On the Maximal Index of Graphs with a Prescribed Number of Edges

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ABSTRACT

Among the graphs with a prescribed number of edges, those with maximal index are determined. The result confirms a conjecture of Brualdi and Hoffman.

1. INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges. The largest eigenvalue of a (0,1) adjacency matrix of a graph G is called the *index* of G. For $e > 0$, let $\mathscr{S}(e)$ denote the set of all graphs with precisely e edges. The problem of finding the graphs in $\mathscr{S}(e)$ with maximal index was posed by Brualdi and Hoffman in 1976 (cf. [1, p. 438]), and their results appeared some ten years later [2]. They showed that if $f(e)$ denotes the maximal index of a graph with e edges, and if $d > 1$, then

$$f\left(\binom{d}{2}\right) = d - 1$$

with equality precisely when the only nontrivial component of the graph is K_d (the complete graph on d vertices). They conjectured that when

$$\binom{d}{2} < e < \binom{d+1}{2}$$

the maximal index is attained precisely when the only nontrivial component

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is the graph G_e obtained from K_d by adding one new vertex of degree

$$t = e - \binom{d}{2}.$$

By applying perturbation-theoretic methods to adjacency matrices, Friedland [3] proved that there exists $K(t) > 0$ such that the conjecture is true for $d \geqslant K(t)$. He also proved that the conjecture is true for $t = d - 1$. Subsequently, Stanley [6] proved that $f(e) \leqslant \frac{1}{2}(-1 + \sqrt{1 + 8e})$, with equality precisely when $e = \binom{d}{2}$. Friedland [4] refined Stanley's inequality and thereby proved the conjecture for $t = 1$, $t = d - 3$, and $t = d - 2$. Here we prove that the conjecture is true in general.

2. SOME PRELIMINARY RESULTS

As in [2], $\mathscr{S}(n, e)$ denotes the set of adjacency matrices of graphs with n vertices and e edges, and $\mathscr{S}^*(n, e)$ denotes the subset of $\mathscr{S}(n, e)$ consisting of those matrices $A = (a_{ij})$ satisfying

(*) if $i < j$ and $a_{ij} = 1$, then $a_{hk} = 1$ whenever $h < k \leqslant j$ and $h \leqslant i$.

In view of the distribution of nonzero entries in a matrix belonging to $\mathscr{S}^*(n, e)$, we call such a matrix a *stepwise* matrix. Note that a nonzero stepwise matrix A is the adjacency matrix of a graph with a unique nontrivial component. It follows from the theory of nonnegative matrices [5, Chapter XIII] that if $\rho(A)$ is the largest eigenvalue of the stepwise matrix A, then there exists a unique nonnegative unit vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ such that $A\mathbf{x} = \rho(A)\mathbf{x}$: moreover, $x_i = 0$ if and only if vertex i is isolated.

LEMMA 1. *Let $A \in \mathscr{S}^*(n, e)$, $e > 0$. If $(x_1, \ldots, x_n)^T$ is a nonnegative eigenvector corresponding to $\rho(A)$, then $x_1 \geqslant \cdots \geqslant x_n$.*

Proof. We give a detailed proof of this straightforward result in order to establish notation and equations for subsequent use. Note that there exists $m \leqslant n$ such that $x_i > 0$ for $1 \leqslant i \leqslant m$ and $x_i = 0$ for $m < i \leqslant n$. We suppose that $e < \binom{n}{2}$, for otherwise $x_1 = \cdots = x_n$. Let $A = (a_{ij})$, and let c be least such that $a_{c,c+1}=0$. Writing $\rho=\rho(A)$, we have

$$x_1+\cdots+x_{r_i}=\begin{cases}(\rho+1)x_i & \text{if } 1\leqslant i\leqslant c,\\ \rho x_i & \text{if } c+1\leqslant i\leqslant m,\end{cases} \tag{1}$$

where $m\geqslant r_1\geqslant r_2\geqslant\cdots\geqslant r_c=c>r_{c+1}\geqslant r_{c+2}\geqslant\cdots\geqslant r_m>0$. Hence $x_1\geqslant x_2\geqslant\cdots\geqslant x_c$ and $x_{c+1}\geqslant x_{c+2}\geqslant\cdots\geqslant x_m$. Moreover, $\rho(x_c-x_{c+1})=(x_1+\cdots+x_{c-1})-(x_1+\cdots+x_{r_{c+1}})$, which is nonnegative because $r_{c+1}\leqslant c-1$. Since $\rho>0$, we have $x_c\geqslant x_{c+1}$. ■

