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CUP-*i* PRODUCTS AND THE ADAMS SPECTRAL SEQUENCE†‡

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§1. INTRODUCTION

THE COHOMOLOGY $H(A)$ of the mod 2 Steenrod algebra A is isomorphic to the E_2 term of the mod 2 Adams spectral sequence $\{E_r(S^0)\}$ [2]. In this paper we extend the results of [5] in relating cup-i products in $H(A)$ to the structure of the Adams spectral sequence.

The main result of the paper is contained in

THEOREM (1.1). *Suppose that* $\alpha \in E_2^{s,s+t}(S^0)$ *is a permanent cycle and that both*
(i) $(k+1) \leq \rho(t+k+1)$ *and*
(ii) $(t+k) \equiv 2^{i+1} - 1 \bmod 2^{i+1}$, $i = 0, 1, 2$ *or* 3.
Then $\mathrm{Sq}_k\alpha$ *is a permanent cycle and* $\mathrm{Sq}_{k+2^i}\alpha$ *survives to* $E_{2^i+1}(S^0)$ *where*

$$\delta_{2^i+1}(\mathrm{Sq}_{k+2^i}\alpha) = h_i(\mathrm{Sq}_k\alpha).$$

(1.1) occurs in this paper as Theorems (5.1) and (5.4). In the statement of (1.1), $\rho((2a+1)2^{c+4d}) = 2^c + 8d$ where $0 \leq c < 4$. Also, $\mathrm{Sq}_k\alpha = \alpha \cup_k \alpha$. ($\mathrm{Sq}_k\alpha$ should not be confused with $\mathrm{Sq}^k\alpha$.)

For $k = 0$, the differentials given in (1.1) correspond to identities in $\pi_*^S(S^0) = G_*$. For example, if $\alpha \in G_t$, then $2\alpha^2 = 0$ if t is even and $\eta\alpha^2 = 0$ if $t \equiv 3(4)$. Identities such as these are proved using the quadratic construction which is also the main tool in the present study. The quadratic construction is a functor from pointed spaces to filtered spaces which has been studied by J. F. Adams, M. G. Barratt and M. Mahowald (unpublished). Theorem (4.4), our main technical result, was an early conjecture in the study of the quadratic construction. The author is indebted to J. F. Adams, M. G. Barratt and M. Mahowald for conversations and correspondence which were helpful in the present work.

This paper is organized as follows:
§2 recalls the construction of the cup-i products in the cohomology of the Steenrod algebra. §3 outlines the quadratic construction. The main technical results are proved in §4, and the applications of Theorem (4.4) to the Adams spectral sequence are carried out in §5. The quadratic construction can also be used to define cup-i products in G_*. §6 discusses the relation of some of these homotopy operations to the cup-i products in $H(A)$.

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§2. CUP-i PRODUCTS IN $H(A)$

In this section, we recall the definition of squaring operations in the cohomology of A, the mod 2 Steenrod algebra [6; Chapt. II, §5]. The construction itself applies to any connected co-commutative Hopf algebra over Z_2, but our only concern is with the Steenrod algebra.

The cohomology of A may be calculated from the bar construction $B(A)$[1, p. 32]. Thus, $H^{s,t}(A)$ is a subquotient of $\mathrm{Hom}_A{}^t(B_s(A), Z_2)$. $B(A) \otimes B(A)$ is made into an A-module via the diagonal map $\psi : A \to A \otimes A$. Since $B(A) \otimes B(A)$ is acyclic, $\exists$ an A-map $D_0 : B(A) \to B(A) \otimes B(A)$ commuting with the augmentation. D_0 may be used to compute cup products in $H(A)$.

Since A is co-commutative, the twisting map $\rho : B(A) \otimes B(A) \to B(A) \otimes B(A)$ given by $x \otimes y \to y \otimes x$ is an A homomorphism commuting with augmentation. Again using the acyclicity of $B(A) \otimes B(A)$, there exists a chain homotopy

$$D_1 : B(A) \to B(A) \otimes BA$$

such that $\partial D_1 + D_1 \partial = \rho D_0 + D_0$. (No signs are needed since we are working mod 2.) Continuing as in the definition of the Steenrod operations in ordinary topology [8; V, §9], we obtain a sequence of A homomorphisms

$$D_j : B_n(A) \to (B(A) \otimes B(A))_{n+j}$$

such that

(2.1) D_0 commutes with augmentation, and

(2.2) for $j > 0$, $\partial D_j + D_j \partial + D_{j-1} + \rho D_{j-1} = 0$.

