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A SELF-CORRECTING POINT PROCESS

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Suppose a point process is attempting to operate as closely as possible to a deterministic rate ρ, in the sense of aiming to produce ρt points during the interval $(0, t]$ for all t. This can be modelled by making the instantaneous rate of t of the process a suitable function of $n - \rho t$, n being the number of points in $[0, t]$. This paper studies such a self-correcting point process in two cases; when the point process is Markovian and the rate function is very general, and when the point process is arbitrary and the rate function is exponential. In each case it is shown that as $t \to \infty$ the mean number of points occurring in $(0, t]$ is $\rho t + O(1)$ while the variance is bounded; further, in the Markov case all the absolute central moments are bounded. An application to the outputs of stationary D/M/s queues is given.

Comparison method	Markov birth process
controlled variability	point process
count-conditional intensity	self-correcting

1. Introduction

Consider a point process $N(\cdot)$ on $[0, \infty)$. Denote by $N(t)$ the number of points in $(0, t]$, with $N(0) = 0$, and let $\mu(t)$ and $V(t)$ represent its mean and variance, respectively.

Initially, suppose $N(\cdot)$ is also *Markovian* so that, in more usual language, it is a pure birth process. Assume it has an instantaneous birth rate $\lambda_n(t)$ at time t, given $N(t) = n$.

In this paper, we study the case

$$\lambda_n(t) = \rho\phi(n - \rho t) \quad (n = 0, 1, 2, \ldots; t \geq 0), \tag{1}$$

where ρ is a positive finite constant and $\phi(\cdot)$ has the following properties:

(A1) $\quad 0 \leq \phi(x) < \infty \quad (x \in \mathbf{R}$, the real line),

(A2) $\quad (\exists \alpha > 0)\phi(x) \geq \alpha \quad (x < 0)$,

(A3) $\quad \liminf_{x \to -\infty} \phi(x) > 1, \limsup_{x \to +\infty} \phi(x) < 1.$

Informally, we may describe the situation as follows. There is a target for $N(\cdot)$, namely that it operates at a rate ρ in the strong sense of aiming to produce ρt points over $(0, t]$. (The weaker aim of merely operating at an instantaneous birth rate ρ can always be achieved by a suitable change of time scale). The process is *self-correcting* in that if the difference between $N(t)$ and the target ρt strays too far from zero the birth rate automatically compensates to force this difference back towards zero; mathematically, this is contained in (A3). Property (A2) serves to ensure that $N(\cdot)$ does not get stranded below ρt. A simple vivid special case is $\phi(\cdot)$ nonconstant and nonincreasing over $\mathbf{R}$ with $\phi(0) = 1$.

Because of the strong nature of the correction mechanism it seems almost certain that $\mu(t)$ is approximately ρt. It is also plausible that the variability of the process about its mean, as exemplified by the absolute central moments of $N(t)$, is bounded as t increases. We prove these results in Section 2 of the paper, thus establishing in particular that $N(\cdot)$ is a point process of controlled variability [9].

In Section 3 we no longer assume $N(\cdot)$ to be Markovian. Let its complete intensity function be $\lambda(t|F_t)$, where F_t is the σ-field generated by $\{N(s), 0 \leq s \leq t\}$ ([4]; $\lambda(\cdot|\cdot)$ is also known as the conditional intensity function). Following Snyder [13] we define the *count-conditional intensity* $\hat{\lambda}_n(t)$ by

$$\hat{\lambda}_n(t) = \mathbf{E}\{\lambda(t|F_t)|N(t) = n\},$$

where the expectation is over the history of $N(\cdot)$ to the left of t. It is now an exercise in conditional probability to show, under obvious regularity conditions, that the probabilities $p_n(t) = \mathbf{P}\{N(t) = n\}$ satisfy the usual Kolmogorov forward equations for a birth process with birth rate $\hat{\lambda}_n(t)$, the boundary condition being $p_0(0) = 1$ [13, Theorem 5.2.1]. That is, the marginal counting distributions over $[0, t)$ of the non-Markov point process and the "corresponding" (Markov) birth process, given the initial conditions, are identical. Were $N(\cdot)$ in fact Markov then clearly $\hat{\lambda}_n(\cdot) \equiv \lambda_n(\cdot)$.

Although this perspective may be unfamiliar the idea is extensively used by Snyder. Of course, in general $\hat{\lambda}_n(\cdot)$ does not help us to determine other properties of $N(\cdot)$ and it may be very difficult to calculate in any particular case. However, it was consideration of situations where $\hat{\lambda}_n(\cdot)$ itself might have a simple and meaningful form which led to the present investigation, since it is reasonable that a self-correcting mechanism with the target considered in this paper should operate purely from the current value of $N(t)$ even for a non-Markov point process (we owe this remark to Professor D.R. Cox).

