Degrees of irreducible morphisms in standard components

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We study the left degree of an irreducible morphism \( f : X \to \bigoplus_{i=1}^{r} Y_i \) with \( X \) and \( Y_i \) indecomposable modules in a standard component of the Auslander–Reiten quiver, for \( 1 \leq i \leq r \). Two criteria to determine whether the left degree of these irreducible morphisms is finite or infinite are given, for standard algebras. We also study which of them has left degree two.

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The notion of left and right degree of an irreducible morphism was introduced by Liu, in [14]. Using this concept, he succeeded in describing the shape of the components of the Auslander–Reiten quiver for algebras of infinite representation type.

This notion has been a fundamental tool in the study of irreducible morphisms. It allows us to solve the problem of finding necessary and sufficient conditions for the existence of \( n \) irreducible morphisms between indecomposable modules, with a non-zero composite lying in \( \mathcal{N}^{n+1} \) (see [6–8,10]). It has also played an important role in describing the shape of the Auslander–Reiten components with sectional bypasses (see [1]). Because of its close connection with the above mentioned problems and since it has shown to be a very helpful tool, we are interested in developing the degree theory, here.

Throughout this paper, unless otherwise stated, all algebras are going to be finite dimensional algebras over an algebraically closed field. Furthermore, we will assume that all algebras are basic.

In [9], the degree of irreducible morphisms between indecomposable modules in generalized standard convex components with length of the Auslander–Reiten quiver \( \Gamma_A \) of an artin algebra \( A \), has been studied. Two criteria to determine whether the degree of an irreducible morphism between indecomposable modules is finite or infinite, were given. In this work, we are going to study the irreducible morphisms \( f : X \to \bigoplus_{i=1}^{r} Y_i \), with \( X, Y_i \), indecomposable modules, for \( 1 \leq i \leq r \), in a standard component of \( \Gamma_A \), not necessarily with length. Two similar characterizations such as those stated in [9], to determine the left degree of such an irreducible morphism \( f : X \to \bigoplus_{i=1}^{r} Y_i \), with \( X, Y_i \), indecomposable modules, for \( 1 \leq i \leq r \), in a standard algebra, are given. We will state the results for left degrees since the ones concerning right degrees follow by duality.

Actually, our first result states the following:

**Theorem A.** Let \( \Gamma \) be a standard component of \( \Gamma_A \) and \( f : X \to \bigoplus_{i=1}^{r} Y_i \) an irreducible morphism with \( X, Y_i \in \Gamma \), for \( 1 \leq i \leq r \). The following conditions are equivalent:

1. \( d_l(f) \) is finite.
2. There exists a module \( M \in \Gamma \), an irreducible morphism \( h : X \to \bigoplus_{i=1}^{r} Y_i \) and a non-zero morphism \( \varphi : M \to X \) such that \( h \varphi = 0 \).

As an immediate consequence of **Theorem A**, we will get that an injective left minimal almost split morphism between modules over a standard component is of infinite left degree. We will also prove that for a directed standard component \( \Gamma' \) of
Let $A$ be a standard algebra and $f : X \to \bigoplus_{i=1}^r Y_i$ an irreducible morphism with $X, Y_i \in \Gamma_A$ for $1 \leq i \leq r$. Then, the following conditions are equivalent:

(a) $d_i(f)$ is finite.
(b) There exists a module $M \in \Gamma_A$ and a non-zero morphism $\varphi : M \to X$ such that $f\varphi = 0$.

The first characterization for a standard algebra $A$ is given in terms of the existence of a non-zero morphism between modules in $\Gamma_A$, such that the composite with $f$ is zero. Precisely, we will prove:

**Theorem B.** Let $A$ be a standard algebra and let $f : X \to \bigoplus_{i=1}^r Y_i$ an irreducible morphism with $X, Y_i \in \Gamma_A$ for $1 \leq i \leq r$. Then, the following conditions are equivalent:

Let $\Gamma$ be a standard component of $\Gamma_A$ and let $f : X \to \bigoplus_{i=1}^r Y_i$ be an irreducible morphism with $X, Y_i \in \Gamma$, for $1 \leq i \leq r$. Then $d_i(f)$ is finite if and only if $\text{Ker}(f) \supseteq \Gamma_A$.

Finally, we will study the irreducible morphisms of left degree two and we will give a characterization for irreducible morphisms in standard components of $\Gamma_A$. Precisely, we will prove:

**Theorem D.** Let $\Gamma$ be a standard component of $\Gamma_A$ and $f : X \to \bigoplus_{i=1}^r Y_i$ be an irreducible morphism with $X, Y_i \in \Gamma$. Then $d_i(f) = 2$ if and only if $r \leq 2$, $f$ is not a surjective right minimal almost split morphism and there exists $M \in \Gamma_A$ and a path $\varphi \in \mathcal{H}(M, X) \setminus \mathcal{H}(M, X)$, such that $f\varphi = 0$.

This paper is organized as follows: In Section 1, we recall and prove some notions needed throughout this paper and give some notations. In Section 2, we prove Theorems A, B, and C and give some consequences. We dedicate Section 3 to study the irreducible morphisms with left degree two. In particular, we prove Theorem D.

1. **Preliminaries**

1.1

A **quiver** $\Gamma$ is given by two sets $\Gamma_0$ and $\Gamma_1$ together with two maps $s, e : \Gamma_1 \to \Gamma_0$. The elements of $\Gamma_0$ are called **vertices** and the elements of $\Gamma_1$ are called **arrows**. A quiver $\Gamma$ is said to be **locally finite** if each vertex of $\Gamma_0$ is the starting and ending point of at most finitely many arrows in $\Gamma$.

For each arrow $\alpha : y \to x$ with $x$ non-projective, we denote by $\sigma\alpha$ the arrow $\tau x \to y$.

In this work, we are going to consider translations quiver with possible multiple arrows. We observe that this definition is slightly different from the one given in [4].

**Definición 1.1.** A pair $(\Gamma, \tau)$ is said to be a **translation quiver** provided $\Gamma$ is a quiver without loops and locally finite; and $\tau : \Gamma_0' \to \Gamma_0''$ is a bijection whose domain $\Gamma_0'$ and codomain $\Gamma_0''$ are both subsets of $\Gamma_0$, and if for every $x \in \Gamma_0$ such that $\tau x$ exists there exists a bijection $\alpha \to \sigma\alpha$ from the set $x^-$ of arrows arriving at $x$ to the set $(\tau x)^+$ of arrows starting from $\tau x$. The vertices of $\Gamma$ which are not in $\Gamma_0'$ (or which are not in $\Gamma_0''$) are called projective (or injective, respectively).

1.2

A **$k$-category** $A$ over an algebraically closed field $k$ is a category where for each pair of objects $x, y$ in $A$, the set of morphisms $A(x, y)$ is a $k$-vector space and the composite of morphisms is $k$-bilinear.

1.3

Consider a locally finite and connected translation quiver $\Gamma$. The full subquiver of $\Gamma$ given by a non-projective vertex $x$, the non-injective vertex $\tau x$ and by the set $(\tau x)^+ = x^-$ is called the **mesh** starting at $\tau x$ and ending at $x$. The **mesh-ideal** is the ideal $I$ of the category $k\Gamma$ generated by the elements

$$\mu_x = \sum_\alpha \sigma\alpha \in k\Gamma(\tau x, x)$$

where $x$ is not projective and $\alpha$ are the arrows of $\Gamma$ ending at $x$. The **mesh-category** of $\Gamma$ is the quotient category $k(\Gamma) = k\Gamma/I$.

Following [4,(1.2)], we define the **homotopy relation** in $\Gamma$ as the smallest equivalence relation $\sim$ on the set of unoriented paths in $\Gamma$ satisfying the following conditions:

(a) If $\alpha : x \to y$ is a single arrow then $\alpha^{-1}\alpha \sim e_x$ and $\alpha\alpha^{-1} \sim e_y$.
(b) If $x$ is non-projective, then we have $\sigma\alpha \sim \beta\sigma\beta$ for all arrows $\alpha, \beta$ ending at $x$ (see (1)).
(c) If $u \sim v$, then $uwv' \sim wuv'$ whenever these compositions make sense.
Let $x \in \Gamma$ be arbitrary. The set $\pi_1(\Gamma, x)$ of equivalence classes $\overline{u}$ of closed unoriented paths $u$ starting and ending at $x$ has a group structure defined by the operation $\overline{u} \cdot \overline{v} = \overline{uv}$. Since $\Gamma$ is connected, then this group does not depend on the choice of $x$. We denote it by $\pi_1(\Gamma, x)$ and call it the fundamental group of $(\Gamma, x)$.

A translation quiver is called simply connected if it is connected and $\pi_1(\Gamma, x) = 1$ for some $x \in \Gamma$.

