Quantum features of a charged particle in ionized plasma controlled by a time-dependent magnetic field

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Quantum characteristics of a charged particle traveling under the influence of an external time-dependent magnetic field in ionized plasma are investigated using the invariant operator method. The Hamiltonian that gives the radial part of the classical equation of motion for the charged particle is dependent on time. The corresponding invariant operator that satisfies Liouville-von Neumann equation is constructed using fundamental relations. The exact radial wave functions are derived by taking advantage of the eigenstates of the invariant operator. Quantum properties of the system is studied using these wave functions. Especially, the time behavior of the radial component of the quantized energy is addressed in detail.

Keywords: ionized plasma, charged particle, quantum energy, wave function, time-dependent magnetic field

1. INTRODUCTION

On account of the importance of plasma and plasma physics on materials science and nuclear fusion, the dynamical characteristics of plasma have been increasingly studied until now. Not only plasma reveal diverse properties during their process but also the features taken place in the plasma are so complex that it is very hard to control their behaviors and reactions. In a static magnetic field, charged particles go round in circles when their velocity vector is perpendicular to magnetic field lines. They however go round in helix in case they have a velocity component parallel to the lines of B-field as well as perpendicular. If the external magnetic field varies in time or in space, the motion of ionized particles becomes more random and both its treatment and analytical analysis require high technology.

The influence of magnetic fields on the motion of a charged particle involves the essential properties of acceleration and the transport of highly ionized particles. The analysis of classical and quantum behaviors of charged particles is important in connection with a well known application of the confinement of magnetized plasma. Charged particles are accelerated and decelerated as they cross a magnetic lens in a magneto optical trapping of ionized plasma. Although the charged particles are trapped both in the high-field region and low-field region, the plasma profile does not follow the naive magnetic field lines. More precisely, the radius of circling particle in the high-field region is smaller than the one that would be obtained by simply tracking the field line from the low-field radial edge toward the high field region [1]. Another application of the external magnetic field in ionized plasma is the use of it in reducing the effect of splash in pulsed laser deposition technique in plasma surface science [2, 3].

Exact theoretical description for quantum and classical properties of plasma may play a pivotal role for understanding the physics of plasma. Lewins studied the motion of a charged particle in a time-dependent magnetic field considering the conservation of magnetic moment about the circling center [4]. Stimulated by this work, we study in this paper quantum features of a charged particle moving under a time-dependent magnetic field in plasma. As magnetic field varies with time, the new electric potential would be created according to the Maxwell's equation. Hence, the motion of charged particle is more complex in the situation characterized by a varying magnetic field than by a static one.

The exact Hamiltonian for the motion of charged particle will be constructed considering the time dependence of the magnetic field. The complete quantum solutions of the system will be derived with the help of a quadratic invariant operator that is a potential tool for treating quantum systems that have time-dependent parameters. The introduction of the invariant operator is the main idea that enables us to overcome the difficulty in quantizing the system that is somewhat complicated. According to reports of Lewis and Riesenfeld [5, 6], a Schrödinger solution $\psi(r, t)$ of a system that has time-dependent parameters is given in terms of an eigenstate $\phi(r, t)$ of the invariant operator. In fact, we can obtain $\psi(r, t)$ by multiplying $\phi(r, t)$ by an appropriate phase factor. The Schrödinger solution $\psi(r, t)$ plays a major role in investigating the quantum characteristics of the system. Quantized energy of the particle will be evaluated in this work using $\psi(r, t)$ and its time behavior will be analyzed in detail in some situations that the time dependence of the magnetic field is chosen differently.

