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Cop-win graphs with maximum capture-time

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ABSTRACT

We present an upper bound $n-4$ for the maximum length of a cop and robber game (the *capture-time*) on a cop-win graph of order n. This bound matches the known lower bound. We analyze the structure of the class of all graphs attaining this maximum and describe an inductive construction of the entire class.

A *cop and robber game* is a two-player vertex-to-vertex pursuit combinatorial game where the players stand on the vertices of a graph and alternate in moving to adjacent vertices. Cop's goal is to capture the robber by occupying the same vertex as the robber, robber's goal is to avoid capture.

1. Preliminaries

In this article we consider only undirected simple loopless finite graphs. A graph is a pair (V, E) of a set of vertices and a set of edges, respectively, with $E \subseteq \binom{V}{2}$. We use $N_G(v)$ to denote the *neighborhood* of a vertex v in G, that is, the set of all the vertices sharing an edge with v. The *closed neighborhood* of a vertex v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. The *degree* of v denoted by $\deg_G(v)$ is defined as $|N_G(v)|$. We write $\Delta(G)$ for the maximum among the degrees of vertices of G.

A *path* on n vertices, denoted P_n, is a graph on n vertices labeled $v_1, \ldots, v_n$ with edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. A *cycle* on n vertices, denoted C_n, is a graph P_n with an additional edge between the endpoints of the path. *Distance* of u and v, or $\mathrm{dist}_G(u, v)$ is the length of a shortest path connecting u and v. When e is an edge of a graph G, $G - e$ denotes the graph obtained from G by removing e. When v is a vertex of a graph G, $G - v$ denotes the graph G obtained from G by removing v and all the edges containing v.

Graph G' is a *subgraph* of (or *contained* in) G if $V'_G \subseteq V_G$ and $E'_G \subseteq E_G$. Graph G' is an *induced subgraph* of G if G' is a subgraph of G and contains all the possible edges. A *component* C of a graph G is an inclusion-maximal subgraph such that for every two vertices u and v in C, C contains a path from u to v. Graph G is connected if it has only one component. A *bridge* is an edge of graph G whose removal increases the number of components of the graph. The *ratio* of a bridge is the ratio of the orders of the two components containing the endpoints of the bridge after its removal.

We say that a vertex v of G is *majorized* by a vertex u if $N_G[v] \subseteq N_G[u]$ and $u \neq v$, in other words u and v are adjacent and every neighbor of v is also a neighbor of u. The set of the vertices majorizing a vertex v in G is denoted by $\mathrm{Maj}_G(v)$. Any $u \in \mathrm{Maj}_G(v)$ is called a *major* of v.

Where no confusion can arise, we drop the subscript G indicating the graph.

2. The game

Given a graph G, the *cop and robber game* on G is a game for two players—the cop and the robber. First, the cop selects his starting vertex, then the robber selects hers, and then they alternate in moving along the edges of G. In their turn, both

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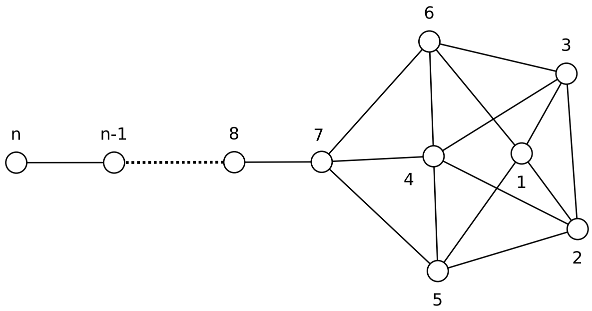

Fig. 1. Graph H_n.

players can either move to some adjacent vertex or stay at their current vertex. They both see each other and have complete information about the game. The cop wins if at some time he shares a vertex with the robber, the robber wins if she is never captured.

The described combinatorial game on a given graph G has a winning strategy for one of the players by von Neumann's theorem. If that player is the cop, we call the graph *cop-win*, otherwise we call it *robber-win*. For detailed information about the existence of winning strategies in various combinatorial games with complete information we recommend the general paper by Banaschewski and Pultr [2]. Note that every disconnected graph is clearly robber-win and therefore we consider only connected graphs from now on.