If $\{D_j\}$ and $\{D_j'\}$ are any two such sequences, there exists a sequence of A maps

$$E_j : B_n(A) \to (B(A) \otimes B(A))_{n+j}, j \geq 0$$

such that

(2.3) $E_0 = 0$, and

(2.4) for $j \geq 0$, $\partial E_{j+1} + E_{j+1} \partial + E_j + \rho E_j + D_j + D_j' = 0$

We now define, for $u \in \mathrm{Hom}_A{}^t(B_n(A), Z_2)$, an element $u \cup_i u \in \mathrm{Hom}_A{}^{2t}(B_{2n-i}(A), Z_2)$ by

$$(u \cup_i u)(\sigma) = (u \times u)(D_i \sigma),$$

where $\sigma \in B_{2n-i}(A)$. Using (2.1) and (2.2), we see that if u is cocycle, so also is $u \cup_i u$ and that the cohomology class of $u \cup_i u$ depends only on the cohomology class of u. Using (2.3) and (2.4) we see that the cohomology class of $u \cup_i u$ does not depend on the choice of the sequence $\{D_j\}$ satisfying (2.1) and (2.2). Thus we have defined $\alpha \cup_i \alpha \in H^{2s-i,2t}(A)$ for $\alpha \in H^{s,t}(A)$. We will also write

$$\mathrm{Sq}_i \alpha = \mathrm{Sq}^{2s-i} \alpha = \alpha \cup_i \alpha.$$

Remark (2.5). In order to define $\alpha \cup_i \alpha$, it suffices to define D_j satisfying (2.1) and (2.2) only for $j \leq i$.

§3. THE QUADRATIC CONSTRUCTION

We describe here the quadratic construction, a functor from pointed spaces to filtered spaces which has been studied by Adams, Barratt and Mahowald (unpublished). The results of this section are well known to these authors. For convenience, we shall only discuss the construction for finite CW complexes with base point.

Let X be a finite CW complex with base point. Denote by $Q'^n(X)$ the space $S^n |\times (X \wedge X)$. (If B is a space with base point b_0, $A |\times B$ shall mean $(A \times B)/(A \times b_0)$.) Define the involution T: $Q'^n(X) \to Q'^n(X)$ by $T(x, y, z) = (-x, z, y)$, where $-x$ is the point of S^n antipodal to x. This defines an action of Z_2 on $Q'^n(X)$ and we define $Q^n(X) = Q'^n(X)/Z_2$. $Q(X)$ shall mean $Q^\infty(X)$ and is called the *quadratic construction* on X. $Q(X)$ is naturally filtered by the subspaces $\{Q^n(X)\}$.

If $f: X \to Y$ is a map of pointed complexes, then $Q'(f) : Q'(X) \to Q'(Y)$ defined by $(x, y, z) \to (x, f(y), f(z))$ induces a map $Q(f) : Q(X) \to Q(Y)$ and $Q(f) \mid Q^n(X) = Q^n(f) : Q^n(X) \to Q^n(Y)$. Thus each Q^n is a functor and if $n < m$, Q^n is a subfunctor of Q^m.

Example (3.1). $Q^n(S^m)$ is homeomorphic with $S^m \wedge P_m{}^{m+n} = \Sigma^m P_m{}^{m+n}$. (By $P_a{}^b$ we mean the stunted projective space RP^b/RP^{a-1}.)

Proof. $Q^n(S^m)$ may also be described as the one point compactification of $(S^n \times R^m \times R^m)/Z_2$ where the action of Z_2 on $S^n \times R^m \times R^m$ is given by $T'(x, y, z) = (-x, z, y)$. Now $S^m \wedge P_m{}^{m+n}$ can be described as the one point compactification of $(S^n \times R^m \times R^m)/Z_2$ where the action of Z_2 on $S^n \times R^m \times R^m$ is given by $T''(x, y, z) = (-x, y, -z)$[4, p. 205]. Since, as is easily seen, $(S^n \times R^m \times R^m, T')$ and $(S^n \times R^m \times R^m, T'')$ are equivariantly homeomorphic, the result follows.

