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The Study of Existence of Equilibria for Generalized Games without Lower Semicontinuity in Locally Topological Vector Spaces*

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The aim of this paper is to establish general existence results of maximal elements for $\mathscr{L}$-majorized mappings, which are, in turn, used to establish the general existence theorems of equilibria for generalized games (resp., abstract economics) without lower semicontinuity for both constraint and preference mappings (correspondences) and in which strategic (resp., commodity) spaces may not be compact, the set of players (resp., agents) may not be countable, and underlying spaces are either finite or infinite dimensional locally topological vector spaces. Our results unify or improve corresponding results in the literature. © 1998 Academic Press

Key Words: equilibria; generalized game; paracompactness; approximation; maximal elements; $\mathscr{L}$-majorized mapping; locally convex space.

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1. INTRODUCTION

Since the existence of equilibria in an abstract economy with compact strategy sets in $\mathbb{R}^n$ was proved in a seminal paper of Debreu [4] which is classical Arrow–Debreu–McKenzie model of exchange economies under perfect information competition, there have been many generalizations of Debreu's theorem by Aliprantis and Brown [1], Borglin and Keiding [3], Ding and Tan [5], Gale and Mas-Colell [9], Kim [11], Kim and Lee [12], Shafer and Sonnenschein [14], Tan and Yuan [15], Tarafdar [16], Toussaint [17], Tulcea [18], Wu [19], Wu and Shen [20], Yannelis and Prabhakar [22], Yannelis [21], Yuan [23], Yuan and Tarafdar [24], Zheng [25], Zhou [26], and many others. Even several different kinds of mappings have been introduced such as class of L_F (resp., $L^*, \mathscr{L}, \mathscr{L}(X, Y, \theta)$), L_F-majorized (resp., $L^*, \mathscr{L}, \mathscr{L}(X, Y, \theta)$-majorized) (e.g., see [5], [15], [18], [22], and others), and a number of existence theorems of equilibria for generalized games (resp., abstract economics) have been established; however, note that almost all of those results so far established in the literature need some assumptions such as either preference mappings or constraint mappings are lower semicontinuous, or the set of players (resp., agents) is countable and strategic (resp., commodity) spaces need to be compact. On the other hand, in order to allow people to study general equilibrium problems involving infinite time horizons, we need to study general existence problems of equilibria for generalized games (resp., abstract economics) in which the strategic (resp., commodity) spaces are infinitely dimensional, and the set of players (resp., agents, or states of world, or varieties of commodity characteristics) is infinite and even uncountable. Moreover we do know that there exist many constraint mappings which are only upper semicontinuous instead of having lower semicontinuity and also there do exist preference mappings which do not have lower semicontinuity (e.g., see [1], [2], [21], and related references therein). Motivated by those facts, by using approximate techniques for upper semicontinuous set-valued mappings in locally topological vector spaces, the purpose of this paper is to establish general existence results of maximal elements for $\mathscr{L}$-majorized mappings, which are, in turn, used to establish the general existence theorems of equilibria for generalized games (resp., abstract economics) without lower semicontinuity for both constraint and preference mappings (correspondences) and in which strategic (resp., commodity) spaces may not be compact, the set of players (resp., agents) may not be countable, and underlying spaces are either finite or infinite dimensional locally topological vector spaces. Our results unify or improve corresponding results in the literature.

In order to expose clearly the idea and method we used in this paper, we only focus on the study of so-called $\mathscr{L}$-majorized mappings which have been used and discussed extensively in mathematical economics in the last more than two decades, though the method used in this paper might be applicable for a more broad class of mappings such as, for example, *KF* mappings discussed by Yuan and Tarafdar [24] and others.

Now we recall or introduce some notions and definitions. Throughout this paper, all topological spaces are assumed to be Hausdorff unless otherwise specified. Let A be a subset of a topological space X. We denote by 2^A the family of all subsets of A, by $\mathrm{int}_X(A)$ the interior of A in X, and by $\mathrm{cl}_X(A)$ the closure of A in X. The subset A is said to be compactly open in X if for each nonempty compact subset C of X, $A \cap C$ is open in C. If A is a subset of a vector space E, we shall denote by $\mathrm{co}\, A$ the convex hull of A. If $S, T\colon A \to 2^E$ are two set-valued mappings (or, say, correspondences), then $\mathrm{co}\, T$, $T \cap S\colon A \to 2^E$ are correspondences defined by $(\mathrm{co}\, T)(x) = \mathrm{co}\, T(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively, and $\mathrm{dom}\, S$ denotes the domain of S, i.e., $\mathrm{dom}\, S := \{x \in X\colon S(x) \neq \emptyset\}$. If X and Y are topological spaces and $T\colon X \to 2^Y$ is a correspondence, the graph of T, denoted by $\mathrm{graph}\, T$, is the set $\{(x, y) \in X \times Y\colon y \in T(x)\}$ and the correspondence $\overline{T}\colon X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in Y\colon (x, y) \in \mathrm{cl}_{X \times Y} \mathrm{Graph}\, T\}$ (the set $\mathrm{cl}_{X\times Y} \mathrm{Graph}\, T$ is called the adherence of the graph of T), and $\mathrm{cl}\, T\colon X \to 2^Y$ is defined by $\mathrm{cl}\, T(x) = \mathrm{cl}_X(T(x))$ for each $x \in X$. It is easy to see that $\mathrm{cl}\, T(x) \subset \overline{T}(x)$ for each $x \in X$.

In this paper, by using the so-called approximate method which has been initially used by Tan and Yuan [15], we will focus on the study of generalized games in which the preference mappings are so-called $\mathscr{L}$-majorized mappings, whose roots could be found from Borglin and Keiding's paper [3] and Yannelis and Prabhakar [22] (for more recent studies, see [17], [18], [5], [15], and references therein). By following the idea of Yannelis and Prabhakar [22], we first have the notion of $\mathscr{L}$-majorized mappings as follows. Let X be a topological space, let Y be a nonempty subset of a vector space E, let $\theta\colon X \to E$ be a (single-valued) mapping, and let $\phi\colon X \to 2^Y$ be a correspondence. Then we have that:

(1) ϕ is said to be of class $\mathscr{L}_{\theta, C}$ if (a) for each $x \in X$, $\mathrm{co}\, \phi(x) \subset Y$ and $\theta(x) \notin \mathrm{co}\, \phi(x)$ and (b) for each $y \in Y$, $\phi^{-1}(y)$ is compactly open in X;

(2) (ϕ_x, N_x) is an $\mathscr{L}_{\theta, C}$-majorant of ϕ at x if $\phi_x\colon X \to 2^Y$ and N_x is an open neighborhood of x in X such that (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$, (b) for each $z \in N_x$, $\theta(z) \notin \mathrm{co}\, \phi_x(z)$, and (c) for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X;

(3) ϕ is said to be $\mathscr{L}_{\theta, C}$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $\mathscr{L}_\theta$-majorant (ϕ_x, N_x) of ϕ at x.

