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ITERATIVE VERSUS NONITERATIVE DERIVATION OF MOVING AVERAGE PARAMETERS OF ARMA PROCESSES

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Abstract – A noniterative approach to deriving the moving average coefficients of a mixed ARMA process is suggested and compared to iterative methods. Results of a Monte Carlo study indicate that the noniterative method compares favorably to the commonly used iterative procedures.

1. INTRODUCTION

Iterative methods are commonly used to derive the moving average (MA) coefficients of autoregressive moving average (ARMA) processes from autocovariances. Box and Jenkins [1] propose a linearly convergent procedure. A quadratically convergent solution is given in Tunnicliffe Wilson [2]. Here we suggest a noniterative approach, which is derived from a matrical expression given in Mittnik [3] relating the autocovariances of an ARMA process to its parameters, and compare it to the iterative procedures.

2. NONITERATIVE DERIVATION

Let a univariate stationary zero mean nondeterministic time series $\{y_t\}$ be generated by the ARMA(p,q) process

$$a(L)y_t = b(L)\varepsilon_t \qquad (1)$$

where the noise is such that $E(\varepsilon_t)=0$ and $E(\varepsilon_s\varepsilon_t)=\delta_{st}\sigma^2$ and $a(L)$ and $b(L)$ are polynomials in the lag operator L defined by $a(L)=1-a_1L-\ldots-a_rL^r$ and $b(L)=b_0+b_1L+\ldots+b_rL^r$, with $r=\max(p,q)$ and $a_i=0$ for $i=p+1,p+2,\ldots,r$, if $r>p$ or $b_i=0$ for $i=q+1,q+2,\ldots,r$, if $r>q$. Note that either b_0 or σ^2 could be set equal to one. To maintain generality we do not impose such a restriction unless stated otherwise. An analytical expression relating the initial autocovariances of $\{y_t\}$, $\gamma_\tau=E(y_ty_{t-\tau})$ $(\tau=0,1,\ldots,r)$, is derived in Mittnik [3]

$$\gamma = (I-M)^{-1}N(I-M_T)^{-1}b\sigma^2 \tag{2}$$

where $\gamma=(\gamma_0\ \gamma_1\ldots\gamma_r)^T$, $b=(b_0\ b_1\ldots b_r)^T$, and the $(r+1)\times(r+1)$ matrix M is the sum of two matrices, $M=M_H+M_T$

$$M_H = \begin{bmatrix} 0 & H \\ 0 & 0 \end{bmatrix} \qquad M_T = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$$

T is the lower triangular Toeplitz matrix whose first column consists of the autoregressive (AR) coefficients $a_1,\ldots,a_r$; $H=(h_{ij})$ and $N=(n_{ij})$ denote Hankel matrices whose first columns are given by $a_1,\ldots,a_r$ and $b_0,\ldots,b_r$, respectively, and all entries below the main counterdiagonal are zero, i.e. $h_{ij}=0$ for $i+j>r$ and $n_{ij}=0$ for $i+j>r+1$. Note that the stationarity assumption guarantees the existence of $(I-M)^{-1}$ and that $\det(I-M_T)=1$.

Since N is Hankel with zeros below the main counterdiagonal and $(I-M_T)^{-1}$ is a lower triangular Toeplitz matrix, (2) can be rewritten as

$$\gamma = (I-M)^{-1}(I-M_T^T)^{-1}Nb\sigma^2 \tag{3}$$

Consequently, the MA parameters and the noise variance satisfy

$$Nb\sigma^2 = (I-M_T^T)(I-M)\gamma \tag{4}$$

Given the AR coefficients (or their estimates) it is standard practice to derive the MA coefficients in a two step procedure: first, compute the autocovariances of the AR process $a(L)y_t$; second, use the linearly convergent (LC) iterative algorithm in [1] or the quadratically convergent (QC) one in [2] to derive b and σ^2. Equation (4) represents, however, an implicit expression relating b and σ^2 to the autocovariances and the AR parameters of an ARMA process which makes the first step unnecessary.

A noniterative (NI) approach can be adopted to accomplish the second step. Let $\delta_0,\ldots,\delta_q$ denote the first q+1 elements of the vector defined by the RHS of (4). Following Mittnik [4], we can specify polynomials $f_q(b_q)=0$, $f_{q-1}(b_q,b_{q-1})=0$, $\ldots$, $f_1(b_q,\ldots,b_1)=0$ which, depending on the value of q, can be solved either analytically or numerically in the order listed. Note that we may set $b_0=1$ and have $\sigma^2=\delta_q/b_q$. The polynomial coefficients are functions of δ_i, $i=0,1,\ldots q$ (for details see Mittnik [4]). In practical estimation problems q is usually small. All ARMA models that Box and Jenkins ([1], p. 293), for example, estimate are such that $q\le 2$.