We leave it to the reader to verify the following three elementary properties of the quadratic construction.

PROPOSITION (3.2). *If $f, g : X \to Y$ are homotopic relative base points, then there exists a homotopy $H : Q(X) \times I \to Q(Y)$ of $Q(f)$ to $Q(g)$ such that $H \mid Q^n(X) \times I$ gives a homotopy of $Q^n(f)$ to $Q^n(g)$.*

COROLLARY (3.3). *If X and Y have the same homotopy type as pointed spaces, then $Q(X)$ and $Q(Y)$ have the same homotopy type, preserving filtration.*

PROPOSITION (3.4). *$Q^n(X)/Q^{n-1}(X) \approx S^n \wedge X \wedge X$ and $Q^n(f)/Q^{n-1}(f) \approx S^n \wedge f \wedge f$.*

§4. THE MAIN THEOREM

Our discussion of the Adams spectral sequence will follow the exposition given in [2], with the exception that we will use the smash product rather than the join in treating products. Finite CW approximations through the stable range respecting base points will be used. We will omit specific mention of the skeletons on which various stable constructions may be carried out.

Let $W_0 \supset W_1 \supset \ldots$ be a realization for $B(A)$ with W_0 having the homotopy type of S^{2n}. Let $Y_0 \supset Y_1 \supset \ldots$ be a realization for $B(A)$ with Y_0 having the homotopy type of S^n.

Then $Z_0 = Y_0 \wedge Y_0$ with the product filtration is a realization for $B(A) \otimes B(A)$. The switching map $\tau : Y \wedge Y \to Y \wedge Y$ is a realization of $\rho : B(A) \otimes B(A) \to B(A) \otimes B(A)$, the switching map for $B(A) \otimes B(A)$.

LEMMA (4.1): *If $n = 2^q$ where $q \geq \phi(r)$ [3], $S^n = P_n{}^n$ is a retract of $P_n{}^{n+r}$.*

Proof. [3, 4].

Let $\phi_r : \Sigma^n P_n{}^{n+r} = Q^r(S^n) \to S^{2n}$ be the n-fold suspension of such a retraction. Since $Q^r(Y_0) \simeq Q^r(S^n)$, there exists a map $\Phi_r : Q^r(Y_0) \to W_0$ equivalent to ϕ_r.

Let $\theta_t : S^t \,|\!\times (Y_0 \wedge Y_0) \to Q^t(Y_0)$ be the identification map. Denote by $E_+{}^t$ $(E_-{}^t)$ the upper (lower) hemisphere of S^t. Let

$$\psi_t{}^{\pm} : (E_\pm{}^t \,|\!\times\; (Y_0 \wedge Y_0) \to S^t \,|\!\times (Y_0 \wedge Y_0)$$

be the inclusion map.

PROPOSITION (4.2). *Let $n = 2^q$, $q \geq \phi(r)$. Then there exists a map $\Theta_r : S^r \,|\!\times (Y_0 \wedge Y_0) \to W_0$ such that*

A) $\Theta_r \circ T = \Theta_r$,

B) *there is an equivariant homotopy H^r of Θ_r to $\Phi \circ \theta_r$ (yielding $\Theta_r/Z_2 \simeq \Phi_r : Q^r (Y_0) \to W_0$), and*

C) $\Theta_r(S^t \,|\!\times Z_s) \subset W_{s-t}$ *for $t \leq r$.*

Proof. We proceed by induction, denoting by A_t, B_t, C_t conditions A, B, C of (4.2) for $r = t$.

Since $\Phi : Q^0(Y_0) = Y_0 \wedge Y_0 \to W_0$ is equivalent to the identity map $S^{2n} \to S^{2n}$, we may use Lemma 3.4 of [2] to obtain a map

$$\Theta_0{}^+ : \psi_0{}^+(E_+{}^0 \,|\!\times (Y_0 \wedge Y_0)) \approx Y_0 \wedge Y_0 \to W_0$$

such that $\Theta_0{}^+\psi_0{}^+(E_+{}^0 \,|\!\times Z_s) \subset W_s$ and such that there is a homotopy $H_+{}^0$ of $\Theta_0{}^+$ with Φ_0. Define $\Theta_0{}^- : \psi_0{}^-(E_-{}^0 \,|\!\times (Y_0 \wedge Y_0)) \to W_0$ by $\Theta_0{}^- = \Theta_0{}^+ \circ T \circ \psi_0{}^-$. This defines Θ_0. We define $H_-{}^0$ by the composite

$$\psi_0{}^-(E_-{}^0 \times |(Y_0 \wedge Y_0)) \times | I \xrightarrow{T\times|} > \psi_0{}^+\_(E_+^0 \,|\!\times (Y_0 \wedge Y_0)) \xrightarrow{H_+{}^0} W_0 .$$

This defines H^0. Conditions A_0, B_0 and C_0 are readily verified.