We note that our notions of the mapping P being of class $\mathscr{L}_{\theta, C}$ and $\mathscr{L}_{\theta, C}$-majorized have been used by Ding and Tan [5], Tan and Yuan [15], and Zhou [26], which in turn generalize the notions of $\phi \in C(X, Y, \theta)$ and C-majorized mappings introduced by Tulcea [18] and a related one of Yannelis and Prabhakar [22] as special cases.

In this paper, we shall deal mainly with either case (I) $X = Y$, which is a nonempty convex subset of the topological vector space E and $\theta = I_X$, the identity map on X, or case (II) $X = \prod_{i \in I} X_i$ and $\theta = \pi_j$: $X \to X_j$ is the projection of X onto X_j and $Y = X_j$ is a nonempty convex subset of a topological vector space. In both cases (I) and (II), we shall write $\mathscr{L}$ in place of $\mathscr{L}_{\theta, C}$. Let I be a (possibly uncountable) set of players. For each $i \in I$, let its choice or strategy set X_i be a nonempty subset of a topological vector space. Let $X = \prod_{i \in I} X_i$. For each $i \in I$, let P_i: $X \to 2^{X_i}$ be a correspondence. Following the notion of Gale and Mas-Colell [9], the collection $\Gamma = (X_i, P_i)_{i \in I}$ is said to be a qualitative game. A point $\hat{x} \in X$ is said to be an equilibrium of the game Γ if

$$P_i(\hat{x}) = \varnothing$$

for all $i \in I$. For each $i \in I$, let A_i be a nonempty subset of X_i. Then, for each fixed $k \in I$, we define $\prod_{j \in I, j \neq k} A_j \otimes A_k = \{x = (x_i)_{i \in I}: x_i \in A_i$ for all $i \in I\}$.

A generalized game (resp., abstract economy) is a family of quadruples

$$\Gamma = (X_i; A_i, B_i; P_i)_{i \in I},$$

where I is a (possibly uncountable) set of players (resp., agents) such that, for each $i \in I$, X_i is a nonempty subset of a topological vector space and A_i, B_i: $X = \Pi_{j \in I} X_j \to 2^{X_i}$ are constraint correspondences and P_i: $X \to 2^{X_i}$ is a preference correspondence. When $I = \{1, \dots, N\}$, where N is a positive integer, $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ is also called an N-person game.

An equilibrium of Γ is a point $\hat{x} \in X$ such that, for each $i \in I$,

$$\hat{x}_i = \pi_i(\hat{x}) \in \overline{B}_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \varnothing.$$

We remark that when $\overline{B}_i(\hat{x}) = \mathrm{cl}_{X_i} B_i(\hat{x})$ (which is the case when $\mathrm{cl}_{X_i} B_i$ has a closed graph in $X \times X_i$; in particular, when B_i is upper semicontinuous with closed values), our definition of an equilibrium point coincides with that of Ding and Tan [5]; and if, in addition, $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point coincides with the standard definition, e.g., in Borglin and Keiding [3], Tarafdar [16], Tulcea [18], and Yannelis and Prabhakar [22].

2. THE EXISTENCE OF SELECTIONS FOR $\mathscr{L}$-MAJORIZED MAPPINGS

In this section, we first have the following existence of $\mathscr{L}$-class mappings for $\mathscr{L}$-majorized mappings in topological vector spaces.

THEOREM 2.1. *Let X be a regular topological vector space and let Y be a nonempty subset of a vector space E. Let θ: $X \to E$ and P: $X \to 2^Y$ be $\mathscr{L}_{\theta, C}$-majorized. If the domain set of the mapping P is open and paracompact, then there exists an $\mathscr{L}_{\theta, C}$-class mapping ϕ: $X \to 2^Y$ such that $P(x) \subset \phi(x)$ for each $x \in X$.*

Proof. Since P is $\mathscr{L}_{\theta, C}$-majorized, for each $x \in B$, let N_x be an open neighborhood of x in X and let ϕ_x: $X \to 2^Y$ be a set-valued mapping such that (1) for each $z \in N_x$, $\theta(z) \notin \operatorname{co}(\phi_x(z))$; (2) for each $z \in N(x)$, $P(z) \subset \phi_x(z)$, and $\operatorname{co} \phi_x(z) \subset Y$; (3) for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X. Note that X is regular; for each $x \in B$, there exists an open neighborhood G_x of x in X such that $\operatorname{cl}_X G_x \subset N_x$. Let $G = \bigcup_{x \in B} G_x$. Then G is an open subset of X which contains $B = \{x \in X: P(x) \neq \emptyset\}$. Note that B is open in X, without loss of generality, we may assume that $G = B$ (otherwise, taking the set $G \cap B$, which is indeed the same set as B). By the paracompactness assumption on B and Theorem VIII.1.4 of Dugundji [6, p. 162], the open covering $\{G_x\}$ of G has an open precise neighborhood-finite refinement $\{G'_x\}$. Given any $x \in B$, we define a set-valued mapping ϕ'_x: $G \to 2^Y$ by

$$\phi'_x(z) = \begin{cases} \phi_x(z), & \text{if } z \in G \cap \operatorname{cl}_X G'_x, \\ Y, & \text{if } z \in G \setminus \operatorname{cl}_X G'_x, \end{cases}$$

then we have that (i) for each $y \in Y$, the set

$$\begin{aligned} (\phi'_x)^{-1}(y) &= \{z \in G: y \in \phi'_x(z)\} \\ &= \{z \in G \cap \operatorname{cl}_X G'_x: y \in \phi'_x(z)\} \cup \{z \in G \setminus \operatorname{cl}_X G'_x: y \in \phi'_x(z)\} \\ &= \{z \in G \cap \operatorname{cl}_X G'_x: y \in \phi_x(z)\} \cup \{z \in G \setminus \operatorname{cl}_X G'_x: y \in Y\} \\ &= \left[(G \cap \operatorname{cl}_X G'_x) \cap \phi_x^{-1}(y)\right] \cup (G \setminus \operatorname{cl}_X G'_x) \\ &= \left(G \cap \phi_x^{-1}(y)\right) \cup (G \setminus \operatorname{cl}_X G'_x). \end{aligned}$$

