challenging to obtain a simple and useful unbiased estimator of $\text{tr}(\gamma \otimes^4)$.

The following theorem shows that under some mild conditions, the estimators, $\hat{\beta}$ and $\hat{\kappa}$, of $\beta$ and $\kappa$ are consistent.
2.2 Review of the $L^2$-norm Based Test

**Theorem 2.1.** Under Assumptions A1~A3 and the null hypothesis (2.1.2), as $n \to \infty$, we have $\hat{\beta} \xrightarrow{p} \beta$, $\hat{\kappa} \xrightarrow{p} \kappa$ for both the naive and bias-reduced methods. In addition, $\hat{T}_n(\alpha) \xrightarrow{p} \tilde{T}_0(\alpha)$, where $\hat{T}_n(\alpha) = \hat{\beta} \chi^2_{k-1}(\alpha)$ is the estimated critical value of $T_n$ and $\tilde{T}_0(\alpha) = \beta \chi^2_{k-1}(\alpha)$ is its approximate theoretical critical value.

By some simple algebra, we have $\beta < \lambda_{\text{max}} < \infty$ and $\kappa \leq m$ where $\lambda_{\text{max}}$ is the largest eigenvalue of $\varpi[(s_1, t_1), (s_2, t_2)]$ and $m$ is the number of all the positive eigenvalues. Then it is easy to verify that $\tilde{T}_0(\alpha) < \infty$ when $m$ is a finite number.

However, under the null hypothesis, when the sample sizes $n_i$, $i = 1, 2, \cdots, k$ of the $k$ samples (3.2.1) are small, Theorem 2.1 is no longer valid so that the Welch-Satterthwaite $\chi^2$-approximation is also no longer applicable. To overcome this difficulty, a random permutation method is proposed to approximate the critical values of $T_n$. This method can also be used when the data are non-Gaussian. The random permutation method can be described as follows.

Firstly, we randomly reorder the pooled estimated subject-effect functions (2.2.2) so that a random permutation sample $\hat{v}_l^*(t)$, $l = 1, 2, \cdots, n$ is obtained where $n$ is the total sample size as defined before. We then use the first $n_1$ permuted subject-effect functions to form the first permutation sample $\hat{v}_{1j}^*(t)$, $j = 1, 2, \cdots, n_1$, the next $n_2$ permuted subject-effect functions to form the second permutation sample $\hat{v}_{2j}^*(t)$, $j = 1, 2, \cdots, n_2$ and so on. The permutation test statistic $T_n^*$ is computed similarly to the computation of the original $L^2$-norm based test statistic $T_n$ as described in (2.2.4) but now based on the $k$ permuted functional samples. That is,

$$T_n^* = \sum_{i=1}^{k} (n_i - 1) \int_T \int_T [\hat{\gamma}_{i}^*(s, t) - \hat{\gamma}^*(s, t)]^2 ds dt,$$
where
\[ \hat{\gamma}_i^*(s, t) = (n_i - 1)^{-1} \sum_{j=1}^{n_i} \hat{v}_{ij}^*(s) \hat{v}_{ij}^*(t), \quad i = 1, 2, \ldots, k, \]
\[ \hat{\gamma}^*(s, t) = \sum_{i=1}^{k} (n_i - 1) \hat{\gamma}_i^*(s, t)/(n - k). \]
Repeating the above process a large number of times, we can get a sample of \( T_n^* \) and use the sample upper 100\( \alpha \)-percentile \( \hat{T}_n^*(\alpha) \) to estimate the critical value of \( T_n^* \). Using this critical value, we then conduct the associated random permutation test. If \( T_n > \hat{T}_n^*(\alpha) \), we reject the null hypothesis (2.1.2).

The following theorem shows that under the null hypothesis the permutation test statistic \( T_n^* \) converges in distribution to the same limit test statistic \( T_0 \) of \( T_n \) where \( T_0 \) is defined in Lemma 2.4 and hence \( \hat{T}_n^*(\alpha) \) will also tend to \( T_0(\alpha) \) in distribution as \( n \to \infty \). Thus the size of the permutation test tends to the nominal size.

**Theorem 2.2.** Under Assumptions A1∼A3 and the null hypothesis (2.1.2), as \( n \to \infty \), we have \( T_n^* \overset{d}{\to} T_0 \), \( \hat{T}_n^*(\alpha) \overset{d}{\to} T_0(\alpha) \). Hence, the size of the random permutation test \( P(T_n > \hat{T}_n^*(\alpha)) \to P(T_n > T_0(\alpha)) \) where \( \hat{T}_n^*(\alpha) \) is the estimated upper 100\( \alpha \)-percentile of \( T_n^* \) based on permutation samples and \( T_0(\alpha) \) is the theoretical critical value of \( T_n \).

**2.3. Asymptotic Power of the \( L^2 \)-norm Based Test**

Zhang (2013) did not study the asymptotic power of the \( L^2 \)-norm based test \( T_n \). In this section, we study its asymptotic power under the following local alternative:

\[ H_1 : \gamma_i(s, t) = \gamma(s, t) + (n_i - 1)^{-1/2} d_i(s, t), \quad i = 1, 2, \ldots, k, \]
where \( d_1(s, t), d_2(s, t), \ldots, d_k(s, t) \) are some fixed bivariate functions, independent of \( n \) and \( \gamma(s, t) \) is some covariance function.
2.3 Asymptotic Power of the $L^2$-norm Based Test

For further study, we can re-write the $L^2$-norm based test statistic $T_n$ (2.2.4) as

$$T_n = \int_T \int_T \text{SSB}(s,t) ds dt,$$

where

$$\text{SSB}(s,t) = \sum_{i=1}^{k} (n_i - 1)[\hat{\gamma}_i(s,t) - \tilde{\gamma}(s,t)]^2,$$

which summarizes the squared differences between the individual sample covariance functions $\hat{\gamma}_i(s,t), i = 1,2,\cdots,k$ and the pooled sample covariance function $\tilde{\gamma}(s,t)$ for any given $(s,t) \in T^2$.

Before we state the main results, we give an alternative expression of SSB$(s,t)$ which is helpful for deriving the asymptotic power of $T_n$. For any $s,t \in T$, SSB$(s,t)$ can be expressed as

$$\text{SSB}(s,t) = z_n(s,t)^T [I_k - b_n b_n^T/(n-k)] z_n(s,t) = z_n(s,t)^T W_n z_n(s,t),$$

where

$$z_n(s,t) = [z_1(s,t), z_2(s,t), \cdots, z_k(s,t)]^T, \quad W_n = I_k - b_n b_n^T/(n-k),$$

with

$$z_i(s,t) = \sqrt{n_i - 1}[\hat{\gamma}_i(s,t) - \gamma(s,t)], \quad i = 1,2,\cdots,k,$$

$$b_n = [\sqrt{n_1 - 1}, \sqrt{n_2 - 1}, \cdots, \sqrt{n_k - 1}]^T.$$

Since $b_n^T b_n/(n-k) = 1$, it is easy to verify that $W_n$ is an idempotent matrix with rank $k-1$. In addition, as $n \to \infty$, we have

$$W_n \to W := I_k - bb^T, \quad \text{with} \quad b = [\sqrt{\tau_1}, \sqrt{\tau_2}, \cdots, \sqrt{\tau_k}]^T,$$

where $\tau_i, i = 1,2,\cdots,k$ are given in Assumption A2. Note that $W$ in (2.3.6) is also an idempotent matrix of rank $k-1$, which has the following singular value
decomposition:

\[ W = U \begin{pmatrix} I_{k-1} & 0 \\ 0^T & 0 \end{pmatrix} U^T, \quad (2.3.7) \]

where the columns of \( U \) are the eigenvectors of \( W \).

Let \( \tilde{d}(s, t) = [I_{k-1}, 0]U^T d(s, t) \) where \( d(s, t) = [d_1(s, t), d_2(s, t), \cdots, d_k(s, t)]^T \) with \( d_i(s, t), i = 1, 2, \cdots, k \) given in (2.3.1). Let \( \lambda_r, r = 1, 2, \cdots \) be the eigenvalues of \( \varpi[(s_1, t_1), (s_2, t_2)] \) with only the first \( m \) eigenvalues being positive and \( \phi_r(s, t), r = 1, 2, \cdots, \) be the associated eigenfunctions. Define

\[ \delta_r^2 = || \int_\mathcal{T} \int_\mathcal{T} \tilde{d}(s, t)\phi_r(s, t)dsdt||^2, r = 1, 2, \cdots, \quad (2.3.8) \]

which measure the information of \( \tilde{d}(s, t) \) projected on the eigenfunctions, \( \phi_r(s, t), r = 1, 2, \cdots, \) of \( \varpi[(s_1, t_1), (s_2, t_2)] \). Theorem 2.3 below gives the asymptotic distribution of \( T_n \) under the local alternative (2.3.1).

**Theorem 2.3.** Under Assumptions A1\( \sim \)A3 and the local alternative (2.3.1), as \( n \to \infty \), we have \( T_n \overset{d}{\to} T_1 \) with

\[ T_1 \overset{d}{=} \sum_{r=1}^{m} \lambda_r A_r + \sum_{r=m+1}^{\infty} \delta_r^2, \]

where \( A_r \sim \chi^2_{k-1}(\lambda_r^{-1} \delta_r^2), r = 1, 2, \cdots, m \), are independent and \( \delta_r^2, r = m + 1, m + 2, \cdots, \infty \) are defined in (2.3.8).

Theorem 2.4 below shows that under the local alternative (2.3.1), \( T_n \) is asymptotically normal. Theorem 2.5 below shows that the \( L^2 \)-norm based test can detect the local alternative (2.3.1) with probability 1 as long as the information provided by \( d(s, t) \) diverges. That is, the \( L^2 \)-norm based test is root-\( n \) consistent. In both Theorems 2.4 and 2.5, the quantities \( \delta_r^2, r = 1, 2, \cdots \) have been defined in (2.3.8).
Let these quantities satisfy the following condition:

\[
\max_{r} \delta_r^2 \to \infty.
\]  

(2.3.9)

This condition describes a situation when the information projected onto at least one eigenfunction tends to \( \infty \).

**Theorem 2.4.** Under Assumptions A1\( \sim \)A3, the local alternative (2.3.1), and condition (2.3.9), as \( n \to \infty \), we have \( \frac{T_n - E(T_n)}{\sqrt{\text{Var}(T_n)}} \overset{d}{\to} N(0, 1) \).

**Theorem 2.5.** Under Assumptions A1\( \sim \)A3, the local alternative (2.3.1), and condition (2.3.9), as \( n \to \infty \), the proposed \( L^2 \)-norm based test has asymptotic power 1, i.e., \( P(T_n > \hat{T}_n(\alpha)) \to 1 \), where \( \hat{T}_n(\alpha) \) is the estimated critical value of \( T_n \) defined in Theorem 2.1.

We now study the consistency property of the random permutation test. Theorem 2.6 shows that the random permutation test is also root-\( n \) consistent.

**Theorem 2.6.** Under Assumptions A1\( \sim \)A3, the local alternative (2.3.1), and condition (2.3.9), as \( n \to \infty \), the power \( P(T_n > \hat{T}_n^*(\alpha)) \to 1 \).

**2.4. Simulation Studies**

In Section 2.2, we described three methods for approximating the null distribution of the \( L^2 \)-norm based test: a naive method, a bias-reduced method, and a random-permutation method. The associated \( L^2 \)-norm based tests may be denoted as \( L^2_{nv}, L^2_{br}, \) and \( L^2_{rp} \) respectively. Recently, Fremdt et al. (2013) described two dimension-reduction methods for testing the equality of the covariance functions of two functional samples. Their first test can be applied to both Gaussian and non-Gaussian functional data while the second one can only be used for Gaussian functional data. For convenience, we refer to these two tests as \( FHK_D \) and \( FHK_G \)
respectively. In this section, we shall present two simulations. In Simulation 1, we shall compare the performances of $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ and in Simulation 2, we shall compare $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ against $FHK_D$ and $FHK_G$.

2.4.1. Data Generating. In the simulations, for $i = 1, 2, \cdots, k$, the $i$-th functional sample will be generated from the following model:

$$y_{ij}(t) = \eta_i(t) + v_{ij}(t), \quad \eta_i(t) = c^T_i [1, t, t^2, t^3], \quad v_{ij}(t) = b^T_{ij} \Psi_i(t), \quad t \in [0, 1],$$

$$b_{ij} = [b_{ij1}, b_{ij2}, \cdots, b_{ijq}]^T, \quad b_{ijr} = \sqrt{\lambda_r} z_{ijr}, \quad r = 1, 2, \cdots, q; j = 1, 2, \cdots, n_i,$$

where the parameter vectors $c_i = [c_{i1}, c_{i2}, c_{i3}, c_{i4}]^T$ for the group mean function $\eta_i(t)$ can be flexibly specified, the random variables $z_{ijr}, r = 1, 2, \cdots, q$ are i.i.d. with mean 0 and variance 1, $\Psi_i(t) = [\psi_{i1}(t), \psi_{i2}(t), \cdots, \psi_{iq}(t)]^T$ is a vector of $q$ basis functions and the variance components $\lambda_r, r = 1, 2, \cdots, q$ are positive and decreasing in $r$, and the number of the basis functions, $q$, is an odd positive integer. These tuning parameters help specify the group mean functions $\eta_i(t) = c_{i1} + c_{i2} t + c_{i3} t^2 + c_{i4} t^3$ and the covariance function $\gamma_i(s, t) = \Psi_i(s)^T diag(\lambda_1, \lambda_2, \cdots, \lambda_q) \Psi_i(t) = \sum_{r=1}^{q} \lambda_r \psi_{ir}(s) \psi_{ir}(t), i = 1, 2, \cdots, k$. Note that our test statistic is translation invariant, so in fact the mean functions have little influence on the results of simulation studies. For simplicity, we assume that the design time points for all the functions $y_{ij}(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ are the same and are specified as $t_j = (j - 1)/(J - 1), j = 1, 2, \cdots, J$, where $J$ is some positive integer. In practice, these functions can be observed at different design time points. In this case, some smoothing technique, such as those discussed in Zhang and Chen (2007), Zhang et al. (2010a) can be used to reconstruct the functions $y_{ij}(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ and then to evaluate them at a common grid of time points. The latter simulation setup will be time-consuming.
to carry out and we did not explore it in this simulation. When calculating the test statistics and the estimators, the integrals involved in them are obtained discretely.

We now specify the parameters in (2.4.1). To specify the group mean functions \( \eta_1(t), \eta_2(t), \ldots, \eta_k(t) \), we set \( \mathbf{c}_1 = [1, 2.3, 3.4, 1.5]^T \) and \( \mathbf{c}_i = \mathbf{c}_1 + (i - 1)\delta \mathbf{u} \), \( i = 1, 2, \ldots, k \), where the constant vector \( \mathbf{u} \) specifies the direction of these differences. We set \( \delta = 0.1 \) and \( \mathbf{u} = [1, 2, 3, 4]^T/\sqrt{30} \) which is a unit vector. Then we specify the covariance function \( \gamma_i(s, t) \). For simplicity, we set \( \lambda_r = a\rho^{r-1}, \ r = 1, 2, \ldots, q \), for some \( a > 0 \) and \( 0 < \rho < 1 \). Notice that the tuning parameter \( \rho \) not only determines the decay rate of \( \lambda_1, \lambda_2, \ldots, \lambda_q \), but also determines how the simulated functional data are correlated: when \( \rho \) is close to \( 0 \), \( \lambda_1, \lambda_2, \ldots, \lambda_q \) will decay very fast, indicating that the simulated functional data are highly correlated; and when \( \rho \) is close to \( 1 \), \( \lambda_1, \lambda_2, \ldots, \lambda_q \) will decay slowly, indicating that the simulated functional data are nearly uncorrelated. To define the basis functions \( \Psi_i(t) \), we firstly generate a vector of \( q \) basis functions \( \phi(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_q(t)]^T \) and we select \( \phi_1(t) = 1, \phi_{2r}(t) = \sqrt{2}\sin(2\pi rt), \phi_{2r+1}(t) = \sqrt{2}\cos(2\pi rt), \ t \in [0, 1], \ r = 1, 2, \ldots, (q - 1)/2. \) Then we specify our basis functions \( \Psi_i(t) \) via the following relationship: \( \psi_{ir}(t) = \phi_r(t), \ r = 1, 3, 4, \ldots, q \) but \( \psi_{i2}(t) = \phi_2(t) + (i - 1)\omega, \ i = 1, 2, \ldots, k \). That is, we obtain \( k \) different bases via shifting the second basis function of the \( i \)-th basis with \( (i - 1)\omega \) steps. This allows that the differences of the \( k \) covariance functions \( \gamma_i(s, t), i = 1, 2, \ldots, k \) are controlled by the tuning parameter \( \omega \) since we actually have

\[
\gamma_i(s, t) = \gamma_1(s, t) + (i - 1)\lambda_2(\phi_2(s) + \phi_2(t))\omega + (i - 1)^2\lambda_2\omega^2, \ i = 1, 2, \ldots, k. \quad (2.4.2)
\]

Further, we set \( a = 1.5, \ q = 11 \) and the number of design time points \( J = 180 \). Finally, we specify two cases of the distribution of the i.i.d. random variables \( z_{ijr}, \ r = 1, 2, \ldots, q; \ j = 1, 2, \ldots, n_i; \ i = 1, 2, \ldots, k; \ z_{ijr} \overset{i.i.d.}{\sim} N(0, 1) \) and
$z_{ijr} \overset{i.i.d.}{\sim} t_{4/\sqrt{2}}$, allowing to generate Gaussian and non-Gaussian functional data respectively with $z_{ijr}$ having mean 0 and variance 1. Notice that the $t_{4/\sqrt{2}}$ distribution is chosen since it has nearly heaviest tails among the $t$-distributions with finite first two moments.

### 2.4.2. Simulation 1: a comparison of $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$

In this simulation, to check the finite sample performance of $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$, we let the number of groups $k = 5$. We consider three cases of the sample size vector: $n_1 = [20, 25, 22, 18, 16]$, $n_2 = [35, 30, 40, 32, 38]$ and $n_3 = [80, 75, 85, 82, 70]$, representing the small, moderate, and large sample size cases. We also consider five correlation cases, i.e., $\rho = 0.1, 0.3, 0.5, 0.7, 0.9$, representing the highly, moderately highly, moderately, less correlated and nearly independent situations. For given model configurations, the required functional samples are generated. The p-values of $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ are then computed. Notice that the p-values of $L^2_{rp}$ is obtained via 500 runs of permutations. We reject the null hypothesis if the calculated p-values are smaller than the nominal significance level $\alpha = 5\%$. We repeat the above simulation process 10000 times to get the empirical sizes or powers of $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$.

Table 2.4.1 shows the empirical sizes ($\omega = 0$) and powers ($\omega > 0$) in percentages of $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ when $z_{ijr}$, $r = 1, 2, \cdots, q$; $j = 1, 2, \cdots, n_i$; $i = 1, 2, \cdots, k$ \overset{i.i.d.}{\sim} N(0, 1). We have the following conclusions:

- In terms of size controlling, $L^2_{nv}$ works well when the functional data are highly correlated but it becomes rather conservative (with the empirical size to be as small as 3.01%) when the correlation of functional data is reduced. $L^2_{br}$ generally works well for various settings and it becomes better with increasing the sample sizes. $L^2_{rp}$ is quite liberal (with the
2.4 Simulation Studies

Table 2.4.1. Empirical sizes and powers (in percentages) of $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ for Simulation 1 when $z_{ijr}$, $r = 1, \cdots, q; j = 1, \cdots, n_i; i = 1, \cdots, k$ are i.i.d. $N(0,1)$.

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<th>$L^2_{rp}$</th>
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empirical size to be as large as 12.51%), especially when the functional data are nearly independent. However, it performs better with increasing the sample sizes.

- In terms of powers, $L^2_{br}$ is comparable or have higher powers than $L^2_{nv}$ and $L^2_{rp}$ when their empirical sizes are comparable.
- Overall, when the functional data are Gaussian, $L^2_{br}$ outperforms $L^2_{nv}$ and $L^2_{rp}$. 25
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Table 2.4.2. Empirical sizes and powers (in percentages) of $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ for Simulation 1 when $z_{ijr}$, $r = 1, \cdots, q; \ j = 1, \cdots, n_i; \ i = 1, \cdots, k$ are i.i.d. $t_4/\sqrt{2}$.

<table>
<thead>
<tr>
<th>$n_1$ = [2, 25, 22, 18, 16]</th>
<th>$n_2$ = [35, 30, 40, 32, 38]</th>
<th>$n_3$ = [80, 75, 85, 82, 70]</th>
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</table>

Note that the inflated sizes of $L^2_{rp}$ may be due to the small number of runs of permutations, which is 500. However, increasing this number requires much more computational efforts and we did not adopt this strategy for time saving. But in real data analysis, the random permutation times can be 10000.

Table 2.4.2 shows the empirical sizes and powers of $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ when $z_{ijr}$, $r = 1, 2, \cdots, q; \ j = 1, 2, \cdots, n_i; \ i = 1, 2, \cdots, k$ i.i.d. $t_4/\sqrt{2}$, representing...
2.4 Simulation Studies

the cases when the functional data are non-Gaussian. We have the following conclusions:

- In terms of size controlling, both $L^2_{nv}$ and $L^2_{br}$ do not work since their empirical sizes are too large compared with the nominal size 5%. This is expected since the formulas (2.2.9) used for computing the approximated null distributions are based on the Gaussian assumption A1.

- The performance of $L^2_{rp}$, on the other hand, is comparable with those cases presented in Table 2.4.1. That is, its empirical sizes are liberal when the functional data are less correlated but they are getting better with increasing the sample sizes.

- Thus, when the functional data are not Gaussian, $L^2_{rp}$ may work for large samples but $L^2_{nv}$ and $L^2_{br}$ do not work at all.