Assume now that Θ_t is defined satisfying A_t, B_t and C_t where $t < r$. Now since $t < r$, $\Theta_t/Z_2 : Q^t(Y_0) \to W_0$ extends to a map $\alpha : Q^{t+1}(Y_0) \to W_0$ with H^t/Z_2 extending to a homotopy K of α with Φ_{t+1}. Consider the composite

$$\alpha \circ \theta_{t+1} \circ \psi_{t+1}^+ : E_+{}^{t+1} \,|\!\times (Y_0 \wedge Y_0) \to W_0 .$$

Using an argument similar to one given in Lemma 3.5 of [2], we see that $\alpha \circ \theta_{t+1} \circ \psi_{t+1}^+$ is homotopic relative to $S^t \,|\!\times (Y_0 \wedge Y_0)$ to a map $\overline{\Theta}_{t+1}^+$ such that $\overline{\Theta}_{t+1}^+(E_+^{t+1} \,|\!\times Z_s) \subset \subset W_{s-(t+1)}$. Define Θ_{t+1}^+ by the equation $\Theta_{t+1} = \overline{\Theta}_{t+1}^+ \circ \psi_{t+1}^+$. This also yields a homotopy $H_+{}^{t+1}$ extending $H^t \,|\, S^t \,|\!\times (Y_0 \wedge Y)$ of Θ_{t+1}^+ to $\Phi_{t+1} \circ \theta_{t+1}$. Now set $\Theta_{t+1}^- = \Theta_{t+1}^+ \circ T$ and $H_-{}^{t+1} = H_+{}^{t+1} \circ (T \times | \, 1)$. One checks that conditions A_{t+1}, B_{t+1} and C_{t+1} are satisfied. This completes the induction.

Identifying $H^*(W_k, W_{k+1})$ with $B_k(A)$ and $H^*(Z_m, Z_{m+1})$ with $(B(A) \otimes B(A))_m$, we define homomorphisms $D_j : B_n(A) \to (B(A) \otimes B(A))_{n+j}$, $0 \le j \le r$ as follows: Let $\Theta = \Theta_r : S^r | \times (Y_0 \wedge Y_0) \to W_0$ be a map given by Proposition (4.2). The Künneth Theorem yields an isomorphism

$$\chi_\pm : H^*((E_\pm{}^j, S^{j-1}) | \times (Z_m, Z_{m+1})) \approx H^{*-j}(Z_m, Z_{m+1}) \approx (B(A) \otimes B(A))_m$$

Consider the composites

$$(E_\pm{}^j, S^{j-1} | \times (Z_m, Z_{m+1}) \xrightarrow{\psi_j{}^\pm} (S^j, S^{j-1}) | \times (Z_m, Z_{m+1}) \xrightarrow{\Theta} (W_{m-j}, W_{m-j+1}) \approx B_{m-j}(A).$$

Then we define $D_j : B_{m-j}(A) \to (B(A) \otimes B(A))_m$ by

$$D_j = \chi_+ \circ (\psi_j{}^+)^* \circ \Theta^*.$$

LEMMA (4.3). *The homomorphisms D_j satisfy* (2.1) *and* (2.2) *for $j \le r$.*

Proof. Since we are working mod 2, it follows by (4.2A) that

$$\rho D_j = \chi \circ (\psi_j{}^-)^* \circ \Theta^*.$$

The result now follows from the cell structure of S^r.

In the following theorem, we retain the notation of this section.