2. HAMILTONIAN DYNAMICS

Let us consider non-relativistic motion of a charged particle in ionized plasma controlled by a magnetic field. The magnetic force acting on a particle that has charge q under the static magnetic

field is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity of the particle. However, if the magnetic field varies with time, it produces a new electric field according to the Maxwell's equation:

$$\nabla \times \mathbf{E}_{\text{produced}}(t) = -\frac{\partial \mathbf{B}(t)}{\partial t}. \quad (1)$$

Then, the overall force exerting on the charged particle is

$$\mathbf{F} = q[\mathbf{E}_{\text{produced}}(t) + \mathbf{v} \times \mathbf{B}(t)]. \quad (2)$$

This gives the following Newtonian equation of motion for the particle

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{q}{m}\left[\frac{1}{2}\mathbf{r} \times \frac{d\mathbf{B}(t)}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{B}(t)\right], \quad (3)$$

where m is the mass of the particle. Lewins showed that the radial part of the above equation in cylindrical coordinate described by a set of variables (r, θ, z) becomes [see Equation (22) of Lewins [4]]

$$\frac{d^2r}{dt^2} + \omega^2(t)r = \frac{r_0^4 K^2}{r^3}, \quad (4)$$

where $r_0 = r(0)$, $\omega(t)$ is a time-dependent frequency of the form

$$\omega(t) = |q||\mathbf{B}(t)|/(2\,m). \quad (5)$$

and K is a constant expressed as $K = \frac{d\theta}{dt}\Big|_{t=0} - \omega(0)$. We can see that angular momentum of the particle is conserved in variable magnetic fields as well as in the static limit [7]. Several interesting phenomena that take place by the presence of magnetic field in an ionized plasma include plume confinement, particle acceleration and deceleration, dissipation of kinetic energy into thermal energy, Debris mitigation, and instability of plasma [8–10].

The difficulty in the study of the quantum motion of charged particles in a "time-dependent" magnetic field is insisted many times in the literature [4, 11–13] because of the production of electric field. We, in this work, may need to deal the problem of a time-dependent Hamiltonian system (TDHS) which is not easy to handle. There are several mathematical techniques available for rigorous quantum treatment of TDHSs, such as invariant operator method [5, 6], reduction method [14], propagator method [15], and canonical transformation method [16]. Among them, we will use invariant operator method as mentioned in the introductory part.

The Hamiltonian that yields the equation of motion given in Equation (4) can be written as

$$\hat{H}(\hat{r}, \hat{p}, t) = \hat{H}_{\text{HO}}(\hat{r}, \hat{p}, t) + \frac{1}{2}mr_0^4 K^2 \frac{1}{\hat{r}^2}, \quad (6)$$

where $\hat{p} = -i\hbar\partial/\partial r$ and $\hat{H}_{\text{HO}}$ is the Hamiltonian of the harmonic oscillator with the time-dependent frequency $\omega(t)$, that is represented as

$$\hat{H}_{\text{HO}}(\hat{r}, \hat{p}, t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{r}^2. \quad (7)$$

Even if a general harmonic oscillator in one dimension is defined through entire region for r, $(-\infty, \infty)$, Equation (7) is

meaningful only in the positive r. In the next section, we will solve Schrödinger equation of the system that is described by the Hamiltonian (6) and quantum features of the system will be studied.

3. THEORY AND RESULTS
3.1. INVARIANT OPERATOR AND QUANTUM SOLUTIONS

The Hamiltonian given in the last section is explicitly dependent on time as the magnetic field varies. Hence the system is a kind of TDHSs that have attracted wide interest in the physical society [5, 6, 17–26]. To derive quantum solutions of a TDHS, it is convenient to introduce an invariant operator [5, 6] because the quantum properties of such system can be investigated via the eigenstates of the invariant operator. From the Liouville-von Neumann equation $d\hat{I}/dt = \partial\hat{I}/\partial t + [\hat{I}, \hat{H}]/(i\hbar) = 0$, it is possible to derive a quadratic invariant operator $\hat{I}$. Thus, considering Equation (6), we have the invariant operator as

$$\hat{I} = \hat{I}_{\text{HO}} + mK^2 r_0^4 \chi(t)\chi^*(t)\frac{1}{\hat{r}^2}, \quad (8)$$

where $\chi(t)$ is a complex classical solution of the following differential equation

$$\ddot{\chi}(t) + \omega^2(t)\chi(t) = 0, \quad (9)$$

and $\hat{I}_{\text{HO}}$ is the invariant operator of the system described by $\hat{H}_{\text{HO}}$ [24]:

$$\hat{I}_{\text{HO}} = \frac{\chi^*(t)\chi(t)}{m}\hat{p}^2 - \dot{\chi}^*(t)\chi(t)\hat{r}\hat{p}$$
$$- \chi^*(t)\dot{\chi}(t)\hat{p}\hat{r} + m\dot{\chi}^*(t)\dot{\chi}(t)\hat{r}^2. \quad (10)$$

One can check, by direct differentiation of Equation (8) with respect to time, that $\hat{I}$ does not vary with time.