1.4

A morphism of translation quivers $(\Gamma, \tau) \rightarrow (\Gamma', \tau')$ is a morphism of quivers $f: \Gamma' \rightarrow \Gamma$ such that $fx$ is non-projective (or non-injective) if $x$ is non-projective (or non-injective, respectively) and $\tau fx = f \tau x$ whenever $x$ is non-projective. This defines the category of translation quivers. Following [4, (1.4)] we say that a morphism of translation quivers $f: (\Gamma', \tau') \rightarrow (\Gamma, \tau)$ is a covering

(i) both $\Gamma$ and $\Gamma'$ are connected.
(ii) for each point $x \in \Gamma_0$, the induced applications $x^- \rightarrow (fx)^-$ and $x^+ \rightarrow (fx)^+$ are bijective; and
(iii) a point $x \in \Gamma_0$ is non-projective (or non-injective) if $fx$ is non-projective (or non-injective, respectively). In view of the definition of a morphism of translation quivers, we say that $x$ is projective (or injective) if and only if $fx$ is projective (or injective, respectively).

As in [4, (1.4)], the homotopy relation of a connected translation quiver $(\Gamma, \tau)$ defines a covering $\tilde{\Gamma} \rightarrow \Gamma$ which factors through any covering of $\Gamma$. For that reason $\tilde{\Gamma}$ is called the universal covering of $\Gamma$. Recall that $\Gamma$ is simply connected [4, (1.4)] and any two paths in $\Gamma$ which have the same source and the same target have the same length [4, (1.6)].

We refer the reader to [4,13,17,18] for a detailed account on covering theory.

1.5

Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. We denote by $\text{mod} A$ the category of all finitely generated left $A$-modules, and by $\text{ind} A$ the full subcategory of $\text{mod} A$ consisting of one representative of each isomorphism class of indecomposable $A$-module. We denote by $\Gamma_A$ the Auslander–Reiten quiver of $A$ and by $\tau$ and $\tau^-$ the Auslander–Reiten translations $\text{DTr}$ and $\text{TrD}$, respectively. Note that a component of the Auslander–Reiten quiver can be seen as a translation quiver with multiple arrows.

Let $X$ be an indecomposable $A$-module. If $X$ is not projective, denote by $\epsilon(X)$ the almost split sequence ending at $X$ and by $\alpha(X)$ the number of indecomposable summands of the middle term of $\epsilon(X)$.

1.6

Next, we calculate the universal cover of a translation quiver with multiple arrows, using the construction given in [4, (1.3)].

Example 1.2. Consider $A$ the Kronecker algebra given by the quiver:

\[
1 \rightarrow 2
\]

The preprojective component $\Gamma$ of $\Gamma_A$ is the following:

\[
\begin{array}{cccccccc}
P_1 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
P_2 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
\end{array}
\]

Observe that $\Gamma$ is a translation quiver with multiple arrows. The fundamental group of such component is $\mathbb{Z}$, that is not simply connected. The universal cover of $\Gamma$ is a quiver of type $\mathbb{NA}_\infty$.

\[
\begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\bullet & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
\bullet & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
\bullet & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
\bullet & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
\bullet & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
\bullet & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{array}
\]

and it is simply connected.
1.7

Let $X \to Z$ be an arrow in a component $\Gamma$ of $\Gamma_\alpha$. Any path $X = Y_0 \to Y_1 \to \cdots \to Y_n = Z$ in $\Gamma$ of length $n \geq 2$ with $Y_1 \neq Y_0$ and $Y_0 \neq Y_{n-1}$ is called a bypass of an arrow $X \to Z$. If this path is sectional, then it is called a sectional bypass, otherwise, it is called a non-sectional bypass.

1.8

A component $\Gamma$ is said to be a directed component of $\Gamma_\alpha$ if there is no sequence $X_0 \to X_1 \to \cdots \to X_n$ of non-zero non-isomorphisms between indecomposable modules, with $X_0 = X_n$.

1.9

Let $\Gamma$ be a component of $\Gamma_\alpha$ and $\Gamma'$ the full subcategory of ind $\alpha$ generated by the modules of $\Gamma$.

A component $\Gamma'$ of $\Gamma_\alpha$ is said to be standard if the category ind $\Gamma'$ is equivalent to the mesh category $k(\Gamma)$ of $\Gamma$ (see [19]). This means, there is an isomorphism $\phi : k(\Gamma) \to \text{ind} \; \Gamma'$ which is the identity on the objects and for each arrow $\alpha \in \Gamma_1$, $\phi(\alpha) = f$ where $f$ is an irreducible morphism in ind $\Gamma'$. Moreover, if we consider the mesh ending at $[M]$

$$
\begin{array}{c}
[x_1] \\
\vdots \\
[x_n]
\end{array}
\begin{array}{c}
\tau M
\end{array}
\begin{array}{c}
[M]
\end{array}
$$

then $\phi(\text{mesh})$ is an almost split sequence ending at $M$ in ind $\Gamma'$.

In [15], it was proven that if $\Gamma'$ is a standard component of $\Gamma_\alpha$ then $\Gamma'$ is generalized standard, that is, $\mathcal{H}_\infty(X, Y) = 0$ for all $X, Y \in \Gamma'$.

1.10

Let $F : \mathcal{C} \to \mathcal{C}'$ be a $k$-linear functor between two $k$-categories. $F$ is called a covering functor if the maps

$$
\bigoplus_{z/b} \mathcal{C}(x, z) \to \mathcal{C}'(a, b) \quad \text{and} \quad \bigoplus_{t/a} \mathcal{C}(t, y) \to \mathcal{C}'(a, b)
$$

which are induced by $F$, are bijective for any $a$ and $b$ in $\mathcal{C}'$. Here $t$ and $z$ range over all objects of $\mathcal{C}$ such that $Ft = a$ and $Fz = b$ respectively.

1.11

Let $\pi : \tilde{\Gamma} \to \Gamma$ be the universal cover of $\Gamma$. Consider the induced functor $k(\pi) : k(\tilde{\Gamma}) \to k(\Gamma)$ where $k(\tilde{\Gamma})$ and $k(\Gamma)$ are the mesh category of $\tilde{\Gamma}$ and $\Gamma$, respectively. We are interested in proving that $k(\pi)$ induces a covering of $k$-categories that induces isomorphisms between the radical layers as stated in [5, p. 27]. We start proving the following more general result:

**Lemma 1.3.** Let $\Gamma$ be a standard component of $\Gamma_\alpha$ and $\pi : \tilde{\Gamma} \to \Gamma$ be the universal cover of $\Gamma$. Then there is a covering functor $F : k(\tilde{\Gamma}) \to \text{ind} \; \Gamma$.

**Proof.** Let $\pi : \tilde{\Gamma} \to \Gamma$ be the universal cover of $\Gamma$. Consider the induced functor $k(\pi) : k(\tilde{\Gamma}) \to k(\Gamma)$ where $k(\tilde{\Gamma})$ and $k(\Gamma)$ are the mesh category of $\tilde{\Gamma}$ and $\Gamma$ respectively. Since $\Gamma$ is standard component of $\Gamma_\alpha$ then $k(\tilde{\Gamma}) \simeq \text{ind} \; \Gamma'$ and we can consider $F : k(\tilde{\Gamma}) \to \text{ind} \; \Gamma'$. Moreover, $F$ is a $k$-linear functor which maps an object $Y$ of $\tilde{\Gamma}$ onto $\pi(Y)$ and a morphism $\beta$ associated with the arrow $\beta$ of $\Gamma'$ onto an irreducible morphism in $\text{ind} \; \Gamma'$. By [18, Proposition 2.2], we know that $F$ is a covering functor. \qed

**Remark 1.4.** If $\Gamma$ is standard component of $\Gamma_\alpha$ then the category $k(\Gamma)$ is equivalent to the category $\text{ind} \; \Gamma$. By the above lemma we conclude that $k(\pi)$ is also a covering functor.

Our next result will prove that both functors $k(\pi)$ and $F$ induces isomorphisms between the radical layers as stated in [5, p. 27].
Proposition 1.5. Let $\Gamma$ be a connected translation quiver and $\pi : \tilde{\Gamma} \to \Gamma$ be the universal cover. Let $x, y$ be vertices in $\tilde{\Gamma}$. Then, $k(\pi)$ induces a linear isomorphism

$$
\bigoplus_{F_2 = F_y} \mathcal{N}^n k(\tilde{\Gamma})(x, z) / \mathcal{N}^{n+1} k(\tilde{\Gamma})(x, z) \to \mathcal{N}^n k(\Gamma)(\pi x, \pi y) / \mathcal{N}^{n+1} k(\Gamma)(\pi x, \pi y)
$$

for all $n \in \mathbb{N}$.

If $\Gamma$ is a standard component of $\Gamma_\lambda$, then the covering functor $F : k(\tilde{\Gamma}) \to \text{ind } \Gamma$ obtained by composing $k(\pi)$ with any isomorphism $k(\Gamma) \cong \text{ind } \Gamma$ induces similarly a linear isomorphism

$$
\bigoplus_{F_2 = F_y} \mathcal{N}^n k(\tilde{\Gamma})(x, z) / \mathcal{N}^{n+1} k(\tilde{\Gamma})(x, z) \to \mathcal{N}^n F_x(Fy) / \mathcal{N}^{n+1} F_x(Fy)
$$

for all $n \in \mathbb{N}$.