THEOREM (4.4). *Let $\alpha \in E_2{}^{s,s+t}(S^0)$ be a permanent cycle and r an integer. Then there is an integer n and a mapping $\overline{\Theta} : \Sigma^{n+t} P_{n+t}^{n+t+r} \to W_0$ such that*

A) $\overline{\Theta}(\Sigma^{n+t} P_{n+t}^{n+t+k}) \subset W_{2s-k}$, $0 \le k \le r$, *and*

B) *the composite*
$(E^{2(n+t)+k}, S^{2(n+t)+k-1}) \xrightarrow{\theta} (\Sigma^{n+t} P_{n+t}^{n+t+k}, \Sigma^{n+t} P_{n+t}^{n+t+k-1}) \xrightarrow{\overline{\Theta}} (W_{2s-k}, W_{2s-k+1})$ *represents a cycle of $E_1{}^{2s-k,2(s+t)}(S^0)$ whose image in $E_2{}^{2s-k,(s+t)}(S^0)$ is $\alpha {}_k\bigcup \alpha$. ($\theta$ denotes the characteristic map of the top cell of $\Sigma^{n+t} P_{n+t}^{u+t+k}$.)*

Remark (4.5). The integer n may be chosen to be as large as one likes. In fact, there is an integer M so that we may choose $n = 2^q$, $q \ge M$.

Proof. Choose $n = 2^q$ where $q \ge \phi(r)$. We also choose q large enough so that the relevant portion of $B(A)$ is realizable. Then we may find a map $\Theta : S^r | \times (Y_0 \wedge Y_0) \to W_0$ which satisfies conditions A, B and C of (4.2).

Now, since α is a permanent cycle, it may be represented by a mapping $f : S^{n+t} \to Y_s$. We define the map $\overline{\Theta}$ to be the composite $(\Theta/Z_2) \circ Q^r(f) : Q^r(S^{n+t}) \to W_0$.

By (3.1), $Q^r(S^{n+t})$ is homeomorphic with $\Sigma^{n+t} P_{n+t}^{n+t+r}$. Proposition (4.2C) implies (4.4A). Note that $f \wedge f : S^{n+t} \wedge S^{n+t} \to (Z_{2s}, Z_{2s-1})$ represents $\alpha \times \alpha$. (4.4B) now follows from (3.4) and (4.3).

§5. SOME DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE

In this section, we use information concerning the cell structure of $P_n{}^{n+k}$ and Theorem (4.4) to relate the differential in the Adams spectral sequence to some of the squaring operations on permanent cycles. The results for Sq_1 were obtained earlier in [5].

THEOREM (5.1). *Let* $\alpha \in E_2^{s+t}(S^0)$ *be a permanent cycle. Then if* $k+1 \leq \rho(t+k+1)$, $\mathrm{Sq}_k\,\alpha$ *is also a permanent cycle.*

Proof. Recall that if $m = (2a+1)2^{c+4d}$, $0 \leq c \leq 3$, then $\rho(m) = 2^c + 8d$. Observe that if $t+k+1 = 2^r(2b+1)$ and $q > r$, then $\rho(2^q + t + k + 1) = \rho(t+k+1)$.

It follows from (4.4) and (4.5) that we need only show $P_{2^q+t}^{2^q+t+k}$ is stably reducible if q is large enough. But it is known that this is true if and only if $(k+1) \leq \rho(2^q + t + k + 1) = \rho(t+k+1)$ [3].

We shall use the symbols $\bar{h}_0$, $\bar{h}_1$, $\bar{h}_2$ and $\bar{h}_3$ to denote any elements in G_* which correspond to the elements h_0, h_1, h_2 and h_3 in $E_2(S^0)$, respectively. This means that $\bar{h}_i$ is a generator of the 2-component of G_{2^i-1}.

Now let $f_i : S_{n+t}^{n+t+2^i-1} \to P^{n+t+2^i-1}$ denote the attaching map of the top cell of $P_{n+t}^{n+t+2^i}$. Let $g : S^{n+t} = P_{n+t}^{n+t} \to P_{n+t}^{n+t+2^i-1}$ denote the inclusion map.

LEMMA (5.2). *Let* $n = 2^q$, *q large enough, and* $i = 0, 1, 2$ *or* 3. *Then* $[f_i] = g_*\bar{h}_i$ *if* $t \equiv 2^{i+1} - 1$ *mod* 2^{i+1}.

Proof. This follows from the fact that Sq^{2^i} is non-zero in $P_{n+t}^{n+t+2^i}$ and that $P_{n+t+1}^{n+t+2^i}$ is S-reducible under the hypothesis of (5.2).

Notation. δ_r denotes the differential in $E_r(S^0)$.