Because in principle the moments of $N(t)$ are determined by the $p_n(t)$ we could hope to study them quite generally by solving the forward equations when $\hat{\lambda}_n(\cdot)$ takes the form (1) to get an explicit expression for these moments. Unfortunately the equations do not typically have a closed-form solution then; the only exception of which we are aware is when $\phi(x) = e^{-\theta x}$ ($\theta > 0$, $x \in \mathbf{R}$). The main result of Section 3 is that the corresponding point process is one of controlled variability, thus demonstrating that the Markovian theorem is still valid, in one case at least, even for non-Markov self-correcting point processes. However, the method of proof, a direct analytical attack on the variance formula, is in marked contrast to the technique for the earlier result which relies essentially on the Markov property. A similar conclusion is no doubt true for the higher absolute central moments, but the algebra seems prohibitively complicated and in any case the variance is of principal interest for most purposes.

Finally, in Section 4 we briefly discuss the necessity of the assumptions (A1)–(A3) and give an application to the outputs of stationary D/M/s queues.

2. The Markov case

In this section we assume that the point process $N(\cdot)$ described in the Introduction is Markovian. Denote its jump times by $\{S_j\}$ ($j = 1, 2, \ldots$), with $S_0 = 0$. Clearly $\{S_j\}$ is a nonhomogeneous Markov chain. We also define $Y(t) = N(t) - t$, a homogeneous jump-linear Markov process with unit jumps upwards, at $\{S_j\}$, and a linear deterministic drift downwards between jumps. Its instantaneous jump rate at t, given $Y(t) = y$, is $\phi(y)$. Note that henceforth we take $\rho = 1$ without loss of generality.

Theorem 1. *If $N(\cdot)$ is a Markov point process on $[0, \infty)$ with birth rate given by (1), then as $t \to \infty$,*

(i) $\limsup |\mu(t) - t| < \infty$,
(ii) $\limsup \mathbf{E}|N(t) - \mu(t)|^r < \infty$ $(r > 0)$.

Note. It will become clear during the proof why neither quantity actually converges.

Proof. This is conveniently divided into the following sequence of lemmas.

Lemma 1. *$Y(\cdot)$ is regenerative, one set of regeneration points being those at which $Y(t) = 0$. The associated renewal process is periodic, with period 1.*

The definition of a regenerative stochastic process may be found in [11, 12]. Given this, the lemma is obvious and needs no proof. Note that $Y(\cdot)$ regenerates during a downward drift, i.e. reaches 0 from above, with probability one.

Let the mean interval between regeneration points of $Y(\cdot)$ be μ_0.

Lemma 2. *We have $\mu_0 < \infty$, whence*

$$\lim_{n\to\infty} \mathbf{P}\{Y(n) \in A \mid Y(0) = 0\} = \frac{1}{\mu_0} \sum_{n=0}^{\infty} \mathbf{P}\{Y(n) \in A,\ Y(t) \neq 0\ (0 < t \leq n) \mid Y(0) = 0\}$$
$$= W(A),\ \textit{say},$$

for any Borel set A on $\mathbb{R}$.

Proof. Because the process regenerates during a downward drift, it is necessary to show $\mu_0 < \infty$ in two stages; first prove that the expected time until $Y(\cdot)$ first enters $(0, \infty)$ is finite, then do the same for the time until next $Y(t) = 0$. Both results are very intuitive, since from (A3) the average "drift" of $Y(\cdot)$ is eventually towards the axis, and formally they are consequences of Theorems 4 and 4′ of Tweedie and Westcott [14]; see Section 3.3 of that paper for a detailed treatment. Note that the persistence of the regeneration points follows from the finite mean.

Having established $\mu_0 < \infty$ the rest of the lemma is Theorem 3 of Smith [11].

Lemma 3. *For any $r > 0$ $\lim_{n\to\infty} \mathbf{E}\{|Y(n)|^r\} < \infty$ if and only if the probability distribution induced by $W(\cdot)$ has finite r^{th} moment.*

Proof. Clearly $\{Y(n)\}$ is an irreducible ergodic Markov chain on $\{0, 1, \ldots\}$ so the result follows from Theorem 2.1 and Remark 2.2(2) of Holewijn and Hordijk [8].