It follows that, for each nonempty compact subset C of X, $(\phi'_x)^{-1}(y) \cap C = (G \cap \phi_x^{-1}(y) \cap C) \cup ((G \setminus \operatorname{cl}_X G'_x) \cap C)$ is open in C by (3). Now define $\phi\colon X \to 2^Y$ by

$$\phi(z) = \begin{cases} \bigcap_{x \in B} \phi'_x(z), & \text{if } z \in G, \\ \varnothing, & \text{if } z \in X \setminus G. \end{cases}$$

Let $z \in X$ be given. Clearly (2) implies $P(z) \subset \phi(z)$ and $\operatorname{co} \phi(z) \subset Y$ for all $z \in X$. As $\theta(z) \notin \operatorname{co} \phi'_x(z)$ by (1) we must have $\theta(z) \notin \operatorname{co} \phi(z)$. Therefore $\theta(z) \notin \operatorname{co} \phi(z)$ for all $z \in X$. Now we show that, for each $y \in Y$, $\phi^{-1}(y)$ is compactly open in X. Indeed, let $y \in Y$ be such that $\phi^{-1}(y) \neq \varnothing$ and let C be a compact subset of X; fix an arbitrary $u \in \phi^{-1}(y) \cap C = \{z \in X\colon\ y \in \phi(z)\} \cap C = \{z \in G\colon\ y \in \phi(z)\} \cap C$. Since $\{G'_x\}$ is a neighborhood-finite refinement, there exists an open neighborhood M_u of u in G such that $\{x \in B\colon\ M_u \cap G'_x \neq \varnothing\} = \{x_1, \dots, x_n\}$. Note that for each $x \in B$ with $x \notin \{x_1, \dots, x_n\}$, $\varnothing = M_u \cap G'_x = M_u \cap \operatorname{cl}_X G'_x$ so that $\phi'_x(z) = Y$ for all $z \in M_u$. Thus we have $\phi(z) = \bigcap_{x \in B} \phi'_x(z) = \bigcap_{i=1}^n \phi'_{x_i}(z)$ for all $z \in M_u$. It follows that

$$\begin{aligned} \phi^{-1}(y) &= \{z \in X\colon y \in \phi(z)\} \\ &= \left\{z \in G\colon y \in \bigcap_{x \in B} \phi'_x(z)\right\} \supset \left\{z \in M_u\colon y \in \bigcap_{x \in B} \phi'_x(z)\right\} \\ &= \left\{z \in M_u\colon y \in \bigcap_{i=1}^n \phi'_{x_i}(z)\right\} \\ &= M_u \cap \left\{z \in G\colon y \in \bigcap_{i=1}^n \phi'_{x_i}(z)\right\} \\ &= M_u \cap \left[\bigcap_{i=1}^n (\phi'_{x_i})^{-1}(y)\right]. \end{aligned}$$

But $M'_u = M_u \cap [\bigcap_{i=1}^n (\psi'_{x_i})^{-1}(y)] \cap C$ is an open neighborhood of u in C such that $M'_u \subset \phi^{-1}(y) \cap C$ since $(\psi'_{x_i})^{-1}(y)$ is compactly open in X. This shows that, for each $y \in Y$, $\phi^{-1}(y)$ is compactly open in X. Thus ϕ is of class $\mathscr{L}_\theta$ and the proof is complete.

Condition (C). Let X be a nonempty convex subset of a topological vector space E. Throughout this paper, a set-valued mapping $P\colon X \to 2^X$ is said to satisfy the following noncompact condition: If there exist a nonempty compact and convex subset X_0 and a nonempty compact (not necessarily convex) subset K of X such that, for each $y \in X \setminus K$, we have that $\operatorname{co} P(y) \cap \operatorname{co}(X_0 \cup \{y\}) \neq \emptyset$.

To establish the existence of maximal elements, we first have the following generalization of the Fan–Browder fixed point theorem which is, indeed, Lemma 1 of Ding and Tan [5, p. 228]. For the convenience of readers, we give the outline of its proof as applications of Fan's geometric lemma in [8] which is an infinite dimensional version of the famous KKM principle in [13].

THEOREM 2.2. *Let X be a nonempty convex subset of a topological vector space* (*not necessarily Hausdorff*) *and let G: $X \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping with compactly open inverse* (*i.e., the set $G^{-1}(y)$ is compactly open in X for each $y \in X$*) *and the mapping G satisfies the noncompact Condition* (C), *i.e., there exist a nonempty compact and convex subset X_0 and a nonempty compact* (*not necessarily convex*) *subset K of X such that, for each $x \in X \setminus K$, we have that* $\operatorname{co}(X_0 \cup \{x\}) \cap \operatorname{co} G(x) \neq \emptyset$.

Then there exists $x \in X$ such that $x \in \operatorname{co} G(x)$.

Proof. Suppose the conclusion were false. Then, for each $x \in X$, we have that $x \notin \operatorname{co} G(x)$. Let F: $X \to 2^X$ be a set-valued mapping defined by $F(x) = \operatorname{co} G(x)$ for each $x \in X$. Then $F(x)$ is nonempty and convex for each $x \in X$. By our assumption, it is easy to verify that $F^{-1}(y)$ is compactly open in X as $G^{-1}(y)$ is compactly open in X for each $y \in X$. It then follows that

$$X = \bigcup_{y \in X} F^{-1}(y).$$

By the noncompact condition, there exists a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that, for each $y \in X \setminus K$, there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ with $x \in F(y)$. Now we define a set-valued mapping T: $X \to 2^X$ by

$$T(x) = \left(X \setminus F^{-1}(x)\right) \cap K$$

for each $x \in X$. We shall prove that the family $\{T(x): x \in X\}$ has the finite interaction property. Let $\{x_1, \ldots, x_n\}$ be any nonempty finite subset of X. Let $D = \operatorname{co}(X_0 \cup \{x_1, \ldots, x_n\})$. Then D is also compact and convex and define T_0: $D \to 2^D$ by

$$T_0(x) = \left(X \setminus F^{-1}(x)\right) \cap D$$

for each $x \in D$. Then we have that (a) for each $x \in D$, $x \in T_0(x)$ as $x \notin F(x)$, so that $T_0(x)$ is nonempty and $T_0(x)$ is compact in D; (b) we can also show that T_0 is a KKM mapping; i.e., the convex hull of each finite subset $\{u_1, \dots, u_m\}$ of D is contained in the corresponding union $\bigcup_{i=1}^m T_0(u_i)$. Now by Lemma 1 of Fan [8] (as we note that the assumption "Hausdorff" is not necessary in its proof), it follows that

$$\bigcap_{x \in D} T_0(x) = \bigcap_{x \in D} \left(X \setminus F^{-1}(x)\right) \cap D \neq \emptyset.$$