Let $\mathscr{S}^{**}(n,e)$ be the subset of $\mathscr{S}^{*}(n,e)$ consisting of those matrices $A=(a_{ij})$ satisfying

(* *) if

(i) $h<p<q<k$, and
(ii) $a_{hk}=1$, $a_{hj}=0$ whenever $j>k$, $a_{ik}=0$ whenever $i>h$, and
(iii) $a_{pq}=0$, $a_{pj}=1$ whenever $p<j<q$, $a_{iq}=1$ whenever $i<p$,
then $p+q\leqslant h+k+1$.

Figure 1 shows a matrix in $\mathscr{S}^{**}(32,224)$: for this matrix, the values of (h,k) $(h<k)$ which satisfy condition (ii) are (3,29), (10,25), (11,21), and (13,16); the values of (p,q) $(p<q)$ which satisfy condition (iii) are (4,26), (11,22), (12,17), and (14,15).

Let $f(n,e)$, $f^{*}(n,e)$, $f^{**}(n,e)$ denote the maximum value of $\rho(A)$ attained by a matrix in $\mathscr{S}(n,e)$, $\mathscr{S}^{*}(n,e)$, $\mathscr{S}^{**}(n,e)$ respectively. Brualdi and Hoffman [2] proved that $f(n,e)=f^{*}(n,e)$; moreover if $A\in\mathscr{S}(n,e)$ and $\rho(A)=f^{*}(n,e)$, then there exists a permutation matrix P such that $P^{-1}AP\in\mathscr{S}^{*}(n,e)$. The next result shows that these statements remain true when $\mathscr{S}^{*}(n,e)$ is replaced by $\mathscr{S}^{**}(n,e)$ and $f^{*}(n,e)$ is replaced by $f^{**}(n,e)$.

LEMMA 2. *If $A\in\mathscr{S}^{*}(n,e)$ and $A\notin\mathscr{S}^{**}(n,e)$, then there exists $A'\in\mathscr{S}^{*}(n,e)$ such that $\rho(A)<\rho(A')$.*

Proof. In respect of the matrix $A=(a_{ij})$, there exist indices h,p,q,k satisfying conditions (i), (ii), and (iii) of property (* *) such that $p+q\geqslant h+k+2$. Since $e<\binom{n}{2}$, there exists a least c such that $a_{c,c+1}=0$. Note that $h<p\leqslant c<q<k$ and $e>0$. Let $\rho=\rho(A)$, and let $\mathbf{x}=(x_1,\dots,x_n)^T$ be the nonnegative unit eigenvector of A corresponding to ρ. In the notation of Lemma 1, we have $r_h=k$, $r_p=q-1$, $r_q=p-1$, and $r_k=h$.

$$
\begin{bmatrix}
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0\\
1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0\\
1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
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1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
$$

FIG. 1. *A matrix in* $\mathscr{S}^{**}(32, 224)$.

Let A' be the matrix obtained from A by interchanging the (h,k) and (p,q) entries and interchanging the (k,h) and (q,p) entries. Then $A' \in \mathscr{S}^*(n,e)$. Since $\rho(A') = \sup\{\mathbf{z}^T A' \mathbf{z} : \mathbf{z}^T\mathbf{z} = 1\}$ and $\mathbf{x}^T A \mathbf{x} = \rho$, it suffices to prove that $\mathbf{x}^T A' \mathbf{x} > \mathbf{x}^T A \mathbf{x}$. Now $\mathbf{x}^T(A' - A)\mathbf{x} = 2(x_p x_q - x_h x_k)$, and from the equations (1) we have