For the next two lemmas, note that from (A3) we may find $k > 0, \beta_1 > 1, \beta_2 < 1$ such that

$$\phi(x) \geq \beta_1 \quad (x < -k), \qquad \phi(x) \leq \beta_2 \quad (x > k).$$

Lemma 4. $\mathbf{P}\{Y(n) \leq -y,\ Y(u) < 0\ \ (0 < u \leq n)\ \mid\ Y(0) = 0\} \leq a_n\, \mathrm{e}^{-\zeta y}\quad (y > 0)$, $n = 1, 2, \ldots$), *where ζ is a positive constant, $a_n \geq 0$ and $\sum_1^\infty a_n < \infty$.*

Proof. Corresponding to $N(\cdot)$, define a new point process $N^*(\cdot)$ on $[0, \infty)$ by its jump times $\{S_j^*\}(j = 1, 2, \ldots)$, where $S_0^* \equiv 0$ and

$$\begin{aligned} S_j^* &= \max(S_j, j) \quad (S_{j-1}^* > j-1), \\ &= S_{j-1}^* \quad (\text{otherwise}) \end{aligned} \qquad (j = 1, 2, \ldots); \tag{2}$$

so $\{S_j^*\}$ is also a Markov chain. Note that $Y^*(\cdot)$ "explodes" to $+\infty$ instantaneously as soon as it becomes nonnegative.

Let the transition functions for S_j be

$$p_j(x; z) = \mathbf{P}\{S_j^* \leq z \mid S_{j-1}^* = x\} \quad (j = 1, 2, \ldots; x, z \in \mathbf{R});$$

note that $p_j(\cdot, \cdot)$ is a distribution function in its second argument. Then

$$\begin{aligned} p_j(x; z) &= 1 - \exp\left\{-\int_x^z \phi(j-1-t)\,\mathrm{d}t\right\} && (x > j-1, z \geq j, x \leq z), \\ &= H(z-x) && (x \leq j-1), \\ &= 0 && \text{otherwise}, \end{aligned} \tag{3}$$

where $H(\cdot)$ is the Heaviside unit function. Further, for each fixed z and j,

$$p_j(x; z) \leq p_j(y; z) \quad (y \leq x). \tag{4}$$

If now we have another self-correcting point process $N_1(\cdot)$ with jump times $\{T_j\}$ and

$$\phi_1(x) \leq \phi(x) \quad (x \leq 0), \tag{5}$$

denote the transition functions of the corresponding $\{T_j^*\}$, defined as in (2), by $q_j(x; z)$; $q_j(\cdot|\cdot)$ thus takes the form (3) with ϕ_1 instead of ϕ. Then from (5) and (4), for each fixed z and j,

$$q_j(x; z) \leq p_j(y; z) \quad (y \leq x). \tag{6}$$

We take

$$\phi_1(x) = \begin{cases} \alpha & 0 > x \geq -k, \\ \beta_1 & x < -k. \end{cases} \tag{7}$$

We may now apply a comparison technique due to O'Brien [10] to conclude that

$$\mathbf{P}(T_j^* \leq z) \leq \mathbf{P}(S_j^* \leq z) \quad (z \in \mathbf{R}; j = 1, 2, \ldots). \tag{8}$$

(Strictly, O'Brien's results are all for homogeneous chains, but as he points out the modifications required to extend them to the nonhomogeneous case are straightforward and are left to the reader. Our (6) is the nonhomogeneous analogue of O'Brien's (3.2)). Then the fundamental duality between intervals and counts in a point process let us deduce from (8) that

$$\begin{aligned} &\mathbf{P}\{Y^*(n) \leq -y,\ Y^*(u) < 0\ (0 < u \leq n) \mid Y^*(0) = 0\} \leq \\ &\quad \leq \mathbf{P}\{Y_1^*(n) \leq -y,\ Y_1^*(u) < 0 (0 < u \leq n) \mid Y_1^*(0) = 0\} \end{aligned} \tag{9}$$

($n = 1, 2, \ldots, y > 0$).

The point of this manoeuvre is that the right side of (9) effectively involves only a Poisson process $N_1^*(\cdot)$, rate either α or β_1, which is easier to handle. For consider a last-exit decomposition [3, p. 46] for $Y_1^*(\cdot)$ based on a *downward* passage across $-k$. Clearly this is a delayed recurrent event and by Lemma 2 it is transient, so the associated renewal sequence $\{U_m\}$ satisfies $\sum_1^\infty U_m < \infty$. The decomposition gives, if $y > k$,

$$\begin{aligned} &\mathbf{P}\{Y_1^*(n) \leq -y,\ Y_1^*(u) < 0 (0 < u \leq n) \mid Y_1^*(0) = 0\} = \\ &\quad = \sum_{m=k}^{n+k-y} U_m \mathbf{P}\{Y_1^*(n) \leq -y,\ Y_1^*(u) < 0\ (0 < u \leq n) \mid \\ &\qquad\qquad \text{last exit from } [-k, 0) \text{ at } m,\ Y_1^*(0) = 0\}. \end{aligned} \tag{10}$$