The *capture-time*[1] $\mathtt{ct}(G)$ is the smallest number of cop's moves (excluding the initial placement) always sufficient to capture the robber (regardless of her strategy). The maximum possible capture-time among all cop-win graphs on n vertices is denoted by $\mathtt{ct}_{\mathtt{max}}(n)$.

This game was first proposed and the cop-win graphs were characterized independently by Nowakowski and Winkler [7] and Quilliot [8,9]. A polynomial algorithm determining whether a given graph is cop-win and also determining its capture-time follows from the paper of Nowakowski and Winkler.

Later, many similar games were proposed, adding more cops, allowing the cops to fly, making the robber invisible, faster, or lazy, allowing to stay at edges, and many more. Some of these games have very nice properties and some relate to interesting graph parameters such as *tree-width*, *path-width*, etc. See papers [1,4] for extensive surveys of the topic.

3. Main results

In their paper, Bonato et al. [3] consider the capture-time of finite and infinite graphs and also prove some upper bounds for the capture-time both in general and for various graph classes. They proved that on every cop-win graph on n vertices, the cop wins in at most $n-3$ steps, and they constructed an example showing that $n-4$ steps are needed. Here we close the gap by improving the upper bound on $\mathtt{ct}_{\mathtt{max}}(n)$ to $n-4$ for $n \geq 7$.

Theorem 1. *For $n \geq 7$, we have $\mathtt{ct}_{\mathtt{max}}(n) = n-4$, and for $n \leq 7$ we have $\mathtt{ct}_{\mathtt{max}}(n) = \lfloor \frac{n}{2} \rfloor$.*

Let $\mathcal{M}$ be the class of the graphs with the maximum capture-time among all the cop-win graphs of the same order. The graphs in $\mathcal{M}$ have a nice recursive structure that allows for a complete characterization. We shall describe the structure after the proof of the main theorem.

First we define the graph H_n for each $n > 7$ to be the graph on n vertices obtained from the graph H_7 by attaching a path on $n-7$ vertices to vertex 7 of H_7; see Fig. 1.

The following theorem describes the structure of the graphs in $\mathcal{M}$ on at least 8 vertices. These conditions are not only necessary but also sufficient.

Theorem 2. *A graph G on $n \geq 8$ vertices G belongs to $\mathcal{M}$ if and only if these conditions hold:*

- *The graph G contains H_n as a subgraph. The subgraph is unique up to automorphisms of H_n. The H_7 contained in H_n is an induced subgraph of G.*
- *The graph G has exactly one majorized vertex v, and the graph $G - v$ belongs to $\mathcal{M}$. The graph $G - v$ has exactly one majorized vertex u, and the vertex v is adjacent to u but to no vertex of $\mathrm{Maj}_{G-v}(u)$.*

These conditions lead to an efficient inductive construction of the entire class $\mathcal{M}$. Algorithm 1 in Section 5.1 describes this construction.

[1] The capture-time is sometimes called the *search-time* or the *cop-time*.

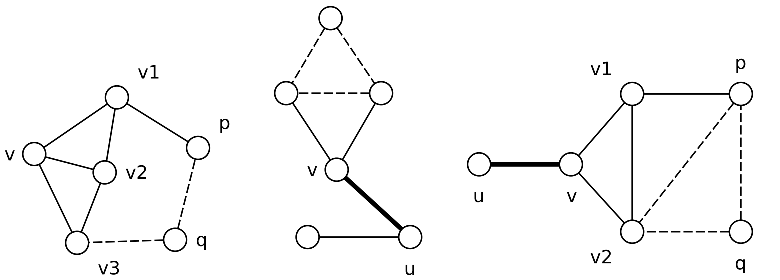

Fig. 2. The cases of $\Delta(G) = 3$ and no bridge, a bridge with the ratio 2:4, and a bridge with the ratio 1:5.

4. Cop-win graphs and capture-time

Before we prove Theorem 1, let us state a simple characterization of cop-win graphs.

Lemma 3 (*[7], [3, Theorem 1]*). *A graph G is cop-win if and only if it is either a single vertex or is obtained from a cop-win graph G' by attaching one new vertex v such that v is majorized in G by some vertex from G'.*
In that case, $\mathtt{ct}(G) \le \mathtt{ct}(G') + 1$.