THEOREM (5.3). *Let* $i = 0, 1, 2$ *or* 3. *If* $\alpha \in E_2{}^{s,s+t}(S^0)$ *is a permanent cycle and* $t \equiv 2^{i+1} - 1$ *mod* 2^{i+1}, *then* $\mathrm{Sq}_{2^i}\alpha$ *survives to* $E_{2^i+1}(S^0)$ *and*

$$\delta_{2^i+1}(\mathrm{Sq}_{2^i}\alpha) = h_i\alpha^2.$$

Proof. Let $\bar{\Theta} : \Sigma_{n+t}^{n+t} P_{n+t}^{n+t+2^i} \to W_0$ be a map given by (4.4) with $n = 2^q$, q large. Then by (5.2) it follows that $\mathrm{Sq}_{2^i}\alpha$ is represented by a map $(E^{2(n+t)+2^i}, S^{2(n+t)+2^i-1}) \xrightarrow{f} (W_{2s-2^i}, W_{2s})$ such that $f \mid S^{2(n+t)+2^i-1} = f'$ is homotopic in W_{2s} to the composite of $\bar{h}_i$ with a map representing α^2. It follows that f' is homotopic through W_{2s} to a map $f'' : S^{2(n+t)+2^i-1} \to W_{2s+1}$ which represents $h_i\alpha^2$. (For $i = 0$, this is Lemma (2.2) of [5]. The proof of this statement for $i = 1, 2$ and 3 is similar.)

A specific homotopy through W_{2s} of f' to f'' can be used to alter f so as to obtain a mapping

$$g : (E^{2(n+t)+2^i}, S^{2(n+t)+2^i-1}) \to (W_{2s-2^i}, W_{2s+1})$$

so that g represents $\mathrm{Sq}_{2^i}\alpha$ and $g \mid S^{2(n+t)+2^i-1}$ represents $h_i\alpha^2$. This completes the proof of (5.3).

THEOREM (5.4). *Suppose that* $\alpha \in E_2{}^{s,s+t}(S^0)$ *is a permanent cycle and that both*

(i) $k+1 \leq \rho(t+k+1)$ *and*

(ii) $t + k \equiv 2^{i+1} - 1$ *mod* 2^{i+1}, $i = 0, 1, 2$ *or* 3.

Then $\mathrm{Sq}_{k+2^i}\alpha$ *survives to* $E_{2^i+1}(S^0)$ *and*

$$\delta_{2^i+1}(\mathrm{Sq}_{k+2^i}\alpha) = h_i(\mathrm{Sq}_k\alpha).$$

Proof. Let $n = 2^q$, q large enough. Condition (i) implies that (stably) P_{n+t}^{n+t+k} has the same homotopy type as $P_{n+t}^{n+t+k-1} \vee S^{n+t+k}$. Let

$$f : S^{n+t+k+2^i-1} \to P_{n+t}^{n+t+k+2^i-1}$$

denote the attaching map of the top cell of $P_{n+t}^{n+t+k+2^i}$. Lemma (5.2) implies that f is homotopic to a map $g : S^{n+t+k+2^i-1} \to P_{n+t}^{n+t+k}$. Now

$$\pi_{n+t+k+2^i-1}(P_{n+t}^{n+t+k}) \approx \pi_{n+t+k+2^i-1}(S^{n+t+k}) \oplus \pi_{n+t+k+2^i-1}(P_{n+t}^{n+t+k-1}).$$

Using this decomposition write $[g] = ([g_1], [g_2])$. Condition (ii) implies that $[g_1] = \bar{h}_i$.

We proceed as in the proof of (5.3). Let $\overline{\Theta} : \Sigma^{n+t}P_{n+t}^{n+t+k+2^i} \to W_0$ be a map given by (4.4) with $n = 2^q$, q large. Then, as in (5.3), $\delta_{2^i}(\mathrm{Sq}_{k+2^i}\alpha)$ is represented by $\overline{\Theta}_*[\Sigma^{n+t}g_1] + \overline{\Theta}_*[\Sigma^{n+t}g_2] \in \pi_*(W_{2s-k})$. Now $\overline{\Theta}_*[\Sigma^{n+t}g_1]$ is homotopic in W_{2s-k} to an element of $\pi_*(W_{2s-k+1})$ which represents $h_i(\mathrm{Sq}_k\alpha)$. Because $\overline{\Theta}_0(\Sigma^{n+t}P_{n+t}^{n+t+k-1}) \subset W_{2s-k+1}$, $\overline{\Theta}_0(\Sigma^{n+t}g_2)(S^{2(n+t)+2^i-1})$ is contained in W_{2s-k+1}, and since $\bar{H}^*(g_2 ; Z_2) = 0$, it follows by Lemma (3.3) of [2] that $\overline{\Theta}_0\Sigma^{n+t}g_2$ is homotopic in W_{2s-k+1} to a map carrying $S^{2(n+t)+k+2^i-1}$ into W_{2s-k+2}.