Take any $\hat{y} \in \bigcap_{x \in D}(X \setminus F^{-1}(x_i)) \cap D$. Then we have that $\hat{y}$ must be in K. Hence, $\hat{y} \in \bigcap_{i=1}^n (X \setminus F^{-1}(x_i)) \cap K = \bigcap_{i=1}^n T(x_i)$; i.e., the family $\{T(x): x \in X\}$ has the finite intersection property. By the compactness of K, it follows that $\bigcap_{x \in X} T_0(X) \neq \emptyset$, that is, $\bigcap_{x \in X}(X \setminus F^{-1}(x)) \cap K \neq \emptyset$. Take any $y \in \bigcap_{x \in X}(X \setminus F^{-1}(x))$. It follows that $y \in X \setminus F^{-1}(x) = \emptyset$, which is impossible as we get that $X = \bigcup_{y \in X} F^{-1}(y)$ above. Therefore our claim "$x \notin \operatorname{co} G(x)$ for all $x \in X$" is impossible and the conclusion must hold; i.e., there exists at least one $x \in X$ such that $x \in \operatorname{co} G(x)$. ∎

As applications of both Theorems 2.1 and 2.2, we have the following existence of maximal elements for $\mathscr{L}$-majorized mappings in topological vector spaces which will play the most important role in the study of general existence of equilibria for generalized games throughout this paper.

THEOREM 2.3. *Let X be a nonempty paracompact convex subset of a topological vector space. Suppose the mapping P: $X \to 2^X$ is $\mathscr{L}$-majorized and satisfies Condition* (C). *Then there exists an $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$; i.e., $\hat{x}$ is a maximal element of P.*

Proof. Suppose that the conclusion does not hold, then $P(x) \neq \emptyset$ for all $x \in X$ and $\operatorname{dom} P = X$ is paracompact. By Theorem 2.1, there exists an $\mathscr{L}$-class mapping ϕ: $X \to 2^X$ such that (1) $P(x) \subset \phi(x)$ for each $x \in X$; (2) $x \notin \operatorname{co} \phi(x)$ for each $x \in X$; and (3) $\phi^{-1}(y)$ is compactly open in X for all $y \in X$. Note that P satisfies Condition (C), so does ϕ. Therefore the mapping ϕ satisfies all hypotheses of Theorem 2.2. By Theorem 2.2, there exists a point $x \in X$ such that $x \in \operatorname{co} \phi(x)$, which is a contradiction. Thus there must exist $x_0 \in X$ such that $P(x_0) = \emptyset$. By the noncompact Condition (C), it follows that $\hat{x}$ must be in K. ∎

3. THE EXISTENCE OF EQUILIBRIA FOR GENERALIZED GAMES WITHOUT LOWER SEMICONTINUITY FOR CONSTRAINT MAPPINGS

To apply the approximate method initially used by Tan and Yuan [15] (see also [18]), we first recall some definitions and some results about approximation for upper semicontinuous set-valued mappings, due to Tulcea [18].

Let X be a subset of a topological vector space E. By following Tulcea [18], the set X is said to have the property (K) if, for every compact subset B of X, the convex hull $\operatorname{co} B$ is relatively compact in E. It is clear that each compact convex set in a Hausdorff (resp., locally) topological vector space always has property (K). Let X and Y be two topological spaces and let $F: X \to 2^Y$ be a set-valued mapping. Then the mapping F is said to be compact if, for any $x \in X$, there exists an open neighborhood $N(x)$ such that $F(N(x)) = \bigcup_{y \in N(x)} F(y)$ is relatively compact in Y.

Let X be a nonempty set, let Y be a nonempty subset of a topological vector space E, and let $F: X \to 2^Y$. As in Tulcea [18, p. 269], a family $(f_j)_{j \in J}$ of correspondences between X and Y, indexed by a nonempty filtering set J (denote by $\leq$ the order relation in J), is an upper approximating family for F if

(1) $F(x) \subset f_j(x)$ for all $x \in X$ and all $j \in J$;

(2) for each $j \in J$ there is a $j^* \in J$ such that, for each $h \geq j^*$ and $h \in J$, $f_h(x) \subset f_j(x)$ for each $x \in X$;

and

(3) for each $x \in X$ and $V \in \mathscr{B}$, where $\mathscr{B}$ is a base for the zero neighborhood in E, there is a $j_{x,V} \in J$ such that $f_h(x) \subset F(x) + V$ if $h \in J$ and $j_{x,V} \leq h$.

From (1)–(3), it is easy to deduce that:

(4) for each $x \in X$ and $k \in J$, $F(x) \subset \bigcap_{j \in J} f_j(x) = \bigcap_{k \leq j,\, k \in J} f_j(x) \subset \operatorname{cl} F(x) \subset \overline{F}(x)$.

A set-valued mapping $F: X \to 2^Y$ is said to be quasi-regular if

(a) it has open lower sections, i.e., $F^{-1}(y)$ is open in X for each $y \in Y$;

(b) $F(x)$ is nonempty and convex for each $x \in X$;

(c) $\overline{F}(x) = \operatorname{cl} F(x)$ for each $x \in X$.

The set-valued mapping F is said to be regular if it is quasi-regular and has an open graph.

By observing Theorem 3 and the Remark of Tulcea [18, p. 280 and pp. 281–282], we have the following:

LEMMA 3.1. *Let $(X_i)_{i \in I}$ be a family of paracompact spaces and let $(Y_i)_{i \in I}$ be a family of nonempty closed convex subsets, each in a locally convex Hausdorff topological vector space and each has property* (K). *For each $i \in I$, let $F_i: X_i \to 2^{Y_i}$ be compact and upper semicontinuous with* *nonempty and compact convex values. Then there is a common filtering set J (independent of $i \in I$) such that, for each $i \in I$, there is a family $(f_{ij})_{j \in J}$ of correspondences between X_i and Y_i with the following properties*:

(i) *for each $j \in J$, f_{ij} is regular*;

(ii) $(f_{ij})_{j \in J}$ *and* $\overline{(f_{ij})}_{j \in J}$ *are upper approximating families for F_i*;

(iii) *for each $j \in J$, $\overline{f_{ij}}$ is continuous if Y_i is compact.*

Now we shall first establish the following existence for equilibria for a so-called one-person game.