Since in (10) we only consider events in the $Y_1^*(\cdot)$ process before it "explodes", the same equation holds for $Y_1(\cdot)$. Further, by the construction of $N_1(\cdot)$, it may be treated as a Poisson process of rate β_1 for the event whose probability is sought in (10). Since the Poisson process has independent increments, we find

$$\begin{aligned}\text{RHS of (10)} &\leqslant \sum_{m=k}^{n+k-y} U_m \mathrm{P}\{N_1(n-m)-(n-m)<-y+k \mid N_1(0)=0\}\\ &\leqslant \sum_{m=k}^{n+k-y} U_m\, \mathrm{e}^{\zeta(-y+k+n-m)}\, \mathrm{e}^{\beta_1(n-m)(\mathrm{e}^{-\zeta}-1)} \quad (\zeta>0),\end{aligned}$$

from the elementary inequality $\mathbf{P}(X\leqslant x)\leqslant \mathrm{e}^{\zeta x}\, \mathbf{E}(\mathrm{e}^{-\zeta X})(\zeta, x>0)$ for any nonnegative random variable X,

$$\leqslant \sum_{m=k}^{n} U_m C\, \mathrm{e}^{-\zeta y}\, \mathrm{e}^{-\gamma(n-m)},$$

where $0<C<\infty$, $\zeta>0$, $\gamma>0$; we may make $\gamma\equiv\beta_1(1-\mathrm{e}^{-\zeta})-\zeta$ positive by suitable choice of ζ,

$$\equiv a_n\, \mathrm{e}^{-\zeta y},$$

say, where $a_n = C\sum_{m=k}^{n} U_m\, \mathrm{e}^{-\gamma(n-m)}$. As

$$\sum_{n=1}^{\infty} a_n \leqslant C \sum_{n=1}^{\infty}\sum_{m=1}^{n} U_m\, \mathrm{e}^{-\gamma(n-m)} = C\Big(\sum_1^{\infty} U_m\Big)\Big(\sum_1^{\infty} \mathrm{e}^{-\gamma m}\Big)<\infty,$$

the lemma is proved when $y>k$. For $0<y\leqslant k$, take

$$a_n = \mathrm{e}^{\zeta k}\mathbf{P}\{Y(u)<0\ (0<u\leqslant n) \mid Y(0)=0\};$$

then $\mathrm{e}^{-\zeta k}\sum_1^{\infty} a_n$ is the mean first-passage time across zero which is finite by Lemma 2.

Lemma 5. $\mathbf{P}\{Y(n)>y,\ Y(u)\neq 0\ (0<u\leqslant n) \mid Y(0)=0\}\leqslant a'_n\, \mathrm{e}^{-\zeta y}\ (y>0, n=1,2,\ldots)$, *where ζ is a positive constant, $a'_n\geqslant 0$ and $\sum_1^{\infty} a'_n<\infty$.*

Proof. This is in many respects similar to the previous proof so will only be sketched. The principal difference is that, as in Lemma 2, we must consider the sojourns of $Y(\cdot)$ below and above the axis separately. We have

$$\begin{aligned}&\mathbf{P}\{Y(n)>y,\ Y(u)\neq 0\ (0<u\leqslant n) \mid Y(0)=0\} =\\ &= \int_0^n\int_{-1}^0 \mathbf{P}\{Y(s)=x,\ Y(v)<0\ (0<v\leqslant s) \mid Y(0)=0\}\phi(x)\\ &\qquad \times \mathbf{P}\{Y(n)>y,\ Y(u)\neq 0\ (s<u\leqslant n) \mid Y(s+)=x+1\}\, \mathrm{d}x\, \mathrm{d}s \qquad (11)\\ &\leqslant M\int_0^n\int_{-1}^0 \psi(s)\mathbf{P}\{Y(n)>y,\ Y(u)\neq 0\ (s<u\leqslant n) \mid Y(s+)=x+1\}\, \mathrm{d}x\, \mathrm{d}s,\end{aligned}$$

where $M\equiv\sup_{x\in(-1,0)}\phi(x)<\infty$, by (A1), and $\psi(s)=\mathbf{P}\{Y(u)<0, 0<u\leqslant s\}$. By Lemma 2, $\int_0^{\infty}\psi(s)\, \mathrm{d}s<\infty$.