Note that Lemma 3 and the fact that $\mathtt{ct}(1) = 0$ immediately give the upper bound $\mathtt{ct}_{\mathtt{max}}(n) \le n-1$. A similar observation was used in the paper by Bonato et al. [3].

As a technical tool to be used later in case analysis, we prove a local property of all cop-win graphs.

Lemma 4 (*Triangle Lemma*). *If G is a cop-win graph, then each edge uv of G is either a bridge or u and v have a common neighbor w. Such a vertex w is said to form a triangle over uv.*

Proof. If G has a non-bridge edge uv such that no C_3 in G contains uv, then there is a shortest path $P = v_0v_1 \cdots v_{k-1}$ with $u = v_0$ and $v = v_{k-1}$ in $G - uv$. The edge set $P \cup uv$ forms a cycle C in G of the length $k \ge 4$.

We construct a winning strategy for the robber:

Let $R(w) = (\mathrm{dist}_{G-uv}(w, v_0) + 2) \mod k$. Note that the distance is measured in the graph without uv, so $R(v_j) = (j + 2) \mod k$. Whenever the cop is on a vertex c, the robber should move to $r = v_{R(c)}$ on C, where she cannot be attacked, because r is in a distance at least 2 from c. When the cop moves to c', his distance from v_0 changes at most by 1, so the robber can always move to an appropriate vertex on C. When the cop moves from the vertex c with $R(c) = k - 2$ to c' with $R(c') = k - 1$, the robber should move across uv from v_{k-1} to v_0. When the cop decides to start at c_0, the robber starts at $v_{R(c_0)}$. □

4.1. Lower bound

The lower bound follows from the following lemmas and the observation in Lemma 3.

Lemma 5. *We have* $\mathtt{ct}_{\mathtt{max}}(7) = 3$.

Proof. By Lemmas 4 and 3, every cop-win graph on 7 vertices with $\mathtt{ct}(G) \ge 4$ must be a P_6 extended by attaching a new majorized vertex v to some interval of P_6. It is straightforward to check that all such graphs have $\mathtt{ct}(G) \le 3$ and therefore $\mathtt{ct}_{\mathtt{max}}(7) \le 3$.

On the other hand, a simple robber's strategy gives $\mathtt{ct}(P_7) = 3$. □

Lemma 6. *The only cop-win graph on* 6 *vertices with* $\mathtt{ct}(G) \ge 3$ *is* P_6.

Proof. We analyze all connected cop-win graphs on 6 vertices separately according to their maximum degree. We estimate capture-time of every case and also give some of the optimal starting vertices for the cop to prove the low capture-time of the case. Dashed edges in the illustrative figures represent possible edges.

- The analysis of graphs with $\Delta(G) \le 2$ is trivial.
- A connected graph G with $\Delta(G) = 2$ is either a path P_6 or a cycle C_6. Cycle C_6 is not a cop-win graph and $\mathtt{ct}(P_6) = 3$.
- If G with $\Delta(G) = 3$ has no bridge, then every edge has to be in some triangle by Lemma 4. Select an arbitrary v with $\deg(v) = 3$ and denote the neighbors of v by v_1, v_2 and v_3. The edge v_1v must be in a triangle, but v can have no more neighbors. Therefore v_1 is adjacent to, say, v_2. A similar argument shows that v_3 must be adjacent to v_1 or v_2, without loss of generality suppose it is adjacent to v_2. The situation with the remaining vertices p, q is drawn in Fig. 2. From the picture it is obvious that no such cop-win graph exists.

If G with $\Delta(G) = 3$ has a bridge that divides the vertices in the ratio 3:3, G clearly has $\mathtt{ct} = 2$, the cop can start at either endpoint of the bridge. If G has no bridge with the ratio 3:3, but has a bridge uv with the ratio 2:4 with 2 vertices on the