Proof. If $\Gamma$ is a standard component of $\Gamma_\lambda$ then any isomorphism induces an isomorphism $\mathcal{N}^n k(\Gamma)(x, y) \cong \mathcal{N}^n k(\Gamma, x, y)$ for all $n$ and all $x, y \in \Gamma$. Therefore we only need to prove the first assertion.

Let $n \geq 0$. Given vertices $x, y$ in $\tilde{\Gamma}$ we write $k\tilde{\Gamma}(x, y)_n$, the subvector space of $k\tilde{\Gamma}(x, y)$ having as basis the family of paths of length $n$. In this way $k\tilde{\Gamma}$ is an $\mathbb{N}$-graded category, that is $k\tilde{\Gamma}(x, y)$ is a direct sum $\bigoplus_{n \in \mathbb{N}} k\tilde{\Gamma}(x, y)_n$ of homogeneous components and the composition of homogeneous morphisms of degree $m$ and $n$ respectively is homogeneous of degree $m + n$. Now, the mesh-ideal is a homogeneous ideal because it is generated by linear combination of paths of length 2. Hence, the $\mathbb{N}$-grading on $k\tilde{\Gamma}$ induces an $\mathbb{N}$-grading on $k(\tilde{\Gamma})$ in such a way that homogeneous components of degree $n \in \mathbb{N}$ are vector spaces generated by classes (modulo the mesh-ideal) of paths of length $n$.

By definition of the radical $\mathcal{N}$, we therefore have

$$
\mathcal{N}^n k(\tilde{\Gamma})(x, y) = \bigoplus_{m \geq n} k(\tilde{\Gamma})(x, y)_m
$$

and

$$
\mathcal{N}^n k(\tilde{\Gamma})(x, y) / \mathcal{N}^{n+1} k(\tilde{\Gamma})(x, y) = k(\tilde{\Gamma})(x, y)_n.
$$

The same construction may be done over $k(\Gamma)$ and, because $\pi$ is a morphism of quivers, the functor $k(\pi) : k(\tilde{\Gamma}) \to k(\Gamma)$ is homogeneous of degree 0. Now, since $\pi$ is a covering functor it induces an isomorphism

$$
\bigoplus_{\pi x = \pi y} k(\tilde{\Gamma})(x, z) \to k(\Gamma)(\pi x, \pi y)
$$

that induces an isomorphism between the associated homogeneous components

$$
\bigoplus_{\pi x = \pi y} k(\tilde{\Gamma})(x, z)_n \to k(\Gamma)(\pi x, \pi y)_n,
$$

for all $n \in \mathbb{N}$.

Finally, if we replace (2) in (3) and since

$$
k(\Gamma)(\pi x, \pi y)_n = \mathcal{N}^n k(\Gamma)(\pi x, \pi y) / \mathcal{N}^{n+1} k(\Gamma)(\pi x, \pi y),
$$

we get that the first linear map stated in this proposition is an isomorphism. $\square$

1.12

A finite dimensional algebra over an algebraically closed field $k$ is a standard algebra if the category $\text{ind } A$ is equivalent to the mesh category $k(\Gamma_\lambda)$ of $\Gamma_\lambda$. It is known that in this case the algebra is of finite representation type.

1.13

Let $A$ be an artin algebra and $f : X \to Y$ be an irreducible morphism in $\text{mod } A$. Following [14], we say that the left degree of $f$ is infinite if for any positive integer $n$, any $Z \in \text{mod } A$ and any morphism $g \in \mathcal{N}^n(Z, X) \setminus \mathcal{N}^{n+1}(Z, X)$, we have $fg \notin \mathcal{N}^{n+2}(Z, Y)$. Otherwise, we say that the left degree of $f$ is the smallest positive integer $m$ such that there exists a morphism $g \in \mathcal{N}^m(Z, X) \setminus \mathcal{N}^{m+1}(Z, X)$, for some $Z \in \text{mod } A$, such that $fg \in \mathcal{N}^{m+2}(Z, Y)$. We denote the left degree of $f$ by $d_l(f)$.

Dually, the right degree of $f$ is defined, denoting by $d_r(f)$. We refer the reader to [9,14,16] for a detailed account of these degrees.
1.14

Finally, we observe that [11, Proposition 2.1], can be stated for a translation quiver with multiple arrows. One can easily check that the proof given there works in our situation.

**Proposition 1.6.** Let $\Gamma$ be a translation quiver with length. Let $X, Y \in k(\Gamma)$ such that $\ell(X, Y) = n$ with $n \geq 1$. Then:

(a) $\mathfrak{r}_{k(\Gamma)}^{n+1}(X, Y) = 0$.
(b) If $g : X \to Y$ is a non-zero morphism in $k(\Gamma)$ then $g \in \mathfrak{r}_{k(\Gamma)}^n(X, Y) \setminus \mathfrak{r}_{k(\Gamma)}^{n+1}(X, Y)$.
(c) $\mathfrak{r}_{k(\Gamma)}^j(X, Y) = \mathfrak{r}_{k(\Gamma)}^n(X, Y)$, for each $j = 1, \ldots, n - 1$.

2. On the degree of irreducible morphisms

In this section, we will consider finite dimensional algebras over an algebraically closed field, unless otherwise stated.

We shall study the left degree of irreducible morphisms $f : X \to \bigoplus_{i=1}^r Y_i$, with $X$ and $Y_i$ in a standard component of the Auslander–Reiten quiver, not necessarily with length, for $1 \leq i \leq r$.

We will give two different characterizations of the left degree of an irreducible morphism $f$ to be finite in the context of standard algebras, similar to the ones given in [9]. The first one is related to the existence of a non-zero morphism $g$ such that their composite is zero. The other characterization depends on whether the kernel of $f$ belongs to the component $\Gamma^A$.

To achieve such characterizations, we are going to reduce the study of a standard component $\Gamma^A$ to the study of a simply connected component of a $k$-category, passing from $\Gamma$ to its universal covering $\Gamma$. Since $\Gamma$ is simply connected then by [4] it is a component with length, and we will get our results applying the ones proven in [9, 11], for translations quivers with length.

We start recalling the notion of component with length given in [9].

2.1

A component $\Gamma$ of $\Gamma^A$ is said to be a **component with length** if parallel paths in $\Gamma$ have the same length. In [9], we extended this notion to translation quivers. It is useful to observe that a component with length has no oriented cycles. By [11, Corollary 2.2], we know that if $\Gamma$ is a translation quiver with length and $f_i : X_i \to X_{i+1}$ are irreducible morphisms with $X_i \in k(\Gamma)$, for $1 \leq i \leq n + 1$, then $f_n \cdots f_1 \in \mathfrak{r}_{k(\Gamma)}^{n}(X_1, X_{n+1})$ if and only if $f_n \cdots f_1 = 0$. We observe that the proof given there works for translations quivers with multiple arrows.

Bongartz and Gabriel considered the homotopy given by the mesh relations, and defined simply connected translation quivers. They had implicitly proven in the proof of [4, Proposition 1.6], that if $\Gamma$ is a component of a simply connected translation quiver, then $\Gamma$ is a component with length.

The notion of radical has been extended to $k$-categories as the powers of the radical (see [4, p. 337]). Given a $k$-category, an irreducible morphism between indecomposable objects is a morphism $f$ such that $f \in \mathfrak{r} \setminus \mathfrak{r}^2$.

Now, we are going to show an example of an irreducible morphism of infinite left degree and another of finite left degree, using the results proven in [9].

**Example 2.1.** Let $A$ be the hereditary $k$-algebra of type $\tilde{E}_7$, given by the quiver:

```
1 ← 2 → 3 → 4 ← 5 → 6 → 7
```

The preprojective component $\Gamma$ of the Auslander–Reiten quiver is:

```
P_7 \quad \tau^{-1}P_7 \quad \tau^{-2}P_7 \quad f \quad \tau^{-1}P_7 \quad \tau^{-2}P_7
P_6 \quad \tau^{-1}P_6 \quad \tau^{-2}P_6 \quad \tau^{-1}P_6 \quad \tau^{-2}P_6
P_5 \quad \tau^{-1}P_5 \quad \tau^{-2}P_5
P_4 \quad \tau^{-1}P_4 \quad \tau^{-1}P_4 \quad \tau^{-2}P_4
P_3 \quad \tau^{-1}P_3 \quad \tau^{-1}P_3
P_2 \quad f_1 \quad \tau^{-1}P_2
P_1 \quad \tau^{-1}P_1
```
We observe that $\Gamma$ is a generalized standard convex component of $\Gamma_A$ with length (see [9, Proposition 2.6]). Consider the irreducible morphisms $f : P_q \to P_h$ and $f_1 : \tau^{-1} P_q \to \tau^{-1} P_h$.

Applying the results proven in [9, Corollary 3.8], any irreducible monomorphism between indecomposable modules has infinite left degree. In particular, since $f$ is an irreducible monomorphism, then it has infinite left degree.

On the other hand, since $f_1$ is an irreducible epimorphism and $\ker(f_1) = P_1$ then by [9, Theorem 3.11], $d_i(f_1)$ is finite. It is not hard to prove that the left degree of $f_1$ is two.

Below, we will show that for finite dimensional algebras over an algebraically closed field, the degree is determined by the same morphism.