Since the eigenstates of the invariant operator play a crucial role in the development of the quantum theory of TDHS, it is necessary to compute them from fundamental relations. Let us write the eigenvalue equation of the invariant operator as

$$\hat{I}\phi(r, t) = \lambda\phi(r, t). \quad (11)$$

We will derive the eigenstates $\phi(r, t)$ by evaluating this equation. The substitution of Equation (8) with Equation (10) into the above equation yields

$$\left[\frac{\partial^2}{\partial r^2} + \frac{m\beta(t)}{i\hbar}r\frac{\partial}{\partial r} - \frac{m^2\dot{\chi}^*\dot{\chi}}{\chi^*\chi\hbar^2}r^2\right.$$
$$\left. - \frac{m^2 K^2 r_0^4}{\hbar^2}\frac{1}{r^2} + \Pi(t)\right]\phi = 0, \quad (12)$$

where $\beta(t)$ and $\Pi(t)$ are time functions of the form

$$\beta(t) = \frac{\dot{\chi}^*(t)}{\chi^*(t)} + \frac{\dot{\chi}(t)}{\chi(t)}, \quad (13)$$

$$\Pi(t) = \frac{m\lambda}{\chi^*(t)\chi(t)\hbar^2} + \frac{m\dot{\chi}(t)}{i\chi(t)\hbar}. \quad (14)$$

Notice that $\beta(t)$ is always real. In case of $\chi(t) = c(t)e^{iy(t)}$ where $c(t)$ and $y(t)$ are time-dependent real values, we have $\beta(t) = 2\dot{c}(t)/c(t)$. On the other hand, for $\chi(t) = c_1 e^{iy(t)} + c_2 e^{-iy(t)}$ where c_1 and c_2 are real constants, $\beta(t)$ becomes

$$\beta(t) = -\frac{4c_1 c_2 \dot{y}(t) \sin[2y(t)]}{c_1^2 + c_2^2 + 2c_1 c_2 \cos[2y(t)]}, \tag{15}$$

which is a more complicated expression. By putting $r = \sqrt{\rho}$ from Equation (12), we can rewrite the eigenvalue equation in the form

$$\frac{\partial^2 \phi(\rho, t)}{\partial \rho^2} + \frac{1}{2}\left(\frac{m\beta}{i\hbar} + \frac{1}{\rho}\right)\frac{\partial \phi(\rho, t)}{\partial \rho}$$
$$+ \left(\frac{\Pi}{4}\frac{1}{\rho} - \frac{m^2 K^2 r_0^4}{4\hbar^2}\frac{1}{\rho^2} - \frac{m^2 \dot{\chi}^* \dot{\chi}}{4\chi^* \chi \hbar^2}\right)\phi(\rho, t) = 0. \tag{16}$$

Now we let

$$\phi(\rho, t) = \rho^s e^{-\gamma(t)\rho} F(\rho, t), \tag{17}$$

where a constant s and a time function $\gamma(t)$ is given by

$$s = \frac{1}{4}\left[1 + \left(1 + \frac{4m^2 K^2 r_0^4}{\hbar^2}\right)^{1/2}\right], \tag{18}$$

$$\gamma(t) = \frac{m}{2\hbar}\left[\frac{\beta(t)}{2i} + \left(\frac{\dot{\chi}^*(t)\dot{\chi}(t)}{\chi^*(t)\chi(t)} - \frac{\beta^2(t)}{4}\right)^{1/2}\right]. \tag{19}$$

Then, the substitution of Equation (17) in Equation (16) leads to

$$\rho\frac{\partial^2 F(\rho, t)}{\partial \rho^2} + \left[2s + \frac{1}{2} + \left(\frac{m\beta}{2i\hbar} - 2\gamma\right)\rho\right]\frac{\partial F(\rho, t)}{\partial \rho}$$
$$+ \left(\frac{m\beta s}{2i\hbar} - 2\gamma s - \frac{\gamma}{2} + \frac{\Pi}{4}\right)F(\rho, t) = 0. \tag{20}$$