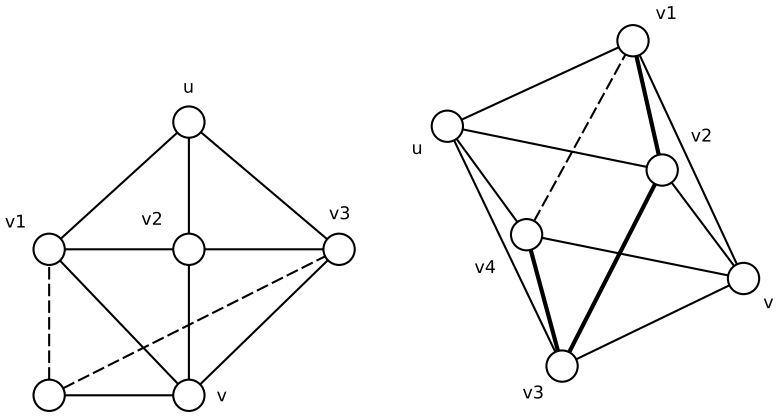

Fig. 3. Cases of $\Delta(G) = 4$ with $\deg(u) = 3$ and $\deg(u) = 4$.

side of u, then v must have $\deg(v) = 3$, otherwise G would have a bridge with the ratio 3:3. The situation is drawn in Fig. 2. Whichever of the remaining possible edges exist, $\mathtt{ct}(G) = 2$ with the cop starting at v.

If G with $\Delta(G) = 3$ has a bridge uv with the ratio 1:5 with a single vertex on the side of u, but has no bridges with the ratios 2:4 or 3:3 then $\deg(v) = 3$. Denote the other neighbors of v by v_1 and v_2. The vertices v_1 and v_2 must be adjacent (otherwise there would be ridges with forbidden ratios). The situation is drawn in Fig. 2. There are several possibilities how the remaining vertices p and q can be connected to v_1 and v_2, but all the possible graphs are either robber-win or have $\mathtt{ct}(G) = 2$ with the cop starting at v_1 or v_2.

• In G with $\Delta(G) = 4$, denote by v one vertex with $\deg(v) = 4$ and by u the single vertex nonadjacent to v. If u has one or two neighbors, then it is easy to check that $\mathtt{ct}(G) = 2$ with the cop starting at v. If u has three neighbors v_1, v_2 and v_3, then there must be at least two edges induced by $\{v_1, v_2, v_3\}$ to form triangles over uv_1, uv_2, and uv_3. This is illustrated in Fig. 3 and in this case we have $\mathtt{ct}(G) = 2$ with the cop starting at v.

In the last case, when u has four neighbors v_1, v_2, v_3 and v_4, there must be at least two edges e_1 and e_2 among the neighbors of u to form triangles over uv_1, uv_2, uv_3, and uv_4. If e_1 and e_2 share an endpoint, there must be an additional e_3 forming P_4 (together with e_1 and e_2) on the set $\{v_1, v_2, v_3, v_4\}$. If e_1 and e_2 do not share an endpoint, then there must be another edge connecting the endpoints of e_1 and e_2, otherwise G would be robber-win.

Both these situations necessarily lead to a graph isomorphic to that in Fig. 3, where a concrete situation with the path $v_1v_2v_3v_4$ is drawn. If v_1v_4 is an edge, the graph is the net of a regular octahedron, which is robber-win. If v_1v_4 is not an edge, then the cop wins in 2 moves by starting at v_2 or v_3 and therefore $\mathtt{ct}(G) = 2$.

• A graph G with $\Delta(G) = 5$ has $\mathtt{ct}(G) = 1$ because the cop can start in a vertex with the maximum degree and capture the robber with his first move.

This shows that every cop-win graph G on 6 vertices except P_6 has $\mathtt{ct}(G) \le 2$. The fact that $\mathtt{ct}(P_6) = 3$ finishes the proof. □

4.2. Upper bound

Bonato et al. [3] show an explicit construction giving $\mathtt{ct}_{\mathtt{max}}(n) \ge n - 4$ for $n \ge 7$. Here we show a simpler construction with the minimum number of edges. We use the graphs H_n defined in Section 3 in Fig. 1.

Lemma 7. *We have* $\mathtt{ct}(H_n) = n - 4$ *for all* $n \ge 7$.

Proof. We sketch both cop's and robber's time-optimal strategies. The cop's optimal strategy is to start at vertex 2 (or at 3, symmetrically). In this situation the robber may start only at vertices $6, 7, \ldots, n$ to avoid immediate capture. A simple analysis shows that a robber starting at 6 is caught in at most $n - 4$ moves.

The only place where cop can corner and capture a clever robber is the end of the tail, as there is no other majorized vertex. Therefore it suffices to show that the robber can safely play at least one move in H_7 before entering vertex 7 and then the tail. But wherever the cop starts, the robber always has an option to start at some vertex in H_7 other than 7.