**Lemma 2.2.** Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Let $f : X \to Y$ be an irreducible morphism and $\varphi \in \mathfrak{H}^n(M, X) \setminus \mathfrak{H}^{n+1}(M, X)$ with $X, Y, M \in \text{ind } A$ such that $f \varphi \in \mathfrak{H}^{n+2}(M, Y)$. Then $h \varphi \in \mathfrak{H}^{n+2}(M, Y)$, for any irreducible morphism $h : X \to Y$.

**Proof.** Since $d_i(f)$ is finite and $k$ is an algebraically closed field, then by [14, Lemma 1.7], any irreducible morphism from $X$ to $Y$ is of the form $h = af + \mu$ with $a \in k^*$ and $\mu \in \mathfrak{H}^2(X, Y)$. Therefore, $h \varphi = (af + \mu) \varphi = af \varphi + \mu \varphi \in \mathfrak{H}^{n+2}(M, Y)$. \qed

By [14, Corollary 1.8], if $f, g : X \to Y$ are irreducible morphisms between indecomposable modules then $d_i(f) = d_i(g)$.

Next, we will generalize such a result for irreducible morphisms with $Y$ not necessarily an indecomposable $A$-module.

**Lemma 2.3.** Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Let $f, g : X \to \bigoplus_{i=1}^r Y_i$ be irreducible morphisms with $X, Y_i \in \text{ind } A$, for $1 \leq i \leq r$. Then $d_i(f) = n$ if and only if $d_i(g) = n$.

**Proof.** Assume that $d_i(f) = n$. Then, there is an indecomposable $A$-module $M$ and a morphism $\varphi \in \mathfrak{H}^n(M, X) \setminus \mathfrak{H}^{n+1}(M, X)$ such that $f \varphi \in \mathfrak{H}^{n+2}(M, Y)$. Since each $f_i$ is of finite left degree then $g_i = \alpha f_i + \mu_i$ with $\alpha \in k^*$ and $\mu_i \in \mathfrak{H}^2(X, Y_i)$, for $1 \leq i \leq r$. Thus $g \varphi \in \mathfrak{H}^{n+2}(M, \bigoplus_{i=1}^r Y_i)$. Therefore, we have that $d_i(g) \leq n$.

If $d_i(g) = m < n$ then there is a morphism $\varphi' \in \mathfrak{H}^m(M', X) \setminus \mathfrak{H}^{m+1}(M', X)$ such that $g \varphi' \in \mathfrak{H}^{m+2}(M, \bigoplus_{i=1}^r Y_i)$. Then, we infer that $f \varphi' \in \mathfrak{H}^{n+2}(M, \bigoplus_{i=1}^r Y_i)$ contradicting that $d_i(f) = n$.

The converse follows with the same argument. \qed

We state the following lemma which will be used throughout this paper.

**Lemma 2.4.** Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Let $f : X \to \bigoplus_{i=1}^r Y_i$ be an irreducible morphism with $X, Y_i \in \text{ind } A$, for $1 \leq i \leq r$. If $d_i(f) = n > 1$ then there exists an indecomposable $A$-module $M$ and a non-zero morphism $u : M \to X$, that is a sum of compositions of $n$ irreducible morphisms between indecomposable modules, such that $fu \in \mathfrak{H}^{n+2}(M, \bigoplus_{i=1}^r Y_i)$.

**Proof.** By hypothesis $d_i(f) = n$, then there exists $M \in \text{ind } A$ and a morphism $\varphi \in \mathfrak{H}^n(M, X) \setminus \mathfrak{H}^{n+1}(M, X)$ such that $f \varphi \in \mathfrak{H}^{n+2}(M, \bigoplus_{i=1}^r Y_i)$. Since $n > 1$ and $M, X$ are indecomposable by [3, V.7.5], $\varphi = u + v$ where $u$ is a non-zero finite sum of compositions of $n$ irreducible morphisms between indecomposable modules and $v \in \mathfrak{H}^{n+1}(M, X)$. Then, $f \varphi = fu + fv$. Since $f \varphi \in \mathfrak{H}^{n+2}(M, \bigoplus_{i=1}^r Y_i)$ and $fv \in \mathfrak{H}^{n+2}(M, \bigoplus_{i=1}^r Y_i)$ then $fu \in \mathfrak{H}^{n+2}(M, \bigoplus_{i=1}^r Y_i)$. \qed

2.2

We recall that an arrow $M \to N$ of $\Gamma_A$ has **valuation** $(a, a)$ if there is a minimal right almost split morphism $aM \oplus X \to N$, where $M, N$ are indecomposable, $M$ is not a summand of $X$, and there is a minimal left almost split morphism $M \to aN \oplus Y$ where $M, N$ are indecomposable, $N$ is not a summand of $Y$. We say that the arrow has **trivial valuation** if $a = 1$.

In [9, Theorem 3.7], it was proved that for a generalized standard convex component $\Gamma'$ of $\Gamma_A$ with length, the left degree of an irreducible morphism $f$ between modules in $\Gamma'$ is finite if and only if there exists a non-zero morphism $\varphi$ such that $f \varphi = 0$. We observe that for a generalized standard component of $\Gamma'$ the convexity is not necessary since every non-zero morphism between modules in such component is a sum of compositions of sequences of morphisms between modules in the component.

Now, we are in the position to prove the following result:

**Proposition 2.5.** Let $A$ be a finite dimensional algebra over an algebraically closed field and $\Gamma$ a standard component of $\Gamma_A$. Let $f : X \to Y$ be an irreducible morphism with $X, Y \in \Gamma$. If $d_i(f) < \infty$ then there is module $M \in \Gamma$, an irreducible morphism $h : X \to Y$ and a morphism $\varphi \in \mathfrak{H}^n(M, X) \setminus \mathfrak{H}^{n+1}(M, X)$ such that $h \varphi = 0$.

**Proof.** If $d_i(f) = 1$ then by [14, Proposition 1.12], $f$ is a surjective right minimal almost split morphism. Therefore there exists $Z \in \Gamma$ and $g \in \mathfrak{H}(Z, X) \setminus \mathfrak{H}^2(Z, X)$ such that $fg = 0$, proving the result.

Assume that $n \geq 2$. Since $d_i(f) = n$ then there exists $Z \in \Gamma$ and $\varphi \in \mathfrak{H}^n(Z, X) \setminus \mathfrak{H}^{n+1}(Z, X)$ such that $f \varphi \in \mathfrak{H}^{n+2}(Z, Y)$. If $f \varphi = 0$ then there is nothing to do. Otherwise, by **Lemma 2.4** there is a non-zero morphism $u = \sum_{i=1}^n u_i$, that is a finite sum of compositions of $n$ irreducible morphisms between indecomposable modules, such that $fu \in \mathfrak{H}^{n+2}(Z, Y)$. If $fu = 0$ then we are done. Assume that $fu \neq 0$. Let $\pi : \tilde{\Gamma} \to \Gamma$ be the universal cover of $\Gamma$. Then $k(\pi) : k(\tilde{\Gamma}) \to k(\Gamma)$ is a Galois covering.
with group $\Pi_G(\Gamma)$. Moreover, $\pi$ is a translation quiver covering and the ideals are homogeneous. By Proposition 1.5, for each $X, Y \in \Gamma$, $k(\pi)$ induce two isomorphisms:

$$
\bigoplus_{\pi(X') = \pi(X)} \mathfrak{g}^0 k(\Gamma)(X', Y) / \mathfrak{g}^{n+1} k(\Gamma)(X', Y) \simeq \mathfrak{g}^0 k(\Gamma)(\pi(X'), \pi(Y)) / \mathfrak{g}^{n+1} k(\Gamma)(\pi(X'), \pi(Y))
$$

(4)

and its dual.

Since $\Gamma$ is standard then there is an isomorphism $\phi : k(\Gamma) \to \text{ind } \Gamma$. This isomorphism sends an arrow to an irreducible morphism of ind $\Gamma$ and a mesh to an almost split sequence. Let $\alpha$ be an arrow from $X$ to $Y$ in $\Gamma$ and $h$ its image by $\phi$, that is $\phi(\alpha) = h$ where $\alpha$ denotes the residue class of $\alpha$ modulo the mesh ideal $I_{\Gamma}$. By [14, Corollary 1.7], we know that $d_i(f) = d_i(h)$.

On the other hand, since $u = \sum_{i=1}^m u_i$, each $u_i$ is a composite $u_i = u_{i_1} \ldots u_{i_t}$, for $1 \leq i \leq m$. Moreover, each $u_{ij}$ can be written as follows:

$$
u_{ij} = \phi \left( \sum \lambda_j \sigma_j \right) + \mu_j
$$

where $\sigma_j$ is the residue class of the arrows $\alpha_j : X_{ij} \to X_{i+1}$ modulo the mesh ideal $I_{\Gamma}$, $\lambda_j \in k^*$ and $\mu_j \in \mathfrak{g}^2 k(\Gamma)(X_{ij}, X_{i+1})$, for $1 \leq j \leq n$. Then, we can assume that

$$u = \phi \left( \sum \nu_{ij} \delta_i \right) \in \mathfrak{g}^0 k(\Gamma)(Z, X) \backslash \mathfrak{g}^{n+1} k(\Gamma)(Z, X)
$$

where $\delta_i : Z \to X$ are paths of length $n$ and $\delta_i$ the residue class of $\delta_i$ modulo $I_{\Gamma}$. Moreover, by Lemma 2.2, $hu \in \mathfrak{g}^{n+2} k(\Gamma)(Z, Y)$. Let $\tilde{\alpha} : X \to \tilde{Y}$ be a lift of the arrow $\alpha : X \to Y$ by $\pi$, that is to say, $\pi(\tilde{\alpha}) = \alpha$. For each $j$, we consider $\tilde{\delta}_j$ the path that lifts the path $\delta_j : Z \to X$ to $\tilde{Y}$ and such that $\tilde{\delta}_j$ ends at the starting point of $\tilde{\alpha}$.