We easily derive the solution of this equation to be

$$F(\rho, t) = {}_1F_1\left(s - \left(\frac{\Pi(t)}{4} - \frac{\gamma(t)}{2}\right)\left(2\gamma(t) - \frac{m\beta(t)}{2i\hbar}\right)^{-1},\right.$$
$$\left.2s + \frac{1}{2}; \left(2\gamma(t) - \frac{m\beta(t)}{2i\hbar}\right)\rho\right), \tag{21}$$

where ${}_1F_1$ is the hypergeometric series. Thus, we completely identified the solution $\phi(r, t)$ in Equation (12). After some rearrangements, the full expression of the normalized eigenstates becomes

$$\phi_n(r, t) = \left[\frac{2\Gamma(n+1)}{\Gamma(n+\nu+1)}\left(\frac{m\Omega}{2\hbar\chi(t)\chi^*(t)}\right)^{\nu+1}\right]^{1/2} r^{\nu+1/2}$$
$$\times \exp\left(\frac{im\dot{\chi}(t)}{2\hbar\chi(t)}r^2\right){}_1F_1\left(-n, \nu+1; \frac{m\Omega}{2\hbar\chi(t)\chi^*(t)}r^2\right), \tag{22}$$

where

$$n = \frac{\lambda}{2\hbar\Omega} - \frac{\nu}{2} - \frac{1}{4}, \tag{23}$$

$$\nu = \frac{1}{2}\left(1 + \frac{4m^2 K^2 r_0^4}{\hbar^2}\right)^{1/2}, \tag{24}$$

$$\Omega = i[\chi(t)\dot{\chi}^*(t) - \chi^*(t)\dot{\chi}(t)]. \tag{25}$$

Here, it can be easily shown that n should be quantized numbers ($n = 0, 1, 2, \cdots$) from the condition that the physically allowed eigenstates cannot be divergent as r grows [27]. While it is manifest that ν is independent of time, we can easily verify that the Wronskian Ω is also a time-constant real value. For convenience, we choose $\chi(t)$ in a way that Ω to be positive. This can be always done without loss of generality.

We see from Equation (23) that the eigenvalues are given by

$$\lambda_n = \hbar\Omega(2n + \nu + 1/2). \tag{26}$$

According to the invariant operator theory of Lewis-Riesenfeld [5, 6], the wave functions $\psi_n(r, t)$ that satisfy Schrödinger equation are represented in terms of the eigenstates of the invariant operator. Hence, we can write the Schrödinger solutions in the form

$$\psi_n(r, t) = \phi_n(r, t) \exp[i\varphi_n(t)], \tag{27}$$

where $\varphi_n(t)$ are some time-dependent phases. By inserting the above equation together with Equation (6) into Schrödinger equation, we obtain the analytical forms of $\varphi_n(t)$ such that

$$\varphi_n(t) = -(2n + \nu + 1)\frac{\Omega}{2}\int_0^t \frac{dt'}{\chi(t')\chi^*(t')} + \varphi_n(0). \tag{28}$$

Thus, the complete radial wave functions of the system are identified. These wave functions are very useful for investigating quantum characteristics of the system. Recall that the expectation values of quantum observables are obtained via the use of wave functions.

3.2. SPECTRUM OF QUANTIZED ENERGY
We apply the quantization scheme developed previously to particular cases for better understanding of quantum features of the system. As an appropriate quantum observable that is worth to be investigated here, let us consider the radial part of the quantum energy. As is well known, the expectation values of the quantum energy are obtained from

$$E_n = \langle\psi_n(t)|\hat{H}(\hat{r}, \hat{p}, t)|\psi_n(t)\rangle. \tag{29}$$

With the use of Equation (6) and the wave functions in Equation (27), we readily have

$$E_n = \frac{1}{4}\hbar\left(\frac{\Omega}{\chi(t)\chi^*(t)} + \frac{\chi(t)\chi^*(t)}{\Omega}[\beta^2(t) + 4\omega^2(t)]\right)$$
$$\times (2n + \nu + 1). \tag{30}$$

This is the general expression of nth order quantum energy. The time evolution of quantum energy is determined by the type of $B(t)$.