Now exactly as in Lemma 4 we can construct a process $Y_2(\cdot)$ which is "more extreme" than $Y(\cdot)$ in $(0,\infty)$ and is based on a point process $N_2(\cdot)$ which is Poisson of rate $\beta_2<1$ whenever $Y_2>k$. Then in (11) we again use a last-exit (from $(0,k]$) decomposition, this inducing a delayed renewal process in continuous time. Let its renewal function be $H(\cdot)$; since the process is transient we have $H(\infty)<\infty$. The decomposition gives, eventually, if $y>k$,

$$\mathbf{P}\{Y_2(n)>y,\ Y_2(u)\neq 0\ (s<u\leqslant n)\mid Y_2(s+1)=x+1\}\leqslant$$

$$\leqslant \int_s^n\int_k^{k+1}\mathbf{P}\{Y_2(n)>y,\ Y_2(u)\neq 0\ (s<u\leqslant n)\mid \text{last exit from } (0,k] \text{ at } v,$$

$$Y_2(v+)=z,\ Y_2(s+)=x+1\}\,\mathrm{d}z\,\mathrm{d}H(v)$$

$$\leqslant \int_s^n \mathbf{P}\{N_2(n-v)>n-v+y-(k+1)\mid N_2(0)=0\}\,\mathrm{d}H(\mathrm{v}),$$

where $N_2(\cdot)$ is Poisson, rate $\beta_2<1$,

$$\leqslant C\,\mathrm{e}^{-\zeta y}\int_s^n \mathrm{e}^{-\gamma'(n-v)}\,\mathrm{d}H(v), \tag{12}$$

using the same sort of inequalities as before, where $0<C<\infty$, $\zeta>0$, $\gamma'>0$. From (11) and (12), if $y>k$,

$$\mathbf{P}\{Y(n)>y,\ Y(u)\neq 0\ (0<u\leqslant n)\mid Y(0)=0\}\leqslant$$

$$\leqslant MC\,\mathrm{e}^{-\zeta y}\int_0^n\int_s^n \psi(s)\,\mathrm{e}^{-\gamma'(n-v)}\,\mathrm{d}H(v)\,\mathrm{d}s$$

$$\equiv a_n'\,\mathrm{e}^{-\zeta y},\ \text{say}.$$

And

$$\sum_1^\infty a_n' = MC\sum_1^\infty\int_0^n\int_0^v \psi(s)\,\mathrm{e}^{-\gamma'(n-v)}\,\mathrm{d}H(v)\,\mathrm{d}s$$

$$\leqslant C'\int_0^\infty \psi(s)\,\mathrm{d}s\sum_{j=0}^\infty\int_j^{j+1}\mathrm{e}^{-\gamma'(j-v)}\,\mathrm{d}H(v)$$

$$\leqslant C''\int_0^\infty \psi(s)\,\mathrm{d}s\cdot H(\infty)<\infty,$$

as required. For $0<y\leqslant k$, the argument is as before.

To complete the proof of the theorem, we have the identity

$$\mathbf{E}(|X|^r)=r\int_0^\infty x^{r-1}\,\mathbf{P}\{|X|>x\}\,\mathrm{d}x\quad (r>0). \tag{13}$$

So from Lemmas 2, 4, 5 and (13), the absolute rth moment of $W(\cdot)$ is bounded by

$$\frac{r}{\mu_0}\sum_{n=0}^\infty (a_n+a_n')\int_0^\infty y^{r-1}\,\mathrm{e}^{-\zeta y}\,\mathrm{d}y,$$

which is finite. Thus $\lim_{n\to\infty} \mathrm{E}\{|Y(n)|^r\}<\infty$ by Lemma 3. Now if $n \le t < n+1$ (n an integer) we get, by repeated use of $|a+b|^r \le 2^r(|a|^r+|b|^r)$,

$$\begin{aligned}|\mathrm{E}\{|Y(t)|^r\}-\mathrm{E}\{|Y(n)|^r\}| &\le 2^r[\mathrm{E}\{|Y(t)-Y(n)|^r\}+\mathrm{E}\{|Y(n)|^r\}]\\ &\le 2^r[\mathrm{E}\{N(n+1)-N(n)\}^r+1+\mathrm{E}\{|Y(n)|^r\}]\\ &\le 2^{4r}[\mathrm{E}\{|Y(n+1)|^r\}+\mathrm{E}\{|Y(n)|^r\}+1]\end{aligned}$$

which remains bounded by the convergence of $\mathrm{E}\{|Y(n)|^r\}$. Hence, for all $r>0$,

$$\limsup_{t\to\infty} \mathrm{E}\{|Y(t)|^r\}<\infty,$$

which implies

$$\limsup_{t\to\infty} |\mathrm{E}\{Y(t)\}|^r<\infty.$$

Results (i) and (ii) now follow readily.