The set of all possible solutions satisfying (4) reflects the multiplicity of the MA parameters giving rise to the same autocovariance generating function. For a given autocovariance function there will be one solution yielding an invertible MA polynomial, which is the one modellers commonly retain.

3. COMPARISON

To illustrate the usefulness of the noniterative method, we derive theoretical as well as sample autocovariances for the ARMA(1,1) process used in [1] (p. 204) with parameters $a_1=0.8$, $b_0=1.0$, $b_1-0.5$, $\sigma^2=1.0$. The first theoretical autocovariances are $\gamma_0=1.25$, $\gamma_1=0.5$, $\gamma_2=0.4$. The AR coefficient is recovered by using the modified Yule-Walker equation, $a_1=\gamma_2/\gamma_1=0.8$. Applying (4) amounts to solving $b_1^2+2.5b_1+1=0$ and $\sigma^2=-0.5/b_1$, yielding $b_1=-1.25\pm0.75$. The solution with the invertible MA polynomial is $b_1=-0.5$ and $\sigma^2=1$. The LC method requires 8 and the QC method requires 4 iterations to calculate b_1 and σ^2 ([1], pp. 204-5), the NI approach provides the solution directly by solving a quadratic equation.

To compare the efficiency of the estimates obtained by the three approaches, a Monte Carlo study, using the above ARMA(1,1) parameters, was conducted. One hundred replications, each consisting of 300 observations, were generated. For each series $\hat{\gamma}_0$, $\hat{\gamma}_1$, and $\hat{\gamma}_2$ were calculated. The estimates $\hat{a}_1$, $\hat{b}_1$, and $\hat{\sigma}^2$ were computed as described in the deterministic case. The fact that the NI method does not suffer from convergence problems proved to be useful in this investigation. Out of the 100 replications convergence failed in one case when using the LC method and in four cases using the QC method. The NI method yielded twice complex values for b_1. Since in most practical applications real ARMA coefficients are assumed, only the real parts were retained in these cases.

Table 1 summarizes the results of the study in terms of the means and medians of the estimates $\hat{b}_1$ and $\hat{\sigma}^2$. The cases that did not converge were omitted when computing the statistics for the iterative methods. For both estimates the NI method performs best in terms of the mean and standard deviation, while the QC method performs worst. In terms of the median values the three methods do not differ substantially. For b_1 the LC procedure does best, while the NI approach yields the best result for σ^2. Table 2 compares the mean squared errors (MSEs) of the estimates given by $MSE=bias^2+variance$. Apart from the computational simplification, for both parameters the NI method provides more efficient estimates than the iterative approaches do.

Table 1: Comparison of Estimation Approaches

Method	b_1=-0.5		σ^2=1.0	
	mean[a]	median[a]	mean	median
LC[b]	-0.3669	-0.5079	0.9531	0.9549
	(0.2251)	(0.2577)	(0.0828)	(0.1133)
QC[c]	-0.5355	-0.5144	0.9371	0.9460
	(0.2713)	(0.2324)	(0.1306)	(0.1233)
NI[d]	-0.5147	-0.5145	0.9690	0.9588
	(0.1947)	(0.2262)	(0.0824)	(0.1186)

[a] Numbers in parentheses below the means and medians represent standard deviations and interquartile ranges, respectively.

[b] Based on 99 replications, convergence failed in one case.

[c] Based on 96 replications, convergence failed in four cases.

[d] Based on 100 replications, complex $\hat{b}_1$ occurred in two cases.

Table 2: Mean Squared Errors of Estimates

Parameter	LC	QC	NI
b_1	8.141[a]	7.486	3.812
σ^2	0.906	2.101	0.775

[a] All entries are multiplied by 100.

REFERENCES

[1] Box, G.E.P. and G.M. Jenkins, *Time Series Analysis Forecasting and Control*, Holden-Day, 1976.

[2] Tunnicliffe Wilson, G., Factorization of the Covariance Generating Function of a Pure Moving Average Process, *SIAM Journal of Numerical Analysis*, 6, 1-7, 1969.

[3] Mittnik, S., Derivation of Theoretical Autocovariance Sequences of ARMA Processes, submitted, 1987.

[4] Mittnik, S., Noniterative Estimation of Moving Average Parameters of ARMA Processes, Technical Report, 1987.