2.4.3. Simulation 2: a comparison of $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ against $FHK_D$ and $FHK_G$. In this simulation, we shall use the simulation codes kindly provided by Dr. Fremdt via email communication. To compare $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ against $FHK_D$ and $FHK_G$, we set the number of groups $k = 2$ and consider the sample size $n_1 = [25, 22]$, $n_2 = [30, 40]$ and $n_3 = [75, 85]$. We also specify $\rho = 0.1, 0.3, 0.5, 0.7, 0.9$ to consider the five cases when the functional samples have high, moderately high, moderate, low, very low correlations. Since smoothing is needed to conduct $FHK_D$ and $FHK_G$, we choose 49 Fourier basis functions to smooth the simulated functions. Notice that $FHK_D$ and $FHK_G$ require selecting $d$, the number of empirical functional principal components. But the selection of $d$ is a big challenge and beyond the scope of this thesis, we instead just consider: $d = 1, 2, 3, 4$, hoping that the important signals in functional data are located at low principal components. Actually, we shall use the same method as described in
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the beginning of this section to generate functional samples for the simulations presented in this subsection so that the main differences between the two covariance functions are located at the first two basis functions as indicated in (2.4.2).

Table 2.4.3. Empirical sizes and powers (in percentages) of $FHK_D, FHK_G, L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ for Simulation 2 with sample size

$\mathbf{n}_1 = [25, 22]$ when $z_{ijr}, r = 1, \cdots, q; j = 1, \cdots, n_i; i = 1, 2 \overset{i.i.d.}{\sim} N(0, 1)$.

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Tables 2.4.3–2.4.5 display the empirical sizes and powers of $FHK_D, FHK_G, L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ when $z_{ijr}, r = 1, \cdots, q; j = 1, 2, \cdots, n_i; i = 1, 2 \overset{i.i.d.}{\sim} N(0, 1)$. We may make the following conclusions:
2.4 Simulation Studies

Table 2.4.4. Empirical sizes and powers (in percentages) of $FHK_D$, $FHK_G$, $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ for Simulation 2 with sample size $n_2 = [30, 40]$ when $z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, 2 \overset{i.i.d.}{\sim} N(0, 1)$.

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- In terms of size controlling, $FHK_D$ is conservative under both small and large sample size but it is liberal under medium sample size. This result shows that $FHK_D$ test is unstable. In addition, choosing large $d$ may cause negative effects on the performance of $FHK_D$ — larger $d$ makes $FHK_D$ more conservative under small sample size and makes $FHK_D$ more liberal under medium sample size. When $\rho$ is large, $L^2_{nv}$ has conservative empirical sizes while the $L^2_{rp}$ test has inflated empirical sizes. As
Chapter 2. An $L^2$-norm Based Test

Table 2.4.5. Empirical sizes and powers (in percentages) of $FHK_D, FHK_G, L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ for Simulation 2 with sample size $n_3 = [75, 85]$ when $z_{ijr}, r = 1, \cdots, q; j = 1, \cdots, n_i; i = 1, 2 \sim N(0, 1)$.

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the sample size increases, their empirical sizes become closer to the nominal size. Among the five methods, $FHK_G$ and $L^2_{br}$ slightly outperform other tests.

- In most cases, the powers of $FHK_D$ and $FHK_G$ increase with increasing the values of $d$ from 1 to 2 and decrease with increasing the values of $d$ from 2 to 3, 4. This is because $d = 2$ is the correct number of functional principal components with the differences of the two covariance functions.
located at the first two basis functions. This shows that the performances of \( FHK_D \) and \( FHK_G \) strongly depend on if a correct number of principal components is used. When the number of functional principal components is not well chosen, the powers of \( FHK_D \) and \( FHK_G \) may be much smaller than those of \( L_{nv}^2, L_{br}^2 \) and \( L_{rp}^2 \), as shown in the table.

- Unlike \( FHK_D \) and \( FHK_G \), \( L_{nv}^2, L_{br}^2 \) and \( L_{rp}^2 \) do not need to calculate the eigenvalues and choose the number of functional principal components. This could be a big advantage.

Tables 2.4.6 and 2.4.8 display the empirical sizes and powers of \( FHK_D, FHK_G, L_{nv}^2, L_{br}^2 \) and \( L_{rp}^2 \) when \( z_{ijr}, r = 1, 2, \cdots, q; j = 1, 2, \cdots, n_i; i = 1, 2 \overset{i \text{ i.i.d.}}{\sim} t_4/\sqrt{2} \). We may make the following conclusions:

- \( FHK_G, L_{nv}^2 \) and \( L_{br}^2 \) have too large empirical sizes and they do not work at all in this simulation setting. This is expected since they are developed only for Gaussian functional data.

- In terms of sizes, \( FHK_D \) is rather conservative even when the correct number of functional principal components, \( d = 2 \), is used, especially under the small and large sample sizes, while \( L_{rp}^2 \) works reasonably well under the large sample size, and is slightly inflated under the small and medium sample sizes when \( \rho \) is large. In terms of powers, \( FHK_D \) generally outperforms \( L_{rp}^2 \), except some cases when sample sizes are not large and when the data are less correlated.

In some situations, \( L_{nv}^2, L_{br}^2 \) and \( L_{rp}^2 \) can have much higher powers than \( FHK_D \) and \( FHK_G \). We can show this via making a minor change of the previous simulation scheme. We continue to use the data generating model (2.4.1) but we
Chapter 2. An $L^2$-norm Based Test

Table 2.4.6. Empirical sizes and powers (in percentages) of $FHK_D$, $FHK_G$, $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ for Simulation 2 with sample size $n_i = [25, 22]$ when $z_{ijr}$, $r = 1, \cdots, q$; $j = 1, \cdots, n_i$; $i = 1, 2 \overset{i.d.}{\sim} t_4/\sqrt{2}$.

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For $\eta_i(t) = 0$, $i = 1, 2, \cdots, k$ for simplicity. In addition, we increase the number of basis functions to $q = 25$ and set $\lambda_1r = \rho^{-1}$, $r = 1, 2, \cdots, q$, $\lambda_{2r} = \rho^{-1}$, $r = 1, 2, \cdots, q - 1$ and $\sqrt{\lambda_{2q}} = \sqrt{\lambda_{1q}} + \omega$ so that the differences of the covariance functions of the functional samples are located at the space spanned by the last eigenfunction. Since the information is located in high frequencies, $FHK_D$ and $FHK_G$ will be less powerful in detecting the differences of the covariance functions. This is not the case for $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ since these $L^2$-norm based tests
### 2.4 Simulation Studies

Table 2.4.7. Empirical sizes and powers (in percentages) of $FHK_D$, $FHK_G$, $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ for Simulation 2 with sample size $n_2 = [30, 40]$ when $z_{ijr}$, $r = 1, \cdots, q$; $j = 1, \cdots, n_i$; $i = 1, 2^{i.i.d.}$ $t_4/\sqrt{2}$.

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use all the information provided by the data. The simulation results presented in Tables 2.4.9–2.4.11 indeed show this is true. From these tables, we can see that the powers of $FHK_D$ and $FHK_G$ are about the same with increasing values of $\omega$ but powers of $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ become larger as $\omega$ increases.
### 2.4.8. Empirical sizes and powers (in percentages) of $FHK_D$, $FHK_G$, $L_{nv}^2$, $L_{br}^2$ and $L_{rp}^2$ for Simulation 2 with sample size $n = [75, 85]$ when $z_{ijr}$, $r = 1, \cdots, q$; $j = 1, \cdots, n_i$; $i = 1, 2^{i.d.}$ $t_4/\sqrt{2}$.

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### 2.5. Applications to Two Real Data Examples

In this section, we shall present the applications of $L_{nv}^2$, $L_{br}^2$, $L_{rp}^2$ to two real data examples. Throughout this section, the p-values of $L_{rp}^2$ were obtained based on 10000 runs of random permutations.

**2.5.1. The Medfly Data.** In this subsection, we present some applications of $L_{nv}^2$, $L_{br}^2$, $L_{rp}^2$ to check if the cell covariance functions of the medfly data are...
2.5 Applications to Two Real Data Examples

Table 2.4.9. Empirical sizes and powers (in percentages) when 
\( \mathbf{n}_1 = [25, 22], z_{ijr}^{i.i.d.} \sim N(0, 1) \) under new scheme of Simulation 2.

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Table 2.4.10. Empirical sizes and powers (in percentages) when 
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Table 2.4.11. Empirical sizes and powers (in percentages) when $n_3 = [75, 85]$, $z_{ijr} \sim N(0, 1)$ under new scheme of Simulation 2.

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The medfly data, which recorded daily egg-laying numbers of 1000 medflies (Mediterranean fruit flies), have been analyzed by several authors in the literature, including Müller and Stadtmüller (2005) and Fremdt et al. (2013) among others. Thanks to Professor Hans-Georg Müller and Professor Carey’s laboratory, this medfly data set is available at [http://anson.ucdavis.edu/~mueller/data/data.html](http://anson.ucdavis.edu/~mueller/data/data.html). Previous studies indicate that the fecundity may be associated with the individual mortality and longevity.

We picked up 534 medflies who lived at least 34 days and studied both the absolute and relative counts of eggs laid by the 534 medflies in the first 30 days. A relative count is defined as the ratio of absolute count in each day to the total number of eggs laid by each medfly. These medflies are classified into two groups: long-lived and short-lived. The long-lived group includes 278 medflies who lived
2.5 Applications to Two Real Data Examples

44 days or longer and the short-lived group includes 256 medflies who lived less than 44 days.

Fremdt et al. (2013) has considered testing if the covariance functions of the long-lived group and the short-lived group are the same. We can also apply $L^2_{nv}, L^2_{br}, L^2_{rp}$ for this problem. Actually, based on the absolute counts, the test statistic $T_n = 2.9774e8$ and the p-values of $L^2_{nv}, L^2_{br}, L^2_{rp}$ are 0.3017, 0.2999 and 0.1228 respectively. These p-values show that there is no strong evidence against the null hypothesis that the covariance functions of the long-lived group and the short-lived group are the same. This conclusion is consistent with the one made by the $FHK_D$ test described in Fremdt et al. (2013). Based on the relative counts, on the other hand, the associated test statistic $T_n = 0.0191$ and the p-values of $L^2_{nv}, L^2_{br}, L^2_{rp}$ are now 0, 0 and 0.0025 respectively. These p-values show that there is very strong evidence against the null hypothesis. This conclusion is again consistent with the one made by the $FHK_D$ test.

According to Fremdt et al. (2013), both the absolute counts and the relative counts have a strong deviation from normality which can be easily verified by QQ-plots. Therefore, the p-values of $L^2_{rp}$ will be more reliable than those of $L^2_{nv}$ and $L^2_{br}$. Although $L^2_{nv}$ and $L^2_{br}$ are based on the Gaussian assumption, these two tests give consistent results for both the absolute counts and relative counts while $FHK_G$ may give a misleading conclusion because the result of $FHK_G$ varies depending on the selection of empirical functional principal components as shown in Fremdt et al. (2013).

It is also possible to classify the medflies into three groups. The first group consists of the long-lived medflies who lived 50 days or longer, the second group consists of the medium-lived medflies who lived at least 40 days but no longer than 50 days, and the third group consists of the short-lived medflies who lived
less than 40 days. Of the 534 medflies, 180 are long-lived, 180 are medium-lived and 174 are short-lived. Of interest is to test if the covariance functions of the three groups of medflies are the same.

Based on the absolute counts, the associated test statistic $T_n = 5.7069e8$ and the p-values of $L^2_{nv}, L^2_{br}, L^2_{rp}$ are 0.3132, 0.3107 and 0.1030 respectively. Thus, again, there is no strong evidence against that the covariance functions of the three groups are the same. Based on the relative counts, the associated test statistic $T_n = 0.0337$ and the p-values of $L^2_{nv}, L^2_{br}, L^2_{rp}$ are 0, 0, and 0.0123 respectively. Thus, again, there is strong evidence against that the covariance functions of the three groups are the same. These conclusions are consistent with those obtained based on the comparison of the covariance functions of the long-lived and short-lived medflies described above.

2.5.2. The Orthosis Data. In this subsection, we present some applications of $L^2_{nv}, L^2_{br}, L^2_{rp}$ to check if the cell covariance functions of the orthosis data are the same. The orthosis data set was kindly provided by Dr. Brani Vidakovic via email communication. It has been previously studied by a number of authors, including Abramovich et al. (2004), Abramovich and Angelini (2006), Antoniadis and Sapatinas (2007), and Cuesta-Albertos and Febrero-Bande (2010) among others.

To better understand how muscle copes with an external perturbation, the orthosis data were acquired and computed by Dr. Amarantini David and Dr. Martin Luc (Laboratoire Sport et Performance Motrice, EA 597, UFRAPS, Grenoble University, France). The data set recorded the moments at the knee of 7 volunteers under 4 experimental conditions (control, orthosis, spring 1, spring 2), each 10 times at equally spaced 256 time points. Figure 2.5.1 displays the raw curves of the orthosis data set, with each panel showing 10 raw curves. Figure 2.5.2 shows
2.5 Applications to Two Real Data Examples

Figure 2.5.1. Raw curves of the orthosis data.

the 4 estimated cell covariance functions for the fifth volunteer under all the 4 conditions. Based on these two figures, it seems that the cell covariance functions are not exactly the same.

We firstly applied $L^2_{nv}$, $L^2_{br}$, $L^2_{rp}$ to test if all the 28 cell covariance functions are the same. It is easy to obtain that the test statistic $T_n = 1.5661e10$ using (2.2.4). To apply $L^2_{nv}$, by (2.2.8) and (2.2.9), we obtained $\text{tr}(\hat{\omega}) = 2.9198e8$, $\text{tr}(\hat{\omega}^{\otimes 2}) = 5.1118e15$ so that $\hat{\beta} = 1.7507e7, \hat{d} = 450.29$. The resulting p-value of $L^2_{nv}$ is then 0. To apply $L^2_{br}$, we obtained $\text{tr}(\overline{\omega}) = 2.9051e8$, $\text{tr}(\overline{\omega}^{\otimes 2}) = 4.9242e15$ using (2.2.11) and then $\hat{\beta} = 1.6950e7, \hat{d} = 462.75$ using (2.2.10). The resulting p-value of $L^2_{br}$ is also 0. Similarly, the resulting p-value of $L^2_{rp}$ is again 0. These
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Figure 2.5.2. Estimated cell covariance functions of the orthosis data for the fifth volunteer under the 4 treatment conditions.

p-values demonstrate that the 28 cell covariance functions of the orthosis data are unlikely to be the same.

Secondly, we applied $L^2_{nv}, L^2_{br}, L^2_{rp}$ to check if the 4 estimated cell covariance functions for the fifth volunteer under the 4 different conditions are the same. The resulting test statistic is $T_n = 5.5510e9$ and the resulting p-values of $L^2_{nv}, L^2_{br}, L^2_{rp}$ are 0.0040, 0.0011 and 0.0150 respectively. These resulting p-values show that the 4 cell covariance functions under consideration are unlikely to be the same which is consistent with what we observed from Figure 2.5.2.
Finally, we applied $L_{nv}^2, L_{br}^2, L_{rp}^2$ to check if the 4 estimated cell covariance functions for the first volunteer under the 4 different conditions are the same. The resulting test statistic is $T_n = 5.70088$ and the resulting p-values of $L_{nv}^2, L_{br}^2, L_{rp}^2$ are 0.4670, 0.4050 and 0.2076 respectively, showing that there is no strong evidence against that the 4 estimated cell covariance functions for the first volunteer under the 4 different conditions are the same.

2.6. Appendix

In this Appendix, we first present four useful lemmas. The first lemma presents the joint distribution of $\hat{\gamma}_i, i = 1, 2, \cdots, k$ under both the null and alternative hypotheses.

**Lemma 2.1.** Under Assumptions A1 and A2, as $n \to \infty$, we have
\[
\begin{pmatrix}
\sqrt{n_1 - 1} [\hat{\gamma}_1(s,t) - \gamma_1(s,t)] \\
\sqrt{n_2 - 1} [\hat{\gamma}_2(s,t) - \gamma_2(s,t)] \\
\vdots \\
\sqrt{n_k - 1} [\hat{\gamma}_k(s,t) - \gamma_k(s,t)]
\end{pmatrix} \xrightarrow{d} GP_k [0, \text{diag}(\varpi_1, \varpi_2, \cdots, \varpi_k)],
\]
where $\varpi_i [(s_1, t_1), (s_2, t_2)] = \gamma_i(s_1, s_2)\gamma_i(t_1, t_2) + \gamma_i(s_1, t_2)\gamma_i(s_2, t_1), i = 1, 2, \cdots, k$. In particular, when the null hypothesis (2.1.2) holds, we have $\varpi_1 = \varpi_2 = \cdots = \varpi_k = \varpi$ and $\text{diag}(\varpi_1, \varpi_2, \cdots, \varpi_k) = \varpi I_k$ where $\varpi [(s_1, t_1), (s_2, t_2)] = \gamma(s_1, s_2)\gamma(t_1, t_2) + \gamma(s_1, t_2)\gamma(s_2, t_1)$.

Lemma 2.2 shows that $\hat{\gamma}(s,t)$ is asymptotically a Gaussian process and is consistent uniformly under Assumptions A1~A3.

**Lemma 2.2.** Under Assumptions A1~A3 and the null hypothesis (2.1.2), as $n \to \infty$, we have
\[
\sqrt{n - k} \{\hat{\gamma}(s,t) - \gamma(s,t)\} \xrightarrow{d} GP(0, \varpi),
\]
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where $\hat{\gamma}(s,t)$ is defined in (2.2.3) and $\varpi[(s_1, t_1), (s_2, t_2)]$ is defined in Lemma 2.1.

In addition, let $O_{UP}$ denote uniformly bounded in probability, we have

$$\hat{\gamma}(s,t) = \gamma(s,t) + O_{UP} \left[ (n - k)^{-1/2} \right].$$

**Proof of Lemmas 2.1–2.2.** Refer to the proof of Theorems 10.4–10.6 in Zhang (2013) (p.364–365).

Lemma 2.3 guarantees that under Assumption A1 and the local alternative (2.3.1), $\varpi_i[(s_1, t_1), (s_2, t_2)]$ will converge to $\varpi[(s_1, t_1), (s_2, t_2)]$ in a root-n rate. This result will be used to prove the asymptotic distribution of $T_n$ under the local alternative (2.3.1).

**Lemma 2.3.** Under Assumption A1 and the local alternative (2.3.1), we have

$$\varpi_i[(s_1, t_1), (s_2, t_2)] = \varpi[(s_1, t_1), (s_2, t_2)] + (n_i - 1)^{-1/2} h_i[(s_1, t_1), (s_2, t_2)] + O(n_i^{-1}),$$

where $h_i[(s_1, t_1), (s_2, t_2)]$, $i = 1, 2, \ldots, k$ are some fixed functions.

**Proof of Lemma 2.3.** Under Assumption A1 and the local alternative (2.3.1), we have

$$\varpi_i[(s_1, t_1), (s_2, t_2)] = \gamma(s_1, s_2) \gamma(t_1, t_2) + \gamma_i(s_1, t_2) \gamma_i(s_2, t_1)$$

$$= [\gamma(s_1, s_2) + (n_i - 1)^{-1/2} d_i(s_1, s_2)] [\gamma(t_1, t_2) + (n_i - 1)^{-1/2} d_i(t_1, t_2)]$$

$$+ [\gamma(s_1, t_2) + (n_i - 1)^{-1/2} d_i(s_1, t_2)] [\gamma(s_2, t_1) + (n_i - 1)^{-1/2} d_i(s_2, t_1)]$$

$$= \gamma(s_1, s_2) \gamma(t_1, t_2) + \gamma(s_1, t_2) \gamma(s_2, t_1) + (n_i - 1)^{-1/2} h_i[(s_1, t_1), (s_2, t_2)]$$

$$+ (n_i - 1)^{-1} [d_i(s_1, s_2) d_i(t_1, t_2) + d_i(s_1, t_2) d_i(s_2, t_1)]$$

$$= \varpi[(s_1, t_1), (s_2, t_2)] + (n_i - 1)^{-1/2} h_i[(s_1, t_1), (s_2, t_2)] + O(n_i^{-1}),$$

42
where \( h_i[(s_1, t_1), (s_2, t_2)] = \gamma(s_1, s_2) d_i(t_1, t_2) + \gamma(t_1, t_2) d_i(s_1, s_2) + \gamma(s_1, t_2) d_i(s_2, t_1) + \gamma(s_2, t_1) d_i(s_1, t_2) \).

The last lemma states that under Assumptions A1~A3 and the null hypothesis (2.1.2), the test statistic \( T_n \) is asymptotically a \( \chi^2 \)-type mixture.

**Lemma 2.4.** Under Assumptions A1~A3 and the null hypothesis (2.1.2), as \( n \to \infty \), we have \( T_n \xrightarrow{d} T_0 \) with

\[
T_0 = \sum_{r=1}^{m} \lambda_r A_r, \quad A_r \xrightarrow{i.i.d.} \chi^2_{k-1},
\]

where \( \lambda_r, \ r = 1, 2, \cdots, m \) are all the positive eigenvalues of \( \varpi[(s_1, t_1), (s_2, t_2)] \) which is given in Lemma 2.1.

**Proof of Lemma 2.4.** Refer to the proof of Theorem 10.7 in Zhang (2013) (p.365) or the proof of Theorem 2.3 since this theorem can be seen as a special case of Theorem 2.3.

We are now ready to give the technical proofs of the main results.

**Proof of Theorem 2.1.** Under the given conditions, by Lemma 2.2, \( \hat{\gamma}(s, t) \xrightarrow{p} \gamma(s, t) \) uniformly over all \( s, t \in \mathcal{T} \) where \( \text{tr}(\gamma) < \infty \) and hence \( \text{tr}(\gamma^{\otimes k}) < \infty \) for \( k = 1, 2, \cdots \). As \( n \to \infty \), \( \text{tr}(\hat{\gamma}) \xrightarrow{p} \text{tr}(\gamma) \), \( \text{tr}(\hat{\gamma}^{\otimes 2}) \xrightarrow{p} \text{tr}(\gamma^{\otimes 2}) \) and \( \text{tr}(\hat{\gamma}^{\otimes 4}) \xrightarrow{D} \text{tr}(\gamma^{\otimes 4}) \) follow immediately from the continuous mapping theorem for random elements taking values in a Hilbert space (Billingsley 1968, p.34; Cuevas et al. 2004). Therefore, as \( n \to \infty \), we have \( \hat{\beta} \xrightarrow{p} \beta \) and \( \hat{\kappa} \xrightarrow{p} \kappa \) for the naive method. Similarly, we can get \( \hat{\beta} \xrightarrow{p} \beta \) and \( \hat{\kappa} \xrightarrow{p} \kappa \) for the bias-reduced method. Based on Slusky’s theorem, it is easy to get \( \hat{T}_n(\alpha) \xrightarrow{P} \tilde{T}_0(\alpha) \).  