$$\begin{aligned}
\rho^2(x_p x_q - x_h x_k) \\
&= \left[(x_1 + \cdots + x_{h-1}) + (x_h + \cdots + x_{q-1}) - x_p\right] \\
&\quad \times \left[(x_1 + \cdots + x_{h-1}) + (x_h + \cdots + x_{p-1})\right] \\
&\quad - \left[(x_1 + \cdots + x_{h-1}) + (x_{h+1} + \cdots + x_k)\right]\left[(x_1 + \cdots + x_{h-1}) + x_h\right] \\
&= (x_1 + \cdots + x_{h-1})\left[(x_h + \cdots + x_{q-1}) - x_p \right.\\
&\qquad\qquad \left. + (x_h + \cdots + x_{p-1}) - (x_h + \cdots + x_k)\right] \\
&\quad + \left[(x_h + \cdots + x_{q-1}) - x_p\right](x_h + \cdots + x_{p-1}) - x_h(x_{h+1} + \cdots + x_k) \\
&= (x_1 + \cdots + x_{h-1})\left[(x_h + \cdots + x_{q-1}) - x_p - (x_p + \cdots + x_k)\right] \\
&\quad + x_h\left[(x_{h+1} + \cdots + x_{p-1}) - (x_q + \cdots + x_k)\right] \\
&\quad + \left[(x_{h+1} + \cdots + x_{q-1}) - x_p\right](x_{h+1} + \cdots + x_{p-1}) + x_h(x_h - x_p).
\end{aligned}$$

We use Lemma 1 to show that each of these four summands is nonnegative, the last nonzero. First, $(x_h + \cdots + x_{q-1}) - x_p - (x_p + \cdots + x_k) \geqslant (x_{h+1} + \cdots + x_{q-1}) - (x_p + \cdots + x_k)$, which is nonnegative because $x_{h+1} \geqslant x_p$ and $(q-1) - h \geqslant k - (p-1)$. Secondly, $(x_{h+1} + \cdots + x_{p-1}) - (x_q + \cdots + x_k) \geqslant 0$ because $x_{h+1} \geqslant x_q$ and $(p-1) - h \geqslant k - (q-1)$. Thirdly, $(x_{h+1} + \cdots + x_{q-1}) - x_p \geqslant 0$ because $h + 1 \leqslant p \leqslant q - 1$; and fourthly, $x_h(x_h - x_p) = x_h(x_q + \cdots + x_k)/(\rho + 1)$, which is positive because vertices $1, 2, \ldots, k$ lie in the nontrivial component of the graph with adjacency matrix A. ∎

3. THE MAIN RESULT

THEOREM. *Let*

$$e=\binom{d}{2}+t, \qquad \text{where} \quad 0<t<d,$$

and let G_e be the graph obtained from the complete graph K_d by adding one new vertex of degree t. If G is a graph with maximal index among the graphs with e edges, then G has a unique nontrivial component H and $H=G_e$.

Proof. Let $A=(a_{ij})\in\mathscr{S}^{**}(n,e)$, and suppose that the graph G with adjacency matrix A does not satisfy the conclusions of the Theorem. Note that $n\geqslant d+1$. By Lemma 2 it suffices to prove that $\rho(A)<\rho(A')$, where A' is an adjacency matrix of the graph G' obtained from G_e by adding $n-(d+1)$ isolated vertices. We suppose that the vertices of G' are labeled so that $A'\in\mathscr{S}^{**}(n,e)$, and we let $\rho=\rho(A)$, $\rho'=\rho(A')$. Let $\mathbf{x},\mathbf{x}'$ be the unique nonnegative unit eigenvectors of A, A' corresponding to ρ, ρ' respectively, say $\mathbf{x}=(x_1,\ldots,x_n)^T$ and $\mathbf{x}'=(x_1',\ldots,x_n')^T$. Note that $x_1'=\cdots=x_t'$ and $x_{t+1}'=\cdots=x_d'$. In respect of the matrix A' the equations (1) reduce to:

$$(\rho'+1)x_1'=tx_1'+(d-t)x_d'+x_{d+1}', \tag{1a}$$

$$(\rho'+1)x_d'=tx_1'+(d-t)x_d', \tag{1b}$$

$$\rho' x_{d+1}'=tx_1'. \tag{1c}$$

On subtracting (1b) and (1c) from (1a), we obtain

$$x_d'+x_{d+1}'=x_1'+x_1't(\rho'+1)^{-1}. \tag{2}$$

Let c be minimal such that $a_{c,c+1}=0$, and let v be maximal such that $a_{v,d+1}\neq 0$. In investigating the sign of $\rho'-\rho$, we distinguish two cases: (I) $v\leqslant t$, (II) $v>t$. In case (I), we have $\mathbf{x}^T\mathbf{x}'(\rho'-\rho)=\mathbf{x}^T(A'-A)\mathbf{x}'=\alpha-\beta$, where α is the sum of (say) r terms $x_i'x_j+x_ix_j'$ for which $v+1\leqslant i\leqslant d$, $c+1\leqslant j\leqslant d+1$, and $i<j$; and β is the sum of r terms $x_i'x_j+x_ix_j'$ for which $1\leqslant i\leqslant v$, $d+2\leqslant j\leqslant n$, and $i<j$. By Lemma 1, we have $\alpha\geqslant r(x_d'+x_{d+1}')x_{d+1}$; then by Equation (2), $\alpha>rx_1'x_{d+1}$. On the other hand, since $x_j'=0$ for $j\geqslant d+2$, we have $\beta\leqslant rx_1'x_{d+2}$. Hence $\rho'>\rho$ because $\mathbf{x}^T\mathbf{x}'>0$ and $\alpha-\beta>0$.

In case (III), again let $2r$ be the number of entries equal to 1 in the matrix $A' - A$. We distinguish two subcases: (IIIa) $r \geq t$, (IIIb) $r < t$. We show next that in subcase (IIIa) we have $\rho \leq d-1$: since $\rho' > d-1$, we again have $\rho' > \rho$. Let A'' be the matrix obtained from A' by making the $(d+1)$th column a zero column and the $(d+1)$th row a zero row. Then $\rho(A'') = d-1$, and the corresponding nonnegative unit eigenvector $\mathbf{x}'' = (x_1'', \ldots, x_n'')^T$ has $x_1'' = \ldots x_d''$, $x_{d+1}'' = \cdots = x_n'' = 0$. Now $\mathbf{x}^T\mathbf{x}''(d-1-\rho) = \mathbf{x}^T(A''-A)\mathbf{x}'' = \alpha - \beta$, where α is a sum of r terms $x_i''x_j + x_ix_j''$ for which $v+1 \leq i \leq d-1$, $c+1 \leq j \leq d$, and $i < j$; and β is a sum of $r+t$ terms $x_i''x_j + x_ix_j''$ for which $1 \leq i \leq v$, $d+1 \leq j \leq n$, and $i < j$. Thus $\alpha \geq rx_1''(x_{d-1} + x_d) \geq 2rx_1''x_{d+1} \geq (r+t)x_1''x_{d+1} \geq \beta$, establishing the required inequality.

Turning now to subcase (IIIb), suppose first that $n \geq d+2$ and $a_{1,d+2} = 1$, and let h be maximal such that $a_{h,d+2} = 1$. Note that $h \leq t-2$ because $a_{t+1,d+1} = 1$ and $r < t$. Let $a = t-h$ and $b = v-t$. Let k be maximal such that $a_{hk} = 1$, and let q be minimal such that $a_{v+1,q} = 0$ and $v+1 < q$. The numbers a, b, d, h, k, q, t, v are illustrated in Figure 2 for a typical matrix $A' - A$: in the illustration, $q \leq d$, but the arguments which follow embrace also the case $q = d+1$. Invoking property $(**)$ with $p = v+1$, we have $v + q \leq h + k$; equivalently $[k-(d+1)] + [d-(q-1)] \geq a+b$. Suppose by way of contradiction that $d-(q-1) < b$: then $k-(d+1) \geq a+1$. But $t-1 \geq r \geq b + [k-(d+1)]h$, and it follows that $t-1 \geq b+(a+1)(t-a)$, whence $t \leq a+1-(b+1)a^{-1}$ and $h \leq 0$, a contradiction. Hence $d-(q-1) \geq b$: therefore, $v \leq d-b-1$, and we can define the matrix $A'' = (a_{ij}'') \in \mathscr{S}^*(n,e)$ as follows. For $i < j$ we have $a_{ij}'' = 1$ precisely when $j \leq d-1$, or $j = d$ and $1 \leq i \leq d-b-1$, or $j = d+1$ and $1 \leq i \leq v$. The matrix A'' is illustrated in Figure 3.