Let $\tilde{Z}$ be the starting point of $\tilde{\delta}_j$. Now, by isomorphism (4), if we consider the morphism $\tilde{u}$ defined as the sum over the indices $j$ such that $\tilde{\delta}_j : \tilde{Z} \to \tilde{X}$ in $k(\tilde{\Gamma})$ then:

$$
\tilde{u} = \sum_j \nu_{ij} \tilde{\delta}_j \in \mathfrak{g}^0 k(\tilde{\Gamma})(\tilde{Z}, \tilde{X}) \backslash \mathfrak{g}^{n+1} k(\tilde{\Gamma})(\tilde{Z}, \tilde{X})
$$

where $\tilde{\delta}_j$ is the residue class of $\tilde{\delta}_j$ modulo the mesh ideal $I_{\tilde{\Gamma}}$. Since $hu \in \mathfrak{g}^{n+2} k(\Gamma)(Z, Y)$ using again (4) we have that

$$
\tilde{\alpha} \tilde{u} = \sum_j \nu_{ij} \tilde{\alpha} \tilde{\delta}_j \in \mathfrak{g}^{n+2} k(\tilde{\Gamma})(\tilde{Z}, \tilde{Y})
$$

in $k(\tilde{\Gamma})$. By [11, Proposition 2.1] $\mathfrak{g}^{n+2} k(\tilde{\Gamma})(\tilde{Z}, \tilde{Y}) = 0$ because $\tilde{\Gamma}$ is with length and there are paths of length $n + 1$ from $\tilde{Z}$ to $X$ in $\tilde{\Gamma}$.

Finally, considering $\varphi' = \phi(k(\pi) \tilde{u})$ by the isomorphism stated in Proposition 1.5 we have that $\varphi' \in \mathfrak{g}^0 (Z, X) \backslash \mathfrak{g}^{n+1} (Z, X)$ and $h \varphi' = 0$, proving the result. □

A dual result holds for the right degree. We will refrain from stating the dual results since it can be easily done.

2.3

If $h : X \to Y$ is an irreducible morphism with $X, Y \in \Gamma$ and $d_i(h) < \infty$ then there is an almost split sequence as follows:

$$
0 \longrightarrow Y \overset{\varphi'}{\longrightarrow} X' \oplus X \overset{\varphi}{\longrightarrow} X \longrightarrow 0
$$

with $X'$ not necessarily indecomposable. Moreover, each $f : X \to Y$ can be written as $f = \alpha h + \mu$ where $\mu = h \varphi_x + t \psi_1$, $\alpha \in k^*$, $\varphi_x \in \mathfrak{g}(X, X)$, and $\psi_1 \in \mathfrak{g}(X, X')$. It would be interesting to know if for any irreducible morphism $f : X \to Y$, we have a non-zero morphism $\varphi$ such that $f \varphi = 0$. We can prove that this is the case when $\mu = h \varphi_x$ and when we consider a directed standard component of $\Gamma_h$. In fact, we prove the following two corollaries:

**Corollary 2.6.** Let $\Gamma$ a standard component of $\Gamma_A$. Let $f : X \to Y$ be an irreducible morphism with $X, Y \in \Gamma$ and such that $f = h + \mu$ with $\mu = h \varphi_x$, and $\varphi_x \in \mathfrak{g}(X, X)$. If $d_i(f) < \infty$ then there exists a morphism $\varphi' \in \mathfrak{g}^0 (M, X) \backslash \mathfrak{g}^{n+1} (M, X)$ such that $f \varphi' = 0$.

**Proof.** By the above proposition, since $d_i(f) < \infty$ there exists an irreducible morphism $h : X \to Y$, and a morphism $\varphi \in \mathfrak{g}^0 (M, X) \backslash \mathfrak{g}^{n+1} (M, X)$ such that $h \varphi = 0$. It is well known that $\mathfrak{g}(X, X)$ is a nilpotent ideal whenever $X$ is an indecomposable module. Then, there exists a positive integer $k$ such that $(\varphi_x)^k = 0$. Then, $\varphi' = \varphi - \varphi_x \varphi + \varphi_x^3 \varphi + \cdots + (-1)^{k-1} \varphi_x^{k-1} \varphi \in \mathfrak{g}^0 (M, X) \backslash \mathfrak{g}^{n+1} (M, X)$ since $\varphi \in \mathfrak{g}^0 (M, X) \backslash \mathfrak{g}^{n+1} (M, X)$. Computing the composite $f \varphi'$ we get that $f \varphi' = 0$. □

**Corollary 2.7.** Let $\Gamma$ be a directed standard component of $\Gamma_A$ and $f : X \to Y$ an irreducible morphism with $X, Y \in \Gamma$. If $d_i(f) < \infty$ then there exists a morphism $\varphi' \in \mathfrak{g}^0 (M, X) \backslash \mathfrak{g}^{n+1} (M, X)$ with $M \in \Gamma$ such that $f \varphi' = 0$. 
Proposition 9. Assume that \( \text{Proposition } 9 \) there is an irreducible morphism \( h : X \to Y \) and a morphism \( \varphi \in \mathcal{M}(M, X) \setminus \mathcal{M}^{n+1}(M, X) \) such that \( h\varphi = 0 \). Moreover, \( f = ah + \mu \) with \( \mu = h\varphi_\alpha + \varphi \varphi_\alpha \in \mathcal{M}(X, X) \) and \( \varphi_\alpha \in \mathcal{M}(X, X') \).

On the other hand, if \( \Gamma' \) is a directed component of \( \Gamma_A \), we claim that \( \mu = 0 \). In fact, the first summand of \( \mu \) is clearly zero. Now, if \( \varphi_\alpha \neq 0 \) then it may be written as a finite sum of compositions of non-sectional paths, since otherwise by [7, Lemma 1.9], we get to a contradiction with the fact that \( d_1(h) < \infty \). Now, if there is a non-sectional bypass of an arrow by [12, Proposition 1], we infer that \( \Gamma' \) has a cycle. Then \( f = ah \) and Proposition 2.5 we get the result. □

Next, we are going to generalize Proposition 2.5, for an irreducible morphism \( f : X \to \bigoplus_{i=1}^r Y_i \) with \( X, Y_i \), for \( 1 < i \leq r \), over a standard component of \( \Gamma_A \).

We will start with a previous result for a generalized standard convex component with length of the Auslander–Reiten quiver of an artin algebra.

Lemma 2.8. Let \( A \) be an artin algebra and \( \Gamma \) a generalized standard convex component of \( \Gamma_A \), with length. Let \( f : X \to \bigoplus_{i=1}^r Y_i \) be an irreducible morphism with \( X, Y_i \in \Gamma \), for \( 1 \leq i \leq r \). If \( d_1(f) = n \) then there is a morphism \( u \in \mathcal{M}(M, X) \setminus \mathcal{M}^{n+1}(M, X) \) for some \( M \in \Gamma \), such that \( fu = 0 \).

Proof. If \( d_1(f) = n \) then there exist \( M \in \Gamma \) and \( \varphi \in \mathcal{M}(M, X) \setminus \mathcal{M}^{n+1}(M, X) \) such that \( f\varphi \in \mathcal{M}^{n+2}(M, Y) \). If \( n = 1 \) then \( f = \) is a surjective right minimal almost split morphism. Moreover, \( r = 1 \) and clearly we get the result.

If \( n > 1 \) then by Lemma 2.4, there is a non-zero morphism \( u = \sum_{i=1}^r u_i \), where each \( u_i \) is a composite of \( n \) irreducible morphisms between indecomposable modules and \( fu \in \mathcal{M}^{n+2}(M, \bigoplus_{i=1}^r Y_i) \). Then, \( fu_i \in \mathcal{M}^{n+2}(M, Y_i) \) for \( 1 \leq i \leq r \).

On the other hand, since \( \Gamma \) is a component with length and \( \ell(M, Y_i) = n + 1 \) then by [9, Proposition 1.2], \( f\mu = 0 \) for \( 1 \leq i \leq r \), proving that \( fu = 0 \). □

With the same techniques as those used in the proof of Proposition 2.5 and using Lemma 2.8, we get the following generalization.

Theorem 2.9. Let \( A \) be a finite dimensional algebra over an algebraically closed field \( k \) and \( \Gamma \) a standard component of \( \Gamma_A \). Let \( f : X \to \bigoplus_{i=1}^r Y_i \) be an irreducible morphism with \( X, Y_i \in \Gamma \), for \( 1 \leq i \leq r \). Then, the following are equivalent:

(a) \( d_1(f) \) is finite.