THEOREM 3.2. *Let (X, F, G, P) be a one-person game and strategic set X a nonempty paracompact convex subset of a Hausdorff locally topological vector space such that*

(i) *for each $x \in X$, $F(x) \neq \emptyset$ and* $\operatorname{co} F(x) \subset G(x)$ *and $F^{-1}(y)$ is compactly open in X for each $y \in X$*;

(ii) *the preference mapping P is such that $F \cap P$ is $\mathscr{L}$-majorized*;

(iii) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that, for each $x \in X \setminus K$, we have that* $\operatorname{co}(X_0 \cup \{x\}) \cap \operatorname{co}(F(x) \cap P(x)) \neq \emptyset$.

Then there exists a point $\hat{x} \in K$ such that $\hat{x} \in \overline{G}(\hat{x})$ and $F(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. Let $M = \{x \in X\colon x \notin \overline{G}(x)\}$. Then M is open in X. Define a set-valued mapping $\Psi\colon X \to 2^X$ by

$$\Psi(x) = \begin{cases} F(x) \cap P(x), & \text{if } x \notin M, \\ F(x), & \text{if } x \in M. \end{cases}$$

Case 1. Suppose $x \in M$ (it follows that $\Psi(x) = F(x) \neq \emptyset$). Let $\phi_x = F$ and $N_x = M$. Then N_x is an open neighborhood of x in X such that (a) for each $z \in N_x$, $\Psi(z) = F(z) = \phi_x(z)$, $z \notin \overline{G}(z)$ so that $z \notin \operatorname{co} F(z) = \operatorname{co} \phi_x(z)$; (b) for each $y \in X$, $\phi_x^{-1}(y) = F^{-1}(y)$ is compactly open in X by condition (i). Thus ϕ_x is an $\mathscr{L}$-majorant of Ψ at the point x in X.

Case 2. Suppose $x \notin M$ and $\Psi(x) \neq \emptyset$. Then $\Psi(x) = F(x) \cap P(x) \neq \emptyset$. As $F \cap P$ is $\mathscr{L}$-majorized, there exist an open neighborhood N_x of x in X and a mapping $\phi'_x\colon X \to 2^X$ such that (a) $\psi(z) = F(z) \cap P(z) \subset \phi'_x(z)$ and $z \notin \operatorname{co}(\phi'_x(z))$ for each $z \in N_x$ and (b) for each $y \in X$, the set $(\phi'_x)^{-1}(y)$ is compactly open in X. Define $\Phi_x\colon X \to 2^X$ by

$$\Phi_x(z) = \begin{cases} F(z) \cap \phi'_x(z), & \text{if } z \notin M, \\ F(z), & \text{if } x \in M. \end{cases}$$

Note that (1) for each $z \in N_x$, clearly $\Psi(z) \subset \Phi_x(z)$ and $z \notin \operatorname{co} \Phi_x(z)$, and (2) for each $y \in X$, $\Phi_x^{-1}(y) = [M \cup (\phi_x')^{-1}(y)] \cap F^{-1}(y)$, which is compactly open in X. Hence Φ_x is an $\mathscr{L}$-majorant of Ψ at the point x in X. Therefore Ψ is an $\mathscr{L}$-majorized mapping. Moreover by condition (iii), for each $x \in X \setminus K$, there exists $y \in \operatorname{co}(X_0 \cup \{x\}) \cap \operatorname{co}(F(x) \cap P(x))$ such that $y \in \operatorname{co}(F(x) \cap P(x)) \subset \operatorname{co} \Psi(x)$. By the paracompact assumption of X and the existence of maximal elements for $\mathscr{L}$-majorized mappings in topological vector spaces, e.g., Theorem 2.3 above, it follows that there exists $\hat{x} \in K$ such that $\Psi(\hat{x}) = \emptyset$. As $F(\hat{x}) \neq \emptyset$ for all $x \in X$, it follows we must have that $\hat{x} \notin M$, i.e., $\hat{x} \in \overline{G}(\hat{x})$ and $F(\hat{x}) \cap P(\hat{x}) = \emptyset$. This completes the proof of Theorem 3.2. ∎

Now as applications of Lemma 3.1 and Theorem 3.2, we have the following existence theorem of equilibria for a one-person game in which the constraint correspondences are only upper semicontinuous.

THEOREM 3.3. *Let the strategic set X be a nonempty paracompact closed and convex with property* (K) *in a Hausdorff locally topological vector space and let F, P: $X \to 2^X$ be two set-valued mappings such that*

(i) *the constraint mapping F is compact and upper semicontinuous with nonempty compact and convex values*;

(ii) *the preference mapping P is $\mathscr{L}$-majorized*;

(iii) *the domain of the mapping $F \cap P$ is open in X, i.e, the set $E := \{x \in X\colon F(x) \cap P(x) \neq \emptyset\}$ is open in X*;

(iv) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset of K of X such that, for each $x \in X \setminus K$, we have that* $\operatorname{co}(X_0 \cup \{x\}) \cap \operatorname{co}(F(x) \cap P(x)) \neq \emptyset$.