3. The non-Markov case with $\phi(x)=\mathrm{e}^{-\theta x}$

As described in Section 1 we now consider $N(\cdot)$ to be a general point process with a rather specific $\hat{\lambda}_n(t)$.

Theorem 2. *If $N(\cdot)$ is a point process on $[0,\infty)$ with count-conditional intensity $\hat{\lambda}_n(t)=\mathrm{e}^{-\theta(n-t)}$ ($t\ge 0$; $n=0,1,2,\dots$), where $\theta>0$ is constant, then as $t\to\infty$,*

(i) $\limsup|\mu(t)-t|<\infty$,

(ii) $\limsup V(t)<\infty$.

Proof. The 'Kolmogorov equations' for $p_n(t)$ are

$$\lambda^t(\mathrm{d}/\mathrm{d}t)p_n(t)=-\lambda^n p_n(t)+\lambda^{n-1}p_{n-1}(t)(1-\delta_{n,0}) \quad (n=0,1,2,\dots), \tag{14}$$

where $\lambda=\mathrm{e}^{-\theta}$ (so $0<\lambda<1$) and $\delta_{n,0}$ is the Kronecker delta, with $p_n(0)=\delta_{n,0}$. But as the coefficient involving time has factored out, by changing the time scale to $\tau=\theta^{-1}(\lambda^{-t}-1)$ (14) can be made homogeneous. The general solution is now well known [1, p. 57] and leads directly to

$$\mu(\tau)=\left\{\sum_{j=0}^{\infty}(-1)^j\lambda^{(1/2)j(j+1)}\pi_j(\lambda)\right\}\left\{\sum_{j=0}^{\infty}(1-\mathrm{e}^{-\lambda^j\tau})\pi_j(\lambda)\right\}, \tag{15}$$

$$\begin{aligned}\mu_{[2]}(\tau)&\equiv\mathrm{E}[N(\tau)\{N(\tau)-1\}]\\ &=2\left\{\sum_{j=0}^{\infty}(-1)^j\lambda^{(1/2)j(j+1)}\pi_j(\lambda)\right\}\left\{\sum_{j=0}^{\infty}j(1-\mathrm{e}^{-\lambda^j\tau})\pi_j(\lambda)\right\}\\ &\quad+2\left\{\sum_{j=0}^{\infty}(-1)^j j\lambda^{(1/2)j(j+1)}\pi_j(\lambda)\right\}\left\{\sum_{j=0}^{\infty}(1-\mathrm{e}^{-\lambda^j\tau})\pi_j(\lambda)\right\},\end{aligned} \tag{16}$$

with

$$V(\tau) = \mu_{[2]}(\tau) + \mu(\tau) - \mu^2(\tau). \tag{17}$$

In these formulae we have written $\pi_j(\lambda)$ for $\{\prod_{l=1}^{j}(1-\lambda^l)\}^{-1}$.

To aid in the dissection of (15)–(17) we state:

Lemma 6. *Consider a sum of the form* $\sum_{j=0}^{\infty} a_j(\tau) b_j$, *where*

(i) $0 < b_j \uparrow b$ *as* $j \to \infty$;

(ii) $0 < a_j(\tau) \uparrow 1$ *as* $\tau \to \infty$ *for each fixed* $j = 0, 1, 2, \ldots$;

(iii) $0 < A_j(\tau) = \sum_{l=0}^{j} a_l(\tau) \uparrow A_\infty(\tau) < \infty$ *as* $j \to \infty$ *for each fixed* τ.

Then as $\tau \to \infty$,

$$\sum_{j=0}^{\infty} a_j(\tau) b_j = A_\infty(\tau) b - \sum_{j=0}^{\infty} (j+1)(b_{j+1} - b_j) + o(1). \tag{18}$$

Lemma 7

$$\prod_{l=1}^{\infty} (1-\lambda^l x) = \sum_{j=0}^{\infty} (-1)^j \lambda^{(1/2)j(j+1)} x^j \pi_j(\lambda), \tag{19}$$

$$\left\{\prod_{l=1}^{\infty} (1-\lambda^l x)\right\}^{-1} = \sum_{j=0}^{\infty} (\lambda x)^j \pi_j(\lambda) \quad (0 \leq x < \lambda^{-1}). \tag{20}$$

Lemma 8. *If* $f(\cdot)$ *is a differentiable function on* $[0, \infty)$ *with derivative* $f'(\cdot)$, *then*

$$\sum_{j=0}^{\infty} f(j) = \int_0^{\infty} f(x)\,dx + f(0) + \int_0^{\infty} (x - [x]) f'(x)\,dx, \tag{21}$$

where $[x] \equiv$ *integer part of* x.

Lemma 6 is proved by routine analysis plus summation by parts; Lemma 7 is just algebra (see [7, p. 489]), while Lemma 8 is a version of the Euler–Maclaurin summation formula.