(b) there exists a module \( M \in \Gamma \), an irreducible morphism \( h : X \to \bigoplus_{i=1}^r Y_i \) and a non-zero morphism \( \varphi : M \to X \) such that \( h\varphi = 0 \).

As an immediate consequence of the above theorem we get this useful result for finite dimensional algebras over an algebraically closed field.

Corollary 2.10. Let \( \Gamma \) a standard component of \( \Gamma_A \) and \( f : X \to \bigoplus_{i=1}^r Y_i \) an irreducible morphism, with \( X, Y_i \in \Gamma \), for \( 1 \leq i \leq \Gamma \). Then, there are equivalent:

(a) \( d_1(f) \) is finite.

(b) there exists a module \( M \in \Gamma \) and a non-zero morphism \( \varphi : M \to X \) such that \( f\varphi = 0 \).

Remark 2.11. Observe that by the above result, \( f \) is an injective left minimal almost split morphism with \( X \) and \( Y_i \) over a standard component of \( \Gamma_A \), for \( 1 \leq i \leq \Gamma \). Thus \( d_1(f) = \Gamma \).

A dual result holds for a surjective right minimal almost split morphism.

By [2, IV, Corollary 1.8], it is known that if \( f \) is an irreducible morphism then \( \text{Ker}(f) \) is an indecomposable module. For standard algebras Theorem 2.9 can be stated as follows:

Theorem 2.12. Let \( A \) be a standard algebra. Let \( f : X \to \bigoplus_{i=1}^r Y_i \) be an irreducible morphism with \( X, Y_i \in \Gamma_A \), for \( 1 \leq i \leq r \). Then, there are equivalent:

(a) \( d_1(f) \) is finite.

(b) there exists a module \( M \in \Gamma_A \) and a non-zero morphism \( \varphi : M \to X \) such that \( f\varphi = 0 \).

Proof. Assume that (a) holds. Then by Corollary 2.10, \( f \) is an epimorphism and therefore \( \text{Ker}(f) \in \Gamma_A \). Then \( j: \text{Ker}(f) \to X \in \mathcal{M}(M, X) \setminus \mathcal{M}^{n+1}(M, X) \) for some positive integer \( n \), since \( \Gamma_A \) is standard and therefore a generalized standard component. Moreover, \( jf = 0 \), proving the implication.

If (b) holds then clearly \( d_1(f) \) is finite since \( \Gamma_A \) is a generalized standard component. □

As a consequence of the above result we have the following lemma.

Lemma 2.13. Let \( A \) be a standard algebra. Let \( f : X \to \bigoplus_{i=1}^r Y_i \) be an irreducible morphism with \( X, Y_i \in \Gamma \), for \( 1 \leq i \leq r \). Then, \( d_1(f) = n \) if and only if the inclusion \( j : \text{Ker}(f) \to X \) such that \( j \in \mathcal{M}(\text{Ker}(f), X) \setminus \mathcal{M}^{n+1}(\text{Ker}(f), X) \).

Proof. Assume that \( d_1(f) = n \). By Theorem 2.12, there is a module \( Z \in \Gamma \) and a morphism \( \varphi \in \mathcal{M}(Z, X) \setminus \mathcal{M}^{n+1}(Z, X) \) such that \( f\varphi = 0 \). Hence, there is a non-zero morphism \( g : Z \to \text{Ker}(f) \) such that \( \varphi = fg \), where \( j : \text{Ker}(f) \to X \) is the inclusion. Since \( \varphi \notin \mathcal{M}^{n+1}(Z, X) \) then \( j \notin \mathcal{M}^{n+1}(\text{Ker}(f), X) \). Now, if \( j \in \mathcal{M}(\text{Ker}(f), X) \setminus \mathcal{M}^{n+1}(\text{Ker}(f), X) \) for some \( r < n \) then \( d_1(f) < n \), since \( jf = 0 \), which is a contradiction. Thus, \( j \in \mathcal{M}(\text{Ker}(f), X) \setminus \mathcal{M}^{n+1}(\text{Ker}(f), X) \), proving one implication.

Assume that \( j \in \mathcal{M}(\text{Ker}(f), X) \setminus \mathcal{M}^{n+1}(\text{Ker}(f), X) \) then \( d_1(f) \leq n \). Suppose that \( d_1(f) = r < n \). By Proposition 2.5, there is a module \( Z \in \Gamma \) and a morphism \( \varphi \in \mathcal{M}(Z, X) \setminus \mathcal{M}^{n+1}(Z, X) \) such that \( f\varphi = 0 \). Getting the contradiction that \( j \in \mathcal{M}(\text{Ker}(f), X) \setminus \mathcal{M}^{n+1}(\text{Ker}(f), X) \), with \( m \leq n \). Then \( d_1(f) = n \). □
In [9], we proved that if $A$ is an artin algebra, $\Gamma$ a generalized standard convex component of $\Gamma_A$ and $f : X \rightarrow Y$ an irreducible morphism with $X, Y \in \Gamma$, then $\ker(f) \in \Gamma$ implies that $d_i(f)$ is finite, or equivalently, $d_i(f) = \infty$ then $\ker(f) \notin \Gamma$. Our next purpose is to give information of the degree of an irreducible morphism $f : X \rightarrow \bigoplus_{i=1}^r Y_i$ between modules in a standard component of $\Gamma_A$, depending on the position of its kernel. We prove the following result.

**Theorem 2.14.** Let $A$ be a finite dimensional algebra over an algebraically closed field $k$ and $\Gamma$ a standard component of $\Gamma_A$. Let $f : X \rightarrow \bigoplus_{i=1}^r Y_i$ be an irreducible morphism with $X, Y_i \in \Gamma$, for $1 \leq i \leq r$. Then, the following conditions are equivalent:

(a) $d_i(f)$ is finite.

(b) There is an irreducible morphism $h : X \rightarrow \bigoplus_{i=1}^r Y_i$ with $\ker(h) \in \Gamma$.

**Proof.** Assume that (a) holds. Then by Theorem 2.9, there exists a module $M$, an irreducible morphism $h : X \rightarrow \bigoplus_{i=1}^r Y_i$ and a non-zero morphism $\varphi : M \rightarrow X$ such that $\varphi h = 0$. Then $\varphi$ factors through $\ker(h)$ and therefore there is a positive integer $n$ such that the inclusion $j : \ker(h) \rightarrow X$ is in $\mathfrak{N}_n(\ker(h), X) \setminus \mathfrak{N}_{n+1}(\ker(h), X)$, proving that $\ker(h) \in \Gamma$.

Now, if $\ker(h) \in \Gamma$ then $h$ is an irreducible epimorphism. Consider the inclusion $j : \ker(h) \rightarrow X$. Since $\Gamma$ is a generalized standard component then there is a positive integer $m$ such that $j \in \mathfrak{N}_m(\ker(h), X) \setminus \mathfrak{N}_{m+1}(\ker(h), X)$. Since $hj = 0$ then it follows by the definition of left degree that $d_i(h)$ is finite and by Lemma 2.3, we have that $d_i(f)$ is finite. □

For standard algebras the above theorem can be stated as follows:

**Theorem 2.15.** Let $A$ be a standard algebra and $f : X \rightarrow \bigoplus_{i=1}^r Y_i$ an irreducible morphism with $X, Y_i \in \Gamma_A$ for $1 \leq i \leq r$. Then, $d_i(f)$ is finite if and only if $\ker(f) \in \Gamma_A$.

The converse of the result proven in Corollary 2.10 follows when $A$ is a standard algebra as we will show below.

**Corollary 2.16.** Let $A$ be a standard algebra. Let $f : X \rightarrow \bigoplus_{i=1}^r Y_i$ be an irreducible morphism with $X, Y_i \in \Gamma_A$, for $1 \leq i \leq r$. Then $d_i(f)$ is finite if and only if $f$ is an epimorphism.

**Proof.** By Corollary 2.10, we only need to prove that if $f$ is an epimorphism then $d_i(f)$ is finite. Since $f$ is an irreducible epimorphism then $\ker(f)$ is an indecomposable module and therefore it belongs to $\Gamma_A$. Then the result follows from Theorem 2.15. □

A dual result holds for the right degree. Our next result will show an useful property of the degrees.

**Lemma 2.17.** Let $A$ be a finite dimensional algebra over an algebraically closed field and $\Gamma$ a standard component of $\Gamma_A$. Let $f : X \rightarrow Y$ be an irreducible morphism with $X, Y \in \Gamma$. Then, if $d_i(f)$ is finite then $d_i(f) = \infty$.