**Proof of Theorem 2.3.** By (2.3.2) and (2.3.4), we have

\[
T_n = \int_T \int_T z_n(s, t)^T W_n z_n(s, t) ds dt,
\]

43
where \( \mathbf{W}_n \) and \( \mathbf{z}_n(s, t) = [z_1(s, t), z_2(s, t), \cdots, z_k(s, t)]^T \) are defined in (2.3.5). We can write

\[
z_i(s, t) = \sqrt{n_i - \bar{1}}[\hat{\gamma}_i(s, t) - \gamma_i(s, t)] + \sqrt{n_i - \bar{1}}[\hat{\gamma}_i(s, t) - \gamma(s, t)], i = 1, 2, \cdots, k.
\]

Under the given conditions, by Lemma 2.1, we have \( \sqrt{n_i - \bar{1}}[\hat{\gamma}_i(s, t) - \gamma_i(s, t)] \xrightarrow{d} \text{GP}(0, \varpi) \). While by the local alternative (2.3.1), we have \( \sqrt{n_i - \bar{1}}[\hat{\gamma}_i(s, t) - \gamma(s, t)] = \delta_i(s, t), i = 1, 2, \cdots, k \). And thus we have \( \mathbf{z}_n(s, t) \xrightarrow{d} \text{GP}_k[\mathbf{d}, \text{diag}(\varpi_1, \varpi_2, \cdots, \varpi_k)] + o_p(1) \) with \( \mathbf{d}(s, t) = [d_1(s, t), d_2(s, t), \cdots, d_k(s, t)]^T \).

It follows from Lemma 2.3 that as \( n \rightarrow \infty \), we have \( \mathbf{z}_n(s, t) \xrightarrow{d} \mathbf{z}(s, t) \sim \text{GP}_k[\mathbf{d}, \varpi \mathbf{I}_k] \). By (2.3.6), as \( n \rightarrow \infty \), we have \( \mathbf{W}_n \rightarrow \mathbf{W} \) which has the singular value decomposition (2.3.7). Again by the continuous mapping theorem for random elements in a Hilbert space, it follows that \( T_n \xrightarrow{d} T_1 \) with

\[
T_1 = \int_{\mathcal{T}} \int_{\mathcal{T}} ||\mathbf{z}_n(s, t)||^2 \, ds \, dt,
\]

where \( \mathbf{z}_n(s, t) = [\mathbf{I}_{k-1}, \mathbf{0}] \mathbf{U}^T \mathbf{z}(s, t) = [\hat{z}_1(s, t), \hat{z}_2(s, t), \cdots, \hat{z}_{k-1}(s, t)]^T \sim \text{GP}_k(\mathbf{d}, \varpi \mathbf{I}_{k-1}) \) with \( \mathbf{d}(s, t) = [\hat{d}_1(s, t), \hat{d}_2(s, t), \cdots, \hat{d}_{(k-1)}(s, t)]^T \). Thus

\[
T_1 = \sum_{i=1}^{k-1} \int_{\mathcal{T}} \int_{\mathcal{T}} \hat{z}_i^2(s, t) \, ds \, dt.
\]

Since \( \hat{z}_i(s, t) \sim \text{GP}(\hat{d}_i, \varpi) \), along the same lines as the proof of Theorem 4.2 of Zhang (2013), we can show that

\[
\int_{\mathcal{T}} \int_{\mathcal{T}} \hat{z}_i^2(s, t) \, ds \, dt = \sum_{r=1}^m \lambda_r A_{ir} + \sum_{r=m+1}^{\infty} \delta_{ir}^2,
\]

where \( A_{ir} \sim \chi_1^2(\lambda_{ir}^{-1} \delta_{ir}^2) \), \( r = 1, 2, \cdots, m \), and \( \delta_{ir} = \int_{\mathcal{T}} \int_{\mathcal{T}} \hat{d}_i(s, t) \phi_r(s, t) \, ds \, dt \). Thus

\[
T_1 = \sum_{i=1}^{k-1} \int_{\mathcal{T}} \int_{\mathcal{T}} \hat{z}_i^2(s, t) \, ds \, dt = \sum_{i=1}^{k-1} \left\{ \sum_{r=1}^m \lambda_r A_{ir} + \sum_{r=m+1}^{\infty} \delta_{ir}^2 \right\}
\]

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\[ \sum_{r=1}^{m} \sum_{i=1}^{k-1} \lambda_r A_{ir} + \sum_{r=m+1}^{\infty} \sum_{i=1}^{k-1} \delta_{ir}^2 = \sum_{r=1}^{m} \lambda_r A_r + \sum_{r=m+1}^{\infty} \delta_{r}^2, \]

where

\[ A_r = \sum_{i=1}^{k-1} A_{ir} \sim \chi^2_{k-1} (\lambda_r^{-1} \delta_r^2), \quad (2.6.2) \]

\[ r = 1, 2, \ldots, m, \] are independent, and

\[ \delta_{r}^2 = \sum_{i=1}^{k-1} \delta_{ir}^2 = || \int_{T} \int_{T} \dot{d}(s, t) \phi_r (s, t) ds dt ||^2. \]

**Proof of Theorem 2.4.** By Theorem 2.3, we have

\[ E(T_n) = \sum_{r=1}^{m} \lambda_r (k - 1 + \lambda_r^{-1} \delta_r^2) + \sum_{r=m+1}^{\infty} \delta_r^2 + o(1) \]

\[ = \sum_{r=1}^{m} \lambda_r (k - 1) + \sum_{r=1}^{\infty} \delta_r^2 + o(1), \]

\[ \text{Var}(T_n) = \sum_{r=1}^{m} 2\lambda_r^2 (k - 1 + 2\lambda_r^{-1} \delta_r^2) + o(1) \]

\[ = \sum_{r=1}^{m} [2\lambda_r^2 (k - 1) + 4\lambda_r \delta_r^2] + o(1). \]

By Theorem 2.3 again, we have

\[ A_r \overset{d}{=} z_{1r}^2 + \cdots + z_{(k-2)r}^2 + (z_{(k-1)r} + \delta_r / \sqrt{\lambda_r})^2 \]

\[ = B_r + 2z_{(k-1)r} \delta_r / \sqrt{\lambda_r} + \delta_r^2 / \lambda_r, \]

where \( A_r \) is defined in (2.6.2), \( z_{ir} \overset{i.i.d.}{\sim} N(0, 1), i = 1, 2, \ldots, k - 1, \quad r = 1, 2, \ldots, m, \)

and \( B_r \sim \chi^2_{k-1} \). Thus,

\[ T_n \overset{d}{=} \sum_{r=1}^{m} \lambda_r (B_r + 2z_{(k-1)r} \delta_r / \sqrt{\lambda_r} + \delta_r^2 / \lambda_r) + \sum_{r=m+1}^{\infty} \delta_r^2 + o_p(1) \]

\[ = \sum_{r=1}^{m} \lambda_r B_r + 2 \sum_{r=1}^{m} \sqrt{\lambda_r} z_{(k-1)r} \delta_r + \sum_{r=1}^{\infty} \delta_r^2 + o_p(1). \]
Thus,
\[
\frac{T_n - E(T_n)}{\sqrt{\text{Var}(T_n)}} = \sum_{r=1}^{m} \lambda_r [B_r - (k - 1)] + 2 \sum_{r=1}^{m} \sqrt{\lambda_r} \delta_r + o_p(1)
\]
\[
= \frac{\sum_{r=1}^{m} \lambda_r [B_r - (k - 1)]}{\sqrt{\sum_{r=1}^{m} \lambda_r}} + \frac{2 \sum_{r=1}^{m} \sqrt{\lambda_r} \delta_r}{\sqrt{\sum_{r=1}^{m} \lambda_r}} + o_p(1)
\]
\[:= I_1 + I_2 + o_p(1).\]

It is easy to see that \(E(I_1) = 0\), \(\text{Var}(I_1) = \frac{2(k-1) \sum_{r=1}^{m} \lambda_r^2}{\sum_{r=1}^{m} \lambda_r(k-1) + 4 \lambda_r \delta_r^2}\) and \(E(I_2) = 0\), \(\text{Var}(I_2) = \frac{4 \sum_{r=1}^{m} \lambda_r \delta_r^2}{\sum_{r=1}^{m} \lambda_r(k-1) + 4 \lambda_r \delta_r^2}\). Under the local alternative (2.3.1), when \(\max_r \delta_r^2 \to \infty\) one has \(\text{Var}(I_1) \to 0\) and \(\text{Var}(I_2) \to 1\). Therefore, \(\frac{T_n - E(T_n)}{\sqrt{\text{Var}(T_n)}} \overset{d}{\to} N(0, 1).\)

**PROOF OF THEOREM 2.5.** Under Assumptions A1~A3 and the local alternative (2.3.1), similar to Theorem 2.1, as \(n \to \infty\), we also have \(\hat{T}_n(\alpha) \overset{p}{\to} \hat{T}_0(\alpha)\). Then by Theorem 2.4, we have
\[
P(T_n > \hat{T}_n(\alpha)) = P \left( \frac{T_n - E(T_n)}{\sqrt{\text{Var}(T_n)}} > \frac{\hat{T}_n(\alpha) - E(T_n)}{\sqrt{\text{Var}(T_n)}} \right) = \Phi \left( \frac{E(T_n) - \hat{T}_0(\alpha)}{\sqrt{\text{Var}(T_n)}} \right) + o(1).
\]

When \(\max_r \delta_r^2 \to \infty\), we have
\[
\frac{E(T_n) - \hat{T}_0(\alpha)}{\sqrt{\text{Var}(T_n)}} = \frac{\sum_{r=1}^{m} \lambda_r (k - 1) + \sum_{r=1}^{\infty} \delta_r^2 + o(1) - \hat{T}_0(\alpha)}{\sqrt{\sum_{r=1}^{m} \lambda_r(k-1) + 4 \lambda_r \delta_r^2 + o(1)}} \to \infty,
\]
since \(\hat{T}_0(\alpha) < \infty\). It follows that \(P(T_n > \hat{T}_n(\alpha)) \to 1.\)

**PROOF OF THEOREMS 2.2 AND 2.6.** Notice that given the original \(k\) samples, the new \(k\) samples
\[
\hat{\gamma}_{ij}(t) \overset{i.i.d.}{\sim} SP(0, \frac{n-k}{n} \hat{\gamma}), \ j = 1, 2, \ldots, n_i; \ i = 1, 2, \ldots, k,
\]
where \(\hat{\gamma}(s, t)\) is the pooled sample covariance function defined in (2.2.3). Under the null hypothesis, according to Lemma 2.2, we have \(\sqrt{n-k} \{\hat{\gamma}(s, t) - \gamma(s, t)\} \overset{d}{\to}
\]
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$GP(0, \pi)$ and $\frac{n-k}{n} \hat{\gamma}(s, t) \xrightarrow{P} \gamma(s, t)$, which implies that $T_n^* \xrightarrow{d} T_0$ and $\hat{T}_n^*(\alpha) \xrightarrow{d} T_0(\alpha)$. The size $P(T_n > \hat{T}_n^*(\alpha)) \rightarrow P(T_n > T_0(\alpha))$ follows immediately. Note that when the null hypothesis doesn’t hold, by Lemma 2.1, there still exists a $\gamma^*(s, t)$ such that $\hat{\gamma}(s, t) \xrightarrow{P} \gamma^*(s, t)$, and similarly one has $\hat{T}_n^*(\alpha) = \hat{T}_0^*(\alpha) + o_p(1)$, where $\hat{T}_0^*(\alpha)$ is the $\alpha$ quantile of $T_n$ based on the samples from $SP(0, \gamma^*)$. Then we can get $\hat{T}_n^*(\alpha) < \infty$. So under the alternative hypothesis (2.3.1), using similar arguments for proving Theorems 2.4 and 2.5, we will have the power $P(T_n > \hat{T}_n^*(\alpha)) \rightarrow 1$. \hfill \Box
CHAPTER 3

A Supremum Norm Based Test

3.1. Introduction

Chapter 2 has shed some light on how to test the multi-sample equal covariance function (ECF) testing problems using the $L^2$-norm based tests. However, the naive and bias-reduced $L^2$-norm tests can be less powerful when data are highly correlated and are seriously biased when the samples follow non-Gaussian distribution. In this chapter, we propose a supremum norm based test which is good at detecting sparse differences of covariance functions for Gaussian data and can also be applied to non-Gaussian data. Via some simulation studies, we also found that in terms of power, the supremum norm based test outperforms the three $L^2$-norm based tests substantially when the functional data are highly or moderately correlated.

This chapter aims to study the supremum norm based test via theoretical investigation and simulation studies. Two methods are provided to approximate the null distribution of the supremum norm based test. Our purpose is to develop
another useful test for the multi-sample ECF problem which can be used under general conditions. This chapter is organized as follows. Section 3.2 provides the preliminaries. Some theoretical results are presented in Section 3.3, with the proofs given in Section 3.6. Sections 3.4 and 3.5 are simulation studies and real data analysis respectively.

3.2. Preliminaries

Recall the \( k \)-sample ECF testing problem introduced in Chapter 2. Let \( \text{SP}(\eta, \gamma) \) denote a stochastic process with mean function \( \eta(t) \) and covariance function \( \gamma(s, t) \). Let \( y_{i1}(t), y_{i2}(t), \ldots, y_{in_i}(t), \ i = 1, 2, \ldots, k \) be \( k \) independent functional samples over a given finite time interval \( T = [a, b] \), \(-\infty < a < b < \infty\), which satisfy

\[
y_{ij}(t) = \eta_i(t) + v_{ij}(t), \ j = 1, 2, \ldots, n_i,
\]

\[
v_{i1}(t), v_{i2}(t), \ldots, v_{in_i}(t) \overset{i.i.d.}{\sim} \text{SP}(0, \gamma_i); \ i = 1, 2, \ldots, k,
\]

(3.2.1)

where \( \eta_1(t), \eta_2(t), \ldots, \eta_k(t) \) model the unknown group mean functions of the \( k \) samples, \( v_{ij}(t), \ j = 1, 2, \ldots, n_i; \ i = 1, 2, \ldots, k \) denote the subject-effect functions, and \( \gamma_i(s, t), \ i = 1, 2, \ldots, k \) are the associated covariance functions. Throughout this chapter, we assume that \( \text{tr}(\gamma_i) < \infty \) and \( \eta_i(t) \in L^2(T), \ i = 1, 2, \ldots, k \), where \( L^2(T) \) denotes the Hilbert space formed by all the squared integrable functions over \( T \) with the inner-product defined as \( <f, g> = \int_T f(t)g(t)dt, \ f, \ g \in L^2(T) \). Our interest is comparing the second-order properties, i.e., testing the equality of the \( k \) covariance functions:

\[
H_0 : \gamma_1(s, t) \equiv \gamma_2(s, t) \equiv \cdots \equiv \gamma_k(s, t), \ \text{for all } s, t \in T.
\]

(3.2.2)

Given the \( k \) samples, the group mean functions \( \eta_i(t), \ i = 1, 2, \ldots, k \), the covariance functions \( \gamma_i(s, t), \ i = 1, 2, \ldots, k \) and the subject-effect functions \( v_{ij}(t) \),
3.3 Supremum Norm Based Test and the Asymptotic Results

\begin{align*}
\hat{\eta}_i(t) &= \bar{y}_i(t) = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}(t), \; i = 1, 2, \cdots, k, \\
\hat{\gamma}_i(s,t) &= (n_i - 1)^{-1} \sum_{j=1}^{n_i} [y_{ij}(s) - \bar{y}_i(s)][y_{ij}(t) - \bar{y}_i(t)], \; i = 1, 2, \cdots, k, \\
\hat{v}_{ij}(t) &= y_{ij}(t) - \bar{y}_i(t), \; j = 1, 2, \cdots, n_i; \; i = 1, 2, \cdots, k,
\end{align*}

(3.2.3)

which have been introduced in (2.2.1) and (2.2.2) in Chapter 2. It is easy to show that \(\hat{\gamma}_i(s,t), \; i = 1, 2, \cdots, k\) are independent and \(E\hat{\gamma}_i(s,t) = \gamma_i(s,t), \; i = 1, 2, \cdots, k\).

Under the null hypothesis (3.2.2), the common covariance function \(\gamma(s,t)\) of the \(k\) samples can be estimated by the following pooled sample covariance function

\[\hat{\gamma}(s,t) = \sum_{i=1}^{k} (n_i - 1)\hat{\gamma}_i(s,t)/(n - k),\]

(3.2.4)

where \(\hat{\gamma}_i(s,t), \; i = 1, 2, \cdots, k\) are given in (3.2.3). Let \(n = \sum_{i=1}^{k} n_i\) denote the total sample size.

3.3. Supremum Norm Based Test and the Asymptotic Results

The new test we shall propose is based on the following pointwise sum of squares between groups with respect to the \(k\)-sample ECF testing problem (3.2.2):

\[SSB(s,t) = \sum_{i=1}^{k} (n_i - 1)[\hat{\gamma}_i(s,t) - \hat{\gamma}(s,t)]^2,\]

(3.3.1)

where \(\hat{\gamma}(s,t)\) is the pooled sample covariance function of the \(k\) functional samples as defined in (3.2.4). For each given \(s, t \in \mathcal{T}\), \(SSB(s,t)\) measures the variations of the sample covariance functions \(\hat{\gamma}_i(s,t), \; i = 1, 2, \cdots, k\) and can be used to test the null hypothesis (3.2.2) restricted at \(s, t \in \mathcal{T}\). Then to test the whole null hypothesis (3.2.2), we can use the supremum norm of \(SSB(s,t)\) as our test statistic:

\[T_{\text{max}} = \sup_{s,t \in \mathcal{T}} SSB(s,t).\]

(3.3.2)
Chapter 3. A Supremum Norm Based Test

It is expected that when the null hypothesis is valid, $T_{\text{max}}$ will be small and otherwise large.

To derive the asymptotic random expression of $T_{\text{max}}$, we impose the following assumptions as in Chapter 2:

**Assumption A**

1. The $k$ functional samples (3.2.1) are Gaussian.
2. As $n \to \infty$, the $k$ sample sizes satisfy $n_i/n \to \tau_i \in (0, 1)$, $i = 1, 2, \cdots, k$ where $\sum_{i=1}^{k} \tau_i = 1$.
3. The variance functions are uniformly bounded. That is, $\rho_i = \sup_{t \in \mathcal{T}} \gamma_i(t, t) < \infty$, $i = 1, 2, \cdots, k$.

The above assumptions are regular. Assumption A2 requires that the $k$ sample sizes tend to $\infty$ proportionally.

Before we state the main results, we give an alternative expression of $SSB(s, t)$ which is helpful for deriving the main results about $T_{\text{max}}$. For any $s, t \in \mathcal{T}$, $SSB(s, t)$ can be expressed as

$$SSB(s, t) = z_n(s, t)^T[I_k - b_n b_n^T/(n - k)]z_n(s, t), \quad (3.3.3)$$

where

$$z_n(s, t) = [z_1(s, t), z_2(s, t), \cdots, z_k(s, t)]^T,$$

with

$$z_i(s, t) = \sqrt{n_i - 1}[\gamma_i(s, t) - \gamma(s, t)], \quad i = 1, 2, \cdots, k,$$

$$b_n = [\sqrt{n_1 - 1}, \sqrt{n_2 - 1}, \cdots, \sqrt{n_k - 1}]^T.$$

Since $b_n^T b_n/(n - k) = 1$, it is easy to verify that $I_k - b_n b_n^T/(n - k)$ is an idempotent matrix with rank $k - 1$. In addition, as $n \to \infty$, we have

$$I_k - b_n b_n^T/(n - k) \to I_k - bb^T, \quad \text{with} \quad b = [\sqrt{\tau_1}, \sqrt{\tau_2}, \cdots, \sqrt{\tau_k}]^T, \quad (3.3.4)$$

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where $\tau_i$, $i = 1, 2, \ldots, k$ are given in Assumption A2. Note that $I_k - bb^T$ in (3.3.4) is also an idempotent matrix of rank $k - 1$, which has the following singular value decomposition:

$$I_k - bb^T = U \begin{pmatrix} I_{k-1} & 0 \\ 0^T & 0 \end{pmatrix} U^T,$$

where the columns of $U$ are the eigenvectors of $I_k - bb^T$.

For further study, let $\varpi_i [(s_1, t_1), (s_2, t_2)]$ denote the covariance function between $v_{i1}(s_1)v_{i1}(t_1)$ and $v_{i1}(s_2)v_{i1}(t_2)$, $i = 1, 2, \ldots, k$. Then we have

$$\varpi_i [(s_1, t_1), (s_2, t_2)] = E\{v_{i1}(s_1)v_{i1}(t_1)v_{i1}(s_2)v_{i1}(t_2)\} - \gamma_i(s_1, t_1)\gamma_i(s_2, t_2).$$

Under the Gaussian assumption A1, we have

$$\varpi_i [(s_1, t_1), (s_2, t_2)] = \gamma_i(s_1, s_2)\gamma_i(t_1, t_2) + \gamma_i(s_1, t_2)\gamma_i(s_2, t_1), \ i = 1, 2, \ldots, k.$$

(3.3.7)

When the null hypothesis (3.2.2) holds, we have

$$\varpi_i [(s_1, t_1), (s_2, t_2)] = \gamma(s_1, s_2)\gamma(t_1, t_2) + \gamma(s_1, t_2)\gamma(s_2, t_1)$$

$$\equiv \varpi [(s_1, t_1), (s_2, t_2)], \ i = 1, 2, \ldots, k.$$

(3.3.8)

where $\gamma(s, t)$ is the common covariance function of the $k$ functional samples. Under the above assumptions, a natural estimator of $\varpi [(s_1, t_1), (s_2, t_2)]$ is given by

$$\hat{\varpi} [(s_1, t_1), (s_2, t_2)] = \hat{\gamma}(s_1, s_2)\hat{\gamma}(t_1, t_2) + \hat{\gamma}(s_1, t_2)\hat{\gamma}(s_2, t_1),$$

(3.3.9)

where $\hat{\gamma}(s, t)$ is the pooled sample covariance function as given by (3.2.4).