For notational convenience let B_1 and B_2, respectively, represent the first sum in each term in (16). Applying Lemma 6 to the second sum in (15) we have $b = \pi_\infty(\lambda)$ so, from (19) with $x = 1$, $B_1 b = 1$. Thus as $\tau \to \infty$,

$$\mu(\tau) = \sum_{j=0}^{\infty} (1 - e^{-\lambda^j \tau}) - B_1 \sum_{j=1}^{\infty} j\lambda^j \pi_j(\lambda) + o(1), \tag{22}$$

and let $C_1 = B_1 \sum_{j=1}^{\infty} j\lambda^j \pi_j(\lambda)$. But from Lemma 8,

$$\sum_{j=0}^{\infty} (1 - e^{-\lambda^j \tau}) = \int_0^{\infty} (1 - e^{-\tau e^{-\theta x}})\,dx + 1 - e^{-\tau} + R_1(\tau), \tag{23}$$

say, where $R_1(\cdot)$ represents the final 'remainder' term in (21). The integral in (23) is easily seen to be

$$\theta^{-1}\log\tau+\theta^{-1}\gamma+O(e^{-\tau}) \tag{24}$$

as $\tau\to\infty$, where γ is Euler's constant, and further

$$|R_1(\tau)|\leq\int_0^\infty |f'(x)|\,dx=1-e^{-\tau}\leq 1. \tag{25}$$

So from (24)–(25), as $\tau\to\infty$,

$$\mu(\tau)=\theta^{-1}\log\tau+O(1)$$

and we arrive at (i) by changing back to the original time scale.

For the variance, use Lemmas 6 and 7 as above on (16) and (17) to get eventually, as $\tau\to\infty$,

$$V(\tau)=2\sum_{j=0}^{\infty} j(1-e^{-\lambda^j\tau})-\left\{\sum_{j=0}^{\infty}(1-e^{-\lambda^j\tau})\right\}^2+C_2\sum_{j=0}^{\infty}(1-e^{-\lambda^j\tau})+O(1), \tag{26}$$

where $C_2=(2B_2b+1+2C_1-o(\{\log\tau\}^{-1}))$. Apply Lemma 8 to the first sum; the resulting integral may be evaluated to give

$$\sum_{j=0}^{\infty} j(1-e^{-\lambda^j\tau})=(2\theta^2)^{-1}\log^2\tau+\gamma\theta^{-2}\log\tau+$$
$$+(2\theta^2)^{-1}(\tfrac{1}{6}\pi^2+\gamma^2)+R_2(\tau)+O(e^{-\tau}) \tag{27}$$

as $\tau\to\infty$, $R_2(\cdot)$ being the "remainder" term as before. So from (23)–(27), after some simplification, as $\tau\to\infty$,

$$V(\tau)=\theta^{-1}\log\tau\{C_3-2-2R_1(\tau)\}+2R_2(\tau)+O(1),$$

where $C_3=2B_2b+1+2C_1$.

To assess the contribution from the "remainder" terms, consider $D(\tau)\equiv 2\{R_2(\tau)-\theta^{-1}\log\tau R_1(\tau)\}$, with $\tau=e^{\theta[I+\eta]}$, where I is an integer and $0\leq\eta<1$. We find that

$$\begin{aligned}D(\tau)&=2\int_{-(I+\eta)}^{0}(y+\eta-[y+\eta])\,dy\\&\quad-2\int_{-(I+\eta)}^{0}(y+\eta-[y+\eta])\,e^{-e^{-\theta y}}(1+\theta y\,e^{-\theta y})\,dy\\&\quad+2\int_0^\infty(y+\eta-[y+\eta])\{1-e^{-e^{-\theta y}}(1+\theta y\,e^{-\theta y})\,dy\\&=\theta^{-1}\log\tau+O(1)\end{aligned}$$

as $\tau\to\infty$, since clearly the latter two integrals remain bounded. Thus

$$V(\tau)=\theta^{-1}\log\tau(C_3-1)+O(1).$$

Finally, differentiation of (20) at $x = 1$ yields the identity

$$\sum_{j=0}^{\infty} j\lambda^j \pi_j(\lambda) + b^2 B_2 = 0,$$

that is, $C_1 B_1^{-1} + b^2 B_2 = 0$. Combined with $B_1 b = 1$ this shows $C_3 - 1 = 0$, and we have proved (ii) and the theorem.