**Proof.** Assume that $d_i(f) = n$. By the dual of Corollary 2.10, it follows that $f$ is a monomorphism. On the other hand, since $d_i(f)$ is finite then by Theorem 2.9 there is a module $M \in \Gamma$ an irreducible morphism $h : X \rightarrow Y$ and a morphism $\varphi \in \mathfrak{N}_n(M, X) \setminus \mathfrak{N}_{n+1}(M, X)$ such that $h \varphi = 0$. Moreover, since $f$ is an monomorphism then $h$ is a monomorphism and therefore $\varphi = 0$ which is a contradiction. Then, $d_i(f) = \infty$, proving the result. □

A dual result holds for the right degree. Observe that the converse of the above result is not necessarily true. In fact, if we consider a hereditary algebra of type $A_n$ with $n \geq 1$, then by [9, (4.3)], we know that all irreducible morphisms between indecomposable preprojective or preinjective modules have an infinite right and left degree. In particular, when $A$ is a standard algebra the converse of the above lemma holds.

**Corollary 2.18.** Let $A$ be a standard algebra. Let $f : X \rightarrow Y$ be an irreducible morphism with $X, Y \in \Gamma$. Then $d_i(f)$ is finite if and only if $d_i(f) = \infty$.

**Proof.** By the above lemma, we only need to prove that if $d_i(f)$ is infinite then $d_i(f)$ is finite. By Corollary 1.6, it is enough to prove that $f$ is an epimorphism. Assume that $f$ is a monomorphism. Then, $\ker(f) \in \Gamma_A$ and the projection $\pi : Y \rightarrow \ker(f)$ is a non-zero morphism. On the other hand, since $A$ is of finite representation type then $\mathfrak{N}_n(X, Y) = 0$, for each module $X, Y \in \Gamma_A$. Hence, $\pi \in \mathfrak{N}_n(Y, \ker(f)) \setminus \mathfrak{N}_{n+1}(Y, \ker(f))$, for some positive integer $n$. Moreover $\pi f = 0$. Therefore, $d_i(f) = n$ contradicting the hypothesis. Then $f$ is an epimorphism and by Corollary 2.16, we get the result. □

A dual result holds for the right degree.

**Remark 2.19.** The above result allows us to know easily if the degree of any irreducible morphism between indecomposable modules in a standard algebra is finite or infinite, determining only if it is an epimorphism or a monomorphism. We also observe that for irreducible morphisms $f : X \rightarrow \bigoplus_{i=1}^r Y_i$, the implication $d_i(f) = \infty$ then $d_i(f) < \infty$ of Corollary 2.18 holds.

In [14, Lemma 1.2], Liu proved that if $f : X \rightarrow Y$ is an irreducible morphism of finite left degree and $0 \rightarrow \tau Y \rightarrow X' \rightarrow f^{-1} \rightarrow Y$ is an almost split sequence ending at $Y$ then $d_i(g)$ is finite. Using the fact that a injective left minimal almost split morphism has infinite left degree, it is possible to prove the converse of such a lemma for standard components.
Proposition 2.20. Let $\Gamma$ be a standard component of $\Gamma_k$ and $f : X \to Y$ an irreducible morphism with $X, Y \in \Gamma$. Assume that $0 \to \tau Y \xrightarrow{(g, g')} X' \sqcup X \xrightarrow{(f', f)} Y \to 0$ is an almost split sequence ending at $Y$. Then $d_i(f)$ is finite if and only if $d_i(g)$ is finite. Moreover, $d_i(f) = n$ if and only if $d_i(g) = n - 1$.

Proof. By [14, Lemma 1.2], if $d_i(f)$ is finite then $d_i(g)$ is finite. Assume that $d_i(g)$ is finite. Then there is a module $Z \in \Gamma$ and a morphism $\varphi \in \mathfrak{N}^n(Z, \tau Y) \setminus \mathfrak{N}^{n+1}(Z, \tau Y)$ such that $g \varphi \in \mathfrak{N}^{n+2}(Z, X')$. By Corollary 2.11, $d_i((g, g')^i) = \infty$ and since $g \varphi \in \mathfrak{N}^{n+2}(Z, X')$ then $g \varphi \not\in \mathfrak{N}^{n+2}(Z, X)$.

On the other hand, since $f' g + fg' = 0$ then $f' g \varphi = f g \varphi \not\in \mathfrak{N}^{n+2}(Z, Y)$, proving that $d_i(f)$ is finite.

Now, suppose that $d_i(f) = n$. Then, by [14, Lemma 1.2], $d_i(g) \leq n - 1$. If $d_i(g) = t < n - 1$, then clearly $d_i(f) < n$, since by Corollary 2.11, the left degree of an injective left minimal almost split morphism is infinite. Then $d_i(g) = n - 1$.

Conversely, assume that $d_i(g) = n - 1$. Then there is a morphism $\varphi \in \mathfrak{N}^{n-1}(Z, \tau Y) \setminus \mathfrak{N}^n(Z, \tau Y)$ for some $Z \in \Gamma$ such that $g \varphi \in \mathfrak{N}^{n+1}(Z, X')$. Thus, $g \varphi \not\in \mathfrak{N}^{n}(Z, X)$ since $d_i((g, g')^i) = \infty$. Then, $d_i(f) = n$. \qed

3. Morphisms of degree two

In this section we will characterize the irreducible morphisms $f : X \to \bigoplus_{i=1}^r Y_i$ with $X, Y_i$ indecomposable modules over a standard component of the Auslander–Reiten quiver, for $1 \leq i \leq r$, of left degree two.

We will start proving some results for finite dimensional algebras over an algebraically closed field.

Lemma 3.1. Let $A$ be a finite dimensional algebra over an algebraically closed field and $f = (f_1, \ldots, f_r) : X \to \bigoplus_{i=1}^r Y_i$ an irreducible morphism with $Y_i$ indecomposable modules, for $i = 1, \ldots, r$. If $d_i(f) = 2$ then $d_i(f_i) = 1$.

Proof. Since $d_i(f) = 2$ then by [16, Lemma 3.1], $d_i(f_i) \leq 2$. Suppose that for some $i$, $1 \leq i \leq r$, $d_i(f_i) = 2$. Without loss of generality, we may assume that $d_i(f_i) = 2$. Then $Y_i$ is not projective and there is an almost split sequence

$$0 \to \tau Y_i \xrightarrow{(g, g')} Y' \bigoplus X \xrightarrow{(f', f)} Y_i \to 0$$

with $d_i(g_1) = 1$. Then $Y'$ is indecomposable. Moreover, by [7, Lemma 5.1], $Y' \not\cong X$ since $d_i(g_1)$ is finite and therefore $\dim \text{Inr}(\tau Y_i, Y') = 1$. Then, there is a configuration of almost split sequences of the form:

$$\begin{array}{ccc}
\tau Y' & \to & Y' \\
\downarrow & & \downarrow \\
\tau Y_1 & \to & \cdots & \to & Y_1 \\
\downarrow & & \uparrow & & \downarrow \\
X & \to & \cdots & \to & X \\
\end{array}$$

where $(\epsilon(Y'))$ is an almost split sequence with indecomposable middle term.

On the other hand, since $d_i(f) = 2$ then there is an indecomposable module $M$ and a morphism $\varphi \in \mathfrak{N}^2(M, X) \setminus \mathfrak{N}^3(M, X)$ such that $f \varphi \in \mathfrak{N}^3(M, X) \setminus \mathfrak{N}^4(M, X)$ for $1 \leq i \leq r$. Moreover, by [14, Lemma 1.2], there is a morphism $\varphi_1 \in \mathfrak{N}(M, \tau Y_1) \setminus \mathfrak{N}^2(M, \tau Y_1)$ such that $\varphi + g \varphi_1 \in \mathfrak{N}^2(M, \tau Y_1)$ and $g \varphi_1 \varphi_1 \not\in \mathfrak{N}^2(M, X)$. By [9, Theorem 2.2], we infer that $M \cong \tau Y'$. Therefore, $\varphi_1 = \alpha t + \nu$ where $\alpha \in k^*$ and $\nu \in \mathfrak{N}^2(\tau Y', \tau Y_1)$. Thus, we write $\varphi = g \varphi_1 + \mu = \alpha g' t + \mu'$, with $\mu', \mu' \in \mathfrak{N}^2(\tau Y', X)$. Observe that the path $\tau Y' \xrightarrow{\epsilon} \tau Y_1 \xrightarrow{\delta} X$ is sectional. Moreover, the path $\tau Y' \xrightarrow{\epsilon} \tau Y_1 \xrightarrow{\delta} X \xrightarrow{\gamma} Y_2$ is also sectional. Then $f \varphi = \alpha f_2 g' t + f_2 \mu'$ does not belong to $\mathfrak{N}^2(M, Y_2)$, a contradiction. Then $d_i(f_i) = 1$, for all $i$. \qed

A dual result holds for the right degree.

Proposition 3.2. Let $A$ be a finite dimensional algebra over an algebraically closed field, $k$. Let $f : X \to Y_1 \oplus Y_2$ be an irreducible morphism with $Y_1$ and $Y_2$ non-zero indecomposable $A$-modules. Then, $d_i(f) = 2$ if and only if there exist an indecomposable module $M$ and a path $\varphi$ that is a composite of two irreducible morphisms $\varphi \in \mathfrak{N}^2(M, X) \setminus \mathfrak{N}^3(M, X)$ such that $f \varphi = 0$.