Throughout this chapter, let $\overset{d}{\Rightarrow}$ denote “converge in distribution” and $\overset{d}{=} X = Y$” denote “$X$ and $Y$ have the same distribution”. Let $\text{GP}(\eta, \gamma)$ denote a Gaussian process with mean function $\eta$ and covariance function $\gamma$.  

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Theorem 3.1. Under Assumptions A1, A2 and the null hypothesis (3.2.2), as \( n \to \infty \), we have \( T_{\text{max}} \overset{d}{\to} T_0 \) where
\[
T_0 = \sup_{s,t \in T} \left\{ \sum_{i=1}^{k-1} w_i^2(s,t) \right\},
\]
(3.3.10)
where \( w_1(s,t), w_2(s,t), \ldots, w_{k-1}(s,t) \) i.i.d. \( \sim \) \( GP(0, \varpi) \) with \( \varpi[(s_1, t_1), (s_2, t_2)] \) defined in (3.3.8).

Theorem 3.1 motivates us to apply a parametric bootstrap (PB) method to approximate the critical value of \( T_{\text{max}} \). This PB method can be described as follows. We firstly generate \( w_i^j(s,t), \) \( i = 1, 2, \ldots, k - 1; \) \( j = 1, 2, \ldots, N \) i.i.d. from \( GP(0, \hat{\varpi}) \) where \( \hat{\varpi} \) is given in (3.3.9) and \( N \) is a pre-specified large number. Then we compute \( T_0^{(j)} = \sup_{s,t \in T} \sum_{i=1}^{k-1} [w_i^j(s,t)]^2, \) \( j = 1, 2, \ldots, N \) based on the expression (3.3.10). Finally, for any given significance level \( \alpha \), we compute the upper 100\( \alpha \) sample percentile of \( T_0^{(j)}, j = 1, 2, \ldots, N \) and use it as the approximate critical value of \( T_{\text{max}} \). Since the PB method makes use of the expression (3.3.10), it works well only when the group sample sizes \( n_1, n_2, \ldots, n_k \) are large and when the functional data are Gaussian. In addition, the PB method may be time-consuming since we have to generate samples from the Gaussian process \( GP(0, \hat{\varpi}) \) a large number of times. This process usually takes a great deal of time as \( \hat{\varpi}[(s_1, t_1), (s_2, t_2)] \) is a function on \( T^4 \). Actually, we did conduct some preliminary simulations with the PB method. Unfortunately, we found that the above PB method is too computationally intensive even for some small sample sizes so that we have to give up our original plan to include it in our simulation studies presented in Section 3.4.

To overcome this difficulty, we propose a non-parametric bootstrap (NPB) method here. The key idea of the NPB method is to approximate the critical value...
3.3 Supremum Norm Based Test and the Asymptotic Results

of $T_{\text{max}}$ via generating the bootstrapped samples from the estimated subject-effect functions (3.2.3). Suppose $\hat{v}_{ij}^*(t), \ j = 1, 2, \cdots, n_i; \ i = 1, 2, \cdots, k$ are the bootstrapped $k$ samples generated from the estimated subject-effect functions (3.2.3). That is, each $\hat{v}_{ij}^*(t)$ takes any estimated subject-effects function from (3.2.3) equally likely. The NPB supremum norm based test statistic can then be computed as

$$T_{\text{max}}^* = \sup_{s,t \in \mathcal{T}_{SSB}^*}(s,t),$$

where $\mathcal{T}_{SSB}^*(s,t) = \sum_{i=1}^{k}(n_i - 1)[\hat{\gamma}_{ij}^*(s,t) - \hat{\gamma}^*(s,t)]^2$, $\hat{\gamma}^*(s,t) = \sum_{i=1}^{k}(n_i - 1)\hat{\alpha}_{ij}^*(s,t)/(n - k)$, and $\hat{\alpha}_{ij}^*(s,t) = (n_i - 1)^{-1}\sum_{j=1}^{n_i}\hat{v}_{ij}^*(s)\hat{v}_{ij}^*(t), \ i = 1, 2, \cdots, k$. Repeating the above NPB process a large number of times, the sample upper 100$\alpha$-percentile, $C_{\alpha}^*$, of $T_{\text{max}}^*$ can then be computed and used as the approximate upper 100$\alpha$-percentile of $T_{\text{max}}$. The supremum norm based test can then be conducted accordingly.

Compared with the PB method, there are a few advantages for using the NPB method. Firstly, the NPB method can be used for both small and large sample sizes. In addition, this NPB method is also applicable even though the data are not from Gaussian process. This is because the Gaussian assumption is only used in Theorem 3.1 to derive the asymptotic random expression of $T_{\text{max}}$. Last but not least, the computation of the NPB method is simple and thus may save a lot of time compared with the PB method.

**Theorem 3.2.** Under Assumptions A1~A3 and the null hypothesis (3.2.2), as $n \to \infty$, we have $T_{\text{max}}^* \xrightarrow{d} T_0$ and $C_{\alpha}^* \xrightarrow{d} C_\alpha$ where $C_\alpha$ is the theoretical upper 100$\alpha$-percentile of $T_0$.

Theorem 3.2 shows that for large samples, the NPB test statistic $T_{\text{max}}^*$ will converge in distribution to the limit random expression $T_0$ of $T_{\text{max}}$ under the null
hypothesis and hence $C^*_\alpha$ will also tend to $C_\alpha$ in distribution as $n \to \infty$. Thus it is consistent to use the NPB critical value $C^*_\alpha$ to conduct the $T_{\text{max}}$ test.

We now study the asymptotic power of $T_{\text{max}}$, aiming to show that the $T_{\text{max}}$ test is root-$n$ consistent. For this end, we specify the following local alternative:

$$H_1 : \gamma_i(s,t) = \gamma(s,t) + (n_i - 1)^{-1/2}d_i(s,t), \ i = 1, 2, \cdots, k,$$  

where $d_1(s,t), d_2(s,t), \cdots, d_k(s,t)$ are some fixed bivariate functions, independent of $n$, and $\gamma(s,t)$ is some fixed covariance function. Let $d(s,t) = [d_1(s,t), d_2(s,t), \cdots, d_k(s,t)]^T$. The asymptotic distribution of $T_{\text{max}}$ and the root-$n$ consistency property with respect to the local alternative (3.3.11) are given in the following two theorems.

**Theorem 3.3.** Under Assumptions $A1 \sim A3$ and the local alternative (3.3.11), as $n \to \infty$, we have $T_{\text{max}} \overset{d}{\to} T_1$ with

$$T_1 \overset{d}{=} \sup_{s,t \in T} \sum_{i=1}^{k-1} [w_i(s,t) + \zeta_i(s,t)]^2,$$

where $w_1(s,t), w_2(s,t), \cdots, w_{k-1}(s,t)$ i.i.d. $\sim \text{GP}(0, \varpi)$ as in Theorem 3.1 and $\zeta_i(s,t), \ i = 1, 2, \cdots, k-1$ are the $(k-1)$ components of $\zeta(s,t) = (I_{k-1}, 0)U^T d(s,t)$ with $U$ defined in (3.3.5).

Define $\delta^2_r = ||\int_T \int_T \zeta(s,t) \phi_r(s,t) dsdt||^2, \ r = 1, 2, \cdots, \infty$ with $\phi_r(s,t), \ r = 1, 2, \cdots$ being the eigenfunctions of $\varpi [(s_1, t_1), (s_2, t_2)]$ and $\zeta(s,t)$ defined in Theorem 3.3.

**Theorem 3.4.** Under Assumptions $A1 \sim A3$ and the local alternative (3.3.11), as $n \to \infty$ and $\max_r \delta^2_r \to \infty$, the power of the supremum norm based test,
3.4 Simulation Studies

\[ P(T_{\text{max}} \geq C^*_\alpha), \] will tend to 1 where \( C^*_\alpha \) is the sample upper 100\( \alpha \)-percentile of the NPB test statistic \( T_{\text{max}}^* \).

Theorem 3.4 presents the root-\( n \) consistency of \( T_{\text{max}} \). When the information of \( d(s, t) \) projected on the space spanned by the eigenfunctions tends to infinity, the asymptotic power of \( T_{\text{max}} \) will tend to 1. The proof of Theorem 3.4 is based on the following relationship between \( T_{\text{max}} \) and the \( L^2 \)-norm based test statistic \( T_n \) defined in Chapter 2:

\[
T_n = \int_T \int_T \text{SSB}(s, t)dsdt \leq (b-a)^2 T_{\text{max}}, \quad (3.3.13)
\]

where \( T = [a, b] \). It then follows that

\[
P(T_{\text{max}} \geq C^*_\alpha) \geq P(T_n \geq (b - a)^2 C^*_\alpha). \quad (3.3.14)
\]

However, \( (b - a)^2 C^*_\alpha \) may not be equal or smaller than the upper 100\( \alpha \)-percentile of \( T_n \). Thus, (3.3.14) does not guarantee that \( T_{\text{max}} \) has higher powers than \( T_n \). Some simulation studies will be presented in the next section to compare the powers of \( T_{\text{max}} \) and \( T_n \) under various simulation configurations.

3.4. Simulation Studies

For the ECF testing problem (3.2.2), we studied an \( L^2 \)-norm based test whose null distribution can be approximated by a naive method, a bias-reduced method and a random permutation method in Chapter 2. The associated \( L^2 \)-norm based tests may be denoted as \( L^2_{\text{nv}}, L^2_{\text{br}} \) and \( L^2_{\text{rp}} \) respectively for easy reference. In this section, we present some simulation studies, aiming to compare the \( T_{\text{max}} \) test against \( L^2_{\text{nv}}, L^2_{\text{br}} \) and \( L^2_{\text{rp}} \) under various simulation configurations. In the simulation studies,
we generate $k$ functional samples using the following data generating model:

$$y_{ij}(t) = \eta_i(t) + v_{ij}(t), \quad \eta_i(t) = \sum_{r=1}^{q} c_{ir} t^{r-1}, \quad v_{ij}(t) = b_{ij}^T \Psi_i(t), \quad t \in [0, 1],$$

$$b_{ij} = [b_{ij1}, b_{ij2}, \cdots, b_{ijq}]^T, \quad b_{ijr} \overset{d}{=} \sqrt{\lambda_r} z_{ijr},$$

$$r = 1, 2, \cdots, q; \quad j = 1, 2, \cdots, n_i; \quad i = 1, 2, \cdots, k,$$

(3.4.1)

where the parameters $c_{ir}, r = 1, 2, \cdots, q; \quad i = 1, 2, \cdots, k$, can be flexibly specified, the random variables $z_{ijr}, r = 1, 2, \cdots, q; \quad j = 1, 2, \cdots, n_i; \quad i = 1, 2, \cdots, k$ are i.i.d. with mean 0 and variance 1, $\Psi_i(t) = [\psi_{i1}(t), \psi_{i2}(t), \cdots, \psi_{iq}(t)]^T$ is a vector of $q$ basis functions $\psi_{ir}(t), t \in [0, 1], r = 1, 2, \cdots, q$, and the variance components $\lambda_r, r = 1, 2, \cdots, q$ are positive and decreasing in $r$, and the number of the basis functions $q$ is an odd positive integer. These tuning parameters help specify the covariance functions

$$\gamma_i(s, t) = \Psi_i(s)^T \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_q) \Psi_i(t) = \sum_{r=1}^{q} \lambda_r \psi_{ir}(s) \psi_{ir}(t), \quad i = 1, 2, \cdots, k.$$
3.4 Simulation Studies

can be directly specified as 0 since the tests under consideration are independent of the specification of the mean functions \( \eta_i(t), i = 1, 2, \cdots, k \). Then we specify the covariance functions \( \gamma_i(s, t), i = 1, 2, \cdots, k \) in the following way. First, we set \( \lambda_r = \rho^{r-1}, r = 1, 2, \cdots, q \) for \( 0 < \rho < 1 \). Then, we select a vector of \( q \) orthonormal Fourier basis functions, denoted as \( \Phi(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_q(t)] \) where

\[
\phi_1(t) = 1, \quad \phi_{2r}(t) = \sqrt{2} \sin(2\pi rt), \quad \phi_{2r+1}(t) = \sqrt{2} \cos(2\pi rt),
\]

\[ t \in [0, 1], \quad r = 1, 2, \cdots, (q-1)/2. \]

To obtain the \( k \) different basis function vectors \( \Psi_i(t), i = 1, 2, \cdots, k \), we set \( \psi_{ir}(t) = \phi_r(t), r = 1, 3, 4, \cdots q \), and \( \psi_{i2}(t) = \phi_2(t) + (i-1)\omega, i = 1, 2, \cdots, k \) for simplicity. With these basis function vectors \( \Psi_i(t), i = 1, 2, \cdots, k \), we have \( k \) different covariance functions

\[
\gamma_i(s, t) = \gamma_1(s, t) + (i - 1)\lambda_2(\phi_2(s) + \phi_2(t))\omega + (i - 1)^2\lambda_2\omega^2, \quad i = 1, 2, \cdots, k.
\]

Note that the differences of the \( k \) covariance functions are located in the space spanned by the first two basis functions \( \phi_1(t), \phi_2(t), t \in [0, 1] \) of the basis function vector \( \Phi(t) \) and these differences are controlled by the tuning parameter \( \omega \). Notice also that the tuning parameter \( \rho \) not only determines the decay rate of \( \lambda_1, \lambda_2, \cdots, \lambda_q \), but also determines how the simulated functional data are correlated: when \( \rho \) is close to 0, \( \lambda_1, \lambda_2, \cdots, \lambda_q \) will decay very fast, indicating that the simulated functional data are highly correlated; and when \( \rho \) is close to 1, \( \lambda_r, r = 1, 2, \cdots, q \) will decay slowly, indicating that the simulated functional data are nearly uncorrelated. In addition, we set \( q = 21 \) and the number of design time points \( J = 180 \). We also set \( \rho = 0.1, 0.3, 0.5 \) to consider the three correlation cases when the simulated functional data have high, moderately high and moderate correlations and specify three cases of the sample size vector:

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Chapter 3. A Supremum Norm Based Test

\( \mathbf{n}_1 = [n_1, n_2, n_3] = [20, 30, 30], \mathbf{n}_2 = [30, 40, 50] \) and \( \mathbf{n}_3 = [80, 70, 100] \), representing the small, moderate and large sample size cases respectively. We choose those three types of correlation because most functional data have high correlations. Finally, we specify two cases of the distribution of the i.i.d. random variables \( z_{ijr}, r = 1, 2, \cdots, q; j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k \): \( z_{ijr} \overset{i.i.d.}{\sim} N(0, 1) \) and \( z_{ijr} \overset{i.i.d.}{\sim} t_{4/\sqrt{2}}, \) allowing to generate Gaussian and non-Gaussian functional data respectively with \( z_{ijr} \) having mean 0 and variance 1. Notice that the \( t_{4/\sqrt{2}} \) distribution is chosen since it has nearly heaviest tails among the \( t \)-distributions with finite first two moments.

For a given model configuration, the three groups of functional samples (3.4.1) are generated. We then apply \( L^2_{nv}, L^2_{br}, L^2_{rp} \) and \( T_{max} \) to them to test the ECF testing problem (3.2.2) and their p-values are computed respectively. In particular, the p-values of \( L^2_{rp} \) and \( T_{max} \) are obtained via 500 runs of random permutations or nonparametric bootstrapping. We reject the null hypothesis (3.2.2) if the resulting p-value of a testing procedure is smaller than the nominal significance level \( \alpha = 5\% \). Repeat the above simulation process, 10000 times, say, so that the associated empirical sizes or powers can be obtained.

We are now ready to check how \( T_{max} \) performs compared with \( L^2_{nv}, L^2_{br}, \) and \( L^2_{rp} \) in terms of level accuracy and power. Table 3.4.1 displays the empirical sizes and powers (in percentages) of \( L^2_{nv}, L^2_{br}, L^2_{rp} \) and \( T_{max} \) when \( z_{ijr}, r = 1, 2, \cdots, q; j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k \) are i.i.d. \( N(0, 1) \). It is seen that in terms of level accuracy, \( L^2_{rp} \) and \( T_{max} \) are generally comparable with their empirical sizes being slightly liberal while \( L^2_{nv} \) and \( L^2_{br} \) are comparable with their empirical sizes being slightly conservative. However, in terms of power, \( T_{max} \) generally has higher powers than \( L^2_{nv}, L^2_{br}, \) and \( L^2_{rp} \) when the functional data are highly correlated (\( \rho = 0.1, 0.3 \)). This shows that \( T_{max} \) is advantageous since functional data are generally highly...
3.4 Simulation Studies

Table 3.4.1. Empirical sizes and powers (in percentages) of $L_{nv}^2$, $L_{br}^2$, $L_{rp}^2$ and $T_{max}$ when $z_{ijr}$, $r = 1, \cdots, q$; $j = 1, \cdots, n_i$; $i = 1, \cdots, k$ are i.i.d. $N(0, 1)$.

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<th>$L_{br}^2$</th>
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<th>$T_{max}$</th>
<th>$\omega$</th>
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<th>$L_{rp}^2$</th>
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<th>$\omega$</th>
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Correlated. Of course, it is also seen that $T_{max}$ has lower powers than $L_{nv}^2$, $L_{br}^2$, and $L_{rp}^2$ when the functional data are moderately correlated ($\rho = 0.5$) but this situation may be improved with the sample sizes enlarged. Note the fact that $T_{max}$ is less powerful compared with $L_{nv}^2$, $L_{br}^2$ and $L_{rp}^2$ when the functional data are less correlated is not a surprise since when the functional data are less correlated, $T_{max}$ just uses the information at the supremum value while $L_{nv}^2$, $L_{br}^2$ and $L_{rp}^2$ can take more information into account via the $L^2$-norm of the differences between the individual sample covariance functions and the pooled covariance function.

When the functional data are non-Gaussian, similar conclusions can also be obtained except now $L_{nv}^2$ and $L_{br}^2$ are no longer workable since their empirical sizes are too large compared with the nominal size 5%. Table 3.4.2 shows the empirical sizes and powers of $L_{nv}^2$, $L_{br}^2$, $L_{rp}^2$ and $T_{max}$ when $z_{ijr}$, $r = 1, 2, \cdots, q$; $j = 1, 2, \cdots, n_i$; $i = 1, 2, \cdots, k$ i.i.d. $t_4/\sqrt{2}$. It is seen that in terms of level accuracy, $L_{rp}^2$ and $T_{max}$ are generally comparable with their empirical sizes being slightly liberal and $L_{nv}^2$ and $L_{br}^2$ have very large empirical sizes which show that $L_{nv}^2$ and
Table 3.4.2. Empirical sizes and powers (in percentages) of $L^2_{nv}, L^2_{br}, L^2_{rp}$ and $T_{\max}$ when $z_{ijr}, r = 1, \cdots, q; j = 1, \cdots, n_i; i = 1, \cdots, k$ are i.i.d. $t_4/\sqrt{2}$.

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$L^2_{br}$ are not applicable for non-Gaussian functional data and hence it does not make any sense to compare their powers with $T_{\max}$ and $L^2_{rp}$. We thus just compare the empirical powers of $T_{\max}$ with $L^2_{rp}$. We see that $T_{\max}$ generally has higher powers than $L^2_{rp}$ when the functional data are highly correlated ($\rho = 0.1, 0.3$) except when $n_1 = [20, 30, 30]$ which may be too small for $T_{\max}$ to work properly. We also see that $T_{\max}$ has lower powers than $L^2_{rp}$ when the functional data are moderately correlated ($\rho = 0.5$).