Let $\rho'' = \rho(A'')$ with corresponding nonnegative unit eigenvector $\mathbf{x}'' = (x_1'', \ldots, x_n'')^T$, and let $w = d-b-1-v$. The equations (1) yield the following in respect of A'':

$$(\rho''+1)x_1'' = vx_1'' + wx_{d-b-1}'' + bx_{d-1}'' + x_d'' + x_{d+1}'',$$

$$(\rho''+1)x_{d-1}'' = vx_1'' + wx_{d-b-1}'' + bx_{d-1}'',$$

$$\rho''x_d'' = vx_1'' + wx_{d-b-1}''.$$

We deduce that $x_{d-1}'' + x_d'' > x_1''$. Now $\mathbf{x}^T\mathbf{x}''(\rho''-\rho) = \alpha - \beta$, where α is the sum of (say) r' terms $x_i''x_j + x_ix_j''$ for which $i < j \leq d$, and β is a sum of r' terms $x_i''x_j + x_ix_j''$ for which $j \geq d+2$ and $i < j$. Hence $\alpha \geq r'(x_{d-1}''x_d + x_d''x_{d-1}) \geq r'x_d(x_{d-1}'' + x_d'') > r'x_1''x_d \geq r'x_1''x_{d+2}$. On the other hand, $x_j'' = 0$ for $j \geq d+2$ and so $\beta \leq r'x_1''x_{d+2}$, whence $\rho < \rho''$.

$$
\begin{array}{lcccccccccl}
 & & (q) & & (d) & & & & (k) & & \\
 & 0 & 0 & \cdots\cdots & 0 & 0 & -1 & \cdots\cdots & -1 & * & \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
(h) & 0 & 0 & \cdots\cdots & 0 & 0 & -1 & \cdots\cdots & -1 & 0 & \\
 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & a \\
(t) & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & \\
 & 0 & 0 & \cdots\cdots & 0 & -1 & 0 & \cdots\cdots & 0 & 0 & \\
 & & & & \vdots & \vdots & \vdots & & \vdots & \vdots & b \\
(v) & 0 & 0 & \cdots\cdots & 0 & -1 & 0 & \cdots\cdots & 0 & 0 & \\
 & 0 & 1 & \cdots\cdots & 1 & 0 & 0 & \cdots\cdots & 0 & 0 & \\
 & * & 1 & \cdots\cdots & 1 & 0 & 0 & \cdots\cdots & 0 & 0 & \\
 & & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
 & & 1 & \cdots\cdots & 1 & 0 & 0 & \cdots\cdots & 0 & 0 & \\
(q) & & 0 & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
 & & & & 1 & 0 & 0 & \cdots\cdots & 0 & 0 & \\
(d) & & & & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & \\
 & & & & & 0 & 0 & \cdots\cdots & 0 & 0 &
\end{array}
$$

FIG. 2. Part of the matrix $A' - A$, subcase *IIb*.