As an example, we choose a magnetic field that decreases with time in a fashion that

$$B(t) = B_0 \frac{1}{(1+kt)^2}, \tag{31}$$

where B_0 is the initial field and k is a positive constant which is relatively small ($0 < k \ll 1$). If we put $\chi(t)$ as $\chi(t) = \chi_0(1 + kt)z(t)$ where χ_0 is a real constant, we can confirm via the use of Equations (5), (9), and (31) that the differential equation that $z(t)$ should obey is given by

$$\frac{d^2z(t)}{dt^2} + \frac{2k}{1+kt}\frac{dz(t)}{dt} + \frac{q^2 B_0^2}{4m^2(1+kt)^4}z(t) = 0. \tag{32}$$

From a direct evaluation, we see that the solution for $z(t)$ is an exponential function of the form $z(t) = e^{-iqB_0/[2mk(1+kt)]}$. Hence, a complex solution of Equation (9) is given by

$$\chi(t) = \chi_0(1 + kt)\exp\left(-i\frac{qB_0}{2mk}\frac{1}{1+kt}\right). \tag{33}$$

In this case, Equation (30) becomes

$$E_n = \hbar\left[\frac{mk^2}{qB_0} + \frac{qB_0}{2m(1+kt)^2}\right](2n + \nu + 1). \tag{34}$$

While the first term is constant, the second term decreases with time. We see from the above equation that quantum energy is independent of χ_0. In general, the choice of any value for χ_0 does not affect to the time behavior of a quantum system [15]. The time evolution of E_n for this case is plotted in **Figure 1** with various values of k. As the magnetic field gradually disappears with time according to Equation (31), E_n also decay. **Figure 1** shows that E_n decrease more rapidly for large k. If we consider the fact that k determines the rate of the decrease of applied magnetic field, this consequence is natural and corresponds to the classical analysis.

Now, as an another example, let us see the case that the time dependence of the external magnetic field is given by
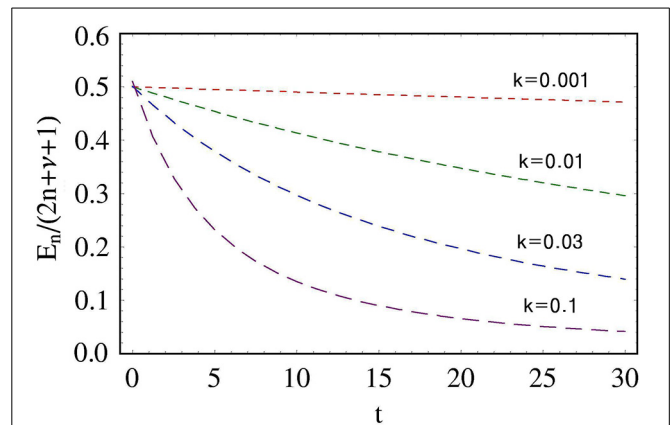
$$B(t) = B_0 e^{kt}. \tag{35}$$

In this case, the magnetic field (exponentially) increases with time whereas the field in the previous case decreases. It is easy to show from a little evaluation that Equation (9) has the form
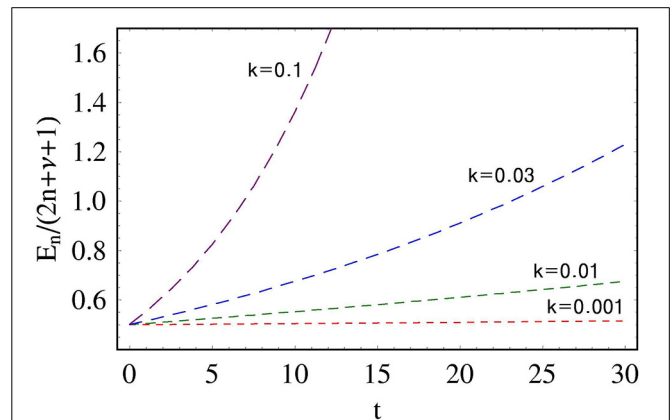
$$\tau^2 \frac{d^2\chi(t)}{d\tau^2} + \tau\frac{d\chi(t)}{d\tau} + \tau^2\chi(t) = 0, \tag{36}$$

where $\tau = qB_0 e^{kt}/(2mk)$. A complex solution for this equation is given by

$$\chi(t) = \chi_0\left[J_0\left(\frac{qB_0 e^{kt}}{2mk}\right) + iN_0\left(\frac{qB_0 e^{kt}}{2mk}\right)\right], \tag{37}$$