Again as in the proof of (5.3), it follows that

$$\delta_{2^i+1}(\mathrm{Sq}_{k+2^i}\alpha) = h_i(\mathrm{Sq}_k\alpha).$$

We conclude this section by observing that the differentials given in (5.3) are "honest". More precisely:

Definition (5.5). Let $\beta \in \pi_*(W_m)$ such that the image of β in $\pi_*(W_0)$ is zero. Denote by β' the image of β in $E_r^{m,*}(S^0)$. Then a relation $\delta_r\gamma' = \beta'$ is called *honest* rel. β if γ' can be represented by an element $\gamma \in \pi_*(W_{m-r}, W_m)$ such that $\partial_*\gamma = \beta$, ∂_* being the boundary homomorphism of the pair (W_{m-r}, W_m).

The condition of honesty of relations occurs in the problem of relating Toda brackets to Massey products. See, for example, the work of Moss [7]. It is clear that the relations of (5.3) are honest.

We shall need another criterion of honesty in the next section.

Definition (5.6). A relation

$$\delta_r^{s-r,t+1+s-r}\gamma' = \beta' \in E_r^{s,s+t}$$

is called *visibly honest* rel. β if

$$\delta_{r+k+l}^{s-r-k,t+1+s-r-k} = 0 \text{ for } k \geq 0, l \geq 1$$

and β is represented by an element $\tilde{\beta} \in \pi_*(W_r)$ whose image in $\pi_*(W_0)$ is zero.

The nomenclature of (5.6) is justified by the following:

LEMMA (5.7). *If $\delta_r\gamma' = \beta'$ is visibly honest rel. β', it is honest rel. β, for any $\beta \in \pi_*(W_s)$ which maps to zero in $\pi_*(W_0)$ and to π' in β_r.*

Proof. [7].

§6. CUP-i PRODUCTS IN HOMOTOPY

The quadratic construction has been used by Adams, Barratt, B. Gray and Mahowald to define cup-i products in G_*. We do not attempt a systematic discussion of these operations. Rather, we just treat some illustrative cases and their relation to the Adams spectral sequence.

Let $f: S^{n+t} \to S^n$ represent $\alpha \in G_t$ where $n = 2^q$, q large. Then from (4.1) we obtain a mapping

$$\overline{\Theta}_r : Q^r(S^{n+t}) = \Sigma^{n+t} P_{n+t}^{n+t+r} \to S^{2n}$$

which is the composition of a retraction $\Psi : \Sigma^n P_n^{n+r} \to S^{2n}$ with $Q^r(f)$. Denote by $g : S^{n+t+r-1} \to P_{n+t}^{n+t+r-1}$ the attaching map of the top cell. Since $\Psi \circ Q^{r-1}(f)$ extends to $\Psi \circ Q^r(f)$, it follows that $\overline{\Theta}_{r-1} \circ \Sigma^{n+t} g$ has a null-homotopy H. Now if for "different" reasons, there exists a null-homotopy K of $\overline{\Theta}_{r-1} \circ \Sigma^{n+t} g$, we define $\alpha \bigcup_r \alpha$ to contain the difference element $d(H, K)$. Thus $\alpha \bigcup_r \alpha \subset G_{2t+r}$. ($\alpha \bigcup_0 \alpha = \alpha^2$.) We consider a few examples.

Examples (6.1). A.) If $r + 1 \leq \rho(t + r + 1)$, then g itself is null-homotopic. In this case $\alpha \bigcup_r \alpha$ is defined by demanding that the null-homotopy K be the composite of Ψ with a null-homotopy K' of $\Sigma^{n+t} g$. The indeterminacy of $\alpha \bigcup_r \alpha$ comes from varying the null-homotopy K' and the retraction Ψ.