Then there exists a point $\hat{x} \in K$ such that $\hat{x} \in F(\hat{x})$ and $F(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. By Lemma 3.1, there exists a family $\{F_j\}$ of regular mappings from X to X such that both $\{F_j\}_{j \in J}$ and $\{\overline{F}_j\}_{j \in J}$ are upper approximating families of F. Let $j \in J$ be arbitrarily fixed. Note that each F_j has an open graph and the mapping $F_j \cap P$ is also $\mathscr{L}$-majorized as P is $\mathscr{L}$-majorized by condition (ii). Thus the game $\mathscr{G}_j = (X, F_j, \overline{F}_j, P)$ satisfies all hypotheses of Theorem 3.1. By Theorem 3.1, it follows that there exists $\hat{x}_j \in K$ such that $F_j(\hat{x}_j) \cap P(\hat{x}_j) = \emptyset$ and $\hat{x}_j \in \overline{F}_j(\hat{x}_j)$. Since $F(\hat{x}_j) \subset F_j(\hat{x}_j)$, it follows that $F(\hat{x}_j) \cap P(\hat{x}_j) = \emptyset$. Note that $\{\hat{x}_j\}_{j \in J}$ is a net in the compact set K; without loss of generality, we may assume that x_j converges to some point $x^* \in K$. By properties (2) and (4) of the upper approximating family defined above, it follows that $x_j^* \in \overline{F}_j(x^*)$ for each $j \in J$ as the graph of $\overline{F}_j$ is closed. By condition (iii), as $F(\hat{x}_j) \cap P(\hat{x}_j) = \emptyset$ for all $j \in J$, i.e., $\hat{x}_j \notin E$ for all $j \in J$ and E is open, it follows that $x^* \notin E$, thus we have that $F(x^*) \cap P(x^*) = \emptyset$. Therefore the proof of Theorem 3.2 is complete. ∎

As a special case of Theorem 3.3, we have the following result which includes corresponding results in the literature as a special case.

THEOREM 3.4. *Let the strategic set X be a nonempty compact and convex set in a Hausdorff locally topological vector space and let $F, P: X \to 2^X$ be two set-valued mappings such that*

(i) *the constraint mapping F is upper semicontinuous with nonempty closed and convex values*;

(ii) *the preference mapping P is $\mathscr{L}$-majorized*;

(iii) *the domain of the mapping $F \cap P$ is open in X.*

Then there exists a point $\hat{x} \in K$ such that $\hat{x} \in F(\hat{x})$ and $F(\hat{x}) \cap P(\hat{x}) = \varnothing$.

Our goal is to establish the general existence of equilibria for any number (countable or uncountable) players (resp., agents) without lower semicontinuity; as an application of Theorem 2.3 we first have the following existence of equilibria for generalized games (resp., abstract economy) in topological vector spaces in which the constraint mapping A_i has compactly open inverse for each $i \in I$.

THEOREM 3.5. *Let $\mathscr{G} = (X_i, A_i, B_i, P_i)_{i \in I}$ be a generalized game (resp., abstract economy), where I is any countable or uncountable set of players. Suppose that the following conditions are satisfied for each $i \in I$:*

(i) *the strategic set X_i is a nonempty convex subset of a topological vector space*;

(ii) *for each $x \in X$, the constraint mapping $A_i(x)$ is nonempty and* $\operatorname{co} A_i(x) \subset B(x)$, *and A_i has compactly open inverse*; *i.e., the set $A_i^{-1}(y) = \{x \in X: y \in A_i(x)\}$ is compactly open in X for each $y \in X_i$*;

(iii) *the preference mapping $P_i: X \to 2^{X_i}$ is such that the mapping $A_i \cap P_i: X \to 2^{X_i}$ is $\mathscr{L}$-majorized*;

(iv) *the set $E_i = \{x \in X: A_i \cap P_i(x) \neq \varnothing\}$ is open in X*;

(v) *the family $\{A_i \cap P_i\}_{i \in I}$ satisfies the noncompact condition*; *i.e., there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that, for each $y \in X \setminus K$, there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ with $x_i \in \operatorname{co}(A_i(y \cap P_i(y))$ for all $i \in I$.*

Then there exists $x \in X$ such that $x_i \in \overline{B}_i(x)$ and $A_i(x) \cap P_i(x) = \varnothing$ for each $i \in I$.

Proof. For each $i \in I$, let $F_i = \{x \in X\colon x_i \in \overline{B}_i(x)\}$. Then the set F_i is closed in X. We now define $Q_i\colon X \to 2^{X_i}$ by

$$Q_i(x) = \begin{cases} (A_i \cap P_i)(x_i), & \text{if } x \in F_i, \\ \operatorname{co} A_i(x), & \text{if } x \notin F_i. \end{cases}$$

By following the same proof as that of Theorem 3 from Ding and Tan [5] and conditions (ii)–(v), we can show that the qualitative game $\Gamma = (X_i, Q_i)_{i \in I}$ satisfies the following properties: (a) the preference mapping $Q_i\colon X \to 2^{X_i}$ is $\mathscr{L}$-majorized; (b) $\bigcup_{i \in I}\{x \in X\colon P_i(x) \neq \emptyset\} = \bigcup_{i \in I} \operatorname{int}\{x \in X\colon P_i(x) \neq \emptyset\}$; and (c) the family $\{Q_i\}_{i \in I}$ satisfies the noncompact Condition (C); i.e., there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that, for each $y \in X \setminus K$, there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ with $x_i \in \operatorname{co} P_i(y)$ for all $i \in I$.

Second, for each $x \in X$, let $I(x) := \{i \in I\colon Q_i(x) \neq \emptyset\}$ and we define a set-valued mapping $P\colon X \to 2^X$ by

$$Q(x) = \begin{cases} \bigcap_{i \in I(x)} Q_i'(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{otherwise}; \end{cases}$$

where $Q_i'(x) := \prod_{j \neq i,\, j \in I} X_j \otimes Q_i(x)$ for each $x \in X$. By employing arguments similar to those used in the proof of Theorem 3 in Ding and Tan [5], we can also show that the mapping Q is $\mathscr{L}$-majorized by using properties (a) and (b) of the qualitative game $\Gamma = (X_i, Q_i)_{i \in I}$ above. By the noncompact condition, it follows that all hypotheses of Theorem 2.3 are satisfied. By Theorem 2.3, it follows that there exists a point $x \in K$ such that $Q(x) = \emptyset$, which implies that $Q_i(x) = \emptyset$ for all $i \in I$. Since, for each $x \in X$, $A_i(x) \neq \emptyset$, therefore we must have that $x_i \in \overline{B}_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$. ∎

In what follows, by Tulcea's approximation theorems for upper semicontinuous set-valued mappings in locally convex topological vector spaces, we will establish one general existence theorem of equilibria for abstract economies in which the constraint mappings are upper semicontinuous in locally convex topological vector spaces. Our result generalizes many results in the existing literature by relaxing the compactness of strategy spaces and the openness of graphs or open lower or open upper sections of constraint correspondences. For example, the following Theorems 3.6 and 3.7 both generalize Corollary 3 of Borglin and Keiding [3, p. 315], the Theorem of Shafer and Sonnenschein [14, p. 374], and the corresponding results given by Zheng [25] and Zhou [26] and related references therein.