FIGURE 1 | Time evolution of the radial energy expectation values divided by ($2n + \nu + 1$) with the choice of $B(t)$ as Equation (31). The values we used are $\hbar = 1$, $m = 1$, $q = 1$, and $B_0 = 1$. All these values are taken to be dimensionless for convenience.

FIGURE 2 | Time evolution of the radial energy expectation values divided by ($2n + \nu + 1$) with the choice of $B(t)$ as Equation (35). The values we used are $\hbar = 1$, $m = 1$, $q = 1$, and $B_0 = 1$. All these values are taken to be dimensionless for convenience.

where J_0 and N_0 are zeroth order Bessel functions. We see from **Figure 2** that the corresponding energy increases with time due to the amplification of the field, as expected. The ratio of energy increase becomes large with time due to the exponential increment of the field.

4. CONCLUSION

Quantum motion of a charged particle in an ionized plasma controlled by a time-dependent external magnetic field is studied using the invariant operator method that is available for TDHSs. If we consider that the time-varying magnetic field produces an electric field that plays the role of an another source of force acting on the moving particle that has some charge, the problem becomes more or less complicated. The radial part of equation of motion for the particle is represented in terms of a time-dependent angular frequency $\omega(t)$ as shown in Equation (4). Hence, the corresponding Hamiltonian given in Equation (6) with Equation (7) is a kind of TDHSs.

To see quantum features of the system, the invariant operator is constructed through the method of Lewis-Riesenfeld [see Equation (8)]. This enabled us to manage the system in more or less simple way by avoiding the direct consideration of the time-dependent problem by means of a constant of motion that is a quadratic form. The normalized radial wave functions derived from the use of the invariant operator are represented as Equation (27) with Equations (22) and (28). An interesting mathematical feature in this case is that the quantum solutions are expressed in terms of the complex classical solutions of Equation (9).

Considering the expression of the phases given in Equation (28), we can also define another invariant operator in the form $\hat{\mathcal{I}} = \hat{I} + \hbar\Omega/2$ which seems a little improved than $\hat{I}$. It is easy to show that the eigenvalue equation of $\hat{\mathcal{I}}$ results in $\hat{\mathcal{I}}\phi(r, t) = \Lambda_n\phi(r, t)$ with

$$\Lambda_n = \lambda_n + \frac{1}{2}\hbar\Omega = \hbar\Omega(2n + \nu + 1). \qquad (38)$$

Although we have used $\hat{I}$ in order to study quantum features of the system, $\hat{\mathcal{I}}$ may be more consistent invariant operator since its eigenvalues are represented in terms of $(2n + \nu + 1)$ which appear in the phases of the wave functions [Equation (28)]. In what follows, it is possible to derive exact quantum states by using either $\hat{I}$ or $\hat{\mathcal{I}}$.

The nth order expectation value of the Hamiltonian is computed by taking advantage of the wave function, as represented in Equation (30). This is the radial part of quantized energy for the particle. To promote the understanding of our development, we considered particular cases characterized by time-dependent magnetic fields appeared in Equations (31) and (35). We confirm from **Figure 1** that E_n for the first example decrease with time as the magnetic field gradually vanishes, whereas, from **Figure 2**, the energy for the second example increases with time as the field grows. These consequences are consistent with the corresponding classical analyses.

All of the results in this work are obtained by treating electromagnetic field as classical backgrounds without incorporating the full quantized Yang-Mill theory. We believe that our theory is valid with high precision so long as we are interested in only the phenomenological quantum behavior of the charged particle, provided that the complex classical solutions χ and χ^* of Equation (9) are found for given types of the time dependence of $B(t)$.

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