B.) Let $t \equiv 2^{i+1} - 1 \mod 2^{i+1}$ and $r = 2^i$ for $i = 0, 1, 2$ or 3. Then we may suppose that $g : S^{n+t+2^i-1} \to S^{n+t}$. Assume that $\alpha^2 = 0$. Let K' be a null-homotopy of $f \wedge f$. In this case, we define $\alpha \bigcup_r \alpha$ by demanding that K be the composite of K' with the cone on g. The indeterminacy comes from varying the compression of g into S^{n+t}, varying K' and again varying the retraction Ψ.

C.) We retain the hypothesis of (6.1B) with the exception that we replace the condition $\alpha^2 = 0$ by the condition that $f \circ g$ is null-homotopic for some choice of compression $g : S^{n+t+2^i-1} \to S^{n+t}$ of the attaching map. Let K' be such a null-homotopy. We define $\alpha \bigcup_r \alpha$ in this case by demanding that K be of the form $K' \wedge f$.

THEOREM (6.2). *Let $\bar{\alpha} \in E_2^{s,s+t}(S^0)$ and suppose that $r + 1 \leq \rho(t + r + 1)$. Then there is an element of $\bar{\alpha} \bigcup_r \bar{\alpha}$ as defined in* (6.1A) *which corresponds to $\alpha \bigcup_r \alpha \in E_2^{2s-r,2s+2t}(S^0)$.*

Proof. Defining $d(H, K) \in \bar{\alpha} \bigcup_r \bar{\alpha}$, by (4.4) H may be chosen so as to represent $\alpha \bigcup_r \alpha$ and K may be compressed into W_{2s-r+1}. Thus $d(H, K)$ represents $\alpha \bigcup_r \alpha$ in $E_2^{2s-r,2s+2t}(S^0)$.

THEOREM (6.3). *Suppose that $\bar{\alpha} \in G_t$ corresponds to $\alpha \in E_2^{s,s+t}(S^0)$ and that $t = 2^{i+1} - 1$ mod 2^{i+1}. Suppose also that $\bar{\alpha}^2 = 0$ and that there is a visibly honest differential $\delta_m \beta = \alpha^2$ rel. α^2. Then $\mathrm{Sq}_{2^i} \bar{\alpha}$ contains an element which corresponds to $\mathrm{Sq}_{2^i} \alpha \mp h_i \beta$.*

Notation. If $\gamma \in E_2^{s,s+k}$ and $\gamma' \in E_2^{s',s'+k}$, then $\gamma \mp \gamma'$ denotes γ, $\gamma + \gamma'$ or γ' as $s < s'$, $s = s'$ or $s > s'$, respectively.

Proof. Let $r = 2^i$. Using (4.4) and the proof of (5.3), we compress $\overline{\Theta}_{r-1} \circ \Sigma^{n+t} g$ into W_{2s+1} with a null-homotopy into W_{2s-r} representing $\alpha \bigcup_r \alpha$. By (5.7), $\delta_m \beta = \alpha^2$ is honest. It follows that $\delta_m h_i \delta = h_i \alpha^2$ is honest rel. $h_i \alpha^2$. Use a representative K of $h_i \beta$ which gives null-homotopy of $\overline{\Theta}_{r-1} \circ \Sigma^{n+t} g$ into W_{2s+1-m}. K exists because of the honesty of the differential. Now $d(H, K)$ represents $\bar{\alpha} \bigcup_r \bar{\alpha}$ in G_* and $\alpha \bigcup_r \alpha \mp h_i \beta$ in $E_2(S^0)$.

THEOREM (6.4). *Suppose that $\bar{\alpha} \in G^t$ corresponds to $\alpha \in E_2^{s,s+t}(S^0)$ and that $t \equiv 2^{i+1} - 1$ mod 2^{i+1}. Suppose also that $\mathrm{Sq}_{2^i} \bar{\alpha}$ is defined as in* (6.1C) *and that there is a visibly honest* *differential* $\delta_m \beta = h_i \alpha$ *rel.* $h_i \alpha$. *Then* $\mathrm{Sq}_{2^i} \tilde{\alpha}$ *contains an element which corresponds to* $\mathrm{Sq}_{2^i} \alpha \mp \beta\alpha$.

The proof is similar to that of (6.3).

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