In some situations, $T_{\max}$ can have much higher powers than $L^2_{nv}, L^2_{br},$ and $L^2_{rp}$ even when functional data are moderately correlated. To show this is the case, we just need to make a small change of the simulation settings used earlier. We continue to use the data generating model (3.4.1) and set $\eta_i(t) = 0$, $i = 1, 2, \cdots, k$. However, we now set $\psi_{1i}(t) = \phi_1(t) + (i - 1)\frac{2}{\sqrt{n}}e^{-4t^2} \omega$, $i = 1, 2, \cdots, k$ so that the differences of the basis function vectors $\Psi_i(t)$, $i = 1, 2, \cdots, k$ are now located at the first basis function. Under this new scheme, we conduct a simulation.
3.4 Simulation Studies

Table 3.4.3. Empirical sizes and powers (in percentages) of \( L^2_{nv}, L^2_{br}, L^2_{rp} \) and \( T_{\text{max}} \) when \( z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k \) are i.i.d. \( N(0,1) \) under the new simulation scheme.

| \( \rho \) | \( \omega \) | \( L^2_{nv} \) | \( L^2_{br} \) | \( L^2_{rp} \) | \( T_{\text{max}} \) | \( \omega \) | \( L^2_{nv} \) | \( L^2_{br} \) | \( L^2_{rp} \) | \( T_{\text{max}} \) | \( \omega \) | \( L^2_{nv} \) | \( L^2_{br} \) | \( L^2_{rp} \) | \( T_{\text{max}} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0   | 0.10  | 4.04  | 5.39  | 4.79  | 0.04  | 4.93  | 5.63  | 5.73  | 4.46  | 4.54  | 5.13  | 5.22  |
| 0.07 | 0.15  | 5.03  | 6.31  | 23.59 | 0.05  | 5.45  | 6.24  | 21.36 | 0.03  | 5.79  | 5.93  | 19.91 |
| 0.1  | 22.04 | 19.92 | 11.92 | 64.79 | 0.10  | 8.90  | 9.53  | 62.05 | 0.05  | 7.91  | 8.17  | 54.40 |
| 0.21 | 18.86 | 19.52 | 21.21 | 84.04 | 0.15  | 16.95 | 17.46 | 18.62 | 0.07  | 11.05 | 11.56 | 82.59 |
| 0.42 | 72.04 | 73.25 | 70.95 | 98.86 | 0.20  | 36.04 | 31.60 | 32.66 | 0.10  | 19.90 | 20.24 | 99.97 |
| 0.09 | 4.25  | 4.53  | 5.45  | 5.30  | 0   | 4.22  | 4.42  | 5.10  | 5.57  | 0   | 5.34  | 5.54  | 5.77  |
| 0.07 | 6.36  | 6.76  | 7.80  | 23.68 | 0.07 | 5.41  | 5.83  | 6.44  | 21.20 | 0.05 | 7.92  | 8.12  | 8.68  |
| 0.17 | 13.15 | 13.99 | 13.75 | 93.99 | 0.11 | 9.13  | 9.44  | 10.60 | 55.02 | 0.07 | 11.71 | 11.89 | 12.25 |
| 0.24 | 25.44 | 26.73 | 28.20 | 84.14 | 0.18 | 24.05 | 24.83 | 25.10 | 91.33 | 0.09 | 16.78 | 17.03 | 17.34 |
| 0.44 | 73.51 | 74.96 | 72.60 | 98.35 | 0.21 | 33.17 | 34.12 | 34.36 | 95.66 | 0.11 | 25.92 | 26.44 | 26.27 |
| 0   | 4.31  | 4.74  | 6.03  | 5.66  | 0   | 4.57  | 5.01  | 5.60  | 5.21  | 0   | 4.51  | 4.66  | 5.00  |
| 0.1  | 6.04  | 6.84  | 8.21  | 14.91 | 0.10 | 5.52  | 9.05  | 9.31  | 24.38 | 0.05 | 8.01  | 8.22  | 8.29  |
| 0.5  | 20.12 | 17.38 | 18.98 | 56.78 | 0.15 | 15.45 | 16.53 | 17.31 | 57.44 | 0.08 | 13.37 | 13.81 | 13.77 |
| 0.5  | 81.75 | 83.20 | 79.06 | 97.92 | 0.29 | 60.08 | 61.41 | 59.54 | 97.68 | 0.15 | 47.73 | 48.49 | 47.81 |

In the above three simulation studies, we see that \( T_{\text{max}} \) generally has higher powers than \( L^2_{nv}, L^2_{br} \), and \( L^2_{rp} \) for \( \rho = 0.1, 0.3, \) and 0.5 as well.

In the above three simulation studies, we see that \( T_{\text{max}} \) generally has higher powers than \( L^2_{nv}, L^2_{br} \) and \( L^2_{rp} \) when the functional data have higher or even moderate correlation and when the sample sizes are large enough, and it has lower powers when the functional data have lower correlation or when the sample sizes are too small.
3.5. Applications to Three Real Data Examples

In this section, we shall present the application of $T_{\text{max}}$, together with $L^2_{nv}, L^2_{br}, L^2_{rp}$, to three real data examples. From these three examples, we shall see that $T_{\text{max}}$ often has higher power than $L^2_{nv}, L^2_{br}, L^2_{rp}$ in detecting the covariance function differences of different functional populations.

3.5.1. Canadian Temperature Data. The Canadian temperature data set has been used for illustrating various methodologies for functional data; see for example, Ramsay and Silverman (2006), Zhang and Chen (2007) and Zhang and Liang (2013) among others. The temperature functional observations consist of daily temperature records of 35 weather stations over 365 days, with each observation being a temperature curve as shown in Figure 3.5.1 where the reconstructed individual temperature functions over a whole year are depicted. These weather stations are located in three different regions over Canada. There are 15 weather stations located in eastern Canada, another 15 in western Canada and the remaining 5 in northern Canada. We are interested in the equality of the covariance functions (variations) of the temperature functions at the three different regions over the whole year and four seasons (spring, summer, autumn and winter). We specify the four seasons as spring (March, April, and May or $J = [60, 151]$), summer (June, July, and August or $J = [152, 243]$), autumn (September, October and November or $J = [244, 334]$) and winter (December, January and February or $J = [1, 59] \cup [335, 365]$).

Table 3.5.1 shows the p-values of $L^2_{nv}, L^2_{br}, L^2_{rp}$ and $T_{\text{max}}$ for testing the equality of the covariance functions (variations) of the Canadian temperature functions of the eastern, western and northern weather stations over the whole year and the four seasons (spring, summer, autumn, and winter). The p-values of $L^2_{rp}$ and $T_{\text{max}}$ were
3.5 Applications to Three Real Data Examples

**Figure 3.5.1.** Reconstructed individual temperature functions for the Canadian temperature data.

![Reconstructed individual temperature functions](image)

**Table 3.5.1.** P-values of $L_{nv}^2$, $L_{br}^2$, $L_{rp}^2$, and $T_{max}$ for testing the equality of the covariance functions for the Canadian temperature data.

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>$L_{nv}^2$</th>
<th>$L_{br}^2$</th>
<th>$L_{rp}^2$</th>
<th>$T_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole</td>
<td>0.0383</td>
<td>0.0323</td>
<td>0.0451</td>
<td>0.0321</td>
<td></td>
</tr>
<tr>
<td>Spring</td>
<td>0.0224</td>
<td>0.0193</td>
<td>0.0621</td>
<td>0.1228</td>
<td></td>
</tr>
<tr>
<td>Summer</td>
<td>0.0010</td>
<td>0.0009</td>
<td>0.0019</td>
<td>0.0199</td>
<td></td>
</tr>
<tr>
<td>Autumn</td>
<td>0.1997</td>
<td>0.1816</td>
<td>0.1896</td>
<td>0.0917</td>
<td></td>
</tr>
<tr>
<td>Winter</td>
<td>0.0266</td>
<td>0.0234</td>
<td>0.0297</td>
<td>0.0339</td>
<td></td>
</tr>
</tbody>
</table>

The testing results are obtained via 10000 runs of random permutations and nonparametric bootstrapping respectively. It can be seen that all the tests suggest that the covariance functions of the three regions over the whole year and in summer and winter are unlikely to be the same but they may be quite similar in autumn. The testing results in spring are not consistent. $L_{nv}^2$ and $L_{br}^2$ suggest that the covariance functions of the three regions in spring are unlikely to be the same but $L_{rp}^2$ and $T_{max}$ are not so sure. Since $L_{nv}^2$ and $L_{br}^2$ only work under the assumption that the functional
data are Gaussian while $L^2_{rp}$ and $T_{\text{max}}$ do not need such an assumption, the testing results of $L^2_{rp}$ and $T_{\text{max}}$ are more reliable than those of $L^2_{nv}$ and $L^2_{br}$.

3.5.2. Nitrogen Oxide Emission Level Data. We now present the application of $T_{\text{max}}$, together with $L^2_{nv}$, $L^2_{br}$, $L^2_{rp}$, to another data set consisting of Nitrogen oxides (NOx) emission levels (in $\mu g/m^3$) measured by an environmental control station close to an industrial area in Poblenou, Barcelona, Spain. The NOx emission level data were kindly made available by Febrero et al. (2008). Each curve of the NOx level data was recorded every hour per day from February 23 to June 26 in 2005. The data set has been studied in Febrero et al. (2008) for illustrating an outlier detection method. In large cities, especially those with heavy traffic and well-developed industries, NOx gases are known to be among the most important pollutants and thus the emission levels of nitrogen oxides (NOx) can be significant. The NOx emission level curves of the data set may be classified into two groups according to the working days and non-working days. The working day group includes 76 NOx emission level curves while the non-working day group has 39 curves. Since the NOx gases are mainly emitted into the atmosphere in sources of motor vehicles and industries, we are wondering if the covariance functions of the working day group and the non-working day group are the same.

Figure 3.5.2 shows the 3-D plots of the estimated covariance functions of the NOx emission level curves of working days and non-working days. It seems that the two sample covariance functions are not the same. We then applied $T_{\text{max}}$, together with $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$, to check if the differences of the covariance functions between the NOx emission level curves of working days and non-working days are significant. The p-values of $T_{\text{max}}$ is 0.006 while those of $L^2_{nv}$, $L^2_{br}$ and $L^2_{rp}$ are 0.1193, 0.1133 and 0.3427 respectively. The p-values of $T_{\text{max}}$ and $L^2_{rp}$ were
3.5 Applications to Three Real Data Examples

Figure 3.5.2. 3-D plots of the estimated covariance functions of the NOx emission level data.

(A) Sample covariance function of non-working days

(B) Sample covariance function of working days
obtained via 10000 runs of nonparametric bootstrapping and random permutations respectively. It is seen that $T_{\text{max}}$ can detect the differences of the covariance functions of the NOx emission level curves of working days and non-working days. This is consistent with what we observed from Figure 3.5.2. However, $L_{\text{nv}}^2$, $L_{\text{br}}^2$ and $L_{\text{rp}}^2$ cannot. This shows that $T_{\text{max}}$ is indeed more powerful than $L_{\text{nv}}^2$, $L_{\text{br}}^2$ and $L_{\text{rp}}^2$ in detecting the covariance function differences between the working day and non-working day groups.

3.5.3. Berkeley Growth Data. We finally present the application of $T_{\text{max}}$, together with $L_{\text{nv}}^2$, $L_{\text{br}}^2$, $L_{\text{rp}}^2$, to the Berkeley growth curve data set which has been extensively studied in Ramsay and Silverman (2006) and Ramsay et al. (2002). This data set contains the heights of 39 boys and 54 girls from age 1 to 18 (Tuddenham and Snyder 1954). It is of interest to check whether the variable “gender” has some impact on the covariance structure of a child’s grow curve. In other words, we want to test the equality of the covariance functions of boys’ and girls’ growth curves.

Figure 3.5.3 depicts the 3-D plots of the estimated covariance functions of the Berkeley growth curve data. It seems that there is a clear difference between the sample covariance structures of boys and girls. To verify if this is the case, we applied $T_{\text{max}}$, together with $L_{\text{nv}}^2$, $L_{\text{br}}^2$ and $L_{\text{rp}}^2$. The p-values of $T_{\text{max}}$ is 0.0453 while those of $L_{\text{nv}}^2$, $L_{\text{br}}^2$ and $L_{\text{rp}}^2$ are 0.4649, 0.4571 and 0.4762 respectively. Again, the p-values of $T_{\text{max}}$ and $L_{\text{rp}}^2$ were obtained via 10000 runs of nonparametric bootstrapping and random permutations respectively. It is seen that $T_{\text{max}}$ can detect the differences of the covariance functions of the growth curves of boys and girls. This is consistent with what we observed from Figure 3.5.3. However, $L_{\text{nv}}^2$, $L_{\text{br}}^2$ and $L_{\text{rp}}^2$ cannot. This shows that $T_{\text{max}}$ is again more powerful than $L_{\text{nv}}^2$, $L_{\text{br}}^2$ and $L_{\text{rp}}^2$. 
3.5 Applications to Three Real Data Examples

Figure 3.5.3. 3-D plots of the estimated covariance functions of the Berkeley growth curve data.

(A) Sample covariance function of heights of boys

(B) Sample covariance function of heights of girls
in detecting the covariance function differences between the growth curves of boys and girls.

3.6. Appendix

Proof of Theorem 3.1. By Lemma 2.1 in Chapter 2 and under the null hypothesis, as \( n \rightarrow \infty \), we have \( z_n(s, t) \overset{d}{\rightarrow} z(s, t) \sim GP_k(0, \varpi I_k) \) uniformly for all \( s, t \in T \) where \( \varpi[(s_1, t_1), (s_2, t_2)] \) is defined in Lemma 2.1 in Chapter 2. By Slusky’s theorem, as \( n \rightarrow \infty \), we have \( SSB(s, t) - R(s, t) \overset{p}{\rightarrow} 0 \) for all \( s, t \in T \) where \( R(s, t) = z(s, t)^T (I_k - \mathbf{b}_n^{\mathbb{T}})z(s, t) \) and \( I_k - \mathbf{b}_n^{\mathbb{T}} \) is the limit matrix of \( I_k - \mathbf{b}_n^{\mathbb{T}} \) as given in (3.3.4). Since \( T^2 \) is a finite interval and \( SSB(s, t) \) is continuous over \( T^2 \), it is also equicontinuous. By Theorem 2.1 in Newey (1991), \( SSB(s, t) - R(s, t) \overset{p}{\rightarrow} 0 \) uniformly over \( T^2 \). Since we always have \( |\sup_{s,t \in T} SSB(s, t) - \sup_{s,t \in T} R(s, t)| \leq \sup_{s,t \in T}|SSB(s, t) - R(s, t)| \), we have \( \sup_{s,t \in T} SSB(s, t) - \sup_{s,t \in T} R(s, t) \overset{p}{\rightarrow} 0 \) which implies that \( \sup_{s,t \in T} SSB(s, t) \overset{d}{\rightarrow} \sup_{s,t \in T} R(s, t) \). That is \( T_{\text{max}} \overset{d}{\rightarrow} T_0 \) where \( T_0 = \sup_{s,t \in T} R(s, t) \). Notice that \( I_k - \mathbf{b}_n^{\mathbb{T}} \) has the singular value decomposition (3.3.5).

Let \( w(s, t) = \mathbf{I}_{k-1}, \mathbf{0} \)U\( ^T \)z(s, t) = [w_1(s, t), w_2(s, t), \ldots, w_{k-1}(s, t)]^T. \)

Then \( w(s, t) \sim GP_{k-1}(0, \varpi I_{k-1}) \) and it follows that \( R(s, t) = w(s, t)^T w(s, t) = \sum_{i=1}^{k-1} w_i^2(s, t) \). □

Proof of Theorem 3.2. First of all, notice that given original \( k \) samples, the bootstrapped \( k \) samples \( v_i^*(t) \), \( j = 1, 2, \ldots, n_i; i = 1, 2, \ldots, k \overset{i.i.d.}{\sim} SP(0, \frac{n-k}{n} \hat{\gamma}) \) where \( \hat{\gamma}(s, t) \) is the pooled sample covariance function (3.2.4). That is to say, the bootstrapped \( k \) samples satisfy the null hypothesis (3.2.2) since they have the same covariance. By Lemma 2.2 in Chapter 2 and under the null hypothesis, as \( n \rightarrow \infty \), we have \( \frac{n-k}{n} \hat{\gamma}(s, t) \overset{p}{\rightarrow} \gamma(s, t) \) uniformly over \( T^2 \) and \( v_i^*(t) \overset{d}{\rightarrow} SP(0, \gamma) \). Applying
Theorem 3.1 leads to the first claim of the theorem and the second claim of the theorem follows immediately. □

**Proof of Theorem 3.3.** Notice that for any \( s, t \in T \),

\[
T_{\text{max}} = \sup_{s, t \in T} z_n(s, t)^T [I_k - b_n b_n^T / (n - k)] z_n(s, t),
\]

where \( z_n(s, t) = [z_1(s, t), z_2(s, t), \ldots, z_k(s, t)]^T \) with \( i = 1, 2, \ldots, k \),

\[
z_i(s, t) = \sqrt{n_i - 1}[\hat{\gamma}_i(s, t) - \gamma(s, t)]
\]

\[
= \sqrt{n_i - 1}[\hat{\gamma}_i(s, t) - \gamma_i(s, t) + \gamma_i(s, t) - \gamma(s, t)]
\]

\[
= \sqrt{n_i - 1}[\hat{\gamma}_i(s, t) - \gamma_i(s, t)] + \sqrt{n_i - 1}[\gamma_i(s, t) - \gamma(s, t)].
\]

From Lemma 2.1 in Chapter 2, we know

\[
\sqrt{n_i - 1}[\hat{\gamma}_i(s, t) - \gamma_i(s, t)] \overset{d}{\to} GP(0, \varpi_i), \; i = 1, 2, \ldots, k.
\]

And from the alternative hypothesis (3.3.11) we get

\[
\sqrt{n_i - 1}[\gamma_i(s, t) - \gamma(s, t)] = d_i(s, t), \; i = 1, 2, \ldots, k.
\]

where the function \( d_i(s, t) \in L^2(T^2) \) and \( T^2 = [a, b] \times [a, b] \).

Then under the alternative hypothesis

\[
z_n(s, t) \overset{d}{\to} GP_k[d, \text{diag}(\varpi_1, \varpi_2, \ldots, \varpi_k)].
\]

Since the data follow the Gaussian process, with the local alternative hypothesis we can prove that \( \varpi_i, \; i = 1, 2, \ldots, k \) has the following property (Lemma 2.3 in Chapter 2):

\[
\varpi_i[(s_1, t_1), (s_2, t_2)] = \varpi[(s_1, t_1), (s_2, t_2)] + (n_i - 1)^{-1/2} h_i[(s_1, t_1), (s_2, t_2)] + O(n^{-1})
\]

(3.6.2)
where \( h_i[(s_1,t_1),(s_2,t_2)], i = 1, 2, \cdots, k \) are some fixed functions. According to (3.6.2), we can easily obtain
\[
\mathbf{z}_n(s,t) \xrightarrow{d} \mathbf{z}(s,t) \sim \text{GP}_k[\mathbf{d}, \varpi \mathbf{I}_k]
\]
with \( \mathbf{d} = [d_1(s,t), d_2(s,t), \cdots, d_k(s,t)]^T \) and \( \mathbf{z}(s,t) \sim \text{GP}_k(0, \varpi \mathbf{I}_k) \).

Similar to the proof of Theorem 3.1, since \( \mathcal{T} \) is a finite interval and SSB\((s,t)\) is equicontinuous over \( \mathcal{T} \), by Slusky’s theorem, Theorem 2.1 of Newey (1991), we can show that as \( n \to \infty \), we have
\[
\sup_{s,t \in \mathcal{T}} \left\{ \mathbf{w}(s,t) + \mathbf{z}(s,t) \right\} \sim \text{GP}_{k-1}(0, \varpi \mathbf{I}_{k-1})
\]
and \( \mathbf{w}(s,t) = (\mathbf{I}_{k-1}, 0) \mathbf{U}^T \mathbf{z}(s,t) = [w_1(s,t), w_2(s,t), \cdots, w_{k-1}(s,t)]^T \) and let \( \mathbf{\zeta}(s,t) = (\mathbf{I}_{k-1}, 0) \mathbf{U}^T \mathbf{d}(s,t) = [\zeta_1(s,t), \zeta_2(s,t), \cdots, \zeta_{k-1}(s,t)]^T \). Then \( \mathbf{w}(s,t) \sim \text{GP}_{k-1}(0, \varpi \mathbf{I}_{k-1}) \) and \( (\mathbf{I}_{k-1}, 0) \mathbf{U}^T \mathbf{z}(s,t) + \mathbf{d}(s,t) = \mathbf{w}(s,t) + \mathbf{\zeta}(s,t) \). Therefore, \( T_1 = \sup_{s,t \in \mathcal{T}} \left\{ \mathbf{w}(s,t) + \mathbf{\zeta}(s,t) \right\} \sim \sup_{s,t \in \mathcal{T}} \sum_{i=1}^{k-1} [w_i(s,t) + \zeta_i(s,t)]^2 \).

**Proof of Theorem 3.4.** By (3.3.14), we first have
\[
P(T_{\text{max}} \geq C^*_\alpha) \geq P(T_n \geq (b-a)^2 C^*_\alpha), \tag{3.6.3}
\]
where the \( T_n \) is the test statistic proposed in Chapter 2. Under the local alternative (3.3.11) and the given conditions, by the proof of Theorems 2.4 and 2.5 in Chapter 2, we have
\[
P(T_{\text{max}} \geq C^*_\alpha) \to 1, \tag{3.6.4}
\]
as \( \max_r \delta_r^2 \to \infty \).
CHAPTER 4

Two Quasi $F$-type Tests

4.1. Introduction

In recent decades, increasing attention has been paid to functional data whose observations are functions, such as curves, surfaces, or images. Such a kind of data arises frequently in various research and industrial areas. How to analyze these functional data becomes a hot topic and novel methodologies to deal with them are in great demand. Many classical statistical methods for multivariate data, such as principal component analysis and canonical correlation analysis among others, have been extended to satisfy this need. Among these methods, hypothesis testing for functional data attracts increasing interests from researchers. Most popular hypothesis testing problems are inferences concerning means or covariances.