These two strategies show that $\mathtt{ct}(H_n) = n - 4$. □

Now we are ready to prove Theorem 1 about the exact value of $\mathtt{ct}_{\mathtt{max}}$.

Proof of Theorem 1. The proof consists of an analysis of all cop-win graphs on at most six vertices. First we note that $\mathtt{ct}_{\mathtt{max}}(1) = 0$, $\mathtt{ct}_{\mathtt{max}}(2) = \mathtt{ct}_{\mathtt{max}}(3) = 1$ and $\mathtt{ct}_{\mathtt{max}}(4) = \mathtt{ct}_{\mathtt{max}}(5) = 2$ by a simple analysis of all small cop-win graphs. We omit the analysis of the cases up to four vertices.

Cop-win graphs on 5 vertices are easily analyzed according to their maximum degree. The only interesting case is $\Delta(G) = 3$. In that case the cop starts at any v with $\deg(v) = 3$ and robber must start at the only safe vertex nonadjacent to v. A simple case analysis using Triangle lemma shows that either $\mathtt{ct}(G) \leq 2$ or G is robber-win.

Lemmas 5 and 6 give $\mathtt{ct}_{\mathtt{max}}(6) = \mathtt{ct}_{\mathtt{max}}(7) = 3$. It follows from Lemma 3 that $\mathtt{ct}_{\mathtt{max}}(n) \leq n - 4$ for all $n \geq 7$.

The graphs P_n (for $n \leq 6$) and H_n (for $n \geq 7$) show that the bound is tight. Note that $\mathtt{ct}(P_n) = \lfloor \frac{n}{2} \rfloor$ and $\mathtt{ct}(H_n) = n - 4$ by Lemma 7. □

5. Graphs with the maximum capture-time

In this section we examine the results about the structure of the class $\mathcal{M}$. We start with some auxiliary observations. Fix a graph G in $\mathcal{M}$.

Lemma 8. *For each majorized vertex v, the vertices $\mathrm{Maj}_G(v)$ induce a complete subgraph of G. For any $u \in N[v]$, the vertices $\mathrm{Maj}_G(v) \cup \{u\}$ also induce a complete subgraph.*

Proof. The observation follows from the definition of majorization. Every neighbor of v must be adjacent to the vertices majorizing v. □

Lemma 9. *Let G be a cop-win graph. If the cop chooses one major vertex m_i for every majorized vertex u_i such that $u_k \neq m_l$ for all k, l, then he has a strategy that either captures the robber in fewer than $\mathtt{ct}(G)$ moves, or such that after $\mathtt{ct}(G) - 1$ moves the cop occupies some m_i with the robber on u_i (robber's turn).*

Proof. Let G' be the graph G without all the majorized vertices u_i. Graph G' is cop-win by Lemma 3. Note that every majorized vertex of G' was not majorized in G.

Observe that $\mathtt{ct}(G') = \mathtt{ct}(G) - 1$. Otherwise the robber would have a strategy to evade the cop for $\mathtt{ct}(G) - 1$ moves in G'. She could also use that strategy in G (playing as if the cop was on m_i whenever he is on u_i) to evade the cop for $\mathtt{ct}(G) - 1$ moves. After $\mathtt{ct}(G) - 1$ moves, the robber would be still in G' and therefore also in a vertex not dominated in G. This would allow her to move to a safe vertex (outside G'), showing $\mathtt{ct}(G) \geq \mathtt{ct}(G) + 1$, a contradiction.

The strategy for the lemma is the following: the cop plays using an optimal strategy on G'; if the robber is at u_i, then the cop plays as if the robber was at m_i. If the robber is still free after $\mathtt{ct}(G) - 1$ cop's moves, she must stand at some u_i with the cop at m_i (remember that he would capture her in at most $\mathtt{ct}(G) - 1$ moves in G'). □

Theorem 2 is a combination of the following lemmas:

Lemma 10. *The only cop-win graph on 7 vertices extendible (by adding a majorized vertex) to a graph $G \in \mathcal{M}$ on 8 vertices with $\mathtt{ct}(G) \geq 4$ is H_7 with $\mathtt{ct}(H_7) = 3$.*

Proof. This has been proved by a computer-based examination of all the non-isomorphic cop-win graphs on at most 8 vertices. We used Nauty software [6] to generate the non-isomorphic graphs of order at most 8 and a minor modification of the algorithm proposed in the paper by Hahn et al. [5] to recognize the cop-win graphs and to calculate their capture-time. The program searches through the state space of the game for every graph generated by Nauty.