Proof. Consider $f = (f_1, f_2) : X \to Y_1 \oplus Y_2$, with $d_i(f) = 2$. By Lemma 3.1, $d_i(f_1) = 1$ and $d_i(f_2) = 1$. Then there are almost split sequences

$$0 \to \tau Y_1 \xrightarrow{g} X \xrightarrow{f_1} Y_1 \to 0$$

for $i = 1, 2$. By [14, Lemma 1.11], $f$ has a left neighbor $g = (g_1, g_2) : \tau Y_1 \oplus \tau Y_2 \xrightarrow{\tau Y_1 \oplus \tau Y_2 \to X}$ with $d_i(g) < d_i(f)$. Thus, $d_i(g) = 1$. Moreover, $g$ is a surjective right minimal almost split morphism. Then, we have a configuration of almost split sequences of the form:

$$\begin{array}{ccc}
\tau Y_1 & \to & Y_1 \\
\downarrow & & \downarrow \\
\tau X & \to & \cdots & \to & \tau X \\
\downarrow & & \uparrow & & \downarrow \\
\tau Y_2 & \to & \cdots & \to & \tau Y_2 \\
\end{array}$$
Let $\tau X \rightarrow \bigoplus_{i=1}^{r} Y_i$ be an irreducible morphism with $Y_i \in \Gamma$, for $1 \leq i \leq r$. Then, there is an indecomposable module $\tau X$ and a path $\varphi \in \End(M) \setminus \End(M, X)$ such that $\varphi d = 0$, proving the result.

Now, if there exists a module $M$ and a morphism $\varphi \in \End(M) \setminus \End(M, X)$ such that $\varphi c = 0$, then $d_i(f) \leq 2$. Since $f : X \rightarrow Y$ is not a surjective minimal right almost split morphism then $d_i(f) = 2$, as we wish to prove.

Next, we are going to characterize the irreducible morphisms of left degree two of the form $f : X \rightarrow \bigoplus_{i=1}^{r} Y_i$, with $X, Y_i$ over a standard component of $\Gamma\chi$. We will start with the following previous result.

**Lemma 3.3.** Let $\Gamma_i$ be a standard component of $\Gamma\chi$ and $f : X \rightarrow \bigoplus_{i=1}^{r} Y_i$ be an irreducible morphism with $X, Y_i \in \Gamma$, for $1 \leq i \leq r$, then $d_i(f) = 2$.

**Proof.** Assume that $r \geq 3$. By Theorem 2.9, since $d_i(f) = 2$ there exists $M \in \Gamma\chi$, an irreducible morphism $h : X \rightarrow \bigoplus_{i=1}^{r} Y_i$ and a morphism $\varphi \in \End(M) \setminus \End(M, X)$ such that $h\varphi = 0$. Since by hypothesis $d_i(f) = 2$ then $d_i(h) = 2$. By Lemma 3.1, $d_i(h_i) = 1$ for each $i$.

Now, for each $1 \leq i \leq r$, consider the almost split sequence

\[ 0 \rightarrow \tau Y_i \rightarrow X \rightarrow Y_i \rightarrow 0 \]

Since $h_i\varphi = 0$ then there are morphisms $\varphi_i \in \End(M, \tau Y_i) \setminus \End(M, \tau Y_i)$ such that $\varphi_i = t_i\varphi_i$, for $1 \leq i \leq r$. Note that $\varphi_i$ are irreducible morphisms, for $1 \leq i \leq r$.

On the other hand, since $h_i\varphi = 0$ then $h_i t_i \varphi_i = 0$ for $i \neq j$. Therefore the path $M \rightarrow \tau Y_j \rightarrow X \rightarrow Y_j$ is not sectional. By [7, Lemma 5.1], since $\dim_{\End}(X, Y_j) = 1$, for $1 \leq i \leq r$ then $Y_i \neq Y_j$ for $i \neq j$. As a consequence, $M \cong \tau X$ or $\tau^{-1}M \cong X$ and $X$ is not projective or $M$ is not injective, respectively.

Consider $X$ to be non-projective and assume that

\[ 0 \rightarrow \tau X \rightarrow \bigoplus_{i=1}^{r} \tau Y_i \rightarrow X \rightarrow 0 \]

is an almost split sequence ending at $X$. Since $M \cong \tau X$ and $A$ is a finite dimensional algebra over an algebraically closed field then $g_i = \alpha_i \varphi_i + \mu_i$ with $\alpha_i \in \mathbb{K}$ and $\mu_i \in \End(\tau X, \tau Y_i)$, for $i = 1, 2, \ldots, r$.

Now, since $h_i t_i g_i = h_i t_i (\alpha_i \varphi_i + \mu_i) = \alpha_i h_i t_i \varphi_i + h_i t_i \mu_i$ with $i \neq j$ and $h_i t_i \varphi_i = 0$ then it follows that $h_i t_i g_i \in \End(\tau X, Y_j)$. Note that, by [6, Theorem 2.2], we have that $h_i t_i \notin \End(\tau Y_i, Y_j)$ and $\varphi_j \in \End(\tau X, X)$.

If $h_i t_i \mu_i \neq 0$ then by [8, Theorem 2.3], it follows that $\alpha(X) \leq 2$ which is a contradiction to the assumption. Otherwise, if $h_i t_i \mu_i = 0$ then $h_i t_i g_i = 0$. Since $g_i$ is not an injective left minimal almost split morphism and $h_i t_i \notin \End(\tau Y_i, Y_j)$ then $d_i(g_i) = 2$. Thus, $\alpha(X) \leq 2$ which is a contradiction to the assumption.

A similar analysis as the one done above, discards the case when $M$ is not injective, proving the result.

As a consequence of the above results we state the following theorem.

**Theorem 3.4.** Let $\Gamma$ be a standard component of $\Gamma\chi$ and $f : X \rightarrow Y$ be an irreducible morphism where $Y = \bigoplus_{i=1}^{r} Y_i$ with $Y_i \in \Gamma$, for $1 \leq i \leq r$. Then, the following conditions are equivalent:

(a) $d_i(f) = 2$.
(b) $r \leq 2$, $f$ is not a surjective right almost split morphism and there exists $M \in \Gamma$ and a path of irreducible morphisms $\varphi \in \End(M) \setminus \End(M, X)$ such that $\varphi c = 0$.
(c) One of these configurations of almost split sequences holds:

(i)
\[
\begin{array}{cccc}
\tau X & \rightarrow & \cdots & \rightarrow X_1 \\
\downarrow f_1 & & & \downarrow g_1 \\
\tau Y & \rightarrow & \cdots & \rightarrow Y \\
\downarrow f_2 & & & \downarrow f \\
& & & \downarrow \\
& & & X \\
\end{array}
\]

(ii)
\[
\begin{array}{cccc}
\tau X & \rightarrow & \cdots & \rightarrow X_1 \\
\downarrow f_1 & & & \downarrow g_1 \\
\tau Y_1 & \rightarrow & \cdots & \rightarrow Y_1 \\
\downarrow f_2 & & & \downarrow f_2 \\
& & & \downarrow \\
& & & \tau Y_2 \\
\end{array}
\]
A dual result holds for the right degree.
Now, we are going to show an algebra having an irreducible morphism of left degree two.

**Example 3.5.** Let $A$ be the algebra given by the quiver:

$$
\begin{array}{ccc}
1 & \alpha & \gamma \\
5 & \beta & \delta \\
4 & \epsilon & 3
\end{array}
$$

with relations $\beta\alpha = 0$ and $\epsilon\delta = 0$. Then $A$ is a triangular algebra of finite representation type and the Auslander–Reiten quiver $\Gamma_A$ of $A$ is the following:

$$
\begin{array}{ccccccc}
P_3 & \ldots & f_1 & \ldots & P_1 & \ldots & I_3 \\
P_4 & \ldots & I_4 & \ldots & P_2 & \ldots & I_2 \\
P_5 & \ldots & 3 & \ldots & 2 & \ldots & I_1
\end{array}
$$

Clearly, by Theorem 3.4 the irreducible morphism $(f_1, f_2)$ has left degree 2. Observe that the irreducible morphism $(g_1, g_2)$ does not have right degree two. It is not hard to prove that $d_l((g_1, g_2)) = 3$.

For the remainder of this section, we will show a property of the irreducible morphisms of left degree two in Auslander–Reiten components with $\alpha(\Gamma') \leq 2$.

**Proposition 3.6.** Let $A$ be a finite dimensional algebra over an algebraically closed field and $\Gamma$ a component of $\Gamma_A$ with $\alpha(\Gamma') \leq 2$. Let $f : X \to Y$ be an irreducible morphism, with either $X$ or $Y$ indecomposable in $\Gamma$. Assume that $d_l(f) = 2$. Then

(a) $X$ is indecomposable.
(b) If $f : X \to Y_1 \oplus Y_2$ with $Y_1, Y_2 \in \Gamma'$ then $X$ is an injective module.

**Proof.** Assume that $X = X_1 \oplus X_2$. Since $d_l(f) = 2$ then $Y$ is not projective. Then, there is an almost split sequence where $f$ has a surjective right minimal almost split sequence. By [14], $d_l(f) = 1$ which is a contradiction. As a result, $X$ is an indecomposable module.

(b) Follows from the fact that $f$ is not an injective left minimal almost split morphism and $\alpha(\Gamma') \leq 2$. □

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**References**