It is well known that in the classical analysis of variance (ANOVA), the $F$-test is a widely used tool which uses the ratio of the sum of squares between subjects (SSB) and the sum of squares due to errors (SSE) as its test statistic. That is $F = \frac{SSB/(k-1)}{SSE/(n-k)}$ where $n$ and $k$ are the sample size and the number of groups.
respectively, SSB and SSE measure the variations explained by the factors involved in the analysis and the variations due to measurement errors. Due to its robustness, the $F$-test is often recommended in practice. In the functional data analysis, we can define SSB and SSE for each time point and denote them as $SSB(t)$ and $SSE(t)$ respectively. The test statistic of the pointwise $F$-test described by Ramsay and Silverman (2006) can be defined as $F(t) = \frac{SSB(t)/(k-1)}{SSE(t)/(n-k)}$ which is a natural extension of the classical $F$-test to the field of functional data analysis; see more details in Section 4.2 below. However, this test is time-consuming and cannot give a global conclusion. To overcome this difficulty, Cuevas et al. (2004) proposed an ANOVA test based on the $L^2$-norm of $SSB(t)$, i.e., the numerator of the pointwise $F$-test statistic but its asymptotic null distribution of the test statistic is not given. Zhang (2013) further investigated this test statistic which is called the $L^2$-norm based test and showed that its null distribution is asymptotically a $\chi^2$-type mixture. Instead of only using the numerator of the pointwise $F$-test, Zhang and Liang (2013) studied a GPF test which is obtained via globalizing the pointwise $F$-test with integration. Alternatively, the pointwise $F$-test can be globalized via using its maximum value as a test statistic, resulting in the so-called $F_{\text{max}}$-test as described by Cheng et al. (2012). It is shown that the $F_{\text{max}}$ test is powerful when the functional data are highly correlated and the GPF test is powerful when the functional data are less correlated. Besides its importance in functional ANOVA problems, the $F$-test can also be applied in functional linear models. In fact, Shen and Faraway (2004) considered an $F$-type test to compare two nested linear models and studied its null distribution. Their test relies on the integrated residual sum of squares proposed in Faraway (1997). Based on their work, Zhang (2011) studied the asymptotic power of this $F$-type test and extended it to a general linear hypothesis testing (GLHT) problem.
4.1 Introduction

In the above, we can see that the pointwise $F$-test is quite useful and powerful in functional data analysis and it can be globalized to yield the so-called $GPF$ and $F_{max}$ tests among others. This chapter aims to develop a similar pointwise test for the equality of the covariance functions of several functional populations, namely, the equal-covariance function (ECF) testing problem which has been studied in Chapters 2 and 3. This task is quite challenging and novel since the pointwise $F$-test is usually defined only for the one-way ANOVA problem or the regression analysis as mentioned above. In fact, it is very difficult to define such a pointwise $F$-test for the ECF testing problem. Instead, we can only mimic the basic idea of the pointwise $F$-test and define a pointwise quasi $F$-test for the ECF testing problem as we shall do in Section 4.2 below. Based on this pointwise quasi $F$-test, we construct two new globalized tests, namely, a quasi GPF test and a quasi $F_{max}$ test. The asymptotic random expressions of the test statistics under both the null and alternative hypotheses are derived. To approximate the null distribution of the quasi GPF test, two methods are proposed. One applies the Welch-Satterthwaite $\chi^2$-approximation and the other applies the random permutation method. For the quasi $F_{max}$ test, we only use the random permutation method. Like the classical $F$-test, these two new tests are scale-invariant. In addition, we show, via simulation studies, that our new tests are more powerful than three existing tests when the covariance functions at different time points have different scales.

The chapter is organized as follows. The main results are presented in Section 4.2. The simulation studies are presented in Section 4.3. A real data example is given in Section 4.4. The technical proofs of our main results are presented in Section 4.5.
4.2. Main Results

Let $y_{i1}(t), y_{i2}(t), \ldots, y_{in_i}(t), i = 1, 2, \ldots, k$ be $k$ independent functional samples over a given finite time interval $T = [a, b], -\infty < a < b < \infty$, which satisfy

\[
y_{ij}(t) = \eta_i(t) + v_{ij}(t), \quad j = 1, 2, \ldots, n_i,
\]

\[
v_{i1}(t), v_{i2}(t), \ldots, v_{in_i}(t) \overset{i.i.d.}{\sim} \text{SP}(0, \gamma_i); \quad i = 1, 2, \ldots, k,
\]

where $\eta_1(t), \eta_2(t), \ldots, \eta_k(t)$ model the unknown group mean functions of the $k$ samples, $v_{ij}(t), j = 1, 2, \ldots, n_i; \ i = 1, 2, \ldots, k$ represent the subject-effect functions, and $\gamma_i(s, t), i = 1, 2, \ldots, k$ are the associated covariance functions.

Throughout this chapter, we assume that $\text{tr}(\gamma_i) < \infty$ and $\eta_i(t) \in L^2(T), i = 1, 2, \ldots, k$, where $L^2(T)$ denotes the Hilbert space formed by all the squared integrable functions over $T$ with the inner-product defined as $\langle f, g \rangle = \int_T f(t)g(t)dt, f, g \in L^2(T)$. We are interested in testing the same ECF problem studied in Chapter 2 and Chapter 3:

\[
H_0 : \gamma_1(s, t) \equiv \gamma_2(s, t) \equiv \cdots \equiv \gamma_k(s, t), \quad \text{for } s, t \in T.
\]  

(4.2.2)

Based on the given $k$ functional samples (4.2.1), the group mean functions $\eta_i(t), i = 1, 2, \ldots, k$ and the covariance functions $\gamma_i(s, t), i = 1, 2, \ldots, k$ can be unbiasedly estimated as

\[
\hat{\eta}_i(t) = \bar{y}_i(t) = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}(t), \quad i = 1, 2, \ldots, k,
\]

\[
\hat{\gamma}_i(s, t) = (n_i - 1)^{-1} \sum_{j=1}^{n_i} [y_{ij}(s) - \bar{y}_i(s)][y_{ij}(t) - \bar{y}_i(t)], \quad i = 1, 2, \ldots, k.
\]  

(4.2.3)

It is easy to show that $\hat{\gamma}_i(s, t), i = 1, 2, \ldots, k$ are independent and $E\hat{\gamma}_i(s, t) = \gamma_i(s, t), \ i = 1, 2, \ldots, k$. Further, the estimated subject-effect functions can be written as

\[
\hat{v}_{ij}(t) = y_{ij}(t) - \bar{y}_i(t), \quad j = 1, 2, \ldots, n_i; \ i = 1, 2, \ldots, k.
\]  

(4.2.4)
4.2 Main Results

When the null hypothesis (4.2.2) holds, let \( \gamma(s, t) \) denote the common covariance function of the \( k \) samples. It can be estimated by the following pooled sample covariance function

\[
\hat{\gamma}(s, t) = \sum_{i=1}^{k} (n_i - 1) \hat{\gamma}_i(s, t)/(n - k),
\]

(4.2.5)

where \( \hat{\gamma}_i(s, t), i = 1, 2, \cdots, k \) are given in (4.2.3).

The tests we shall propose are inspired by the GPF test of Zhang and Liang (2013) and the \( F_{\text{max}} \)-test of Cheng et al. (2012). Both of them are based on the pointwise \( F \)-test as mentioned in the introduction. To better understand how we shall define our new tests, we first review the GPF and \( F_{\text{max}} \) tests. These two tests are designed to test the one-way ANOVA for functional data, i.e., to test if the \( k \) mean functions are equal: \( H_0 : \eta_1(t) = \eta_2(t) = \cdots = \eta_k(t) \). For this end, Zhang and Liang (2013) first defined the pointwise sum of squares between groups (SSB) and the pointwise sum of squares due to errors (SSE):

\[
\begin{align*}
\text{SSB}(t) &= \sum_{i=1}^{k} n_i \left[ \hat{\eta}_i(t) - \hat{\eta}(t) \right]^2, \\
\text{SSE}(t) &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left[ y_{ij}(t) - \hat{\eta}_i(t) \right]^2,
\end{align*}
\]

(4.2.6)

where \( \hat{\eta}(t) = \sum_{i=1}^{k} n_i \hat{\eta}_i(t)/n \) denotes the pooled sample mean function of the \( k \) functional samples. Then the pointwise \( F \)-test statistic can be defined as

\[
F_n(t) = \frac{\text{SSB}(t)/(k - 1)}{\text{SSE}(t)/(n - k)},
\]

(4.2.7)

where and throughout \( n = \sum_{i=1}^{k} n_i \) denotes the total sample size. The test statistics of the GPF and \( F_{\text{max}} \) tests are then given respectively by

\[
T_n = \int_{\mathcal{T}} F_n(t) dt, \quad F_{\text{max}} = \sup_{t \in \mathcal{T}} F_n(t).
\]

(4.2.8)
Our new test statistics can be defined similarly but they are based on a pointwise quasi $F$-test. For the ECF testing problem (4.2.2), we first define the pointwise sum of squares between groups (SSB) and sum of squares due to errors (SSE):

$$SSB(s,t) = \sum_{i=1}^{k}(n_i - 1)[\hat{\gamma}_i(s,t) - \hat{\gamma}(s,t)]^2,$$

$$SSE(s,t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} [\hat{v}_{ij}(s)\hat{v}_{ij}(t) - \hat{\gamma}_i(s,t)]^2, \quad s, t \in T,$$

where $\hat{\gamma}(s,t)$, the pooled sample covariance function of the $k$ functional samples as defined in (4.2.5), $\hat{\gamma}_i(s,t)$, the $i$-th sample covariance function, and $\hat{v}_{ij}(s)\hat{v}_{ij}(t)$ play the roles of $\hat{\eta}(t), \hat{\eta}_i(t)$ and $y_{ij}(t)$ in (4.2.6) respectively. Then the pointwise quasi $F$-test statistic for testing (4.2.2) can be defined as

$$F_n(s,t) = \frac{SSB(s,t)/(k-1)}{SSE(s,t)/(n-k)}, \quad s, t \in T, \quad (4.2.9)$$

which may not have an $F$-distribution and hence $F_n(s,t)$ should not be called a pointwise $F$-test statistic. Then the test statistic obtained via integrating the pointwise quasi $F$-test statistic may be called a quasi GPF test statistic and the test statistic obtained via taking the supremum of the pointwise quasi $F$-test statistic may be called a quasi $F_{\max}$ test statistic. That is, the test statistics of the quasi GPF and $F_{\max}$ tests are then given respectively by

$$T_n = \int_T \int_T F_n(s,t)dsdt, \quad F_{\max} = \sup_{s,t \in T} F_n(s,t). \quad (4.2.10)$$

Notice that when the null hypothesis is valid, it is expected that both $T_n$ and $F_{\max}$ will be small and otherwise large.

For further study, let $\varpi_i[(s_1, t_1), (s_2, t_2)]$ denote the covariance function between $v_{i1}(s_1)v_{i1}(t_1)$ and $v_{i1}(s_2)v_{i1}(t_2)$. Then we have

$$\varpi_i[(s_1, t_1), (s_2, t_2)] = E\{v_{i1}(s_1)v_{i1}(t_1)v_{i1}(s_2)v_{i1}(t_2)\} - \gamma_i(s_1, t_1)\gamma_i(s_2, t_2). \quad (4.2.11)$$
4.2 Main Results

When $\gamma_i(s,t)$ does not depend on $i$, i.e., when $H_0$ holds, we use $\gamma(s,t)$ to denote the common covariance function, and define

$$\varpi [(s_1,t_1), (s_2,t_2)] = n^{-1} \sum_{i=1}^{k} n_i E \{ v_{i1}(s_1)v_{i1}(t_1)v_{i1}(s_2)v_{i1}(t_2) \} - \gamma(s_1,t_1)\gamma(s_2,t_2).$$

(4.2.12)

The natural estimator for $\varpi [(s_1,t_1), (s_2,t_2)]$ is

$$\hat{\varpi} [(s_1,t_1), (s_2,t_2)] = n^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \hat{v}_{ij}(s_1)\hat{v}_{ij}(t_1)\hat{v}_{ij}(s_2)\hat{v}_{ij}(t_2) - \hat{\gamma}(s_1,t_1)\hat{\gamma}(s_2,t_2).$$

(4.2.13)

When the samples are Gaussian, a consistent estimator of $\varpi [(s_1,t_1), (s_2,t_2)]$ is given by

$$\hat{\varpi}[(s_1,t_1), (s_2,t_2)] = \hat{\gamma}(s_1,s_2)\hat{\gamma}(t_1,t_2) + \hat{\gamma}(s_1,t_2)\hat{\gamma}(s_2,t_1).$$

(4.2.14)

To derive the asymptotic random expressions of $T_n$ and $F_{\text{max}}$, we impose the following assumptions as before:

**Assumption A**

1. The $k$ samples are Gaussian.
2. As $n \to \infty$, the $k$ sample sizes satisfy $n_i/n \to \tau_i \in (0,1)$, $i = 1,2,\cdots,k$.
3. The variance functions are uniformly bounded. That is, $\rho_i = \sup_{t \in \mathcal{T}} \gamma_i(t,t) < \infty$, $i = 1,2,\cdots,k$.

Assumption A2 requires that the $k$ sample sizes tend to $\infty$ proportionally.

Before we state the main results, we give an alternative expression of $\text{SSB}(s,t)$ which is helpful for deriving the main results about the quasi GPF and $F_{\text{max}}$ tests. For any $s,t \in \mathcal{T}$, $\text{SSB}(s,t)$ can be expressed as

$$\text{SSB}(s,t) = z_n(s,t)^T [I_k - b_n b_n^T/(n-k)] z_n(s,t),$$

(4.2.15)
where
\[
\mathbf{z}_n(s, t) = [z_1(s, t), z_2(s, t), \ldots, z_k(s, t)]^T,
\]
with
\[
z_i[s, t] = \sqrt{n_i - 1}[\gamma_i(s, t) - \gamma(s, t)], \ i = 1, 2, \ldots, k,
\]
\[
\mathbf{b}_n = [\sqrt{n_1 - 1}, \sqrt{n_2 - 1}, \ldots, \sqrt{n_k - 1}]^T.
\]
Since \(b_n^T b_n/(n - k) = 1\), it is easy to verify that \(I_k - b_n b_n^T/(n - k)\) is an idempotent matrix with rank \(k - 1\). In addition, as \(n \to \infty\), we have
\[
I_k - b_n b_n^T/(n - k) \to I_k - \mathbf{b}\mathbf{b}^T, \text{ with } \mathbf{b} = [\sqrt{\tau_1}, \sqrt{\tau_2}, \ldots, \sqrt{\tau_k}]^T, \quad (4.2.16)
\]
where \(\tau_i, i = 1, 2, \ldots, k\) are given in Assumption A2. Note that \(I_k - \mathbf{b}\mathbf{b}^T\) in (4.2.16) is also an idempotent matrix of rank \(k - 1\), which has the following singular value decomposition:
\[
I_k - \mathbf{b}\mathbf{b}^T = \mathbf{U} \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{U}^T, \quad (4.2.17)
\]
where the columns of \(\mathbf{U}\) are the eigenvectors of \(I_k - \mathbf{b}\mathbf{b}^T\). We now have the following theorem.

**Theorem 4.1.** Under Assumptions A1∼A3 and the null hypothesis (4.2.2), as \(n \to \infty\), we have \(T_n \overset{d}{\to} T_0\) with
\[
T_0 \overset{d}{=} \int_T \int_T (k - 1)^{-1} \sum_{i=1}^{k-1} \omega_i^2(s, t) ds dt
\]
\[
= (k - 1)^{-1} \sum_{r=1}^{\infty} \lambda_r A_r, \quad A_r \overset{i.i.d.}{\sim} \chi_1^2, \quad (4.2.18)
\]
and \(F_{\max} \overset{d}{\to} F_0\) with
\[
F_0 \overset{d}{=} \sup_{s, t \in T} \{(k - 1)^{-1} \sum_{i=1}^{k-1} \omega_i^2(s, t)\}, \quad (4.2.19)
\]
4.2 Main Results

where \( \omega_1(s, t), \omega_2(s, t), \ldots, \omega_{k-1}(s, t) \overset{i.i.d.}{\sim} GP(0, \gamma_\omega) \) with

\[
\gamma_\omega[(s_1, t_1), (s_2, t_2)] = \varpi[(s_1, t_1), (s_2, t_2)] / \sqrt{\varpi[(s_1, t_1), (s_1, t_1)] \varpi[(s_2, t_2), (s_2, t_2)]},
\]

(4.2.20)

and \( \varpi[(s_1, t_1), (s_2, t_2)] \) is defined in (4.2.12), and \( \lambda_r, r = 1, 2, \ldots, \infty \) are the decreasing-ordered eigenvalues of \( \gamma_\omega[(s_1, t_1), (s_2, t_2)] \).

By Theorem 4.1, \( \omega_i(s, t), i = 1, 2, \ldots, k \overset{i.i.d.}{\sim} GP(0, \gamma_\omega) \) which are known except \( \gamma_\omega[(s_1, t_1), (s_2, t_2)] \). The covariance function \( \gamma_\omega[(s_1, t_1), (s_2, t_2)] \) can be estimated by

\[
\hat{\gamma}_\omega[(s_1, t_1), (s_2, t_2)] = \frac{\hat{\varpi}[(s_1, t_1), (s_2, t_2)]}{\sqrt{\hat{\varpi}[(s_1, t_1), (s_1, t_1)] \hat{\varpi}[(s_2, t_2), (s_2, t_2)]}},
\]

(4.2.21)

where \( \hat{\varpi}[(s_1, t_1), (s_2, t_2)] \) is given in (4.2.13) or (4.2.14).

Theorem 4.1 says that the asymptotic distribution of \( T_n \) is the same as that of a \( \chi^2 \)-type mixture. Therefore we can approximate its distribution using the well-known Welch-Satterthwaite \( \chi^2 \)-approximation. That is, we approximate the null distribution of \( T_n \) using that of a random variable

\[
R = \beta \chi^2_d
\]

(4.2.22)

via matching the first two moments of \( T_n \) and \( R \). By some simple algebra, we have

\[
\beta = \frac{\text{tr}(\gamma^{\otimes 2}_\omega)}{(k-1)\text{tr}^2(\gamma_\omega)}, \quad d = \frac{(k-1)\text{tr}^2(\gamma_\omega)}{\text{tr}(\gamma^{\otimes 2}_\omega)},
\]

(4.2.23)

where

\[
\text{tr}(\gamma_\omega) = \int_T \int_T \gamma_\omega[(s, t), (s, t)] \, ds \, dt = (b - a)^2,
\]

\[
\text{tr}(\gamma^{\otimes 2}_\omega) = \int_T \int_T \int_T \int_T \gamma^{\otimes 2}_\omega[(s_1, t_1), (s_2, t_2)] \, ds_1 \, dt_1 \, ds_2 \, dt_2.
\]

The quasi GPF test can be implemented provided that the parameters \( \beta \) and \( d \) are properly estimated. For the given \( k \) samples, we can obtain the following naive estimators of \( \beta \) and \( d \) via replacing \( \gamma_\omega[(s_1, t_1), (s_2, t_2)] \) with its estimator.
\( \hat{\gamma}_\omega [(s_1, t_1), (s_2, t_2)] \) as given in (4.2.21) in the expressions (4.2.23):

\[
\hat{\beta} = \frac{\text{tr}(\hat{\gamma}_\omega \otimes 2 \omega)}{(k - 1)(b - a)^2}, \quad \hat{d} = \frac{(k - 1)(b - a)^4}{\text{tr}(\hat{\gamma}_\omega \otimes 2 \omega)},
\] (4.2.24)

where \( \hat{\gamma}_\omega [(s_1, t_1), (s_2, t_2)] \) is given in (4.2.21). Then we have

\[
T_n \sim \hat{\beta} \chi^2_d \quad \text{approximately},
\] (4.2.25)

so that the quasi GPF test can be conducted accordingly.

Theorem 4.2. Under Assumptions A1~A3 and the null hypothesis (4.2.2), as \( n \to \infty \), we have \( \hat{\beta} \overset{p}{\to} \beta \), \( \hat{d} \overset{p}{\to} d \) and \( \hat{C}_\alpha \overset{p}{\to} C_\alpha \) where \( \hat{C}_\alpha = \hat{\beta} \chi^2_d(\alpha) \) is the estimated critical value of \( T_n \) and \( C_\alpha = \beta \chi^2_d(\alpha) \) is the approximate theoretical critical value of \( T_n \).

Theorem 4.2 shows that the naive estimators \( \hat{\beta} \) and \( \hat{d} \) converge in probability to their underlying values and thus the estimated 100\( \alpha \)-quantile converges to the theoretical 100\( \alpha \)-quantile. The naive estimators are simple to implement and easy to compute. However, it requires large sample sizes so that the asymptotic results of Theorem 4.1 are valid.

Alternatively, we can adopt the following random permutation method for approximating the null distribution of the quasi GPF and \( F_{\max} \) tests. This random permutation method is applicable for both large and small sample sizes. Let

\[
v^{*}_{ij}(t), \quad j = 1, 2, \cdots, n_i; \quad i = 1, 2, \cdots, k,
\] (4.2.26)

be the \( k \) permuted samples generated from the estimated subject-effect functions given in (4.2.4). That is, we first permute the estimated subject-effect functions \( \hat{v}_{ij}(t), \quad j = 1, 2, \cdots, n_i; \quad i = 1, 2, \cdots, k \) and then use the first \( n_1 \) functions as \( v^{*}_{ij}(t), \quad j = 1, 2, \cdots, n_1 \) and use the next \( n_2 \) functions as \( v^{*}_{2j}(t), \quad j = 1, 2, \cdots, n_2 \) and so on. It is obvious that given the original \( k \) functional samples (4.2.1), the \( k \)
permuted samples (4.2.26) are i.i.d with mean function 0 and covariance function
\( \frac{n-k}{n} \hat{\gamma}(s, t) \), where \( \hat{\gamma}(s, t) \) is the pooled sample covariance function given in (4.2.5).
Then the permuted test statistics of the quasi GPF and \( F_{\max} \) tests based on the \( k \) permuted samples can be obtained similarly as we defined \( T_n \) and \( F_{\max} \) based on the \( k \) original functional samples (4.2.1). That is, the permuted test statistics can be obtained as
\[
T_n^* = \int_T \int_T F_n^*(s, t) ds dt, \quad F_{\max}^* = \sup_{s, t \in T} F_n^*(s, t),
\]
where
\[
F_n^*(s, t) = \frac{SSB^*(s, t)/(k-1)}{SSE^*(s, t)/(n-k)},
\]
\[
SSB^*(s, t) = \sum_{i=1}^{k} (n_i - 1) [\hat{\gamma}^*_i(s, t) - \hat{\gamma}^*(s, t)]^2,
\]
\[
SSE^*(s, t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} [\hat{v}_{ij}^*(s) \hat{v}_{ij}^*(t) - \hat{\gamma}^*_i(s, t)]^2,
\]
with
\[
\hat{\gamma}_i^*(s, t) = (n_i - 1)^{-1} \sum_{j=1}^{n_i} \hat{v}_{ij}^*(s) \hat{v}_{ij}^*(t), \quad i = 1, 2, \ldots, k,
\]
\[
\hat{\gamma}^*(s, t) = \sum_{i=1}^{k} (n_i - 1) \hat{\gamma}_i^*(s, t)/(n - k).
\]
The permuted upper 100\( \alpha \)-percentiles \( C_{1\alpha}^* \) and \( C_{2\alpha}^* \) of \( T_n^* \) and \( F_{\max}^* \) can then be obtained via repeating the above random permutation process a large number of times.