The author will gladly provide further details and the source code of the program to the interested reader. □

The structural results about the unique majorized vertex are proved by induction from the lemma above. The inductive step itself does not depend on the exhaustive search, however the base does.

Lemma 11. *Every graph $G \in \mathcal{M}$ on at least 8 vertices has exactly one majorized vertex v_1. Furthermore, this vertex v_1 is adjacent to the unique vertex v_2 majorized in $G' = G - v_1$ and is nonadjacent to every vertex of $\mathrm{Maj}_{G'}(v_2)$.*

Proof. The proof is by induction on $|V_G|$. The verification of the theorem for all graphs of order 8 is a byproduct of the computer-based examination of cop-win graphs on at most 8 vertices described in the proof of Lemma 10. Note that a verification by hand is possible using an approach similar to that of Algorithm 1 below.

Every cop-win graph contains at least one majorized vertex. If G contains only one majorized vertex v_1, then we denote by v_2 the majorized vertex in the graph $G' = G - v_1$. In this case, the vertex v_2 is well defined, as G' has exactly one majorized vertex by the induction hypothesis.

If there is a $G \in \mathcal{M}$ with two majorized vertices v_1 and v_2, let G be a smallest one. Graph G has $|V_G| = n \leq 9$. At least one of the graphs $G - v_1$ and $G - v_2$ has to be in $\mathcal{M}$ (otherwise G could not have $\mathtt{ct}(G) = n - 4$ by Lemma 3); without loss of generality suppose that this is $G' = G - v_1$. By the induction hypothesis $\mathtt{ct}(G') = n - 5$, and G' has exactly one majorized vertex v_2.

We use the notation v_1, v_2 and G' for both the cases, the only difference being that v_2 may or may be not majorized in G. Note that v_2 is always majorized in G' and $\mathrm{Maj}_{G'}(v_2) \subseteq \mathrm{Maj}_G(v_2) \cup \{v_1\}$. The graph $G \in \mathcal{M}$ clearly cannot have more than two majorized vertices. We consider two cases.

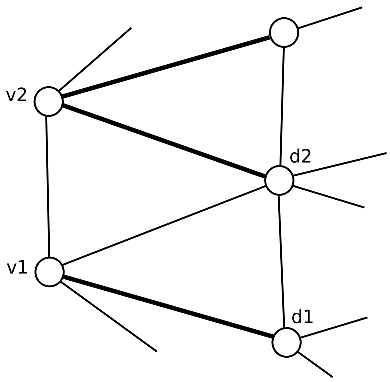

Fig. 4. v_1 is adjacent to $m_2 \in \mathrm{Maj}_{G'}(v_2)$.

If v_1 and v_2 are nonadjacent or v_1 is adjacent to some $m_2 \in \mathrm{Maj}_{G'}(v_2)$, then, by Lemma 9, the cop can use an optimal strategy for G' preferring to capture the robber at v_2 from m_2. When the robber is on v_1, play as if she were on some (fixed) vertex m_1 majorizing v_1 in G.

Using this strategy, in $n - 5$ moves cop either captures the robber, or his strategy makes him move to m_1 with robber at v_1 (his strategy does not distinguish these two vertices), or to m_2 with the robber at v_2. But in each case the robber cannot escape from cop's neighborhood, and the cop will capture her in his next move. The latter situation is illustrated in Fig. 4. Note that m_1 can be equal to m_2.

Both these situations imply that $\mathtt{ct}(G) = \mathtt{ct}(G')$, a contradiction.

If u and v are adjacent and v_1 is not adjacent to any vertex of $\mathrm{Maj}_{G'}(v_2)$, then the vertex v_2 is not majorized in G by any vertex, as it has a neighbor v_1 not adjacent to any vertex of $\mathrm{Maj}_{G'} v_2$ and v_1 also does not majorize v_2. □

Note that Lemma 11 does not hold for graphs on 7 vertices. The graph H_7 has one majorized vertex 7, but in $H_7 - 7$ vertices 5 and 6 are majorized.