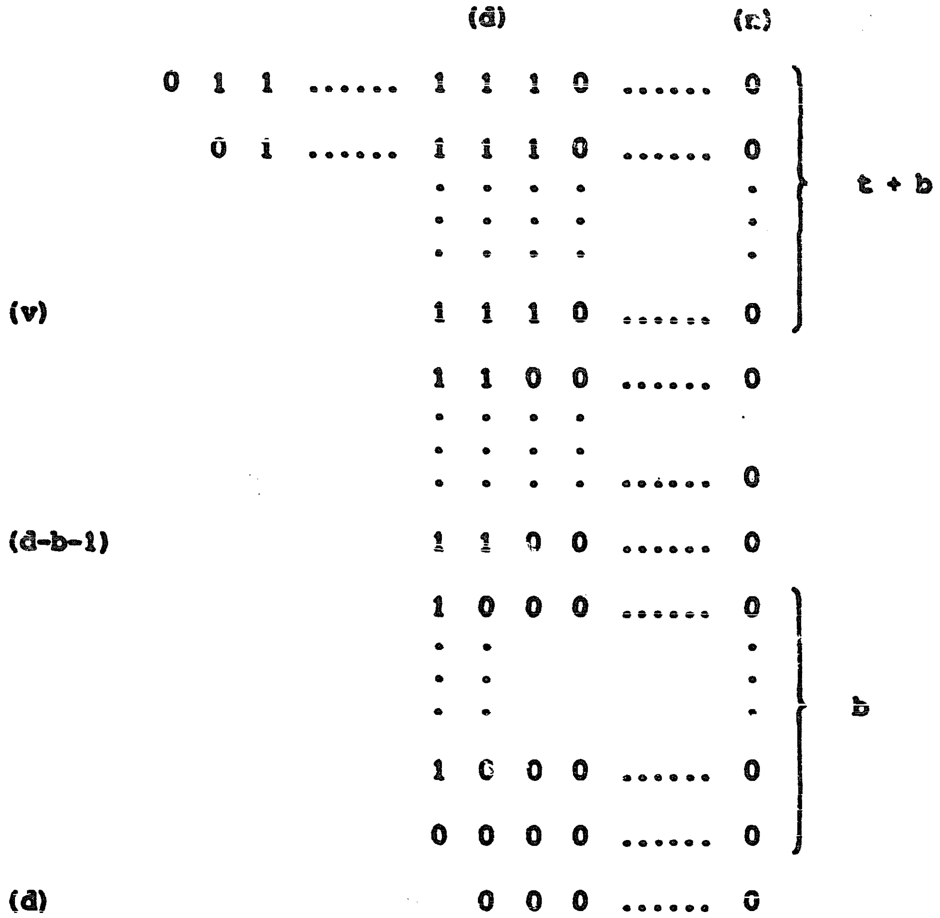

FIG. 3. Part of the matrix A'', subcase *IIb*.

If $A'' \notin \mathcal{S}^{**}(n,e)$ (equivalently, if $d-t-2b \geq 3$), then $\rho'' < \rho'$ because the spectral radius of A'' may be increased by successively transferring the last b entries equal to 1 in column $d+1$ to positions (i,d) $(i=d-b,\dots,d-1)$ as in the proof of Lemma 2. Thus $\rho<\rho'$ when $A''\notin\mathcal{S}^{**}(n,e)$. If $A''\in\mathcal{S}^{**}(n,e)$, then A'' is an instance of a matrix A in $\mathcal{S}^{**}(n,e)$ for which $a_{1,d+2}=0$. Accordingly, to prove that $\rho<\rho'$ in every case it suffices to prove that $\rho(A)<\rho'$ for those matrices $A=(a_{ij})\in\mathcal{S}^{**}(n,e)-\{A'\}$ for which $a_{1,d+2}=0$ and $b=r<t$. Since isolated vertices of the corresponding graphs may be ignored, we may assume that $n=d+1$.

We now have $x_1=\cdots=x_{t+b}$, $t+b<d$ and $\rho x_{d+1}=(t+b)x_1$. Moreover $\mathbf{x}^T\mathbf{x}'(\rho'-\rho)=\alpha-\beta$, where α is the sum of b terms $x_i'x_j+x_ix_j'$ with $i<j\leq d$, and $\beta = x_{d+1}(x'_{t+1}+\cdots+x'_{t+b})+x'_{d+1}(x_{t+1}+\cdots+x_{t+b})$. Hence $\alpha\geq b(x'_{d-1}x_d+x_{d-1}x'_d)=bx'_d(x_d+x_{d-1})$ and $\beta=b(x_{d+1}x'_d+x'_{d+1}x_1)$. Thus $\alpha-\beta\geq b\{x'_d(x_d+x_{d-1}-x_{d+1})-x'_{d+1}x_1\}$, and it suffices to

prove that

$$\frac{x'_d}{x'_{d+1}} > \frac{x_1}{x_d + x_{d-1} - x_{d+1}}.$$

Since $\{\rho' - (d-1-t)\}x'_d = tx'_1 = \rho' x'_{d+1}$, we have

$$\frac{x'_d}{x'_{d+1}} = \frac{\rho'}{\rho' - (d-1-t)}.$$

On the other hand, $\rho(x_d + x_{d-1} - x_{d+1}) \geqslant \rho x_d \geqslant \rho x_{d+1} = (t+b)x_1$, and so

$$\frac{x_1}{x_d + x_{d-1} - x_{d+1}} \leqslant \frac{\rho}{t+b} < \frac{d}{t+b}.$$

Hence it suffices to prove that

$$\frac{\rho'}{\rho' - (d-1-t)} \geqslant \frac{d}{t+b}$$

—equivalently, that

$$\rho' \leqslant d\left(\frac{d-t-1}{d-t-b}\right).$$

This is clear because $\rho' < d$, and the Theorem is now proved. ■

We now know that when

$$e = \binom{d}{2} + t, \qquad 0 < t < d,$$

the graph G_e has the largest index of any graph in $\mathscr{S}(e)$: accordingly $f(e)$ is the largest eigenvalue of the matrix

$$\begin{pmatrix} t-1 & d-t & 1 \\ t & d-t-1 & 0 \\ t & 0 & 0 \end{pmatrix}.$$

Hence $f(e)=d-1+\varepsilon$, where $0<\varepsilon<1$ and $\varepsilon^3+(2d-1)\varepsilon^2+(d^2-d-t)\varepsilon-t^2=0$.

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