Two comments on this result are in order. First, it is perhaps not obvious from the above proof that $\mu(t) - t$ and $V(t)$ do not actually converge, though since the proof is equally valid in the Markov case we know from Theorem 1 that they cannot. In fact, by taking limits of $R_1(\tau)$ down the sequences $e^{\theta[J+\eta]}$ used elsewhere in the argument it can be shown that the limits exist but vary with η; that is, $\mu(t) - t$ oscillates boundedly for large t, and similarly for $V(t)$.

Second, if $N(\cdot)$ is Markovian and $\phi(\cdot)$ exponential a simple alternative proof of Theorem 2 can be based on recent results of Grimmett [6] about point processes with independent intervals having geometrically increasing means. In our notation, he shows (on the τ-scale) that as $\tau \to \infty$ the lim sup and lim inf of $\mathbf{P}\{N(\tau) - \nu(\tau) \leq u\}$ are explicitly calculable distribution functions, where $\nu(\tau)$ is a suitable increasing function which in our case can be $\theta^{-1} \log \tau$. Taking the interval distributions as exponential we thus have Lemma 2 with a specific $W(\cdot)$ and a direct argument such as on p. 682 of Grimmett's paper easily produces the exponential bounds of Lemmas 4 and 5, and hence the controlled variability. However since his paper makes crucial use of the independence of the intervals it cannot assist in the non-Markov case. Incidentally, the nonexistence of a unique limit for the above probability is a consequence of the periodic nature of the underlying regenerative process and not for the reason given on p. 676.

4. Discussion and an application

The conditions (A1)–(A3) on $\phi(\cdot)$ are mild and reasonably intuitive for a self-controlling process. We now discuss briefly the possibilities for further relaxation.

The consequence of (A3) that $\phi(\cdot)$ eventually stays away from 1 is unlikely to be dispensible in general, though in a particular case some weakening may be possible. Certainly our proof uses this consequence crucially in Lemmas 4 and 5, but more generally it is known that unpleasant things happen to the recurrence properties of a state of a Markov chain if the mean "drift" towards that state can become arbitrarily close to zero while remaining positive. Thus without this form of (A3) we may not get the essential positive recurrence of $Y(\cdot)$ in Lemma 2.

As mentioned in the Introduction, (A2) is designed to guarantee $Y(\cdot)$ does not get marooned below zero. Clearly weaker versions would suffice—essentially we need only that $\phi(\cdot)$ is never zero over an interval of length greater than one—but our choice is both simple and convenient.

The restriction $\phi(x)<\infty$ in (A1) is also one of convenience and the reader may convince himself that, with some extra argument, it could be removed. Similarly, we could allow atoms in the birth rate without affecting the conclusions; in fact, on occasions they can make the proof more simple by restricting the range of the process!

Finally, we turn to the following application, suggested by Dr. D.J. Daley. Consider a D/M/s queue ($s\geq 1$) starting idle at time $0-$, with unit-spaced arrivals at $0, 1, 2, \ldots$ and mean service time a^{-1}, with $as>1$. Because of the exponential service distribution the output of the queue is a Markov point process and it is easily checked that the departure rate has the form (1) with

$$\phi(x)=\begin{cases} 0 & (x>0),\\ ja & (-j<x\leq -(j-1);\ j=1,2,\ldots,s),\\ sa & (x\leq -s).\end{cases}$$

Then from Theorem 1 we have that the variance of the number of departures in $[0, t)$ from a non-equilibrium D/M/s queue is bounded. For the stationary queue starting at 0 in equilibrium the number of departures in $[0, t)$, $N(t)$, may be represented as

$$N(t)=N'(t)+N''(t),$$

where $N'(t)$ is the number of departures in $[0, t)$ during the initial busy period and $N''(t)$ is the number during subsequent busy periods (note that $N''(t)$ will not contribute if the initial busy period is still running at t). Now

$$V(t)\leq ([\mathrm{Var}\{N'(t)\}]^{1/2}+[\mathrm{Var}[N''(t)\}]^{1/2})^2. \tag{28}$$

The first term on the right side of (28) is bounded by the second moment of the total number served during a busy period initiated by an equilibrium queue, which is finite, while the above result for the non-equilibrium queue shows that the second term is bounded for all t. This proves:

Theorem 3. *The variance of the number of departures during* $[0, t)$ *from a stationary* D/M/s *queue with unit interarrival intervals is bounded.*

Theorem 3 is clearly related to a conjecture by Daley [5, 404], and when $s=\infty$ it is a special case of a result of Lewis and Govier (see [9]).

To conclude, we mention that the problem of regulating the intensity of a point process to a fixed value ρ over $[0, t]$ by possibly rejecting points which occur is described in [2]. His decision rule is to minimise $\mathbb{E}[\int_0^t \{N(u)-\rho u\}^2\,du]$, which has connexions with our process's controlled variability.

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