Lemma 12. *Every graph $G \in \mathcal{M}$ on at least 8 vertices contains H_7 as an induced subgraph and H_n as a subgraph. The embedding of H_n is uniquely determined (up to symmetry of H_7).*

Proof. From Lemmas 10 and 11 it follows that the only graphs on 8 vertices with $\mathtt{ct}(G) \geq 4$ are those obtained from H_7 by the addition of a majorized vertex v to H_7 such that v is adjacent to vertex 7. All the other graphs in the class are constructed from these by adding vertices and edges from the new vertices, so they also contain H_7 as an induced subgraph and the new vertices form a (generally non-induced) path from vertex 7.

The numbering of vertices by a repeated removal of a majorized vertex with numbers $n \cdots 7$ is unique (as there is always only a single majorized vertex), the remaining 7 vertices are isomorphic to an H_7 in one of two possible ways. This determines at most two embeddings of H_n in G. □

Lemma 13. *If we extend $G' \in \mathcal{M}$ with majorized vertex v_2 by a majorized vertex v_1 adjacent to v_2 and nonadjacent to* $\mathrm{Maj}_{G'}(v_2)$, *then the resulting graph G is also in $\mathcal{M}$.*

Proof. In G', the robber has an optimal strategy where cop captures her at v_2 after she passed her last move. She can use this strategy also in G: when the cop is at v_1, she plays as if he was at some m_1 majorizing v_1. If the cop plays optimally, then this results in the robber standing at v_2 and the cop at some $m_2 \in \mathrm{Maj}_{G'}(v_2)$, but instead of passing her move, she moves to v_1, which is nonadjacent to m_2 and therefore prolongs the game by at least one move. □

Proof of Theorem 2. The above lemmas show the proposed properties of graphs in $\mathcal{M}$ on at least 8 vertices.

Sufficiency of the properties follows by induction on number of vertices by Lemma 13. To prove the base of the induction (for $n = 8$) we use the fact that G contains an embedding of H_8 and the only extra edges lead from the vertex 8. This gives us only 10 graphs, each one with $\mathtt{ct}(G) = 4$ and therefore belonging to $\mathcal{M}$. □

5.1. The construction of $\mathcal{M}$

A straightforward algorithm for an efficient construction of the entire class $\mathcal{M}$ directly follows from Theorem 2. The algorithm is efficient in the sense that every graph in $\mathcal{M}$ is generated at most twice (because of the symmetry of H_7).

Algorithm 1. Each graph in $\mathcal{M}$ on at least 8 vertices is generated by the following algorithm (with appropriate choices of N_{i+1}):

- Start with $G_0 = H_7$ and $u_0 = 7$. Vertex u_i will always be majorized in G_i.
- Repeat for $i = 1, 2, \ldots$.

– Let G_{i+1} be G_i with a new vertex u_{i+1}.
– Select a neighborhood N_{i+1} of u_{i+1} in any way so that u_{i+1} is majorized in G_{i+1}, $u_i \in N_{i+1}$ and $\mathrm{Maj}_{G_i}(u_i) \cap N_{i+1} = \emptyset$.
– Connect u_{i+1} to the vertices N_{i+1}.

Each graph in $\mathcal{M}$ is generated in at most two ways by this algorithm.

Proof. By generating the graphs as described in the algorithm we generate only graphs belonging to $\mathcal{M}$, by Lemma 13.

Suppose that not all the graphs on at least 8 vertices in $\mathcal{M}$ are generated, and the smallest graph omitted is $G \in \mathcal{M}$. By Lemma 10 the graph G has at least 9 vertices. From Theorem 2 it follows that G has a single majorized vertex v. The graph $G - v$ is in $\mathcal{M}$ and has at least 8 vertices, so the algorithm generated it. The contradiction follows from the fact that $N(v)$ satisfies the conditions of the algorithm as a neighborhood for v, so G is generated from $G - v$.

The construction is efficient, since for a fixed target graph G the choices of all the N_i are uniquely determined (up to the symmetry of H_7) by Lemma 12. □

Note that the class $\mathcal{M}$ is very rich, namely it contains at least 2^{n-8} non-isomorphic graphs on n vertices for $n \geq 8$. The simple proof goes by an inductive construction and is omitted due to space limitations.

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