Let \( C_{1\alpha} \) and \( C_{2\alpha} \) denote the upper 100\( \alpha \)-percentiles of \( T_0 \) and \( F_0 \) respectively, where \( T_0 \) and \( F_0 \) are the limit random variables of \( T_n \) and \( F_{\max} \) under the null hypothesis \( H_0 \) as defined in Theorem 4.1. The following theorem shows that the permutation test statistics admit the same limit random expressions of the original test statistics and hence the associated critical values \( C_{1\alpha}^* \) and \( C_{2\alpha}^* \) will tend to \( C_{1\alpha} \) and \( C_{2\alpha} \) in distribution as \( n \to \infty \). Thus we can use the critical values \( C_{1\alpha}^* \) and \( C_{2\alpha}^* \) to conduct the quasi GPF and \( F_{\max} \) tests.

We can get the sample upper 100\( \alpha \)-percentiles \( C_{1\alpha}^* \) and \( C_{2\alpha}^* \) of \( T_n^* \) and \( F_{\max}^* \) via repeating the above random permutation process a large number of times.
Chapter 4. Two Quasi $F$-type Tests

**Theorem 4.3.** Under Assumptions A1$\sim$A3 and the null hypothesis (4.2.2), as $n \to \infty$, we have $T^n \overset{d}{\to} T_0$, $F^{*}_{\text{max}} \overset{d}{\to} F_0$ and $C^{*}_{1\alpha} \overset{d}{\to} C_{1\alpha}$, $C^{*}_{2\alpha} \overset{d}{\to} C_{2\alpha}$.

We now study the asymptotic powers of the quasi GPF and $F_{\text{max}}$ tests under the following local alternative:

$$H_1 : \gamma_i(s, t) = \gamma(s, t) + (n_i - 1)^{-1/2}d_i(s, t), \ i = 1, 2, \ldots, k,$$  \hspace{1cm} (4.2.27)

where $d_1(s, t), d_2(s, t), \ldots, d_k(s, t)$ are some fixed bivariate functions, independent of $n$ and $\gamma(s, t)$ is some covariance function. This local alternative will tend to the null hypothesis in a root-$n$ rate and hence it is difficult to detect. First of all, we derive the alternative distribution of the quasi $F_{\text{max}}$ test in Theorem 4.4 and that of the quasi GPF test in Theorem 4.5 below.

**Theorem 4.4.** Under Assumptions A1$\sim$A3 and the local alternative (4.2.27), as $n \to \infty$, we have $F_{\text{max}} \overset{d}{\to} F_1$ with

$$F_1 \overset{d}{=} \sup_{s, t \in \mathcal{T}} \left\{ (k - 1)^{-1} \sum_{i=1}^{k-1} \left[ \omega_i(s, t) + \zeta_{\omega i}(s, t) \right]^2 \right\},$$

where $\omega_1(s, t), \omega_2(s, t), \ldots, \omega_{k-1}(s, t)$ i.i.d. $\text{GP}(0, \gamma_{\omega})$ as in Theorem 4.1 and $\zeta_{\omega i}(s, t), \ i = 1, 2, \ldots, k - 1$ are the $(k - 1)$ components of $\zeta_{\omega}(s, t) = (I_{k-1}, 0)U^T d(s, t)/\sqrt{\omega([s, t], [s, t])}$ with $U$ given in (4.2.17), $\omega([s, t], [s, t])$ given in (4.2.12) and $d(s, t) = [d_1(s, t), d_2(s, t), \ldots, d_k(s, t)]^T$ with its entries given in (4.2.27).

Let $\lambda_r, r = 1, 2, \ldots, \infty$ be the eigenvalues of $\gamma_{\omega} [(s_1, t_1), (s_2, t_2)]$ with only the first $m$ eigenvalues being positive and $\phi_r(s, t), r = 1, 2, \ldots, \infty$ are the associated eigenfunctions.
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**Theorem 4.5.** Under Assumptions A1∼A3 and the local alternative (4.2.27), as \( n \to \infty \), we have \( T_n \overset{d}{\to} R_1 \) with

\[
R_1 \overset{d}{=} (k-1)^{-1} \int_T \int_T ||x(s,t)||^2 dsdt = (k-1)^{-1} \sum_{i=1}^{k-1} \int_T \int_T x_i^2(s,t) dsdt
\]

\[
= (k-1)^{-\frac{1}{2}} \left[ \sum_{r=1}^{m} \lambda_r A_r + \sum_{r=m+1}^{\infty} \delta_r^2 \right],
\]

where \( A_r \sim \chi^2_{k-1}(\lambda_r^{-1} \delta_r^2) \), \( r = 1, 2, \ldots, m \), are independent, \( x(s,t) = [x_1(s,t), x_2(s,t), \ldots, x_{k-1}(s,t)]^T \sim GP_{k-1}(\zeta_\infty(s,t), \gamma_\omega I_{k-1}) \) with \( \zeta_\infty(s,t) \) defined in Theorem 4.4, and \( \delta_r^2 = || \int_T \int_T \xi(s,t) \phi_r(s,t) dsdt ||^2 \), \( r = 1, 2, \ldots, \infty \).

Theorem 4.5 states the asymptotic normality of the quasi GPF test under the local alternative (4.2.27). Theorems 4.7 and 4.8 show that the quasi GPF and \( F_{\max} \) tests are root-\( n \) consistent. In these three theorems, the quantities \( \delta_r^2 \), \( r = 1, 2, \ldots \) are defined in Theorem 4.5.

**Theorem 4.6.** Under Assumptions A1∼A3 and the local alternative (4.2.27), as \( \max_r \delta_r^2 \to \infty \), we have

\[
\frac{T_n - E(T_n)}{\sqrt{\text{Var}(T_n)}} \overset{d}{\to} N(0,1).
\]

**Theorem 4.7.** Under Assumptions A1∼A3 and the local alternative (4.2.27), as \( \max_r \delta_r^2 \to \infty \), the quasi GPF test has asymptotic power 1. That is, \( P(T_n > C_\alpha) \to 1 \) where \( C_\alpha \) can be \( \hat{C}_\alpha = \hat{\beta}_d \chi^2_d(\alpha) \), the estimated critical value of \( T_n \), or \( C^*_{\alpha} \), the estimated upper 100\( \alpha \)-percentile of \( T_n \) using the random permutation method.

**Theorem 4.8.** Under Assumptions A1∼A3 and the local alternative (4.2.27), as \( n \to \infty \), the power of the quasi \( F_{\max} \) test \( P(F_{\max} \geq C^*_{2\alpha}) \) will tend to 1 as \( \max_r \delta_r^2 \to \infty \) where \( C^*_{2\alpha} \) is the estimated upper 100\( \alpha \)-percentile of the random permuted test statistic \( F^*_{\max} \).
In the proof of Theorem 4.8, we shall use the following relationship between the quasi $F_{\text{max}}$ test statistic and the quasi GPF test statistic defined in (4.2.10) :

$$T_n = \int_T \int_T F_n(s, t) ds dt \leq (b - a)^2 F_{\text{max}},$$

where we use the fact that $T = [a, b]$. It then follows that

$$P(F_{\text{max}} \geq C_{2\alpha}^*) \geq P(T_n \geq (b - a)^2 C_{2\alpha}^*).$$

(4.2.29)

However, we cannot compare the values of $(b - a)^2 C_{2\alpha}^*$ and the upper 100$\alpha$-percentile of the quasi GPF test statistic $T_n$. Thus, the expression (4.2.29) does not guarantee that the quasi $F_{\text{max}}$ test is more powerful than the quasi GPF test. To compare the powers of these two tests, some simulation studies are then needed.

### 4.3. Simulation Studies

For the ECF testing problem, Chapter 2 studied an $L^2$-norm based test. There are three methods to approximate the null distribution of the $L^2$-norm based test statistic: a naive method, a bias-reduced method, and a random permutation method. These tests can be represented by $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$ respectively. When the functional data are Gaussian, $L^2_{br}$ and $L^2_{rp}$ are comparable and they outperform $L^2_{nv}$ in general. Also for the ECF testing problem, in Chapter 3 we proposed a so-called $T_{\text{max}}$ test using the supremum value of the sum of the squared differences between the group sample covariance functions and the associated pooled sample covariance function. When functional data are highly correlated, it is shown that the $T_{\text{max}}$ test has higher powers than $L^2_{nv}, L^2_{br}$ and $L^2_{rp}$. Since we can approximate the null distribution of the quasi GPF test using a naive method and a random permutation method, the associated quasi GPF tests are denoted as $\text{GPF}_{nv}$ and $\text{GPF}_{rp}$ respectively. In this section, we present some simulation studies, aiming
4.3 Simulation Studies

to compare GPF$_{ne}$, GPF$_{rp}$ and $F_{\text{max}}$ against $L^2_{br}$, $L^2_{rp}$ and $T_{\text{max}}$. We exclude $L^2_{nv}$ since its performance is not as good as $L^2_{br}$, $L^2_{rp}$ and $T_{\text{max}}$. In this section, we shall present three different simulation studies for three different goals.

4.3.1. Data generating. We use the following model to generate $k$ functional samples:

$$y_{ij}(t) = \eta_i(t) + v_{ij}(t), \quad \eta_i(t) = c_i^T [1, t, t^2, t^3]^T, \quad v_{ij}(t) = b_{ij}^T \Psi_i(t), \quad t \in [0, 1],$$

$$b_{ij} = [b_{ij1}, b_{ij2}, \ldots, b_{ijq}]^T, \quad b_{ijr} \overset{d}{=} \sqrt{\lambda_r} z_{ijr}, \quad r = 1, 2, \ldots, q;$$

(4.3.1)  

$j = 1, 2, \ldots, n_i; \quad i = 1, 2, \ldots, k$ are the group mean functions with the parameter vectors $c_i = [c_{i1}, c_{i2}, c_{i3}, c_{i4}]^T, \quad i = 1, 2, \ldots, k$, $\Psi_i(t) = [\psi_{i1}(t), \psi_{i2}(t), \ldots, \psi_{iq}(t)]^T$ is a vector of $q$ basis functions $\psi_{ir}(t), \quad t \in [0, 1], \quad r = 1, 2, \ldots, q$, the variance components $\lambda_r, \quad r = 1, 2, \ldots, q$ are positive and decreasing in $r$, and the number of the basis functions $q$ is an odd positive integer and the random variables $z_{ijr}, \quad r = 1, 2, \ldots, q; \quad j = 1, 2, \ldots, n_i; \quad i = 1, 2, \ldots, k$ are i.i.d. with mean 0 and variance 1. Then we have the group mean functions $\eta_i(t) = c_{i1} + c_{i2}t + c_{i3}t^2 + c_{i4}t^3, \quad i = 1, 2, \ldots, k$ and the group covariance functions

$$\gamma_i(s, t) = \Psi_i(s)^T \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_q) \Psi_i(t) = \sum_{r=1}^{q} \lambda_r \psi_{ir}(s) \psi_{ir}(t), \quad i = 1, 2, \ldots, k.$$

In the simulations, the design time points for all the functions $y_{ij}(t), \quad j = 1, 2, \ldots, n_i; \quad i = 1, 2, \ldots, k$ are assumed to be the same and are specified as $t_j = (j - 1)/(J - 1), \quad j = 1, 2, \ldots, J$, where $J$ is some positive integer.

We next specify the model parameters in (4.3.1). We choose the group number $k = 3$. To specify the group mean functions $\eta_1(t), \eta_2(t), \ldots, \eta_k(t)$, we set $c_1 = [1, 2.3, 3.4, 1.5]^T$ and $c_i = c_1 + (i - 1)\delta u, \quad i = 2, 3$, where the tuning parameter $\delta$ specifies the differences $\eta_i(t) - \eta_1(t), \quad i = 2, 3$, and the constant vector $u$ specifies the direction of these differences. We set $\delta = 0.1$ and
\[ \mathbf{u} = [1, 2, 3, 4]^T / \sqrt{30} \] which is a unit vector. Then we specify the covariance functions \( \gamma_i(s, t), i = 1, 2, \ldots, k \). For simplicity, we set \( \lambda_r = a \rho r^{-1}, r = 1, 2, \ldots, q \), for some \( a > 0 \) and \( 0 < \rho < 1 \). Notice that the tuning parameter \( \rho \) not only determines the decay rate of \( \lambda_1, \lambda_2, \ldots, \lambda_q \), but also determines how the simulated functional data are correlated: when \( \rho \) is close to 0, \( \lambda_1, \lambda_2, \ldots, \lambda_q \) will decay very fast, indicating that the simulated functional data are highly correlated; and when \( \rho \) is close to 1, \( \lambda_r, r = 1, 2, \ldots, q \) will decay very slowly, indicating that the simulated functional data are nearly uncorrelated. The functions \( \psi_{ir}(t), i = 1, 2, 3; r = 1, 2, \ldots, q \) in the above model (4.3.1) are carefully specified. First of all, let \( \phi_1(t) = 1, \phi_{2r}(t) = \sqrt{2} \sin(2\pi rt), \phi_{2r+1}(t) = \sqrt{2} \cos(2\pi rt), t \in [0, 1], r = 1, 2, \cdots, (q-1)/2 \) to be a vector of \( q \) orthonormal basis functions \( \Phi(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_q(t)]^T \), and specify \( \psi_{ir}(t) = \phi_r(t), r = 1, 3, 4, \cdots, q \) and \( \psi_{i2}(t) = \phi_2(t) + (i-1)\omega \) respectively where \( \omega \) is some constant. It can be seen the covariance functions are

\[
\gamma_i(s, t) = \gamma_1(s, t) + (i - 1)\lambda_2[\phi_2(s) + \phi_2(t)]\omega + (i - 1)^2\lambda_2\omega^2, \quad i = 1, 2, \cdots, k.
\]

It is seen that the parameter \( \omega \) controls the differences between the three covariance functions. In addition, we set \( a = 1.5, q = 11 \) and \( \rho = 0.1, 0.5, 0.9 \) to consider the three cases when the simulated functional data have high, moderate and low correlations. We generate independent samples with three cases of the sample size vector: \( \mathbf{n}_1 = [20, 30, 30] \), \( \mathbf{n}_2 = [30, 40, 50] \) and \( \mathbf{n}_3 = [80, 70, 100] \), representing the small, medium and large sample size cases respectively, and specify the number of design time points \( J = 80 \). Finally, we consider two cases of the distribution of the i.i.d. random variables \( z_{ijr}, r = 1, 2, \cdots, q; j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k \):

\( z_{ijr} \overset{i.i.d.}{\sim} N(0, 1) \) and \( z_{ijr} \overset{i.i.d.}{\sim} t_4 / \sqrt{2} \), allowing to generate Gaussian and non-Gaussian functional data respectively with \( z_{ijr} \) having mean 0 and variance 1.
4.3 Simulation Studies

Notice that the \( t_4/\sqrt{2} \) distribution is chosen since it has nearly the heaviest tails among the \( t \)-distributions with finite first two moments.

For a given model configuration, the three groups of functional samples are generated from the data generating model (4.3.1). The \( p \)-values of \( L_{br}^2, L_{rp}^2, T_{\text{max}}, \text{GPF}_{nv}, \text{GPF}_{rp}, \) and \( F_{\text{max}} \) are then computed. The \( p \)-value of \( \text{GPF}_{nv} \) is based on the Welch-Satterthwaite \( \chi^2 \)-approximation as given in (4.2.25). To compute the associated parameters \( \hat{\beta} \) and \( \hat{d} \), we need the estimation of \( \varpi \) which is defined in (4.2.12). We use (4.2.13) instead of (4.2.14) in the simulations as (4.2.13) gives similar results to (4.2.14) for Gaussian data and the former can also be used for non-Gaussian data. The \( p \)-values of \( L_{rp}^2, T_{\text{max}} \) and \( F_{\text{max}} \) are obtained via using 500 runs of random permutations. The null hypothesis is rejected if the calculated \( p \)-value of a testing procedure is smaller than the nominal significance level \( \alpha = 5\% \). We repeat the above process for 10000 times. The empirical sizes or powers of the testing procedures can then be obtained as the percentages of rejection in the 10000 runs.

4.3.2. Simulation 1. In Simulation 1, we aim to check whether the random permuted null pdfs of \( \text{GPF}_{rp} \) and \( F_{\text{max}} \) approximate their true null pdfs well. We compare the curves of the simulated null pdfs and the first 50 random permuted null pdfs of \( \text{GPF}_{rp} \) and \( F_{\text{max}} \) under two cases when \( z_{ijr}, r = 1, 2, \ldots, q; j = 1, 2, \ldots, n_i; i = 1, 2, \ldots, k : z_{ijr} \sim i.i.d. N(0,1) \) and when \( z_{ijr} \sim i.i.d. t_4/\sqrt{2} \). For space saving, we only consider the small and large sample sizes (later we will also find that the sample sizes have little effect on the shapes of the curves). Figure 4.3.1 displays the simulated null pdfs (wider solid curves) and the 50 random permuted null pdfs (dashed curves) of \( \text{GPF}_{rp} \) (left 6 panels) and \( F_{\text{max}} \) (right 6 panels). Note that the simulated null pdf of a testing procedure is computed using a kernel
density estimator (KDE) with a Gaussian kernel based on the simulated 10000 test statistics when the null hypothesis is satisfied and a random permuted null pdf of a testing procedure is based on 10000 random permuted test statistics. The associated bandwidths are chosen automatically with the KDE software. It is seen that the random permuted null pdfs of GPF\textsubscript{rp} and \( F_{\text{max}} \) work well in approximating their underlying null pdfs under the Gaussian case.

**Figure 4.3.1.** The simulated null pdfs (wider solid curves) and the first 50 random permuted null pdfs (dashed curves) of GPF\textsubscript{rp} and \( F_{\text{max}} \) when \( z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k \overset{i.i.d.}{\sim} N(0,1) \).

Figure 4.3.2 displays the simulated null pdfs and the first 50 random permuted null pdfs of GPF\textsubscript{rp} and \( F_{\text{max}} \) when \( z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k \overset{i.i.d.}{\sim} t_4/\sqrt{2} \). It is seen that the random permutation method works generally well for GPF\textsubscript{rp} and \( F_{\text{max}} \) but not as well as when \( z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k \overset{i.i.d.}{\sim} N(0,1) \). It is seen that both Figures 4.3.1 and 4.3.2 indicate that the decay rates of the variance components \( \lambda_r, r = 1, 2, \ldots, q \) have
4.3 Simulation Studies

**Figure 4.3.2.** The simulated null pdfs (wider solid curves) and the first 50 random permuted null pdfs (dashed curves) of GPF\(_{rp}\) and \(F_{\text{max}}\) when \(z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k\) i.i.d. \(\sim t_{4}/\sqrt{2}\).

a great effect on the shapes of the null pdf curves of GPF\(_{rp}\) and \(F_{\text{max}}\) while the sample sizes have little effect on them.

**4.3.3. Simulation 2.** In Simulation 2, we aim to compare GPF\(_{nv}\), GPF\(_{rp}\) and \(F_{\text{max}}\) against \(L^2_{br}, L^2_{rp}\) and \(T_{\text{max}}\). Tables 4.3.1 and 4.3.2 present the empirical sizes and powers (in percentages) of \(L^2_{br}, L^2_{rp}, T_{\text{max}}, \) GPF\(_{nv}\), GPF\(_{rp}\) and \(F_{\text{max}}\) when the \(k\) functional samples follow Gaussian and non-Gaussian distributions, respectively.

First of all, it is seen that in terms of size controlling, \(F_{\text{max}}\) works reasonably well under various simulation configurations while GPF\(_{nv}\) and GPF\(_{rp}\) work well only when the functional data are highly correlated or when the sample sizes are large. When the functional data are less correlated or when the sample sizes are too small, the empirical sizes of GPF\(_{nv}\) are too large (for Gaussian functional data) or too small (for non-Gaussian functional data) compared with the nominal size 5%
Table 4.3.1. Empirical sizes and powers (in percentages) of $L^2_{br}$, $L^2_{rp}$, $T_{\text{max}}$, GPF$_{nv}$, GPF$_{rp}$ and $F_{\text{max}}$ when $z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k$ are i.i.d. $N(0, 1)$.

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4.3 Simulation Studies

Table 4.3.2. Empirical sizes and powers (in percentages) of \(L^2_{br}, L^2_{rp}, T_{\max}, \text{GPF}_{nv}, \text{GPF}_{rp}\) and \(F_{\max}\) when \(z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k\) are i.i.d. \(t_4/\sqrt{2}\).

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<td>99.27</td>
<td>99.24</td>
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93
and those of GPF$_{rp}$ are too large for both Gaussian and non-Gaussian functional data. On the other hand, $L^2_{br}$ performs quite well under the Gaussian case but it does not work for non-Gaussian data, $L^2_{rp}$ performs well when the functional data are highly correlated or the sample sizes are large but it is liberal when the functional data are less correlated or when the sample sizes are too small, and $T_{max}$ is good under various simulation configurations. In summary, in terms of size controlling, it seems $F_{max}$ and $T_{max}$ perform similarly while GPF$_{nv}$, GPF$_{rp}$ and $L^2_{br}, L^2_{rp}$ perform similarly. In terms of powers, it seems GPF$_{nv}$, GPF$_{rp}$ and $L^2_{br}, L^2_{rp}$ have comparable powers but they have smaller (or higher) powers than $F_{max}$ and $T_{max}$ when the functional data are highly (or less) correlated.