Now we have the following main existence theorem for equilibria without the assumption on lower semicontinuity for the preference mappings and in which the constraint mappings are only upper semicontinuous.

THEOREM 3.6. *Let $\Gamma = (X_i; F_i; P_i)_{i \in I}$ be a generalized game such that X is paracompact and each X_i has property* (K), *where I may be an uncountable set of players. Suppose that the following conditions are satisfied for each $i \in I$:*

(i) *the strategic set X_i is a nonempty closed convex set in a Hausdorff locally topological vector space*;

(ii) *the constraint mapping F_i is compact and upper semicontinuous with nonempty closed and convex values*;

(iii) *the preference mapping P_i: $X \to 2^{X_i}$ is such that the mapping $F_i \cap P_i$ is $\mathscr{L}$-majorized*;

(iv) *the set $E_i = \{x \in X: F_i(x) \cap P_i(x) \neq \emptyset\}$ is open and paracompact in X*;

(v) *there exist a nonempty compact and convex subset X_0 and a nonempty compact subset K of X such that, for each $x \in X \setminus K$, there exists $y = (y_i)_{i \in I} \in \operatorname{co}(X_0 \cup \{x\})$ such that $y_i \in \bigcap \operatorname{co}(F_i(x) \cap P_i(x))$ for all $i \in I$.*

Then Γ has an equilibrium $\hat{x}$ in K; i.e., there exists $\hat{x} \in K$ such that $\hat{x}_i \in F_i(\hat{x})$ and $F_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for all $i \in I$.

Proof. By Lemma 3.1, there is a common filtering set J such that, for every $i \in I$, there exists a family $(F_{ij})_{j \in J}$ of regular correspondences between X and X_i, such that both $(F_{ij})_{j \in J}$ and $(\overline{F_{ij}})_{j \in J}$ are upper approximating families for F_i. Note that E_i is paracompact and $F_i \cap P_i$ is $\mathscr{L}$-majorized on E_i from E_i to X_i. By Theorem 2.1, it follows that there exists an $\mathscr{L}$-class mapping ϕ_i: $E_i \to 2^{X_i}$ such that (a) $(F_i \cap P_i)(x) \subset \phi_i(x)$ and $x_i \notin \operatorname{co} \phi_i(x)$ for all $x \in X$ and (b) $\phi_i^{-1}(y)$ is compactly open in X for each $y \in X_i$. For each $j \in J$, define a set-valued mapping ϕ_{ij}: $X \to 2^{X_i}$ by

$$\phi_{ij}(x) = F_{ij}(x) \cap \phi_i(x)$$

for each $x \in X$. Note that F_{ij} is regular; it follows that F_{ij} has an open graph and thus the mapping ϕ_{ij} is of class $\mathscr{L}$. Second, for each $i \in I$, the set $\{x \in X: F_{ij}(x) \cap \phi_{ij}(x) \neq \emptyset\} = \bigcup_{y \in X_i} F_{ij}^{-1}(y) \cap \phi_i^{-1}(y)$, which is open in X. Therefore the game $\Gamma_j = (X_i; F_{ij}, \overline{F_{ij}}, \phi_{ij})_{i \in I}$ satisfies all hypotheses of Theorem 3.5. It follows by Theorem 3.5 that Γ_j has an equilibrium $\hat{x}^j \in K$ such that $F_{ij}(\hat{x}^j) \cap \phi_i(\hat{x}^j) = \emptyset$, and $\pi_i(\hat{x}^j) \in \overline{F_{ij}}(\hat{x}^j)$ for all $i \in I$. Since $F_i(\hat{x}^j) \subset F_{ij}(\hat{x}^j)$, it follows that $F_i(\hat{x}^j) \cap \phi_i(\hat{x}^j) = \emptyset$.

Therefore $\{x^j\}_{j\in J} \subset E_i^c$, where the set $E_i^c = \{x \in X: x \notin E_i\}$ is closed in X by condition (iv). On the other hand, note that $(\hat{x}^j)_{j\in J}$ is a net in the compact set K; without loss of generality, we may assume that $(\hat{x}^j)_{j\in J}$ converges to $x^* \in K$. Then, for each $i \in I$, $x_i^* = \lim_{j\in J} \hat{x}_i^j$. As $x^* \in E_i^c$ for all $i \in I$, it follows that $F_i(x^*) \cap P_i(x^*) = \emptyset$. Since $\hat{x}^j$ is an equilibrium point of Γ_j and $\overline{F_{ij}}$ is regular, for each $x \in X$, $\operatorname{cl} F_{ij}(x) = \overline{F_{ij}}(x)$, therefore $(\hat{x}_i^j) \in \operatorname{cl}(B_{ij}(\hat{x}^j)) = \overline{F_{ij}}(\hat{x}^j)$. As $\overline{F_{ij}}$ has a closed graph, it follows that $(x^*, x_i^*) \in \operatorname{Graph} \overline{F_{ij}}$ for every $i \in I$. For each $i \in I$, since $\operatorname{cl}(F_{ij})_{j\in J}$ is an upper approximation family for F_i, it follows that $\bigcap_{j\in J} \overline{F_{ij}}(x) \subset \overline{F}_i(x)$ for each $x \in X$ so that $(x^*, x_i^*) \in \operatorname{Graph} \overline{F}_i$. Therefore, for each $i \in I$, $F_i(x^*) \cap P_i(x^*) = \emptyset$ and $\pi_i(x^*) \in \overline{F}_i(x^*)$ and this completes the proof. ∎

Remark 3.1. Due to the fact that each open subset of a perfectly normal and paracompact set is paracompact by Theorem 5.1.28 of Engelking [7, p. 308], as a special case of Theorem 3.6, we have the following result.

THEOREM 3.7. *Let $\Gamma = (X_i; F_i; P_i)_{i\in I}$ be a generalized game such that X is a perfectly normal and paracompact set and each X_i has property* (K), *where I may be any countable or uncountable set of players. Suppose that the following conditions are satisfied for each $i \in I$*:

(i) *the strategic set X_i is a nonempty closed convex set in a Hausdorff locally convex topological vector space*;

(ii) *the constraint mapping F_i is compact and upper semicontinuous with nonempty closed and convex values*;

(iii) *the mapping $F_i \cap P_i$ is $\mathscr{L}$-majorized*;

(iv) *the set $E_i = \{x \in X: F_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X*;

(v) *there exist a nonempty compact and convex subset X_0 and a nonempty compact subset K of X such that, for each $x \in X \setminus K$, there exists $y = (y_i)_{i\in I} \in \operatorname{co}(X_0 \cup \{x\})$ such that $y_i \in \bigcap \operatorname{co}(F_i(x) \cap P_i(x))$ for all $i \in I$.*

Then Γ has an equilibrium $\hat{x}$ in K; i.e., there exists $\hat{x} \in K$ such that $\hat{x}_i \in F_i(\hat{x})$ and $F_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for all $i \in I$.