4.3.4. Simulation 3. In Simulation 3, we aim to demonstrate that in some situations, the quasi pointwise $F$-test based tests such as GPF$_{nv}$, GPF$_{rp}$ and $F_{max}$ can have much better performance than $L^2_{br}, L^2_{rp}$ and $T_{max}$. For this goal, we can revise the previous data generating model slightly. That is, we specify the subject-effect functions $v_{ij}(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ as in the following new data generating model:

\[
y_{ij}(t) = \eta_i(t) + v_{ij}(t),
\]
\[
\eta_i(t) = c_i^T [1, t, t^2, t^3]^T, \quad v_{ij}(t) = b_{ij}^T \Psi_i(t) / (t + 1/J), \quad t \in [0, 1],
\]
\[
b_{ij} = [b_{ij1}, b_{ij2}, \cdots, b_{ijq}]^T, \quad b_{ijr} \overset{d}{=} \sqrt{\lambda_r} z_{ijr}, \quad r = 1, 2, \cdots, q;
\]
\[
j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k. \] In addition, we modify the second basis function via setting $\psi_{12}(t) = \psi_{32}(t) = \phi_2(t)$ and $\psi_{22}(t) = \phi_2(t) + t\omega$. The term $t\omega$ is used to control the difference between the three covariance functions. In this new data generating model, the covariance functions have different scales at different time points. As GPF$_{nv}$, GPF$_{rp}$ and $F_{max}$ are scale-invariant, we expect that they should have better performance than $L^2_{br}, L^2_{rp}$ and $T_{max}$ which are not scale-invariant. This
4.3 Simulation Studies

Table 4.3.3. Empirical sizes and powers (in percentages) of $L^2_{br}$, $L^2_{rp}$, $T_{\text{max}}$, GPF$_{nv}$, GPF$_{rp}$ and $F_{\text{max}}$ when $z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k$ are i.i.d. $N(0,1)$ under the new data generating model.

<table>
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<tr>
<th>n</th>
<th>$\rho$</th>
<th>$\omega_0$</th>
<th>$L^2_{br}$</th>
<th>$L^2_{rp}$</th>
<th>$T_{\text{max}}$</th>
<th>GPF$_{nv}$</th>
<th>GPF$_{rp}$</th>
<th>$F_{\text{max}}$</th>
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<tr>
<td>20</td>
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<td>5.89</td>
<td>6.49</td>
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<td>6.79</td>
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<td>56.75</td>
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<td>99.00</td>
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<tr>
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<td>4.70</td>
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<td>5.36</td>
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<td>83.13</td>
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<td>5.00</td>
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<td>0.5</td>
<td>4.70</td>
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<td>5.29</td>
<td>99.70</td>
<td>99.70</td>
<td>99.70</td>
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</tr>
</tbody>
</table>
is indeed the case as shown by the simulation results presented in Table 4.3.3 where it is seen that GPF\textsubscript{nv}, GPF\textsubscript{rp} and \( F_{\text{max}} \) are more powerful than \( L_{\text{br}}^2, L_{\text{rp}}^2 \) and \( T_{\text{max}} \) whose empirical powers are always around the nominal sizes.

4.4. Real Data Analysis

In this section, we present a real data example for applications of the quasi GPF tests (GPF\textsubscript{nv}, GPF\textsubscript{rp}) and the quasi \( F_{\text{max}} \) test, together with \( L_{\text{br}}^2, L_{\text{rp}}^2 \) and \( T_{\text{max}} \) tests. The real functional data set was collected by Professor Carey at UC Davis in a medfly rearing facility in Mexico. It recorded the number of alive medflies over a period of time aiming to quantify the effects of nutrition and gender on mortality. The data set was kindly made available online by Professor Hans-Georg Müller and Professor Carey’s laboratory at http://anson.ucdavis.edu/~mueller/data/data.html and has been extensively studied in Müller et al. (1997) and Müller and Wang (1998).

The data set consists of the lifetimes of four groups of medflies over 101 days. Each group has 33 cohorts with each cohort consisting of about 3000–4000 medflies. The four groups of medflies are “1. males on sugar diet”, “2. males on protein plus sugar diet”, “3. females on sugar diet” and “4. females on protein plus sugar diet”. In applications, the cohort survival behavior can be conveniently summarized in the form of a survival function. This survival function can be obtained by dividing the daily number of alive medflies by the total number of medflies in each cohort at the beginning. For simplicity, we only consider the survival functions on the first 2–31 days since on the first day all the survival functions equal 1. It is of interest to check if the covariance structures of the four different groups of medflies are the same.
4.4 Real Data Analysis

Table 4.4.1 shows the p-values (in percentages) of $L_{br}^2$, $L_{rp}^2$, $T_{\text{max}}$, $\text{GPF}_{nv}$, $\text{GPF}_{rp}$ and $F_{\text{max}}$ applied to several selected group comparisons of the survival functions of the four groups of medflies. The p-values of $L_{rp}^2$, $T_{\text{max}}$, $\text{GPF}_{rp}$ and $F_{\text{max}}$ are obtained via 10000 runs of random permutations. For different group comparisons, the goals are different. The comparison “Group 1 vs Group 2” aims to assess the effect of the sugar diet on male medflies, the comparison “Group 3 vs Group 4” aims to assess the effect of the sugar diet on female medflies, the comparison “Group 1 vs Group 3” aims to assess the gender effect of the sugar diet, the comparison “Group 2 vs Group 4” aims to assess the gender effect of the protein plus sugar diet, and “All the four groups” comparison aims to test if all the four groups have the same covariance structure.

It is seen that all the p-values of the tests for the comparison of “Group 1 vs Group 2” suggest that the effect of the sugar diet on male medflies is not significant, showing that the sugar diet may be useless for male medflies. However, it is not the case for the effect of the sugar diet on female medflies since all the p-values of the tests for the comparison of “Group 3 vs Group 4” suggest that the effect of the sugar diet on female medflies is highly significant. Therefore, it is expected that the gender effect of the sugar diet should be significant and it is also expected that the gender effect of the protein plus sugar diet should be significant. However,
Table 4.4.2. Computation time (in seconds) comparison of $L^2_{br}$, $L^2_{rp}$, $T_{\text{max}}$, $\text{GPF}_{nv}$, $\text{GPF}_{rp}$ and $F_{\text{max}}$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$L^2_{br}$</th>
<th>$L^2_{rp}$</th>
<th>$T_{\text{max}}$</th>
<th>$\text{GPF}_{nv}$</th>
<th>$\text{GPF}_{rp}$</th>
<th>$F_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0.64</td>
<td>8.50</td>
<td>4.71</td>
<td>0.57</td>
<td>21.03</td>
<td>21.23</td>
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</tbody>
</table>

only the p-values of $T_{\text{max}}$, $\text{GPF}_{nv}$, $\text{GPF}_{rp}$, $F_{\text{max}}$ for the comparison of “Group 1 vs Group 3” suggest that the gender effect of the sugar diet is highly significant and only the p-value of $F_{\text{max}}$ for the comparison of “Group 2 vs Group 4” suggest that the gender effect of the protein plus sugar diet is highly significant. All the P-values of the tests except $L^2_{br}$ for the comparison “All the four groups” suggest that the covariance structures of the four groups are unlikely the same. The p-values in this table suggest that the supremum based tests such as $T_{\text{max}}$ and $F_{\text{max}}$ are more powerful than other tests, and the pointwise quasi $F$-test based tests such as $\text{GPF}_{nv}$, $\text{GPF}_{rp}$ and $F_{\text{max}}$ are generally more powerful than those $L^2$-norm based tests such as $L^2_{br}$, $L^2_{rp}$. It is also seen that the $F_{\text{max}}$ test is the most powerful test among all the tests under consideration.

In practice, besides the robustness and powerfulness, we also would like to compare the computation costs of the tests. Table 4.4.2 gives us the computation times of the above-mentioned six tests for testing all the four groups of medflies. We can see that $L^2_{br}$ and $\text{GPF}_{nv}$ are much faster than the remaining four random permutation based tests. This is as expected since the random permutation based tests are computationally intensive. Besides the $\text{GPF}_{rp}$ and $F_{\text{max}}$ tests take more time than the $L^2_{rp}$ and $T_{\text{max}}$ tests which may due to the additional calculation of the denominator in the quasi $F$-type tests.
4.5 Appendix

**Proof of Theorem 4.1.** Recall that $\text{SSE}(s, t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} [\hat{v}_{ij}(s) - \hat{v}_{ij}(t)]^2$. Under the given conditions and the null hypothesis, by Lemmas 2.1 and 2.2 in Chapter 2, as $n \to \infty$, we have $\text{SSE}(s, t)/(n - k) = \bar{\omega }[(s, t), (s, t)] \xrightarrow{p} \bar{\omega }[(s, t), (s, t)]$, $z_n(s, t) \xrightarrow{d} z(s, t) \sim GP_k(0, \omega I_k)$ uniformly for all $s, t \in T$ and by (4.2.16), $I_k - b_n^T b_n/(n - k) \to I_k - bb^T$, with $b = [\sqrt{\tau_1}, \sqrt{\tau_2}, \ldots, \sqrt{\tau_k}]^T$ where $\bar{\omega }$ is defined in (4.2.13) and $\omega$ is defined in (4.2.12). Then, by Slusky’s theorem, we can easily get $F_n(s, t) \xrightarrow{p} (k - 1)^{-1} \mathbf{1}_{k-1} \mathbf{1}_{k-1}^T \mathbf{1}_{k-1} \omega [s, s, \ldots, s]/\omega [s, s, \ldots, s]$. Based on (4.2.17), we have $F_n(s, t) \xrightarrow{p} (k - 1)^{-1} \omega^T(s, t)\omega(s, t)$ where $\omega(s, t) = (I_{k-1}, 0)U^Tz(s, t)/\sqrt{\bar{\omega }[(s, t), (s, t)]} = [\omega_1(s, t), \omega_2(s, t), \ldots, \omega_{k-1}(s, t)]^T \sim GP_{k-1}(0, \gamma_\omega I_{k-1})$ and $\gamma_\omega[(s_1, t_1), (s_2, t_2)] = \bar{\omega }[(s_1, t_1), (s_2, t_2)]/\sqrt{\bar{\omega }[(s_1, t_1), (s_1, t_1)]\bar{\omega }[(s_2, t_2), (s_2, t_2)]}$. It is easy to prove that $tr(\gamma_\omega) = b - a < \infty$ since $T = [a, b]$ is a finite interval. The second expression of (4.2.18) can be proved by continuous mapping theorem for random elements taking values in a Hilbert space (Billingsley 1968, p.34; Cuevas et al. 2004) and along the same lines of the proof of the Theorem 4.10 of Chapter 4 in Zhang (2013) (p.90). Similar to the proof of Theorem 3.1 in Chapter 3, (4.2.19) can be obtained. \hfill $\Box$

**Proof of Theorem 4.2.** Based on Assumption A1 and Lemma 2.2 in Chapter 2, we can easily get $\hat{\omega }[(s_1, t_1), (s_2, t_2)] \xrightarrow{p} \bar{\omega }[(s_1, t_1), (s_2, t_2)]$. Then $\gamma_\omega \xrightarrow{p} \gamma_\omega$ and $tr(\gamma_\omega^2) \xrightarrow{p} tr(\gamma_\omega^2)$ follow immediately from the continuous mapping theorem for random elements taking values in a Hilbert space. Therefore, as $n \to \infty$, we have $\hat{\beta} \xrightarrow{p} \beta$ and $\hat{d} \xrightarrow{p} d$. It then follows that $\hat{C}_a \xrightarrow{p} \tilde{C}_a$. \hfill $\Box$

**Proof of Theorem 4.3.** First of all, notice that given original $k$ samples, the random permuted $k$ samples $v_{ij}^*(t)$, $j = 1, 2, \ldots, n_i; i = 1, 2, \ldots, k$ are $\text{IID } SP(0, \frac{1-k}{n} \hat{\gamma})$ where $\hat{\gamma}(s, t)$ is the pooled sample covariance function (4.2.5). That
is to say, the random permuted \( k \) samples satisfy the null hypothesis (4.2.2) since they have the same covariance. By Lemma 2.2 in Chapter 2 and under Assumptions A1\~A3, as \( n \to \infty \), we have \( \frac{n-k}{n} \hat{\gamma}(s,t) \xrightarrow{P} \gamma(s,t) \) uniformly over \( \mathcal{T}^2 \) and \( v_{ij}^*(t) \xrightarrow{d} SP(0,\gamma) \). The same argument for proving Theorem 4.1 leads to the first claim of the theorem and the second claim of the theorem follows immediately. □

**Proof of Theorem 4.4.** Notice that \( \mathbf{z}_n(s,t) = [z_1(s,t), z_2(s,t), \ldots, z_k(s,t)]^T \) can be expressed as

\[
  z_i(s,t) = \sqrt{n_i - 1}[\hat{\gamma}_i(s,t) - \gamma(s,t)], \quad i = 1, 2, \ldots, k
\]

\[
  = \sqrt{n_i - 1}[\hat{\gamma}_i(s,t) - \gamma_i(s,t) + \gamma_i(s,t) - \gamma(s,t)]
\]

\[
  = \sqrt{n_i - 1}[\hat{\gamma}_i(s,t) - \gamma_i(s,t)] + \sqrt{n_i - 1}[\gamma_i(s,t) - \gamma(s,t)].
\]

From Lemma 2.1 in Chapter 2, we know

\[
  \sqrt{n_i - 1}[\hat{\gamma}_i(s,t) - \gamma_i(s,t)] \xrightarrow{d} GP(0, \varpi_i).
\]

And from the alternative hypothesis (4.2.27) we get

\[
  \sqrt{n_i - 1}[\gamma_i(s,t) - \gamma(s,t)] = d_i(s,t),
\]

where the function \( d_i(s,t) \in \mathcal{L}^2(\mathcal{T}^2) \) and \( \mathcal{T}^2 = [a, b] \times [a, b] \).

Then under the alternative hypothesis (4.2.27), we get

\[
  \mathbf{z}_n(s,t) \xrightarrow{d} GP_k[\mathbf{d}, \text{diag}(\varpi_1, \varpi_2, \ldots, \varpi_k)].
\]

Based on Assumption A1, with the local alternative hypothesis we can prove that \( \varpi_i, \quad i = 1, 2, \ldots, k \) has the following property (Lemma 2.3 in Chapter 2):

\[
  \varpi_i[(s_1, t_1), (s_2, t_2)] = \varpi[(s_1, t_1), (s_2, t_2)] + (n_i - 1)^{-1/2} h_i[(s_1, t_1), (s_2, t_2)] + O(n^{-1}),
\]

(4.5.1)
where \( h_i[(s_1,t_1),(s_2,t_2)], i = 1, 2, \cdots, k \) are some fixed functions. According to (4.5.1), we can easily obtain

\[
z_n(s,t) \xrightarrow{d} z(s,t) + d(s,t) \sim GP_k[d, \omega I_k]
\]

with \( d = [d_1(s,t), d_2(s,t), \cdots, d_k(s,t)]^T \) and \( z(s,t) \sim GP_k(0, \omega I_k) \).

Since \( T \) is a finite interval and \( T_n(s,t) \) is equicontinuous over \( T \), by Slusky’s theorem, and Theorem 2.1 of Newey (1991), we can show that as \( n \to \infty \), we have \( F_{\max} \xrightarrow{d} F_1 \) with \( F_1 = \sup_{s,t \in T} \{(k - 1)^{-1}[z(s,t) + d(s,t)]^T(I_k - b b^T)](z(s,t) + d(s,t))/\omega[(s,t),(s,t)]\} \) where the idempotent matrix \( I_k - b b^T \) has the singular value decomposition (4.2.17). We already know \( \omega(s,t) = (I_{k-1}, 0)U^T z(s,t)/\sqrt{\omega[(s,t),(s,t)]} = [\omega_1(s,t), \omega_2(s,t), \cdots, \omega_{k-1}(s,t)]^T \)

and let \( \phi_{s,t} = (I_{k-1}, 0)U^T d(s,t)/\sqrt{\omega[(s,t),(s,t)]} = [\delta_{s,t} \omega(s,t), \cdots, \delta_{s,t} \omega_{k-1}(s,t)]^T \). Then \( \omega(s,t) \sim GP_{k-1}(0, \gamma \omega I_{k-1}) \)

Therefore, \( F_1 = \sup_{s,t \in T} \{(k - 1)^{-1}[\omega(s,t) + \phi_{s,t}]^T[\omega(s,t) + \phi_{s,t}]\} = \sup_{s,t \in T} \{(k - 1)^{-1}\sum_{i=1}^{k-1}[\omega_i(s,t) + \phi_{s,t}]^2\} \). \( \square \)

**Proof of Theorem 4.5.** Similar to the proof of Theorem 4.4 and the continuous mapping theorem for random elements taking values in a Hilbert space, we can easily get

\[
R_1 \times (k - 1)^{-1} \int_T \int_T \frac{|z(s,t) + d(s,t)|^T(I_k - b b^T)](z(s,t) + d(s,t))/\omega[(s,t),(s,t)]|}{ds dt}. \tag{4.5.2}
\]

Denote \( x(s,t) = (I_{k-1}, 0)U^T z(s,t) + d(s,t)/\sqrt{\omega[(s,t),(s,t)]} \sim GP_{k-1}(\phi_{s,t}, \gamma \omega I_{k-1}) \) with \( \phi_{s,t} \) defined in Theorem 4.4 and let \( x(s,t) = [x_1(s,t), x_2(s,t), \cdots, x_{k-1}(s,t)]^T \). Then it follows that

\[
R_1 \times (k - 1)^{-1} \int_T \int_T |x(s,t)|^2 ds dt = (k - 1)^{-1} \sum_{i=1}^{k-1} \int_T \int_T x_i^2(s,t) ds dt. \tag{4.5.3}
\]
Chapter 4. Two Quasi $F$-type Tests

According to the proof of Theorem 2.3 in Chapter 2, we have

$$ R_1 \overset{d}{=} (k-1)^{-1} \left[ \sum_{r=1}^{m} \lambda_r A_r + \sum_{r=m+1}^{\infty} \delta_r^2 \right], \quad (4.5.4) $$

where $A_r \sim \chi^2_{k-1}(\lambda^{-1}_r \delta^2_r)$, $r = 1, 2, \cdots, m$ are independent, $\lambda_r$, $r = 1, 2, \cdots, \infty$ are the eigenvalues of $\gamma_{\omega}[(s_1, t_1), (s_2, t_2)]$ with only the first $m$ eigenvalues being positive, $\phi_r(s, t)$, $r = 1, 2, \cdots, \infty$ are the associated eigenfunctions, and $\delta^2_r = || \int_T \int_T \zeta_\omega(s, t) \phi_r(s, t) dsdt ||^2$, $r = 1, 2, \cdots, \infty$.

\[ \square \]

**Proof of Theorems 4.6 and 4.7.** The proof is similar to the proofs of Theorems 2.4 and 2.5 in Chapter 2. \[ \square \]

**Proof of Theorem 4.8.** By (4.2.29), we first have

$$ P(F_{\max} \geq C^*_{2\alpha}) \geq P(T_n \geq (b-a)^2 C^*_{2\alpha}). \quad (4.5.5) $$

Notice that under Assumptions A1-3 and the local alternatives (4.2.27), similar to Theorem 4.3, we can prove that

$$ (b-a)^2 C^*_{2\alpha} \overset{d}{\rightarrow} (b-a)^2 C_{2\alpha} \quad (4.5.6) $$

with $C_{2\alpha}$ being the upper 100$\alpha$ percentile of $F_0$. Under Assumptions A1-3 and the local alternative (4.2.27), by the proofs of Theorems 2.4 and 2.5 in Chapter 2, we know that if $(b-a)^2 C_{2\alpha} < \infty$ and $\max_r \delta^2_r \rightarrow \infty$, we have

$$ P(T_n \geq (b-a)^2 C^*_{2\alpha}) \rightarrow 1. \quad (4.5.7) $$

Then $P(F_{\max} \geq C^*_{2\alpha}) \rightarrow 1$ follows immediately. \[ \square \]
CHAPTER 5

Conclusion and Future Work

In this thesis, we have studied the multi-sample equal-covariance functional (ECF) testing problem for functional data. We have proposed and studied three types of tests: the $L^2$-norm based test, the supremum norm based test and the quasi $F$-type tests, for this second-order comparison problem. We have demonstrated the asymptotic power of the three testing procedures and all these tests were shown to be root-$n$ consistent. As shown by the simulation studies conducted in each chapter, all these tests worked well under various configurations of sample size and degree of data correlation.

For the $L^2_{nv}$ and $L^2_{br}$ tests, we can approximate the null distribution by the Welch-Satterthwaite $\chi^2$-approximation and thus they are computationally efficient and perform quite well if the functional data come from the Gaussian process although the $L^2_{nv}$ test is slightly conservative compared with the $L^2_{br}$ test. The $L^2_{rp}$ test can work for non-Gaussian data but is computationally intensive. For the supremum norm based test, it is seen that this test is more powerful than the
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$L^2$-norm based tests when data are highly correlated or under a sparse alternative. However, sometimes it is not so robust especially when the data have outliers. For the quasi $F$-type tests, because of their scale-invariant property, they are more powerful than the above-mentioned tests when the covariances have different scales at different time points. In general, every test has its unique advantages and no test performs uniformly better than the others in all the settings.

Although the $L^2$-norm based test, supremum norm based test and the quasi $F$-type tests can work under the non-Gaussian case by random permutation or bootstrap, the theoretical properties of all the above tests established in this thesis are based on the assumption that the functional data are Gaussian. Further, we may want to relax the Gaussian assumption for some of the theoretical results obtained or find new methods that do not need such requirement of the data.

In addition, in Chapter 2, we compared the performance of $L^2_{\text{nu}}, L^2_{\text{br}}$ and $L^2_{\text{rp}}$ against $FHK_D$ and $FHK_G$ proposed in Fremdt et al. (2013) under the two-sample case since Fremdt et al. (2013) only focused on testing the equality of the covariance functions of two functional samples. Actually, their dimension-reduction approach can be extended to the multi-sample case. Besides, it is known that the dimension reduction method needs the selection of the functional principal components, i.e., a suitably-chosen low-dimensional space to project the functional samples. So far most methods on selecting this space are not satisfactory. In the future, we can also try to solve these problems.

Last but not least, the functional data considered in this thesis can be viewed as independent curves drawn from several populations of interest. In many scientific research fields, however, the functional curves are obtained sequentially in time and are in fact so-called functional time series. In this case there exists temporal dependence between the curves. Horváth et al. (2013) studied the inference for
testing the equality of mean functions of two functional time series. Other related contributions are Horváth and Rice (2015), Kokoszka (2012) and Zhang and Shao (2015). Their approaches can be extended to the multi-sample case. We can also consider the problem of testing the equality of covariance functions of functional time series. Further work in this direction is interesting and warranted.