Proof. By condition (iv), the set E_i is open in X for all $i \in I$. Note that X is perfectly normal paracompact; it follows that E_i is also paracompact. Thus all hypotheses of Theorem 3.6 are satisfied. By Theorem 3.6, the conclusion follows. ∎

Note that each metrizable space is perfectly normal and paracompact by Theorem 4.1.13 of Engelking [7, p. 254], as an application of either Theorem 3.6 or Theorem 3.7, we have the following.

THEOREM 3.8. *Let $\Gamma = (X_i; F_i; P_i)_{i \in I}$ be a generalized game such that X is metrizable and each X_i has property* (K). *Suppose that the following conditions are satisfied for each $i \in I$*:

(i) *the strategic set X_i is a nonempty closed convex set* (*indeed, it is metrizable*) *in a Hausdorff locally topological vector space*;

(ii) *the constraint mapping F_i is compact and upper semicontinuous with nonempty closed and convex values*;

(iii) *the mapping $F_i \cap P_i$ is $\mathscr{L}$-majorized*;

(iv) *the set $E_i = \{x \in X: F_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X*;

(v) *there exist a nonempty compact and convex subset X_0 and a nonempty compact subset K of X such that, for each $x \in X \setminus K$, there exists $y = (y_i)_{i \in I} \in \mathrm{co}(X_0 \cup \{x\})$ such that $y_i \in \bigcap \mathrm{co}(F_i(x) \cap P_i(x))$ for all $i \in I$.*

Then Γ has an equilibrium $\hat{x}$ in K such that $\hat{x}_i \in F_i(\hat{x})$ and $F_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for all $i \in I$.

Remark 3.2. We first note that the assumption X is metrizable in Theorem 3.8 is equivalent to saying that "the index set I is countable and X_i is metrizable for each $i \in I$."

In a recent paper by Zhou [26], he also proves some existence theorems of equilibria for generalized games by assuming that constraint mappings have upper semicontinuity plus they also satisfy so-called ϵ-CS and (CS)-properties which look somehow like variant versions of fixed point free property for set-valued mappings; and in this way, a further generalization, the so-called (LSC)-property, has been introduced by Zheng [25]. In general, these properties are not easy to verify. However, our Theorems 3.7 and 3.8 generalize the corresponding Corollary 1 of Zhou [26] in the following ways: that the strategic sets may not be compact and the set of players (resp., agents) may not be countable.

As compact versions of both Theorem 3.6 and Theorem 3.8, we have the following corollaries.

COROLLARY 3.9. *Let $\Gamma = (X_i; F_i; P_i)_{i \in I}$ be a generalized game, where I may be an uncountable set of players. Suppose that the following conditions are satisfied for each $i \in I$*:

(i) *the strategic set X_i is a nonempty compact and convex set in a Hausdorff locally topological vector space*;

(ii) *the constraint mapping F_i is upper semicontinuous with nonempty closed and convex values*;

(iii) *the mapping $F_i \cap P_i$ is $\mathscr{L}$-majorized*;

(iv) *the set $E_i = \{x \in X: F_i(x) \cap P_i(x) \neq \emptyset\}$ is open and paracompact in X.*

Then Γ *has an equilibrium* $\hat{x}$ *in* K *such that* $\hat{x}_i \in F_i(\hat{x})$ *and* $F_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ *for all* $i \in I$.

Corollary 3.9 generalizes Corollary 1 of Zhou [26] to a case that the set of players (resp., agents) may not be countable.

COROLLARY 3.10. *Let* $\Gamma = (X_i; F_i; P_i)_{i \in I}$ *be a generalized game, where the* I *is a countable set of players. Suppose that the following conditions are satisfied for each* $i \in I$:

(i) *the strategic set* X_i *is a metrizable compact and convex set in a Hausdorff locally topological vector space*;

(ii) *the constraint mapping* F_i *is upper semicontinuous with nonempty closed and convex values*;

(iii) *the mapping* $F_i \cap P_i$ *is* $\mathscr{L}$*-majorized*;

(iv) *the set* $E_i = \{x \in X: F_i(x) \cap P_i(x) \neq \emptyset\}$ *is open in* X.

Then Γ *has an equilibrium* $\hat{x}$ *in* K *such that* $\hat{x}_i \in F_i(\hat{x})$ *and* $F_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ *for all* $i \in I$.

Finally, as a special case of Theorem 3.6, we have the following famous Fan−Glicksberg fixed point theorem.

COROLLARY 3.11. *Let* X *be a nonempty compact convex set in a Hausdorff locally topological vector space and let* $F: X \to 2^X$ *be an upper semicontinuous set-valued mapping with nonempty closed values. Then* F *has at least one fixed point in* X.

Proof. Define $P: X \to 2^X$ by $P(x) = \emptyset$ for each $x \in X$. Then the conclusion follows from Theorem 3.6. ∎

Finally we note that Theorems 3.6 and 3.7 show that the corresponding existence results of equilibria for generalized games due to Zhou [26] (e.g., see Corollary 1) still hold when the set of players (resp., agents) is uncountable instead of countable and the strategic (resp., commodity) spaces may not be compact; however, we need one extra hypothesis, so-called paracompactness (of course this hypothesis is automatically satisfied when the space X is metrizable), but we do not know whether the conclusion of Theorem 3.6 (resp., Theorem 3.7) still holds without the paracompact hypothesis on the set E_i for each $i \in I$ in condition (iv).

We also note that by using Michael type continuous selection theorems and their generalized versions plus the Himmelberg fixed point theorem in [10], which is a different approach compared with approximate method used in this paper and by Tulcea [18] and Tan and Yuan [15], some existence of equilibria for generalized games (resp., abstract economics) have been established by Kim and Lee [12], Wu [19], Wu and Shen [20], Zheng [25], and Zhou [26]; however, the shortage or their methods is that, in general, it needs to assume that either constraint and/or preference mappings have lower semicontinuity or even stronger hypotheses such as having lower open sections.

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